Isoparametric Elements

Introduction

In this chapter, we introduce the isoparametric formulation of the element stiffness matrices.

After considering the linear-strain triangular element (LST) in Chapter 8, we can see that the development of element matrices and equations expressed in terms of a global coordinate system becomes an enormously difficult task (if even possible) except for the simplest of elements such as the constant-strain triangle of Chapter 6.

Hence, the isoparametric formulation was developed.

Isoparametric Elements

Introduction

The isoparametric method may appear somewhat tedious (and confusing initially), but it will lead to a simple computer program formulation, and it is generally applicable for two- and three-dimensional stress analysis and for nonstructural problems.

The isoparametric formulation allows elements to he created that are nonrectangular and have curved sides.

Numerous commercial computer programs (as described in Chapter 1) have adapted this formulation for their various libraries of elements.

Isoparametric Elements

Introduction

First, we will illustrate the isoparametric formulation to develop the simple bar element stiffness matrix.

Use of the bar element makes it relatively easy to understand the method because simple expressions result.

Then, we will consider the development of the isoparametric formulation of the simple quadrilateral element stiffness matrix.

Isoparametric Elements

Introduction

Next, we will introduce numerical integration methods for evaluating the quadrilateral element stiffness matrix.

Then, we will illustrate the adaptability of the isoparametric formulation to common numerical integration methods.

Finally, we will consider some higher-order elements and their associated shape functions.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

The term *isoparametric* is derived from the use of the same shape functions (or interpolation functions) [N] to define the element's geometric shape as are used to define the displacements within the element.

Thus, when the interpolation function is $u = a_1 + a_2 s$ for the displacement, we use $x = a_1 + a_2 s$ for the description of the nodal coordinate of a point on the bar element and, hence, the physical shape of the element.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Isoparametric element equations are formulated using a natural (or intrinsic) coordinate system ***s*** that is defined by element geometry and not by the element orientation in the global-coordinate system.

In other words, axial coordinate *s* is attached to the bar and remains directed along the axial length of the bar, regardless of how the bar is oriented in space.

There is a relationship (called a *transformation mapping)* between the natural coordinate systems and the global coordinate system *x* for each element of a specific structure.

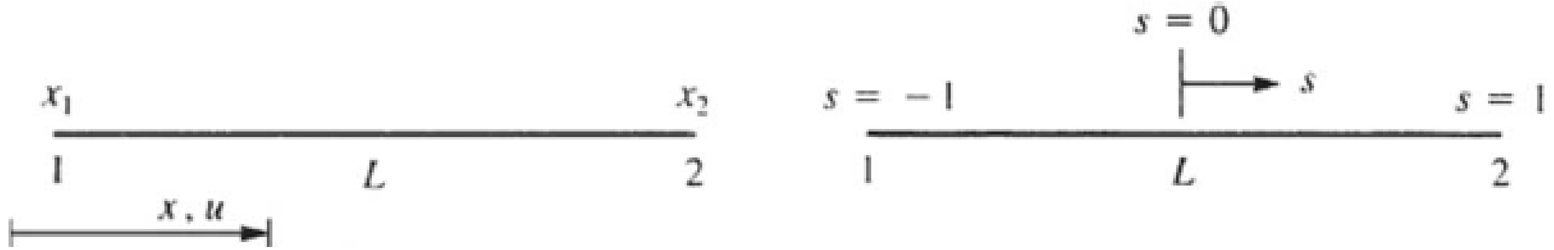
Isoparametric Elements

Isoparametric Formulation of the Bar Element

First, the natural coordinate ***s*** is attached to the element, with the origin located at the center of the element.

The ***s*** axis need not be parallel to the *x* axis-this is only for convenience.

Consider the bar element to have two degrees of freedom-axial displacements u_1 and u_2 at each node associated with the global *x* axis.



Isoparametric Elements

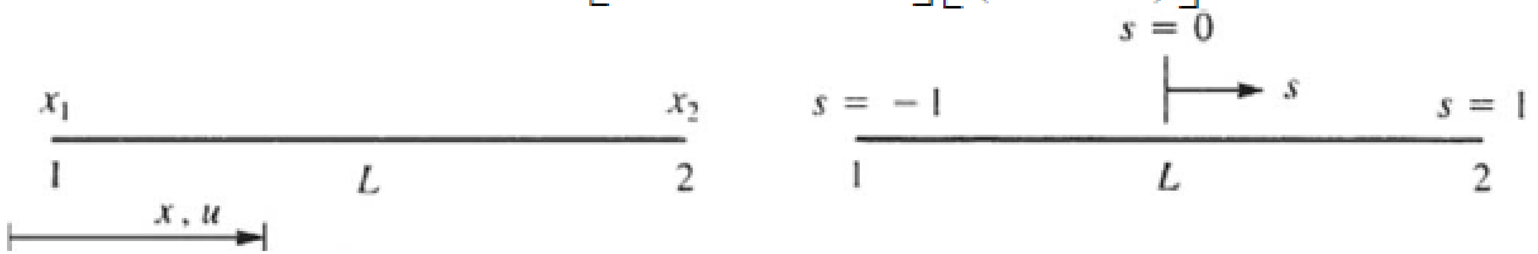
Isoparametric Formulation of the Bar Element

For the special case when the *s* and *x* axes are parallel to each other, the *s* and *x* coordinates can be related by:

$$x = x_c + \frac{L}{2}s$$

Using the global coordinates x_1 and x_2 with $x_c = (x_1 + x_2)/2$, we can express the natural coordinate *s* in terms of the global coordinates as:

$$s = \left[\frac{x - (x_1 + x_2)}{2}\right]\left[\frac{2}{(x_2 - x_1)}\right]$$



Isoparametric Elements

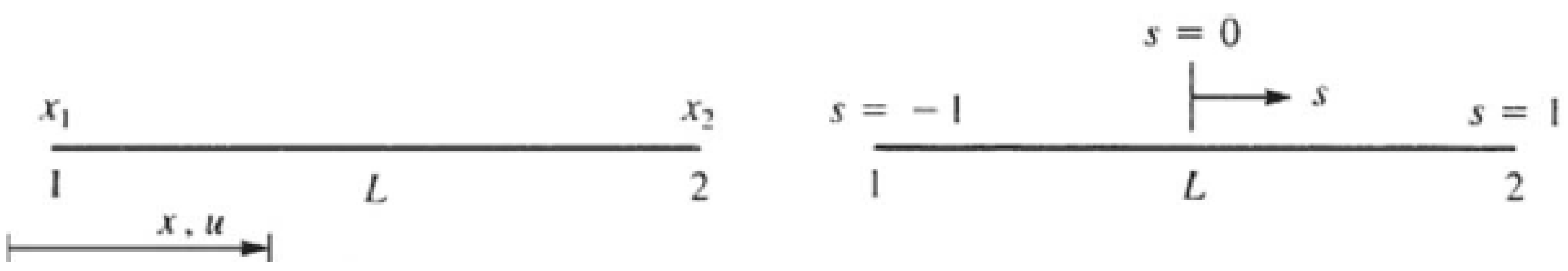
Isoparametric Formulation of the Bar Element

The shape functions used to define a position within the bar are found in a manner similar to that used in Chapter 3 to define displacement within a bar (Section 3.1).

We begin by relating the natural coordinate to the global coordinate by:

$$x = a_1 + a_2 s$$

Note that $-1 \leq s \leq 1$.



Isoparametric Elements

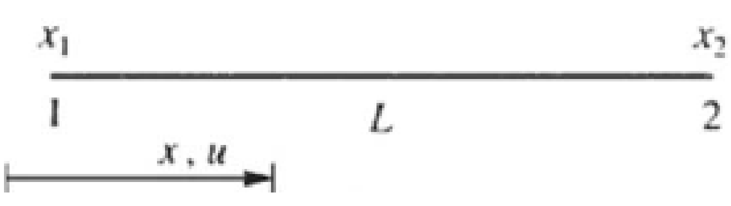
Isoparametric Formulation of the Bar Element

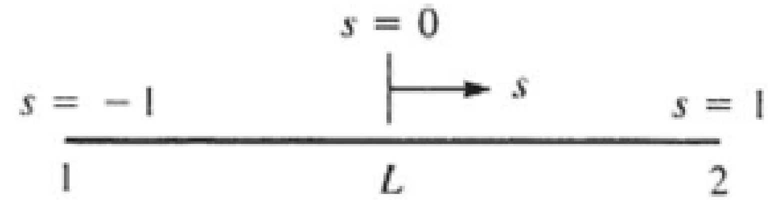
Solving for the *a's* in terms of x_1 and x_2, we obtain:

$$x = \left(\frac{1}{2}\right)\left[(1-s)x_1 + (1+s)x_2\right]$$

In matrix form:

$$\{x\} = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$





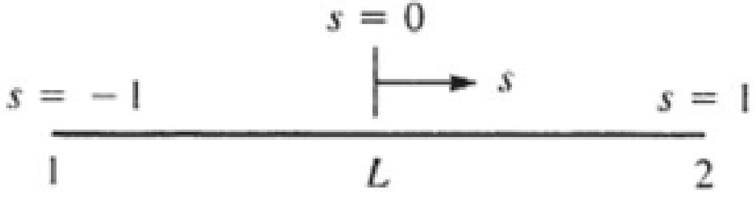
Isoparametric Elements

Isoparametric Formulation of the Bar Element

The linear shape functions map the *s* coordinate of any point in the element to the *x* coordinate.

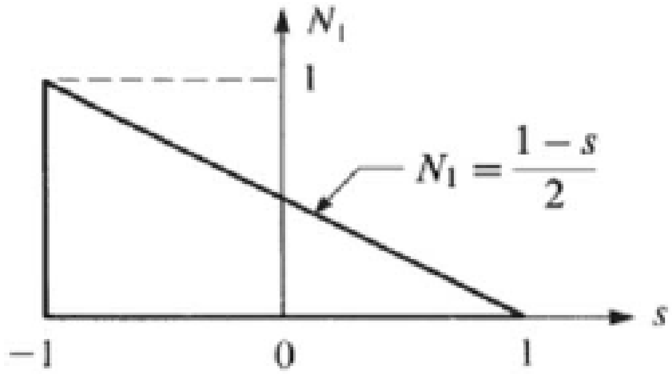
For instance, when $s = -1$, then $x = x_1$ and when $s = 1$, then $x = x_2$

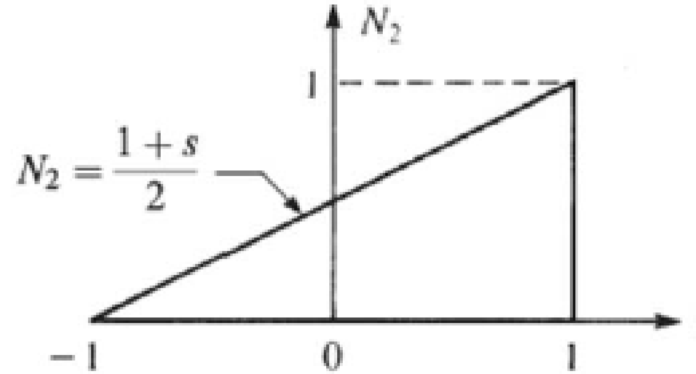
$$\{x\} = [N_1 \quad N_2] \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$



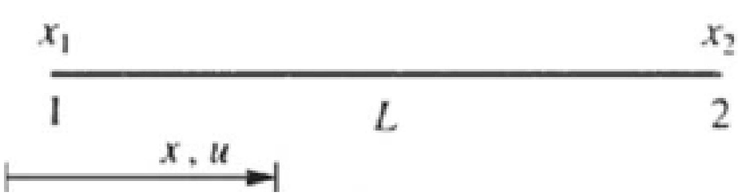
Isoparametric Elements

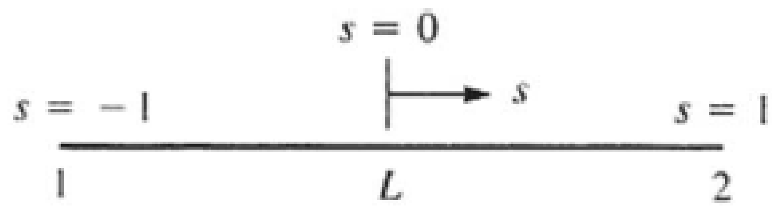
Isoparametric Formulation of the Bar Element





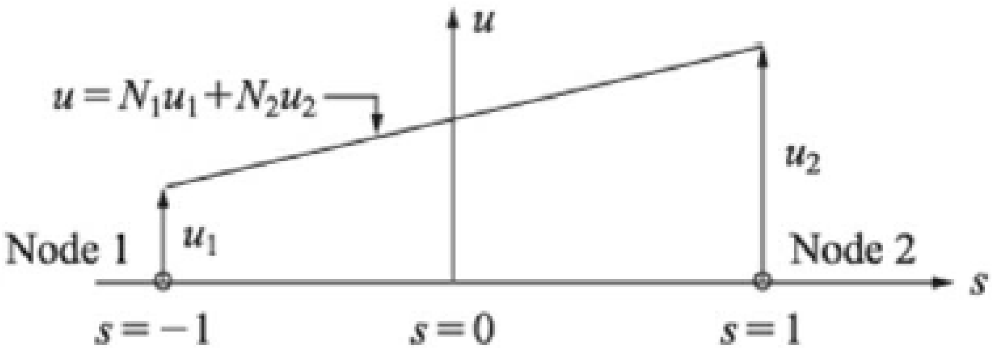
$$\{x\} = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$



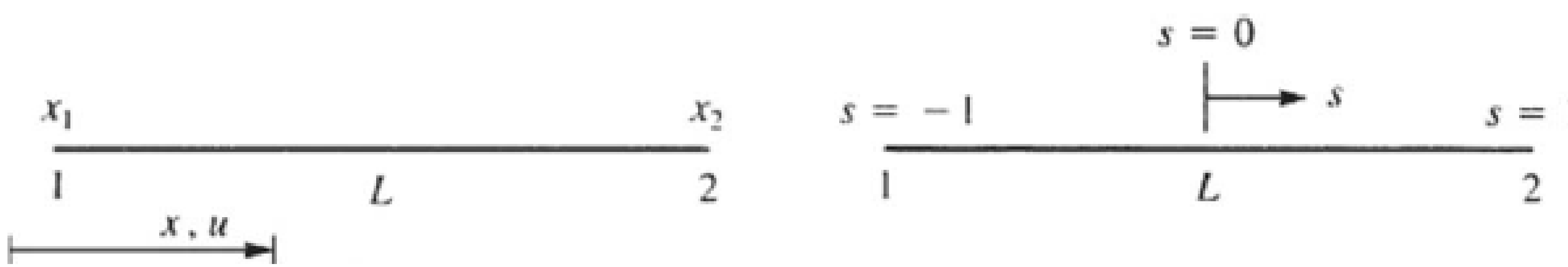


Isoparametric Elements

Isoparametric Formulation of the Bar Element



$$\{x\} = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$



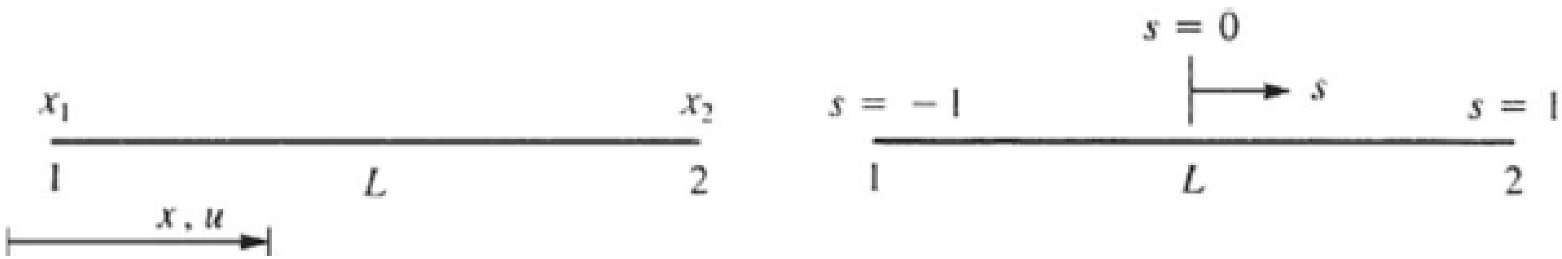
Isoparametric Elements

Isoparametric Formulation of the Bar Element

When a particular coordinate *s* is substituted into [N] yields the displacement of a point on the bar in terms of the nodal degrees of freedom u_1 and u_2.

Since *u* and *x* are defined by the same shape functions at the same nodes, the element is called *isoparametric.*

$$\{x\} = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$



Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 3 - Strain-Displacement and Stress-Strain Relationships

We now want to formulate element matrix [B] to evaluate [k].

We use the isoparametric formulation to illustrate its manipulations.

For a simple bar element, no real advantage may appear evident.

However, for higher-order elements, the advantage will become clear because relatively simple computer program formulations will result.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 3 - Strain-Displacement and Stress-Strain Relationships

To construct the element stiffness matrix, determine the strain, which is defined in terms of the derivative of the displacement with respect to *x*.

The displacement *u,* however, is now a function of *s* so we must apply the chain rule of differentiation to the function *u* as follows:

$$\frac{du}{ds} = \frac{du}{dx}\frac{dx}{ds} \qquad \varepsilon_x = \frac{du}{dx} \quad \Rightarrow \quad \varepsilon_x = \frac{du}{dx} = \frac{du}{ds} \Big/ \frac{dx}{ds}$$

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 3 - Strain-Displacement and Stress-Strain Relationships

The derivative of *u* with respect to *s* is: $\dfrac{du}{ds} = \dfrac{u_2 - u_1}{2}$

The derivative of *x* with respect to *s* is: $\dfrac{dx}{ds} = \dfrac{x_2 - x_1}{2} = \dfrac{L}{2}$

Therefore the strain is: $\{\varepsilon_x\} = \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

Since $\{\varepsilon\} = [B]\{d\}$, the strain-displacement matrix $[B]$ is:

$$[B] = \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 3 - Strain-Displacement and Stress-Strain Relationships

Recall that use of linear shape functions results in a constant [*B*] matrix, and hence, in a constant strain within the element.

For higher-order elements, such as the quadratic bar with three nodes, [*B*] becomes a function of natural coordinates *s*.

The stress matrix is again given by Hooke's law as:

$$\{\sigma\} = E\{\varepsilon\} = E[B]\{d\}$$

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

The stiffness matrix is: $[k] = \int_0^L [B]^T E [B] A\,dx$

However, in general, we must transform the coordinate *x* to *s* because [*B*] is, in general, a function of *s*.

$$\int_0^L f(x)\,dx = \int_{-1}^{1} f(s)\left|[J]\right|\,ds$$

where [*J*] is called the ***Jacobian*** matrix.

In the one-dimensional case, we have |[*J*]| = *J*.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

For the simple bar element: $\left|[J]\right| = \dfrac{dx}{ds} = \dfrac{L}{2}$

The Jacobian determinant relates an element length (dx) in the global-coordinate system to an element length (ds) in the natural-coordinate system.

In general, |[J]| is a function of s and depends on the numerical values of the nodal coordinates.

This can be seen by looking at for the equations for a quadrilateral element.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

The stiffness matrix in natural coordinates is:

$$[k] = \frac{L}{2}\int_{-1}^{1} [B]^T E [B] A \, ds$$

For the one-dimensional case, we have used the modulus of elasticity $E = [D]$.

Performing the simple integration, we obtain:

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Determine the body-force matrix using the natural coordinate system *s*. The body-force matrix is:

$$\{f_{\mathrm{b}}\} = \int_V [N]^{\mathrm{T}} \{X_{\mathrm{b}}\}\, dV \qquad \{f_{\mathrm{b}}\} = \int_V [N]^{\mathrm{T}} \{X_{\mathrm{b}}\}\, A\, dx$$

Substituting for N_1 and N_2 and using $dx = (L/2)ds$

$$\{f_{\mathrm{b}}\} = A\int_{-1}^{1} \begin{Bmatrix} \dfrac{1-s}{2} \\ \dfrac{1+s}{2} \end{Bmatrix} \{X_{\mathrm{b}}\} \frac{L}{2} ds = \frac{ALX_{\mathrm{b}}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

The physical interpretation of the results for $\{f_b\}$ is that since AL represents the volume of the element and X_b the body force per unit volume, then ALX_b is the total body force acting on the element.

The factor ½ indicates that this body force is equally distributed to the two nodes of the element.

$$\{f_b\} = A\int_{-1}^{1} \begin{Bmatrix} \dfrac{1-s}{2} \\ \dfrac{1+s}{2} \end{Bmatrix} \{X_b\} \frac{L}{2} ds = \frac{ALX_b}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Determine the surface-force matrix using the natural coordinate system *s*. The surface-force matrix is:

$$\{f_s\} = \int_S [N_s]^T \{T_x\} \, dS$$

Assuming the cross section is constant and the traction is uniform over the perimeter and along the length of the element, we obtain:

$$\{f_s\} = \int_0^L [N_s]^T \{T_x\} \, dx$$

where we now assume $\{T_x\}$ is in units of force per unit length.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Substituting for N_1 and N_2 and using $dx = (L/2)ds$

$$\{f_s\} = \int_{-1}^{1} \begin{Bmatrix} \dfrac{1-s}{2} \\ \dfrac{1+s}{2} \end{Bmatrix} \{T_x\} \frac{L}{2} ds \quad = \{T_x\} \frac{L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Since $\{T_x\}$ is in force-per-unit-length $\{T_x\}L$ is now the total force.

The ½ indicates that the uniform surface traction is equally distributed to the two nodes of the element.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Substituting for N_1 and N_2 and using $dx = (L/2)ds$

$$\{f_s\} = \int_{-1}^{1} \begin{Bmatrix} \frac{1-s}{2} \\ \frac{1+s}{2} \end{Bmatrix} \{T_x\} \frac{L}{2} ds \quad = \{T_x\} \frac{L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Since $\{T_x\}$ is in force-per-unit-length $\{T_x\}L$ is now the total force.

The ½ indicates that the uniform surface traction is equally distributed to the two nodes of the element.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Substituting for N_1 and N_2 and using $dx = (L/2)ds$

$$\{f_s\} = \int_{-1}^{1} \begin{Bmatrix} \dfrac{1-s}{2} \\ \dfrac{1+s}{2} \end{Bmatrix} \{T_x\} \frac{L}{2} ds \quad = \{T_x\} \frac{L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Note that if $\{T_x\}$ were a function of x (or s), then the amounts of force allocated to each node would generally not be equal and would be found through integration.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Recall that the term *isoparametric* is derived from the use of the same interpolation functions to define the element shape as are used to define the displacements within the element.

The approximation for displacement is:

$$u = a_1 + a_2 s + a_3 t + a_4 st$$

The description of a coordinate point in the plane element is:

$$x = a_1 + a_2 s + a_3 t + a_4 st$$

The natural-coordinate systems *s-t* defined by element geometry and not by the element orientation in the global-coordinate system *x-y*.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Much as in the bar element example, there is a transformation mapping between the two coordinate systems for each element of a specific structure, and this relationship must be used in the element formulation.

We will now formulate the isoparametric formulation of the simple linear plane quadrilateral element stiffness matrix.

This formulation is general enough to be applied to more complicated (higher-order) elements such as a quadratic plane element with three nodes along an edge, which can have straight or quadratic curved sides.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Higher-order elements have additional nodes and use different shape functions as compared to the linear element, but the steps in the development of the stiffness matrices are the same.

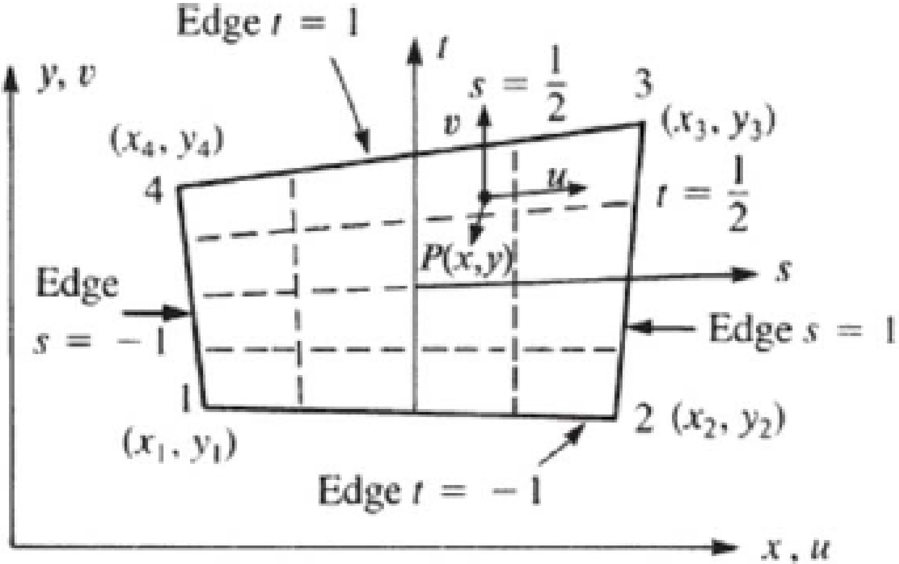
We will briefly discuss these elements after examining the linear plane element formulation.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

Consider the quadrilateral to have eight degrees of freedom u_1, v_1, ... , u_4, and v_4 associated with the global x and y directions. The element then has straight sides but is otherwise of arbitrary shape.

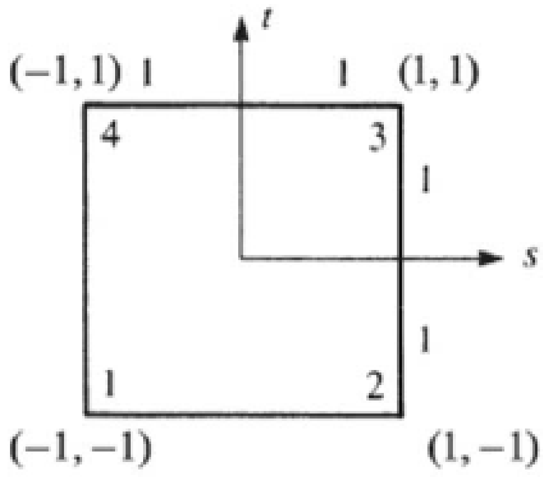


Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

The natural ***s-t*** coordinates are attached to the element, with the origin at the center of the element.



The ***s*** and ***t*** axes need not be orthogonal, and neither has to be parallel to the *x* or *y* axis.

The orientation of ***s-t*** coordinates is such that the four corner nodes and the edges of the quadrilateral are bounded by +1 or -1

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

Assuming global coordinates *x* and *y* are related to the natural coordinates *s* and *t* as follows:

$$x = a_1 + a_2 s + a_3 t + a_4 st \qquad y = a_5 + a_6 s + a_7 t + a_8 st$$

Solving for the *a*'s in terms of x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, we obtain

$$x = \frac{1}{4}\left[(1-s)(1-t)x_1 + (1+s)(1-t)x_2 + (1+s)(1+t)x_3 + (1-s)(1+t)x_4\right]$$

$$y = \frac{1}{4}\left[(1-s)(1-t)y_1 + (1+s)(1-t)y_2 + (1+s)(1+t)y_3 + (1-s)(1+t)y_4\right]$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

In matrix form:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

where:

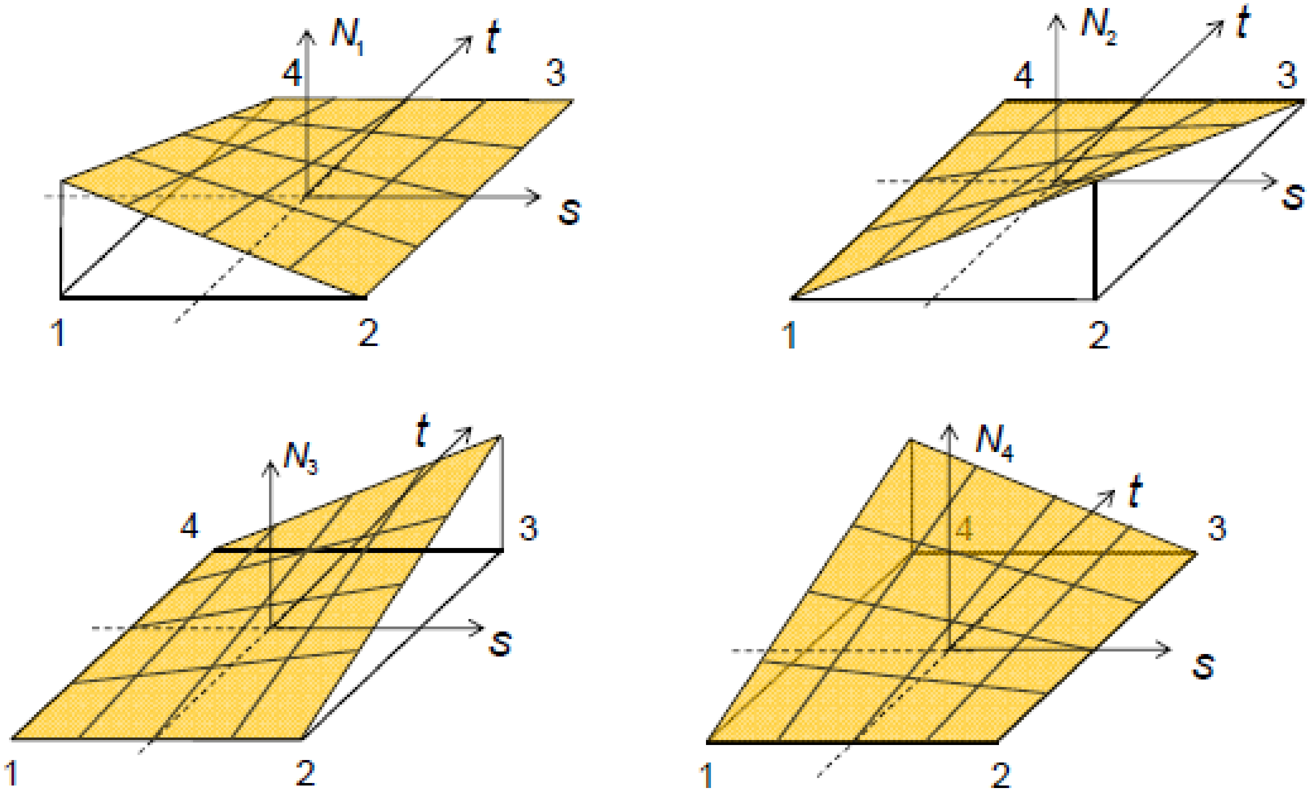
$$N_1 = \frac{(1-s)(1-t)}{4} \qquad N_2 = \frac{(1+s)(1-t)}{4}$$

$$N_3 = \frac{(1+s)(1+t)}{4} \qquad N_4 = \frac{(1-s)(1+t)}{4}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

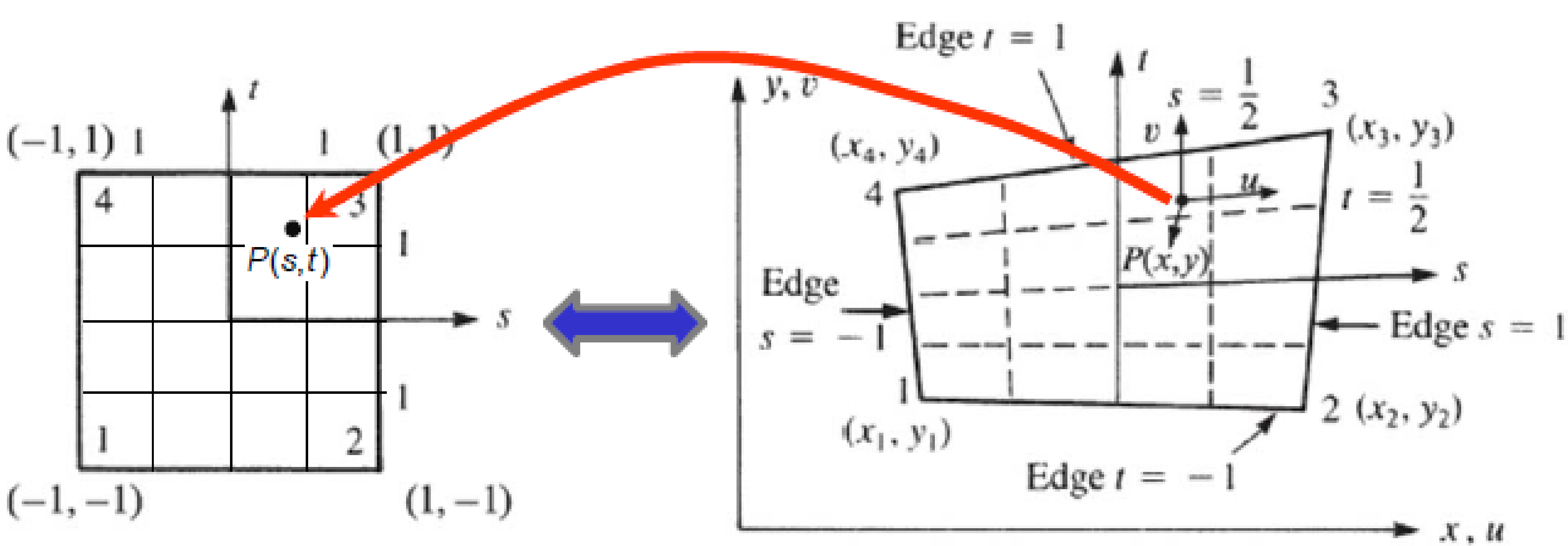


Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

These shape functions are seen to map the ***s*** and ***t*** coordinates of any point in the square element to those ***x*** and ***y*** coordinates in the quadrilateral element.

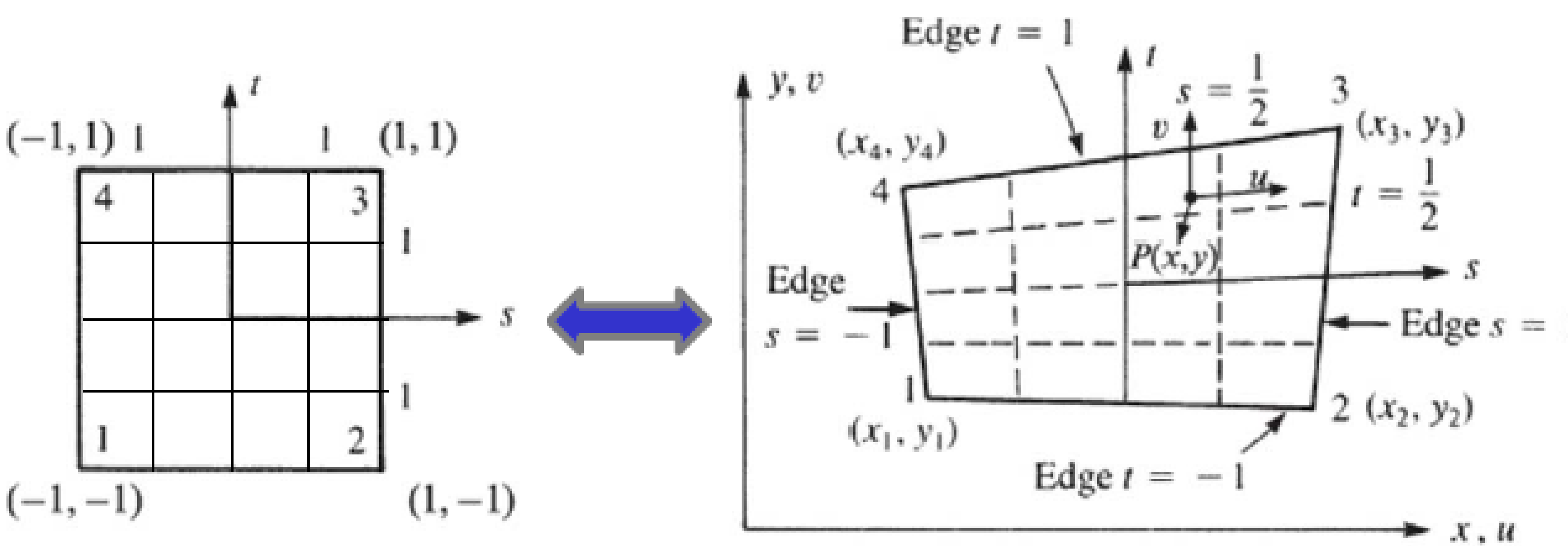


Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

Consider square element node 1 coordinates, where s = -1 and t = -1 then $x = x_1$ and $y = y_1$.

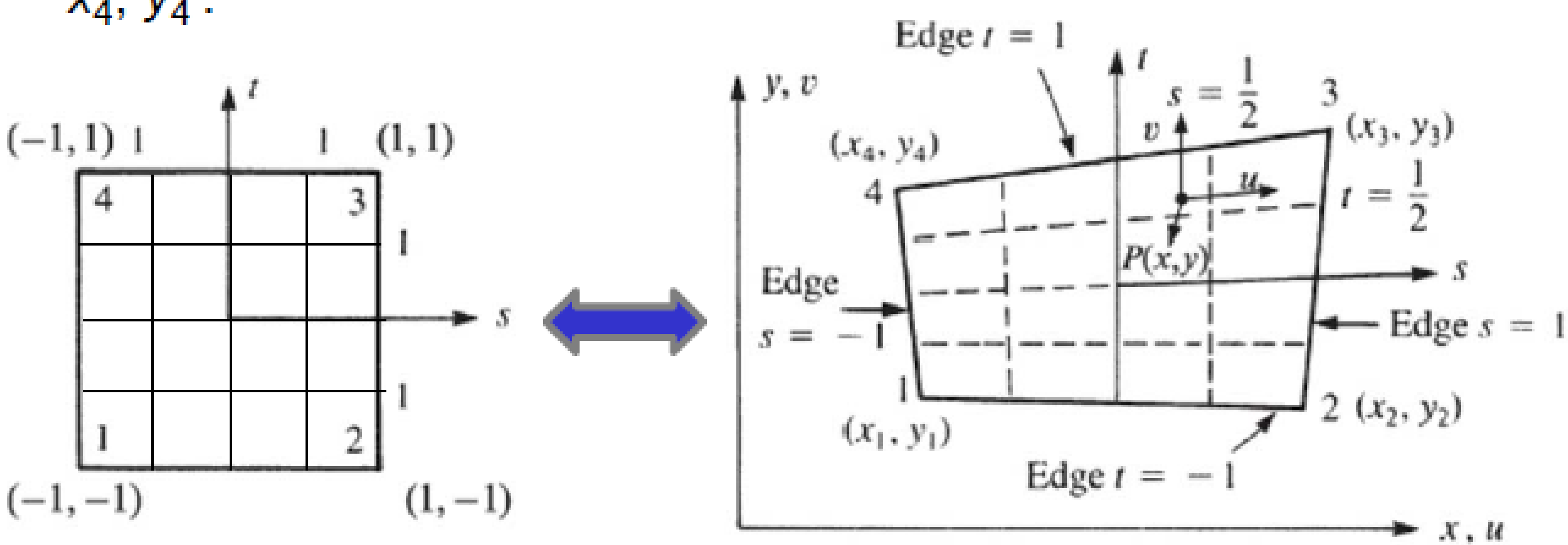


Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

Other local nodal coordinates at nodes 2, 3, and 4 on the square element in *s-t* isoparametric coordinates are mapped into a quadrilateral element in global coordinates x_1, y_1 through x_4, y_4 .

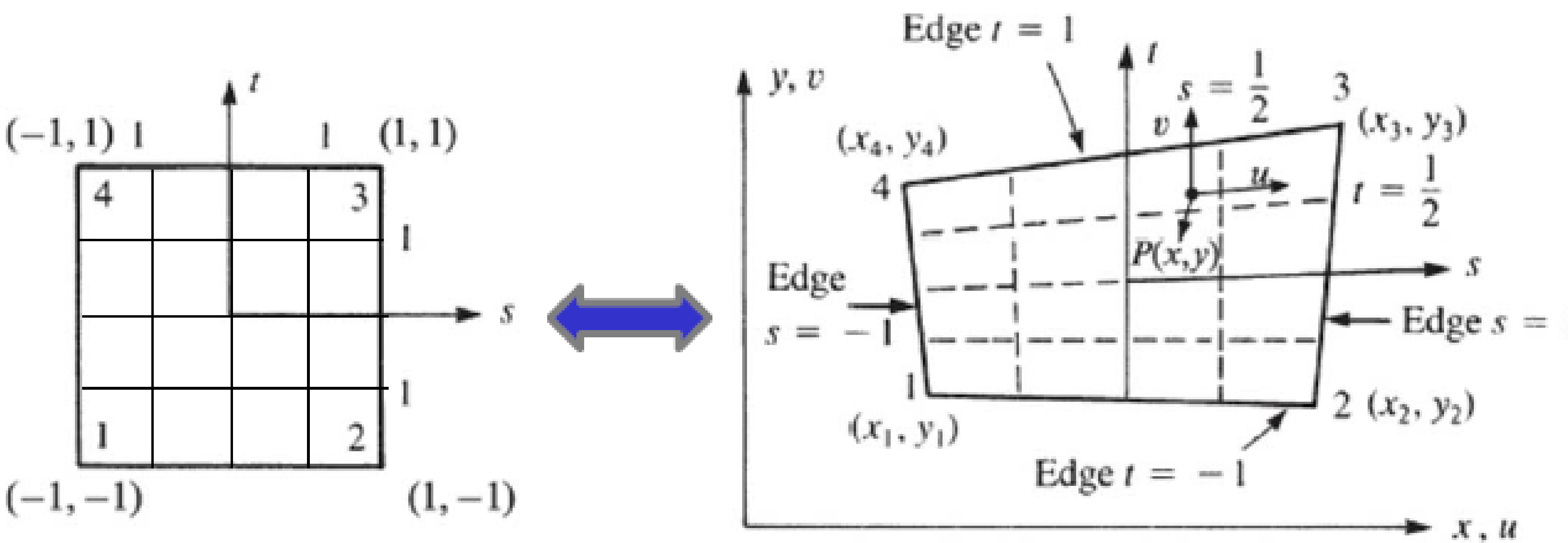


Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

Also observe the property that $N_1 + N_2 + N_3 + N_4 = 1$ for all values of s and t.



Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 2 Select of Displacement Functions

The displacement functions within an element are now similarly defined by the same shape functions as are used to define the element geometric shape:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

We now want to formulate element matrix [B] to evaluate [k].

However, because it becomes tedious and difficult (if not impossible) to write the shape functions in terms of the ***x*** and ***y*** coordinates, as seen in Chapter 8, we will carry out the formulation in terms of the isoparametric coordinates ***s*** and ***t***.

This may appear tedious, but it is easier to use the ***s***- and ***t***-coordinate expressions.

This approach also leads to a simple computer program formulation.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

The usual relationship between strains and displacements given in matrix form as:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial (\ )}{\partial x} & 0 \\ 0 & \dfrac{\partial (\ )}{\partial y} \\ \dfrac{\partial (\ )}{\partial y} & \dfrac{\partial (\ )}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

Where the rectangular matrix on the right side is an *operator matrix;* that is, $\partial(\ )/\partial$**x** and $\partial(\ )/\partial$**y** represent the partial derivatives of any variable we put inside the parentheses.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

To construct an element stiffness matrix, we must determine the strains, which are defined in terms of the derivatives of the displacements with respect to the ***x*** and ***y*** coordinates.

The displacements, however, are now functions of the ***s*** and ***t*** coordinates.

The derivatives $\partial u/\partial x$ and $\partial v/\partial y$ are now expressed in terms of ***s*** and ***t***.

Therefore, we need to apply the chain rule of differentiation.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

The chain rule yields:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

The strains can then be found; for example, $\varepsilon_x = \partial \boldsymbol{u}/\partial \boldsymbol{x}$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Using Cramer's rule, which involves the determinants of matrices, we can obtain:

$$\frac{\partial f}{\partial x} = \frac{\begin{vmatrix} \frac{\partial f}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}} \qquad \frac{\partial f}{\partial y} = \frac{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial f}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial f}{\partial t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

The determinant in the denominator is the determinant of the *Jacobian* matrix [J].

$$|J| = \begin{vmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{vmatrix}$$

We now want to express the element strains as: $\{\varepsilon\} = [B]\{d\}$

Where [B] must now be expressed as a function of ***s*** and ***t***.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Evaluating the determinant in the numerators, we have

$$\frac{\partial(\ )}{\partial x} = \frac{1}{|[J]|}\left[\frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t}\right]$$

$$\frac{\partial(\ )}{\partial y} = \frac{1}{|[J]|}\left[\frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s}\right]$$

Where |[***J***]| is the determinant of [***J***].

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

We can obtain the strains expressed in terms of the natural coordinates (***s-t***) as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|[J]|} \begin{bmatrix} \frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t} & 0 \\ 0 & \frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s} \\ \frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s} & \frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

We can express the previous equation in terms of the shape functions and global coordinates in compact matrix form as:

$$\{\varepsilon\} = [D'][N]\{d\}$$

$$[D'] = \frac{1}{\|[J]\|}\begin{Bmatrix} \dfrac{\partial y}{\partial t}\dfrac{\partial(\ )}{\partial s} - \dfrac{\partial y}{\partial s}\dfrac{\partial(\ )}{\partial t} & 0 \\ 0 & \dfrac{\partial x}{\partial s}\dfrac{\partial(\ )}{\partial t} - \dfrac{\partial x}{\partial t}\dfrac{\partial(\ )}{\partial s} \\ \dfrac{\partial x}{\partial s}\dfrac{\partial(\ )}{\partial t} - \dfrac{\partial x}{\partial t}\dfrac{\partial(\ )}{\partial s} & \dfrac{\partial y}{\partial t}\dfrac{\partial(\ )}{\partial s} - \dfrac{\partial y}{\partial s}\dfrac{\partial(\ )}{\partial t} \end{Bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

The shape function matrix [N] is the 2 x 8 {d} is the column matrix.

$$\underset{3\times 8}{[B]} = \underset{3\times 2}{[D']} \underset{2\times 8}{[N]}$$

The matrix multiplications yield

$$\left[B(s,t)\right] = \frac{1}{\left|[J]\right|}\left[[B_1] \quad [B_2] \quad [B_3] \quad [B_4]\right]$$

$$[B_i] = \begin{bmatrix} a(N_{i,s}) - b(N_{i,t}) & 0 \\ 0 & c(N_{i,t}) - d(N_{i,s}) \\ c(N_{i,t}) - d(N_{i,s}) & a(N_{i,s}) - b(N_{i,t}) \end{bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Here *i* is a dummy variable equal to 1, 2, 3, and 4, and

$$a = \frac{1}{4}\left[y_1(s-1) + y_2(-s-1) + y_3(1+s) + y_4(1-s)\right]$$

$$b = \frac{1}{4}\left[y_1(t-1) + y_2(1-t) + y_3(1+t) + y_4(-1-t)\right]$$

$$c = \frac{1}{4}\left[x_1(t-1) + x_2(1-t) + x_3(1+t) + x_4(-1-t)\right]$$

$$d = \frac{1}{4}\left[x_1(s-1) + x_2(-s-1) + x_3(1+s) + x_4(1-s)\right]$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Using the shape functions, we have

$$N_{1,s} = \frac{1}{4}(t-1) \qquad N_{1,t} = \frac{1}{4}(s-1)$$

where the comma followed by the variable *s* or *t* indicates differentiation with respect to that variable; that is, $N_{1,s} = \partial N_1/\partial \mathbf{s}$ and so on.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

The determinant |[J]| is a polynomial in *s* and *t* and is tedious to evaluate even for the simplest case of the linear plane quadrilateral element.

However, we can evaluate |[J]| as

$$|[J]| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\}$$

$$\{X_c\}^T = [x_1 \quad x_2 \quad x_3 \quad x_4] \qquad \{Y_c\}^T = [y_1 \quad y_2 \quad y_3 \quad y_4]$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

We observe that |[*J*]| is a function of ***s*** and ***t*** and the known global coordinates x_1, x_2, ... , y_4.

Hence, [**B**] is a function of ***s*** and ***t*** in both the numerator and the denominator and of the known global coordinates x_1 through y_4.

The stress-strain relationship is a function of ***s*** and ***t***.

$$\{\sigma\} = [D][B]\{d\}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 4 Derive the Element Stiffness Matrix and Equations

We now want to express the stiffness matrix in terms of ***s-t*** coordinates.

For an element with a constant thickness *h*, we have

$$[k] = \int_A \int [B]^T [D][B] \, h \, dx \, dy$$

However, [*B*] is now a function of ***s*** and ***t***, we must integrate with respect to ***s*** and ***t***.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 4 Derive the Element Stiffness Matrix and Equations

Once again, to transform the variables and the region from ***x*** and ***y*** to ***s*** and ***t***, we must have a standard procedure that involves the determinant of [*J*].

$$\int_A\int f(x,y)\,dx\,dy = \int_A\int f(s,t)\left|[J]\right|ds\,dt$$

where the inclusion of |[J]| in the integrand on the right side of equation results from a theorem of integral calculus.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 4 Derive the Element Stiffness Matrix and Equations

We also observe that the Jacobian (the determinant of the Jacobian matrix) relates an element area (*dx dy*) in the global coordinate system to an elemental area (*ds dt*) in the natural coordinate system.

For rectangles and parallelograms, *J* is the constant value *J* = *A*/4, where *A* represents the physical surface area of the element.

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\, h \left|[J]\right| ds\, dt$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 4 Derive the Element Stiffness Matrix and Equations

The $|[J]|$ and $[B]$ are complicated expressions within the integral.

Integration to determine the element stiffness matrix is usually done numerically.

The stiffness matrix is of the order 8 x 8.

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\, h \left|[J]\right| ds\, dt$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 4 Derive the Element Stiffness Matrix and Equations

Body Forces - The element body-force matrix will now be determined from

$$\underset{(8\times 1)}{\{f_b\}} = \int_{-1}^{1}\int_{-1}^{1} \underset{(8\times 2)}{[N]^T} \underset{(2\times 1)}{\{X_b\}} h\left|[J]\right| ds\,dt$$

Like the stiffness matrix, the body-force matrix has to be evaluated by numerical integration.

Isoparametric Elements

Gaussian Quadrature

Generalization of the formula leads to:

$$\mathrm{I} = \int_{-1}^{1} y\,dx = \sum_{i=1}^{n} W_i y(x_i)$$

That is, to approximate the integral, we evaluate the function at several sampling points *n*, multiply each value y_i by the appropriate weight W_i , and add the terms.

Gauss's method chooses the sampling points so that for a given number of points, the best possible accuracy is obtained.

Sampling points are located symmetrically with respect to the center of the interval.

Table 10–1 Table for Gauss points for integration from minus one to one, $\int_{-1}^{1} y(x)\,dx = \sum_{i=1}^{n} W_i y_i$

Number of Points	Locations, x_i	Associated Weights, W_i
1	$x_1 = 0.000\ldots$	2.000
2	$x_1, x_2 = \pm 0.57735026918962$	1.000
3	$x_1, x_3 = \pm 0.77459666924148$	$\frac{5}{9} = 0.555\ldots$
	$x_2 = 0.000\ldots$	$\frac{8}{9} = 0.888\ldots$
4	$x_1, x_4 = \pm 0.8611363116$	0.3478548451
	$x_2, x_3 = \pm 0.3399810436$	0.6521451549

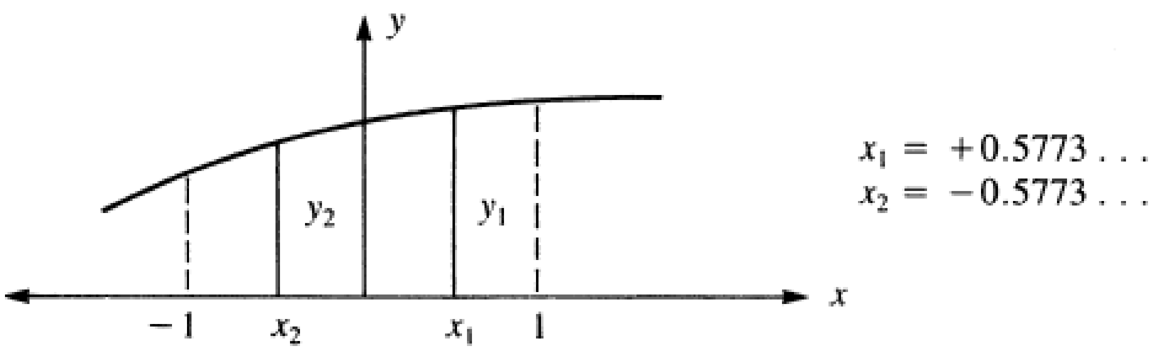


Figure 10–9 Gaussian quadrature using two sampling points

Isoparametric Elements

Gaussian Quadrature

In two dimensions, we obtain the quadrature formula by integrating first with respect to one coordinate and then with respect to the other as

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,ds\,dt \approx \int_{-1}^{1}\left[\sum_{i=1}^{n} W_i f(s_i,t)\right]dt$$

$$\approx \sum_{j=1}^{n} W_j\left[\sum_{i=1}^{n} W_i f(s_i,t_j)\right]$$

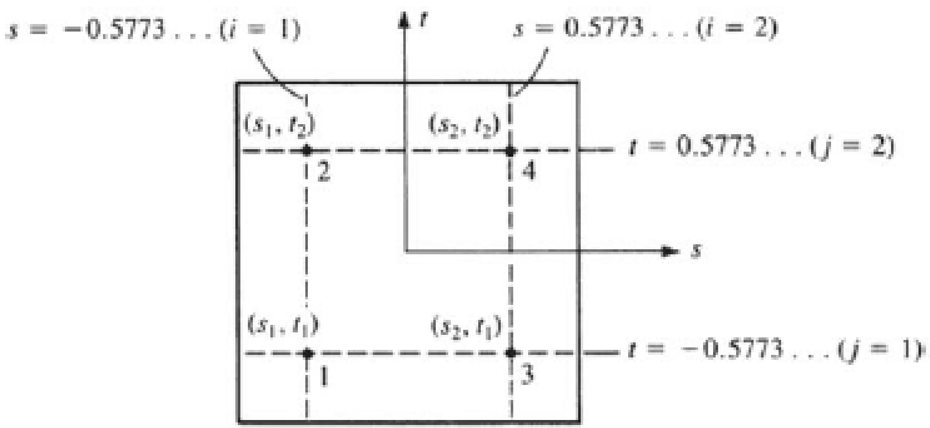
$$\approx \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j f(s_i,t_j)$$

Isoparametric Elements

Gaussian Quadrature

For example, a four-point Gauss rule (often described as a 2 x 2 rule) is shown below with i = 1, 2 and j = 1, 2 yields

$$I \approx \sum_{i=1}^{2}\sum_{j=1}^{2} W_i W_j f(s_i, t_j) \approx W_1 W_1 f(s_1, t_1) + W_1 W_2 f(s_1, t_2) + W_2 W_1 f(s_2, t_1) + W_2 W_2 f(s_2, t_2)$$



The four sampling points are at s_i *and* t_i = ±0.5773... and W_i = 1.0

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

We have shown that [k] for a quadrilateral element can be evaluated in terms of a local set of coordinates ***s*-*t***, with limits from -1 to 1within the element.

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\, h \left|[J]\right| ds\, dt$$

Each coefficient of the integrand $[B]^T [D] [B] |[J]|$ evaluated by numerical integration in the same manner as $f(s, t)$ was integrated.

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

The explicit form for four-point Gaussian quadrature (now using the single summation notation with i = 1, 2, 3, 4), we have

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\, h \left|[J]\right| ds\, dt$$

$$= \left[B(s_1,t_1)\right]^T [D]\left[B(s_1,t_1)\right]\left|\left[J(s_1,t_1)\right]\right| W_1 W_1$$

$$+\left[B(s_2,t_2)\right]^T [D]\left[B(s_2,t_2)\right]\left|\left[J(s_2,t_2)\right]\right| W_2 W_2$$

$$+\left[B(s_3,t_3)\right]^T [D]\left[B(s_3,t_3)\right]\left|\left[J(s_3,t_3)\right]\right| W_3 W_3$$

$$+\left[B(s_4,t_4)\right]^T [D]\left[B(s_4,t_4)\right]\left|\left[J(s_4,t_4)\right]\right| W_4 W_4$$

where s_1=t_1= -0.5773, s_2=-0.5773, t_2=0.5773, s_3=0.5773, t_3=-0.5773, and s_4=t_4=0.5773 and W_1=W_2=W_3=W_4=1.0