Plane Stress and Plane Strain Equations

Line elements are connected only at common nodes, forming framed or articulated structures such as trusses, frames, and grids.

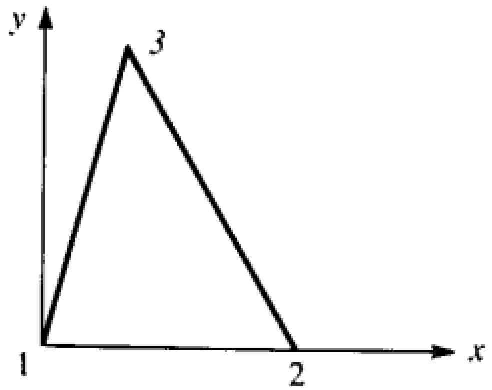
Line elements have geometric properties such as cross-sectional area and moment of inertia associated with their cross sections.

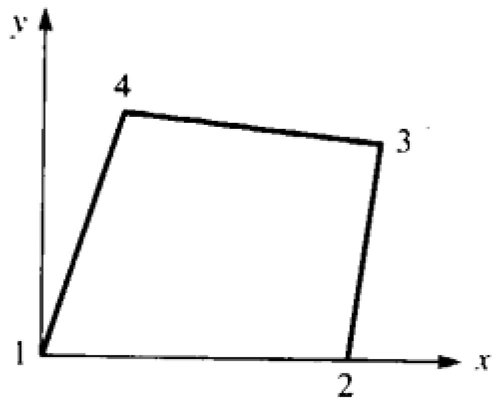
However, only one local coordinate along the length of the element is required to describe a position along the element (hence, they are called ***line elements***).

Nodal compatibility is then enforced during the formulation of the nodal equilibrium equations for a line element.

Two-dimensional (planar) elements are thin-plate elements such that two coordinates define a position on the element surface.

The elements are connected at common nodes and/or along common edges to form continuous structures.



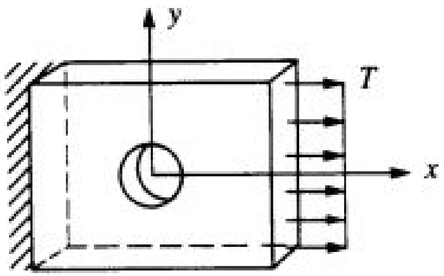


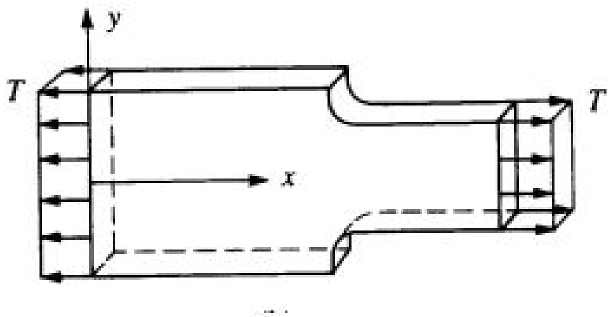
Plane Stress

Plane stress is defined to be ***a state of stress in which the normal stress and the shear stresses directed perpendicular to the plane are assumed to be zero***.

That is, the normal stress σ_z and the shear stresses τ_{xz} and τ_{yz} are assumed to be zero.

Generally, members that are thin (those with a small *z* dimension compared to the in-plane *x* and *y* dimensions) and whose loads act only in the *x*-*y* plane can be considered to be under plane stress.



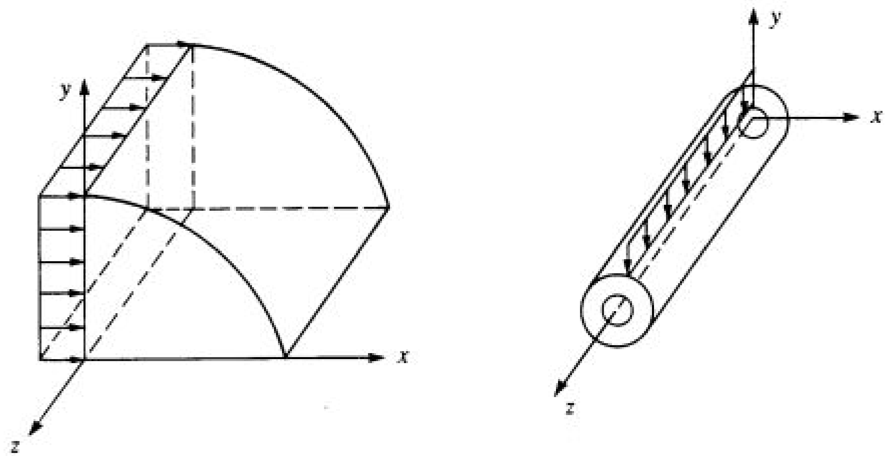


Plane Stress Problems

Plane Strain

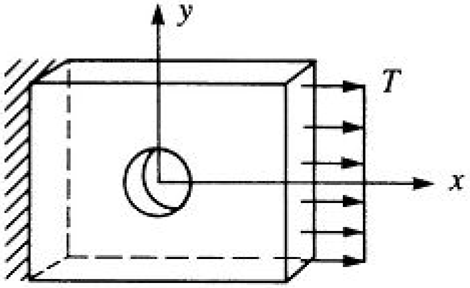
Plane strain is defined to be ***a state of strain in which the strain normal to the x-y plane ε_z and the shear strains τ_{xz} and τ_{yz} are assumed to be zero.***

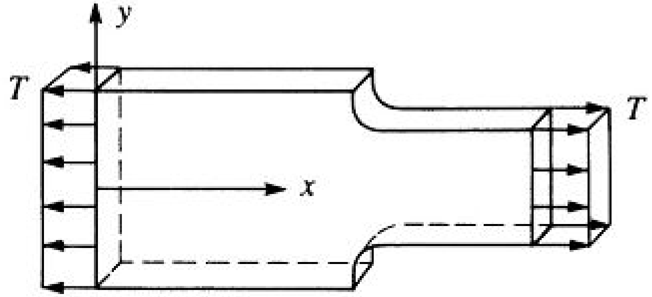
The assumptions of plane strain are realistic for long bodies (say, in the *z* direction) with constant cross-sectional area subjected to loads that act only in the *x* and/or *y* directions and do not vary in the *z* direction.



Plane Strain Problems

Plane Stress:





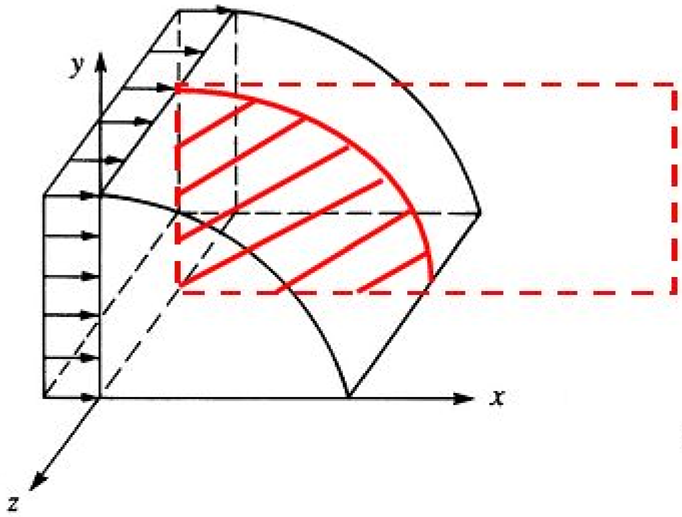
$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

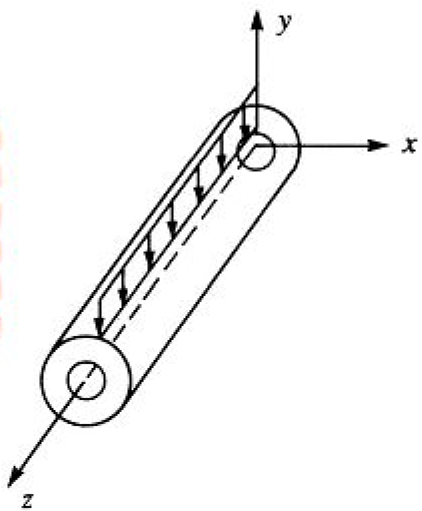
$$\therefore \vec{\sigma} = \left\{ \sigma_x \quad \sigma_y \quad \tau_{xy} \right\}^T$$

$$\downarrow$$

$$\vec{\varepsilon} = \left\{ \varepsilon_x \quad \varepsilon_y \quad \gamma_{xy} \right\}^T$$

Plane Strain:





$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

$$\therefore \vec{\varepsilon} = \left\{ \varepsilon_x \quad \varepsilon_y \quad \gamma_{xy} \right\}^T$$

$$\downarrow$$

$$\vec{\sigma} = \left\{ \sigma_x \quad \sigma_y \quad \tau_{xy} \right\}^T$$

Plane stress:

$$\vec{\sigma} = \underline{D}\vec{\varepsilon} \quad ; \quad \underline{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{(0)}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{(0)}{E}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{(0)}{E} \rightarrow \varepsilon_z \text{ is not independent.}$$

For **plane stress,** the stresses σ_z, τ_{xz}, and τ_{yz} are assumed to be zero. The stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D]\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad [D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}$$

is called the ***stress-strain matrix*** (or the ***constitutive matrix***), E is the modulus of elasticity, and ν is Poisson's ratio.

Plane strain:

$$\vec{\sigma} = \underline{D}\vec{\varepsilon} \quad ; \quad \underline{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$0 = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \rightarrow \sigma_z \text{ is not independent.}$$

For **plane strain**, the strains ε_z, τ_{xz}, and τ_{yz} are assumed to be zero. The stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

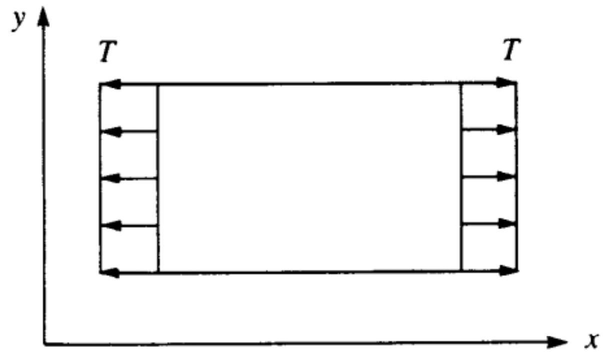
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad [D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

is called the ***stress-strain matrix*** (or the ***constitutive matrix***), E is the modulus of elasticity, and ν is Poisson's ratio.

Plane Stress and Plane Strain Equations

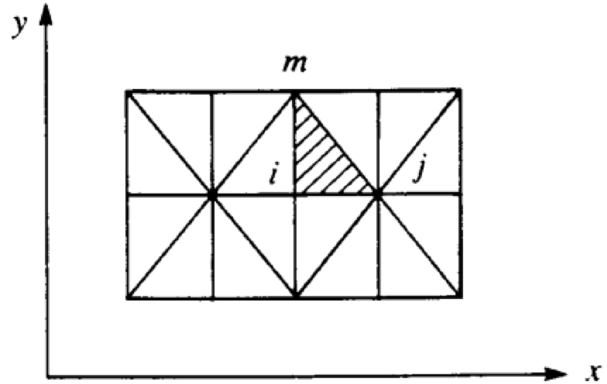
Formulation of the Plane Triangular Element Equations

Consider the problem of a thin plate subjected to a tensile load as shown in the figure below:



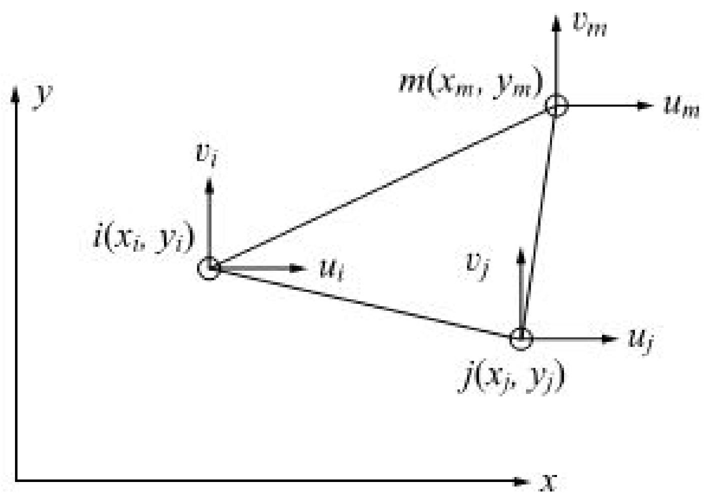
Step 1 - Discretize and Select Element Types

Discretize the thin plate into a set of triangular elements. Each element is define by nodes *i*, *j*, and *m*.



We use triangular elements because boundaries of irregularly shaped bodies can be closely approximated, and because the expressions related to the triangular element are comparatively simple. This discretization is called a ***coarse-mesh generation*** if few large elements are used. Each node has two degrees of freedom: displacements in the *x* and *y* directions. We will let u_i and v_i represent the node *i* displacement components in the *x* and *y* directions, respectively.

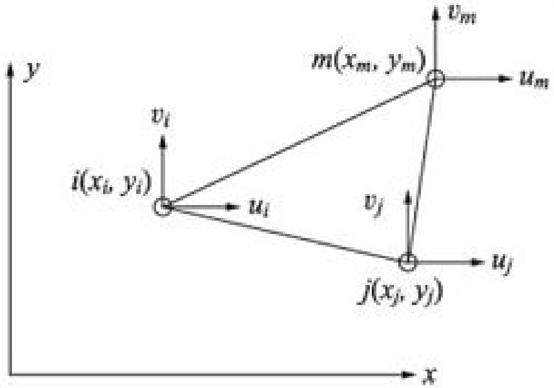
The nodal displacements for an element with nodes *i*, *j*, and *m* are:



Basic triangular element showing degrees of freedom

The nodal displacements for an element with nodes *i*, *j*, and *m* are:

$$\{d\} = \begin{Bmatrix} d_i \\ d_j \\ d_m \end{Bmatrix}$$



where the nodes are ordered ***counterclockwise*** around the element, and

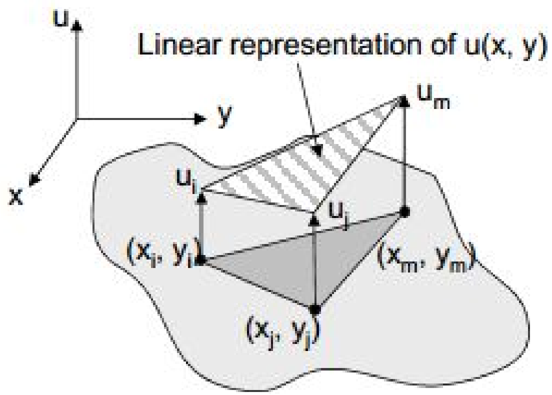
$$\{d_i\} = \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

The nodal displacements for an element with nodes *i*, *j*, and *m* are:

$$\{d\} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

Step 2 - Select Displacement Functions

We will select a linear displacement function for each triangular element, defined as:



$$\{\Psi_i\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$

$$= \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix}$$

A linear function ensures that the displacements along each edge of the element and the nodes shared by adjacent elements are equal.

$$\{\Psi_i\} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

To obtain the values for the *a*'s substitute the coordinated of the nodal points into the above equations:

$$u_i = a_1 + a_2 x_i + a_3 y_i \qquad v_i = a_4 + a_5 x_i + a_6 y_i$$

$$u_j = a_1 + a_2 x_j + a_3 y_j \qquad v_j = a_4 + a_5 x_j + a_6 y_j$$

$$u_m = a_1 + a_2 x_m + a_3 y_m \qquad v_m = a_4 + a_5 x_m + a_6 y_m$$

Solving for the ***a***'s and writing the results in matrix forms gives:

$$\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \quad \Rightarrow \quad \{a\} = [x]^{-1}\{u\}$$

The inverse of the $[x]$ matrix is:

$$[x]^{-1} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}$$

where A is the area of the triangle

$$2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}$$

$$2A = x_i\left(y_j - y_m\right) + x_j\left(y_m - y_i\right) + x_m\left(y_i - y_j\right)$$

Determinant of triangle is 2A

$$\alpha_i = x_j y_m - y_j x_m \qquad \beta_i = y_j - y_m \qquad \gamma_i = x_m - x_j$$

$$\alpha_j = x_i y_m - y_i x_m \qquad \beta_j = y_m - y_i \qquad \gamma_j = x_i - x_m$$

$$\alpha_m = x_i y_j - y_i x_j \qquad \beta_m = y_i - y_j \qquad \gamma_m = x_j - x_i$$

The values of ***a*** may be written matrix form as:

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} v_i \\ v_j \\ v_m \end{Bmatrix}$$

We Know that

$$\{u\} = \begin{bmatrix} 1 & x & y \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Substituting the values for *a* into the above equation gives:

$$\{u\} = \frac{1}{2A}\begin{bmatrix} 1 & x & y \end{bmatrix}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

We will now derive the u displacement function in terms of the coordinates x and y.

$$\{u\} = \frac{1}{2A}\begin{bmatrix} 1 & x & y \end{bmatrix}\begin{bmatrix} \alpha_i u_i + \alpha_j u_j + \alpha_m u_m \\ \beta_i u_i + \beta_j u_j + \beta_m u_m \\ \gamma_i u_i + \gamma_j u_j + \gamma_m u_m \end{bmatrix}$$

Multiplying the matrices in the above equations gives:

$$u(x,y) = \frac{1}{2A}\Big\{\left(\alpha_i + \beta_i x + \gamma_i y\right)u_i + \left(\alpha_j + \beta_j x + \gamma_j y\right)u_j + \left(\alpha_m + \beta_m x + \gamma_m y\right)u_m\Big\}$$

Similarly

$$\{v\} = \frac{1}{2A}\begin{bmatrix} 1 & x & y \end{bmatrix}\begin{bmatrix} \alpha_i v_i + \alpha_j v_j + \alpha_m v_m \\ \beta_i v_i + \beta_j v_j + \beta_m v_m \\ \gamma_i v_i + \gamma_j v_j + \gamma_m v_m \end{bmatrix}$$

Multiplying the matrices in the above equations gives:

$$v(x,y) = \frac{1}{2A}\Big\{\left(\alpha_i + \beta_i x + \gamma_i y\right)v_i + \left(\alpha_j + \beta_j x + \gamma_j y\right)v_j + \left(\alpha_m + \beta_m x + \gamma_m y\right)v_m\Big\}$$

The displacements can be written in a more convenience form as:

$$u(x,y) = N_i u_i + N_j u_j + N_m u_m$$

$$v(x,y) = N_i v_i + N_j v_j + N_m v_m$$

where:

$$N_i = \frac{1}{2A}\left(\alpha_i + \beta_i x + \gamma_i y\right)$$

$$N_j = \frac{1}{2A}\left(\alpha_j + \beta_j x + \gamma_j y\right)$$

$$N_m = \frac{1}{2A}\left(\alpha_m + \beta_m x + \gamma_m y\right)$$

The elemental displacements can be summarized as:

$$\{\Psi_i\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix}$$

In another form the above equations are:

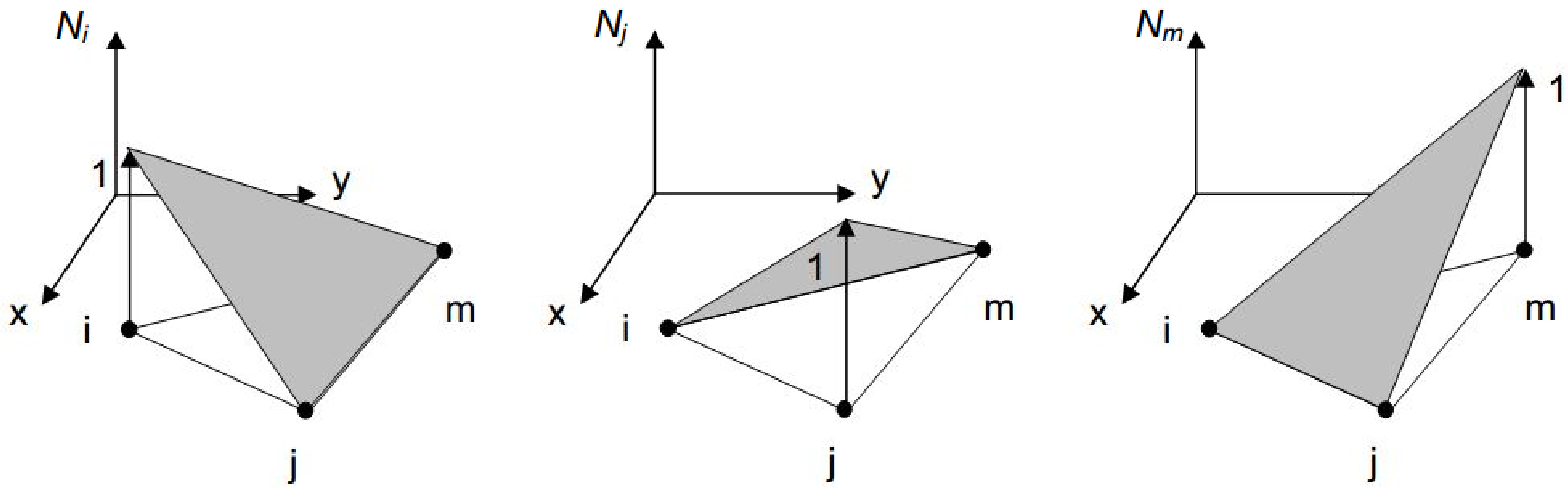
$$\{\Psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

$$\{\Psi\} = [N]\{d\}$$

In another form the equations are: $\{\Psi\} = [N]\{d\}$

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$$

The linear triangular shape functions are illustrated below:



Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Elemental Strains: The strains over a two-dimensional element are:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

Substituting our approximation for the displacement gives:

$$\frac{\partial u}{\partial x} = u_{,x} = \frac{\partial}{\partial x}\left(N_i u_i + N_j u_j + N_m u_m\right)$$

$$u_{,x} = N_{i,x} u_i + N_{j,x} u_j + N_{m,x} u_m$$

where the comma indicates differentiation with respect to that variable.

The derivatives of the interpolation functions are:

$$N_{i,x} = \frac{1}{2A}\frac{\partial}{\partial x}(\alpha_i + \beta_i x + \gamma_i y) = \frac{\beta_i}{2A}$$

$$N_{j,x} = \frac{\beta_j}{2A} \qquad N_{m,x} = \frac{\beta_m}{2A}$$

Therefore:

$$\frac{\partial u}{\partial x} = \frac{1}{2A}\left(\beta_i u_i + \beta_j u_j + \beta_m u_m\right)$$

In a similar manner, the remaining strain terms are approximated as:

$$\frac{\partial v}{\partial y} = \frac{1}{2A}\left(\gamma_i v_i + \gamma_j v_j + \gamma_m v_m\right)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A}\left(\beta_i u_i + \gamma_i v_i + \beta_j u_j + \gamma_j v_j + \beta_m u_m + \gamma_m v_m\right)$$

We can write the strains in matrix form as:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \frac{1}{2A} \left[\begin{array}{cc:cc:cc} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{array} \right] \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

or

$$\{\varepsilon\} = \begin{bmatrix} B_i & B_j & B_m \end{bmatrix} \begin{Bmatrix} d_i \\ d_j \\ d_m \end{Bmatrix}$$

where

$$[B_i] = \frac{1}{2A} \begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{bmatrix} \qquad [B_j] = \frac{1}{2A} \begin{bmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{bmatrix} \qquad [B_m] = \frac{1}{2A} \begin{bmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{bmatrix}$$

These equations can be written in matrix form as:

$$\{\varepsilon\} = [B]\{d\}$$

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Stress-Strain Relationship: The in-plane stress-strain relationship is:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\{\sigma\} = [D][B]\{d\}$$

For plane stress $[D]$ is:

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}$$

For plane strain $[D]$ is:

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 0.5-\nu \end{bmatrix}$$

Step 4 - Derive the Element Stiffness Matrix and Equations

The stiffness matrix can be defined as:

$$[k] = \int_V [B]^T [D][B] dV$$

For an element of constant thickness, t, the above integral becomes:

$$[k] = t \int_A [B]^T [D][B] \, dx \, dy$$

$$[k] = t[B]^T [D][B] \int_A dx \, dy$$

$$[k] = tA \, [B]^T [D][B]$$

Expanding the stiffness relationship gives:

$$[k] = \begin{bmatrix} [k_{ii}] & [k_{ij}] & [k_{im}] \\ [k_{ji}] & [k_{jj}] & [k_{jm}] \\ [k_{mi}] & [k_{mj}] & [k_{mm}] \end{bmatrix}$$

where each $[k_{ij}]$ is a 2 x 2 matrix define as:

$$[k_{ii}] = [B_i]^T [D][B_i]\, tA \qquad [k_{ij}] = [B_i]^T [D][B_j]\, tA$$

$$[k_{im}] = [B_i]^T [D][B_m]\, tA$$

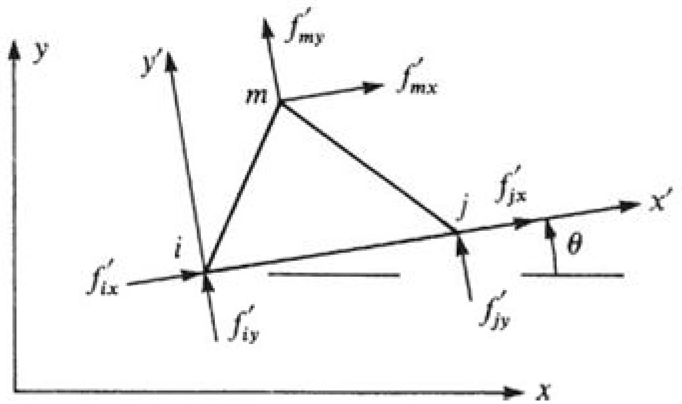
Where

$$[B_i] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \gamma_i & \beta_i \end{bmatrix} \qquad [B_j] = \frac{1}{2A}\begin{bmatrix} \beta_j & 0 \\ 0 & \gamma_j \\ \gamma_j & \beta_j \end{bmatrix} \qquad [B_m] = \frac{1}{2A}\begin{bmatrix} \beta_m & 0 \\ 0 & \gamma_m \\ \gamma_m & \beta_m \end{bmatrix}$$

Step 5 - Assemble the Element Equations to Obtain the Global Equations and Introduce the Boundary Conditions

To relate the local to global displacements, force, and stiffness matrices we will use:

$$d' = Td \qquad f' = Tf \qquad k = T^T k' T$$



The transformation matrix *T* for the triangular element is:

$$T = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & C & S & 0 & 0 \\ 0 & 0 & -S & C & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix}$$

$$C = \cos\theta$$

$$S = \sin\theta$$

Step 6 - Solve for the Nodal Displacements

Step 7 - Solve for Element Forces and Stress

Having solved for the nodal displacements, we can obtain strains and stresses in *x* and *y* directions in the elements by using:

$$\{\varepsilon\} = [B]\{d\} \qquad \{\sigma\} = [D][B]\{d\}$$

Example 6.1

Evaluate the stiffness matrix for the element shown in Figure 6–11. The coordinates are shown in units of inches. Assume plane stress conditions. Let $E = 30 \times 10^6$ psi, $\nu = 0.25$, and thickness $t = 1$ in. Assume the element nodal displacements have been determined to be $u_1 = 0.0$, $v_1 = 0.0025$ in., $u_2 = 0.0012$ in., $v_2 = 0.0$, $u_3 = 0.0$, and $v_3 = 0.0025$ in. Determine the element stresses.

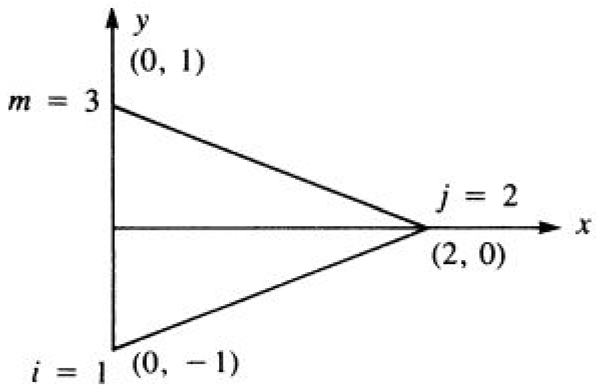
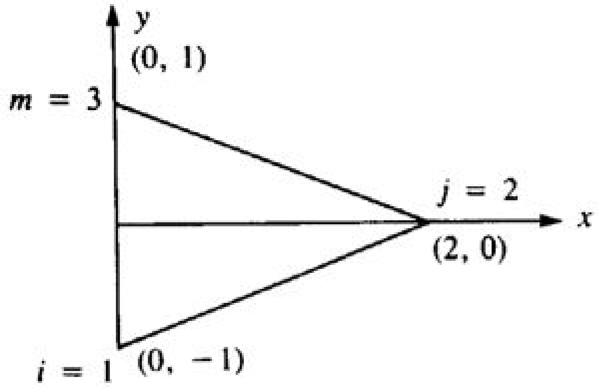


Figure 6–11 Plane stress element for stiffness matrix evaluation

First, we calculate the element β's and γ's as:



$$\beta_i = y_j - y_m = 0 - 1 = -1 \qquad \gamma_i = x_m - x_j = 0 - 2 = -2$$

$$\beta_j = y_m - y_i = 0 - (-1) = 2 \qquad \gamma_j = x_i - x_m = 0 - 0 = 0$$

$$\beta_m = y_i - y_j = -1 - 0 = -1 \qquad \gamma_m = x_j - x_i = 2 - 0 = 2$$

Therefore, the [*B*] matrix is:

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} = \frac{1}{2(2)}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

For plane stress conditions, the [D] matrix is:

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix}$$

$$[D] = \frac{30\times10^6}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

Substitute the above expressions for [D] and [B] into the general equations for the stiffness matrix:

$$[k] = tA\,[B]^T[D][B]$$

$$k = \frac{(2)30\times10^6}{4(0.9375)}\begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}\frac{1}{2(2)}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

Performing the matrix triple product gives:

$$k = 4\times10^{6}\begin{bmatrix} 2.5 & 1.25 & -2 & -1.5 & -0.5 & 0.25 \\ 1.25 & 4.375 & -1 & -0.75 & -0.25 & -3.625 \\ -2 & -1 & 4 & 0 & -2 & 1 \\ -1.5 & -0.75 & 0 & 1.5 & 1.5 & -0.75 \\ -0.5 & -0.25 & -2 & 1.5 & 2.5 & -1.25 \\ 0.25 & -3.625 & 1 & -0.75 & -1.25 & 4.375 \end{bmatrix} {}^{lb}\!/_{in}$$

The in-plane stress can be related to displacements by:

$$\{\sigma\} = [D][B]\{d\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{30\times10^{6}}{0.9375}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \frac{1}{2(2)} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix} \begin{Bmatrix} 0.0 \\ 0.0025\ in \\ 0.0012\ in \\ 0.0 \\ 0.0 \\ 0.0025\ in \end{Bmatrix}$$

The stresses are:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} 19{,}200\ psi \\ 4{,}800\ psi \\ -15{,}000\ psi \end{bmatrix}$$

Example 6.2

For a thin plate subjected to the surface traction shown in Figure 6–16, determine the nodal displacements and the element stresses. The plate thickness $t = 1$ in., $E = 30 \times 10^6$ psi, and $\nu = 0.30$.

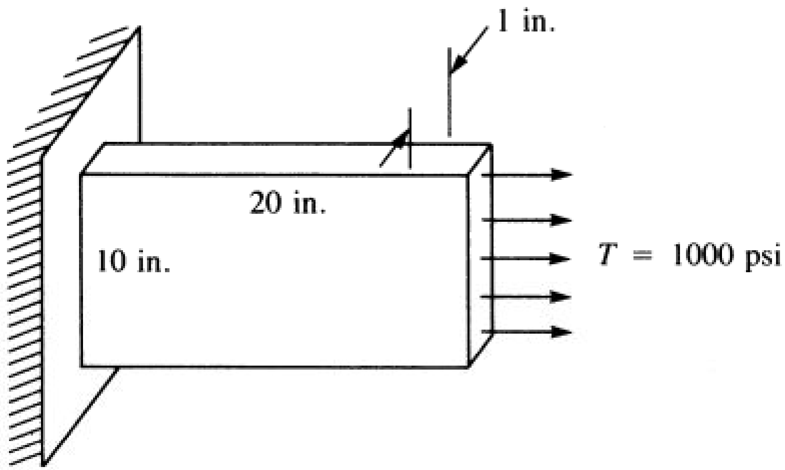
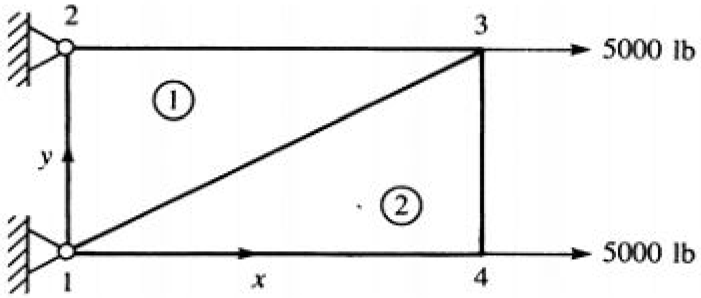


Figure 6–16 Thin plate subjected to tensile stress

Discretization

Let's discretize the plate into two elements as shown below:



This level of discretization will probably not yield practical results for displacement and stresses: however; it is useful example for a longhand solution.

The tensile traction forces can be converted into nodal forces as follows:

$$\{f_s\} = \begin{Bmatrix} f_{s1x} \\ f_{s1y} \\ f_{s2x} \\ f_{s2y} \\ f_{s3x} \\ f_{s3y} \end{Bmatrix} = \frac{pLt}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1{,}000\ psi(1\ in)10\ in}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5{,}000\ lb \\ 0 \\ 0 \\ 0 \\ 5{,}000\ lb \\ 0 \end{bmatrix}$$

The governing global matrix equations are: $\{F\} = [K]\{d\}$

Expanding the above matrices gives:

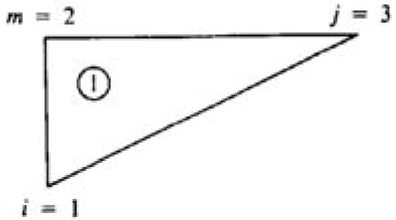
$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 5{,}000\ lb \\ 0 \\ 5{,}000\ lb \\ 0 \end{Bmatrix} = [K]\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix} = [K]\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}$$

Assemblage of the Stiffness Matrix

The global stiffness matrix is assembled by superposition of the individual element stiffness matrices.

The element stiffness matrix is: $[k] = tA[B]^T[D][B]$

For **element 1**: the coordinates are $x_i = 0$, $y_i = 0$, $x_j = 20$, $y_j = 10$, $x_m = 0$, and $y_m = 10$. The area of the triangle is:



$$A = \frac{bh}{2} = \frac{(20)(10)}{2} = 100\ in.^2$$

$$\beta_i = y_j - y_m = 10 - 10 = 0 \qquad \gamma_i = x_m - x_j = 0 - 20 = -20$$

$$\beta_j = y_m - y_1 = 10 - 0 = 10 \qquad \gamma_j = x_i - x_m = 0 - 0 = 0$$

$$\beta_m = y_i - y_j = 0 - 10 = -10 \qquad \gamma_m = x_i - x_j = 20 - 0 = 20$$

Therefore, the [*B*] matrix is:

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \qquad [B] = \frac{1}{200}\begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix} \frac{1}{in}$$

For plane stress conditions, the [D] matrix is:

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} = \frac{30\times10^6}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} psi$$

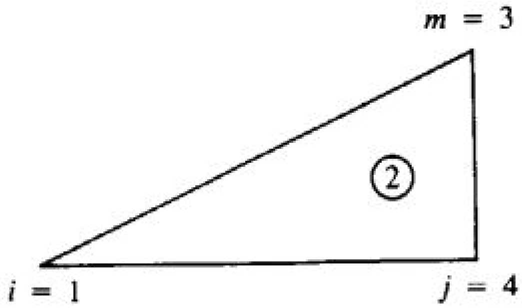
Substitute the above expressions for [D] and [B] into the general equations for the stiffness matrix:

$$[k] = tA\,[B]^T[D][B]$$

Simplifying the above expression gives:

$$[k] = \frac{75{,}000}{0.91}\begin{array}{c} \begin{matrix} u_1 & v_1 & u_3 & v_3 & u_2 & v_2 \end{matrix} \\ \begin{bmatrix} 140 & 0 & 0 & -70 & -140 & 70 \\ 0 & -400 & -60 & 0 & 60 & -400 \\ 0 & -60 & 100 & 0 & -100 & 60 \\ -70 & 0 & 0 & 35 & 70 & -35 \\ -140 & 60 & -100 & 70 & 240 & -130 \\ 70 & -400 & 60 & -35 & -130 & 435 \end{bmatrix} \end{array}$$

For **element 2**: the coordinates are $x_i = 0$, $y_i = 0$, $x_j = 20$, $y_j = 0$, $x_m = 20$, and $y_m = 10$. The area of the triangle is:



$$A = \frac{(20)(10)}{2} = 100\ in.^2$$

We need to calculate the element β's and γ's as:

$$\beta_i = y_j - y_m = 0 - 10 = -10 \qquad \gamma_i = x_m - x_j = 20 - 20 = 0$$

$$\beta_j = y_m - y_1 = 10 - 0 = 10 \qquad \gamma_j = x_i - x_m = 0 - 20 = -20$$

$$\beta_m = y_i - y_j = 0 - 0 = 0 \qquad \gamma_m = x_i - x_j = 20 - 0 = 20$$

Therefore, the [*B*] matrix is:

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \quad [B] = \frac{1}{200}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -10 & -20 & 10 & 20 & 0 \end{bmatrix} \frac{1}{in}$$

For plane stress conditions, the [D] matrix is:

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} = \frac{30\times 10^6}{0.91}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} psi$$

Substitute the above expressions for [D] and [B] into the general equations for the stiffness matrix:

$$[k] = tA\,[B]^T[D][B]$$

Simplifying the above expression gives:

$$[k^{(2)}] = \frac{75{,}000}{0.91}\begin{array}{c} \begin{matrix} u_1 & v_1 & u_4 & v_4 & u_3 & v_3 \end{matrix} \\ \begin{bmatrix} 100 & 0 & -100 & 60 & 0 & -60 \\ 0 & 35 & 70 & -35 & -70 & 0 \\ -100 & 70 & 240 & -130 & -140 & 60 \\ 60 & -35 & -130 & 435 & 70 & -400 \\ 0 & -70 & -140 & 70 & 140 & 0 \\ -60 & 0 & 60 & -400 & 0 & 400 \end{bmatrix} \end{array}$$

Using the superposition, the total global stiffness matrix is:

$$
[k] = \frac{375{,}000}{0.91}
\begin{array}{c}
\begin{matrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{matrix} \\
\begin{bmatrix}
48 & 0 & -28 & 14 & 0 & -26 & -20 & 12 \\
0 & 87 & 12 & -80 & -26 & 0 & 14 & -7 \\
-28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\
14 & 80 & -26 & 87 & 12 & -7 & 0 & 0 \\
0 & -26 & -20 & 12 & 48 & 0 & -28 & 14 \\
-26 & 0 & 14 & -7 & 0 & 87 & 12 & -80 \\
-20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\
12 & -7 & 0 & 0 & 14 & -80 & -26 & 87
\end{bmatrix}
\end{array}
$$

The governing global matrix equations are:

$$
\begin{Bmatrix}
R_{1x} \\ R_{1y} \\ R_{2x} \\ R_{2y} \\ 5{,}000\ lb \\ 0 \\ 500\ lb \\ 0
\end{Bmatrix}
= \frac{375{,}000}{0.91}
\begin{bmatrix}
48 & 0 & -28 & 14 & 0 & -26 & -20 & 12 \\
0 & 87 & 12 & -80 & -26 & 0 & 14 & -7 \\
-28 & 12 & 48 & -26 & -20 & 14 & 0 & 0 \\
14 & 80 & -26 & 87 & 12 & -7 & 0 & 0 \\
0 & -26 & -20 & 12 & 48 & 0 & -28 & 14 \\
-26 & 0 & 14 & -7 & 0 & 87 & 12 & -80 \\
-20 & 14 & 0 & 0 & -28 & 12 & 48 & -26 \\
12 & -7 & 0 & 0 & 14 & -80 & -26 & 87
\end{bmatrix}
\begin{Bmatrix}
d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y}
\end{Bmatrix}
$$

Applying the boundary conditions: $d_{1x} = d_{1y} = d_{2x} = d_{2y} = 0$

The governing equations are:

$$\begin{Bmatrix} 5{,}000\ lb \\ 0 \\ 5{,}000\ lb \\ 0 \end{Bmatrix} = \frac{375{,}000}{0.91} \begin{bmatrix} 48 & 0 & -28 & 14 \\ 0 & 87 & 12 & -80 \\ -28 & 12 & 48 & -26 \\ 14 & -80 & -26 & 87 \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix}$$

Solving the equations gives:

$$\begin{Bmatrix} d_{3x} \\ d_{3y} \\ d_{4x} \\ d_{4y} \end{Bmatrix} = \left(10^{-6}\right) \begin{Bmatrix} 609.6 \\ 4.2 \\ 663.7 \\ 104.1 \end{Bmatrix} in$$

The exact solution for the displacement at the free end of the one-dimensional bar subjected to a tensile force is:

$$\delta = \frac{PL}{AE} = \frac{(10{,}000)20}{10(30 \times 10^{6})} = 670 \times 10^{-6}\ in$$

Plane Stress Example 1

Element 1: $\{\sigma\} = [D][B]\{d\}$

$$\{\sigma\} = \frac{E}{2A(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} \begin{bmatrix} \beta_1 & 0 & \beta_3 & 0 & \beta_2 & 0 \\ 0 & \gamma_1 & 0 & \gamma_3 & 0 & \gamma_2 \\ \gamma_1 & \beta_1 & \gamma_3 & \beta_3 & \gamma_2 & \beta_2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{3x} \\ d_{3y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{30(10^6)(10^{-6})}{0.96(200)} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \begin{bmatrix} 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & -20 & 0 & 0 & 0 & 20 \\ -20 & 0 & 0 & 10 & 20 & -10 \end{bmatrix} \begin{Bmatrix} 0.0 \\ 0.0 \\ 609.6 \\ 4.2 \\ 0.0 \\ 0.0 \end{Bmatrix}$$

Element 2: $\{\sigma\} = [D][B]\{d\}$

$$\{\sigma\} = \frac{E}{2A(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} \begin{bmatrix} \beta_1 & 0 & \beta_4 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_4 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_4 & \beta_4 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{4x} \\ d_{4y} \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{30(10^6)(10^{-6})}{0.96(200)} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \begin{bmatrix} 10 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & 10 & -20 & 10 & 20 & 0 \end{bmatrix} \begin{Bmatrix} 0.0 \\ 0.0 \\ 663.7 \\ 104.1 \\ 609.6 \\ 4.2 \end{Bmatrix}$$