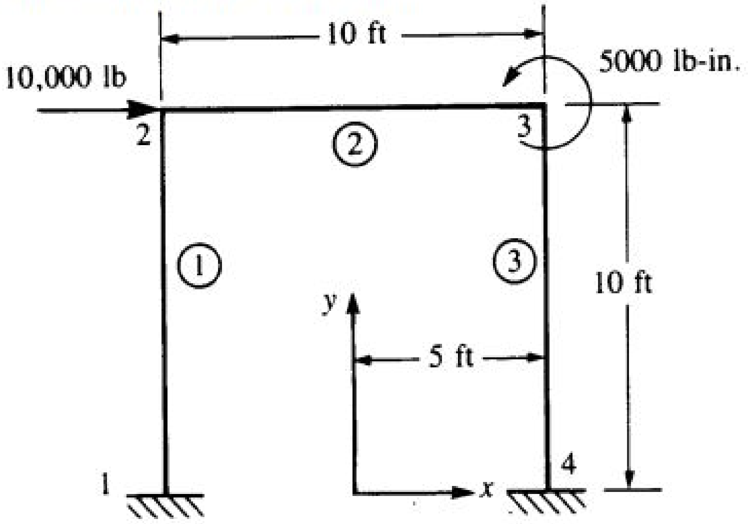
Example #1

Consider the frame shown in the figure below.



The frame is fixed at nodes 1 and 4 and subjected to a positive horizontal force of 10,000 lb applied at node 2 and to a positive moment of 5,000 lb-in. applied at node 3. Let $E = 30 \times 10^6$ psi and $A = 10$ in.^{2} for all elements, and let $I = 200$ in.^{4} for elements 1 and 3, and $I = 100$ in.^{4} for element 2.

Element 1: The angle between x and $\hat{x}$ is 90°

$$C = 0 \qquad S = 1$$

where

$$\frac{12I}{L^2} = \frac{12(200)}{(120)^2} = 0.167\ in^2 \qquad \frac{6I}{L} = \frac{6(200)}{120} = 10.0\ in^3$$

$$\frac{E}{L} = \frac{30 \times 10^6}{120} = 250{,}000\ lb/in^3$$

Therefore, for element 1:

$$k^{(1)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} d_{1x} & d_{1y} & \phi_1 & d_{2x} & d_{2y} & \phi_2 \end{array} \\ \begin{bmatrix} 0.167 & 0 & -10 & -0.167 & 0 & -10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ -10 & 0 & 800 & 10 & 0 & 400 \\ -0.167 & 0 & 10 & 0.167 & 0 & 10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ -10 & 0 & 400 & 10 & 0 & 800 \end{bmatrix} \end{array} \, {}^{lb}\!/_{in}$$

Element 2: The angle between x and $\hat{x}$ is 0°

$$C = 1 \qquad S = 0$$

$$\frac{12I}{L^2} = \frac{12(100)}{(120)^2} = 0.0835\ in^2 \qquad \frac{6I}{L} = \frac{6(100)}{120} = 5.0\ in^3$$

Therefore, for element 2:

$$k^{(2)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} d_{2x} & d_{2y} & \phi_2 & d_{3x} & d_{3y} & \phi_3 \end{array} \\ \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.0835 & 5 & 0 & \boxed{0.0835} & 5 \\ 0 & 5 & 400 & 0 & -5 & 200 \\ -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & \boxed{0.0835} & -5 & 0 & 0.0835 & -5 \\ 0 & 5 & 200 & 0 & -5 & 400 \end{bmatrix} \end{array} {}^{lb}\!/_{in}$$

Element 3: The angle between x and $\hat{x}$ is 270°

$$C = 0 \qquad S = -1$$

$$\frac{12I}{L^2} = \frac{12(200)}{(120)^2} = 0.167\ in^2 \qquad \frac{6I}{L} = \frac{6(200)}{120} = 10.0\ in^3$$

$$\frac{E}{L} = \frac{30 \times 10^6}{120} = 250{,}000\ lb/in^3$$

Therefore, for element 3:

$$k^{(3)} = 250{,}000 \begin{array}{c} \begin{array}{cccccc} d_{3x} & d_{3y} & \phi_3 & d_{4x} & d_{4y} & \phi_4 \end{array} \\ \begin{bmatrix} 0.167 & 0 & 10 & -0.167 & 0 & 10 \\ 0 & 10 & 0 & 0 & -10 & 0 \\ 10 & 0 & 800 & -10 & 0 & 400 \\ -0.167 & 0 & -10 & 0.167 & 0 & -10 \\ 0 & -10 & 0 & 0 & 10 & 0 \\ 10 & 0 & 400 & -10 & 0 & 800 \end{bmatrix} \end{array} {}^{lb}\!/_{in}$$

The boundary conditions for this problem are:

$$d_{1x} = d_{1y} = \phi_1 = d_{4x} = d_{4y} = \phi_4 = 0$$

After applying the boundary conditions the global beam equations reduce to:

$$\begin{Bmatrix} 10{,}000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 5{,}000 \end{Bmatrix} = 2.5\times 10^{5} \begin{bmatrix} 10.167 & 0 & 10 & -10 & 0 & 0 \\ 0 & 10.0835 & 5 & 0 & -0.0835 & 5 \\ 10 & 5 & 1200 & 0 & -5 & 200 \\ -10 & 0 & 0 & 10.167 & 0 & 10 \\ 0 & -0.0835 & -5 & 0 & 10.0835 & -5 \\ 0 & 5 & 200 & 10 & -5 & 1200 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{2y} \\ \phi_2 \\ d_{3x} \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} d_{2x} \\ d_{2y} \\ \phi_2 \\ d_{3x} \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0.211\,in \\ 0.00148\,in \\ -0.00153\,rad \\ 0.209\,in \\ -0.00148\,in \\ -0.00149\,rad \end{Bmatrix}$$

Element 1: The element force-displacement equations can be obtained using $\hat{f} = \hat{k}\overline{T}d$. Therefore, $\overline{T}d$ is:

$$\overline{T}d = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} = 0 \\ d_{1y} = 0 \\ \phi_1 = 0 \\ d_{2x} = 0.211\,in \\ d_{2y} = 0.00148\,in \\ \phi_2 = -0.00153\,rad \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148\,in \\ -0.211\,in \\ -0.00153\,rad \end{Bmatrix}$$

Recall the elemental stiffness matrix is:

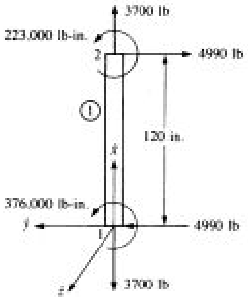
$$\hat{k} = \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{bmatrix}$$

Therefore, the local force-displacement equations are:

$$\hat{f}^{(1)} = \hat{k}\overline{T}d = 2.5\times10^{5}\begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.167 & 10 & 0 & -0.167 & 10 \\ 0 & 10 & 800 & 0 & -10 & 400 \\ -10 & 0 & 0 & 10 & 0 & \boxed{10} \\ 0 & -0.167 & -10 & 0 & 0.167 & -10 \\ 0 & 10 & 400 & 0 & -10 & 800 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.00148\ in \\ -0.211\, in \\ -0.00153\ rad \end{Bmatrix}$$

Simplifying the above equations gives:

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \begin{Bmatrix} -3{,}700\ lb \\ 4{,}990\ lb \\ 376\ k\cdot in \\ 3{,}700\ lb \\ -4{,}990\ lb \\ 223\ k\cdot in \end{Bmatrix}$$



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Element 2: The element force-displacement equations are:

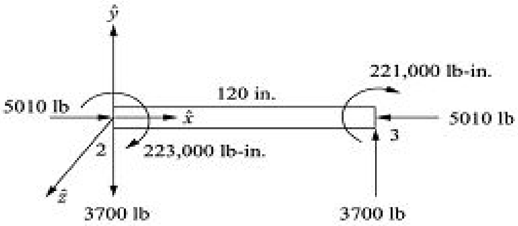
$$\overline{T}d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} d_{2x} = 0.211\, in \\ d_{2y} = 0.00148\ in \\ \phi_2 = -0.00153\ rad \\ d_{3x} = 0.209\ in \\ d_{3y} = -0.00148\ in \\ \phi_3 = -0.00149\ rad \end{Bmatrix} = \begin{Bmatrix} -0.211\ in \\ 0.00148\ in \\ -0.00153\ rad \\ 0.209\ in \\ -0.00148\ in \\ -0.00149\ rad \end{Bmatrix}$$

Therefore, the local force-displacement equations are:

$$\hat{f}^{(2)} = \hat{k}\overline{T}d = 2.5\times10^{5}\begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.0833 & 5 & 0 & -0.0833 & 5 \\ 0 & 5 & 400 & 0 & -5 & 200 \\ -10 & 0 & 0 & 10 & 0 & 0 \\ 0 & -0.0833 & -5 & 0 & 0.0833 & -5 \\ 0 & 5 & 200 & 0 & -5 & 400 \end{bmatrix}\begin{Bmatrix} 0.211\, in \\ 0.00148\ in \\ -0.00153\ rad \\ 0.209\ in \\ -0.00148\ in \\ -0.00149\ rad \end{Bmatrix}$$

Simplifying the above equations gives:

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \\ \hat{f}_{3x} \\ \hat{f}_{3y} \\ \hat{m}_3 \end{Bmatrix} = \begin{Bmatrix} 5{,}010\ lb \\ -3{,}700\ lb \\ -223\ k\cdot in \\ -5{,}010\ lb \\ 3{,}700\ lb \\ -221\, k\cdot in \end{Bmatrix}$$



Element 3: The element force-displacement equations are:

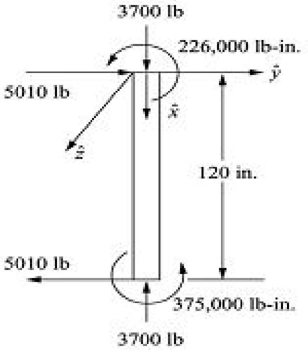
$$\overline{T}d = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{3x} = 0.209\ in \\ d_{3y} = -0.00148\ in \\ \phi_3 = -0.00149\ rad \\ d_{4x} = 0 \\ d_{4y} = 0 \\ \phi_4 = 0 \end{Bmatrix} = \begin{Bmatrix} 0.00148\ in \\ 0.209\ in \\ -0.00149\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Therefore, the local force-displacement equations are:

$$\hat{f}^{(3)} = \hat{k}\overline{T}d = 2.5\times10^5 \begin{bmatrix} 10 & 0 & 0 & -10 & 0 & 0 \\ 0 & 0.167 & 10 & 0 & -0.167 & 10 \\ 0 & 10 & 800 & 0 & -10 & 400 \\ -10 & 0 & 0 & 10 & 0 & 10 \\ 0 & -0.167 & -10 & 0 & 0.167 & -10 \\ 0 & 10 & 400 & 0 & -10 & 800 \end{bmatrix} \begin{Bmatrix} 0.00148\ in \\ 0.209\ in \\ -0.00149\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

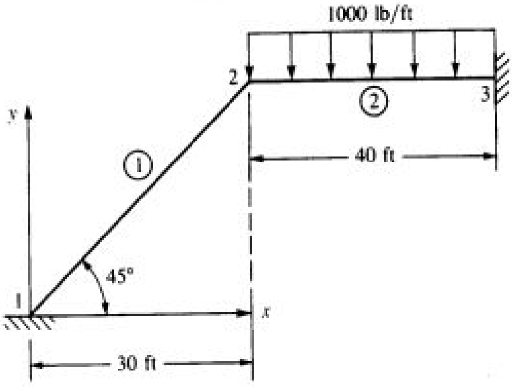
Simplifying the above equations gives:

$$\begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{3y} \\ \hat{m}_3 \\ \hat{f}_{4x} \\ \hat{f}_{4y} \\ \hat{m}_4 \end{Bmatrix} = \begin{Bmatrix} 3{,}700\ lb \\ 5{,}010\ lb \\ 226\ k\cdot in \\ -3{,}700\ lb \\ -5{,}010\ lb \\ 375\ k\cdot in \end{Bmatrix}$$



Example # 2

Consider the frame shown in the figure below.



The frame is fixed at nodes 1 and 3 and subjected to a positive distributed load of 1,000 lb/ft applied along element 2. Let E = 30 x 10^6 psi and A = 100 in.^{2} for all elements, and let I = 1,000 in.^{4} for all elements.

First we need to replace the distributed load with a set of equivalent nodal forces and moments acting at nodes 2 and 3. For a beam with both end fixed, subjected to a uniform distributed load, w, the nodal forces and moments are:

$$f_{2y} = f_{3y} = -\frac{wL}{2} = -\frac{(1{,}000\, lb/ft)40ft}{2} = -20k$$

$$m_2 = -m_3 = -\frac{wL^2}{12} = -\frac{(1{,}000\, lb/ft)(40\, ft)^2}{12} = -133{,}333\ lb \cdot ft = 1{,}600\ k \cdot in$$

If we consider only the parts of the stiffness matrix associated with the three degrees of freedom at node 2, we get:

Element 1: The angle between x and $\hat{x}$ is 45°

$C = 0.707 \qquad S = 0.707$

where

$$\frac{E}{L} = \frac{30 \times 10^6}{509} = 58.93\ k/in^3 \qquad \frac{12I}{L^2} = \frac{12(1{,}000)}{\left(12 \times 30\sqrt{2}\right)^2} = 0.0463\ in^2$$

$$\frac{6I}{L} = \frac{6(1{,}000)}{12 \times 30\sqrt{2}} = 11.78551\, in^3$$

Therefore, for element 1:

$$k^{(1)} = 58.93 \begin{array}{c} \begin{array}{ccc} d_{2x} & d_{2y} & \phi_2 \end{array} \\ \begin{bmatrix} 50.02 & 49.98 & 8.33 \\ 49.98 & 50.02 & -8.33 \\ 8.33 & -8.33 & 4000 \end{bmatrix} \end{array} {}^{k}\!/_{in}$$

Simplifying the above equation:

$$k^{(1)} = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{ccc} d_{2x} & d_{2y} & \phi_2 \\ \end{array}$$

$$k^{(1)} = \left[\begin{array}{ccc} 2{,}948 & 2{,}945 & 491 \\ 2{,}945 & 2{,}948 & -491 \\ 491 & -491 & 235{,}700 \end{array}\right] \frac{k}{in}$$

Element 2: The angle between x and x is 0°

$C = 1 \qquad S = 0$

where

$$\frac{E}{L} = \frac{30\times10^{6}}{480} = 62.5\ k/in^{3} \qquad \frac{12I}{L^2} = \frac{12(1{,}000)}{(12\times40)^2} = 0.0521\, in^2$$

$$\frac{6I}{L} = \frac{6(1{,}000)}{12\times40} = 12.5\ in^{3}$$

Therefore, for element 2:

$$k^{(2)} = 62.50\left[\begin{array}{ccc} d_{2x} & d_{2y} & \phi_2 \\ \hline 100 & 0 & 0 \\ 0 & 0.052 & 12.5 \\ 0 & 12.5 & 4{,}000 \end{array}\right] \frac{k}{in}$$

Simplifying the above equation:

$$k^{(2)} = \left[\begin{array}{ccc} d_{2x} & d_{2y} & \phi_2 \\ \hline 6{,}250 & 0 & 0 \\ 0 & 3.25 & 781.25 \\ 0 & 781.25 & 250{,}000 \end{array}\right] \frac{k}{in}$$

The global beam equations reduce to:

$$\left\{\begin{array}{c} 0 \\ -20\ k \\ -1{,}600\ k\cdot in \end{array}\right\} = \left[\begin{array}{ccc} 9{,}198 & 2{,}945 & 491 \\ 2{,}945 & 2{,}951 & 290 \\ 491 & 290 & 485{,}700 \end{array}\right] \left\{\begin{array}{c} d_{2x} \\ d_{2y} \\ \phi_2 \end{array}\right\}$$

Solving the above equations gives:

$$\left\{\begin{array}{c} d_{2x} \\ d_{2y} \\ \phi_2 \end{array}\right\} = \left\{\begin{array}{c} 0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \end{array}\right\}$$

Element 1: The element force-displacement equations can be obtained using $\hat{f} = \hat{k}\overline{T}d$. Therefore, $\overline{T}d$ is:

$$\overline{T}d = \begin{bmatrix} 0.707 & 0.707 & 0 & 0 & 0 & 0 \\ -0.707 & 0.707 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.707 & 0.707 & 0 \\ 0 & 0 & 0 & -0.707 & 0.707 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00452\ in \\ -0.0092\ in \\ -0.0033\ rad \end{Bmatrix}$$

Recall the elemental stiffness matrix is a function of values C_1, C_2, and L

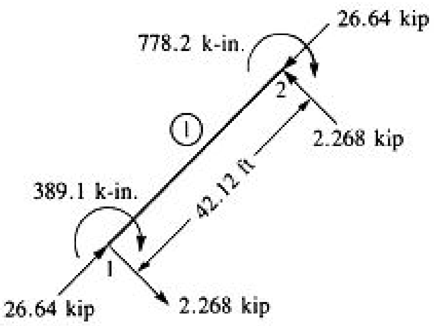
$$C_1 = \frac{AE}{L} = \frac{(100)30\times10^6}{12\times30\sqrt{2}} = 5{,}893\ {}^{k}\!/\!_{in} \qquad C_2 = \frac{EI}{L^3} = \frac{30\times10^6(1{,}000)}{\left(12\times30\sqrt{2}\right)^3} = 0.2273\ {}^{k}\!/\!_{in}$$

Therefore, the local force-displacement equations are:

$$\hat{f}_{(1)} = \hat{k}\overline{T}d = \begin{bmatrix} 5{,}893 & 0 & 10 & -5{,}893 & 0 & 0 \\ 0 & 2.730 & 694.8 & 0 & -2.730 & 694.8 \\ 10 & 694.8 & 117{,}900 & 0 & -694.8 & 117{,}000 \\ -5{,}893 & 0 & 0 & 5{,}983 & 0 & 0 \\ 0 & -2.730 & -694.8 & 0 & 2.730 & -694.8 \\ 0 & 694.8 & 117{,}000 & 0 & -694.8 & 235{,}800 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00452\ in \\ -0.0092\ in \\ -0.0033\ rad \end{Bmatrix}$$

Simplifying the above equations gives:

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \begin{Bmatrix} 26.64\ k \\ -2.268\ k \\ -389.1\ k\cdot in \\ -26.64\ k \\ 2.268\ k \\ -778.2\ k\cdot in \end{Bmatrix}$$



Element 2: The element force-displacement equations are:

$$\overline{T}d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Recall the elemental stiffness matrix is a function of values C_1, C_2, and L

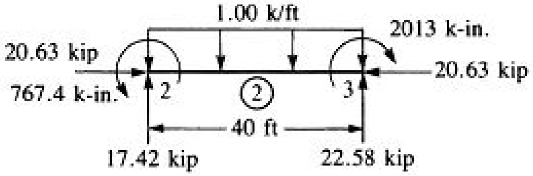
$$C_1 = \frac{AE}{L} = \frac{(100)30\times10^6}{12\times40} = 6{,}250\ {}^{k}\!/_{in} \qquad C_2 = \frac{EI}{L^3} = \frac{30\times10^6(1{,}000)}{(12\times40)^3} = 0.2713\ {}^{k}\!/_{in}$$

Therefore, the local force-displacement equations are:

$$\hat{f}_{(2)} = \hat{k}\overline{T}d = \begin{bmatrix} 6{,}250 & 0 & 0 & -6{,}250 & 0 & 0 \\ 0 & 3.25 & 781.1 & 0 & -3.25 & 781.1 \\ 0 & 781.1 & 250{,}000 & 0 & -781.1 & 125{,}000 \\ -6{,}250 & 0 & 0 & 6{,}250 & 0 & 0 \\ 0 & -3.25 & -781.1 & 0 & 3.25 & -781.1 \\ 0 & 781.1 & 125{,}000 & 0 & -781.1 & 250{,}00 \end{bmatrix} \begin{Bmatrix} -0.0033\ in \\ -0.0097\ in \\ -0.0033\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Simplifying the above equations gives:

$$\hat{k}\hat{d} = \begin{Bmatrix} 20.63\ k \\ -2.58\ k \\ -832.57\ k\cdot in \\ -20.63\ k \\ 2.58\ k \\ -412.50\ k\cdot in \end{Bmatrix}$$

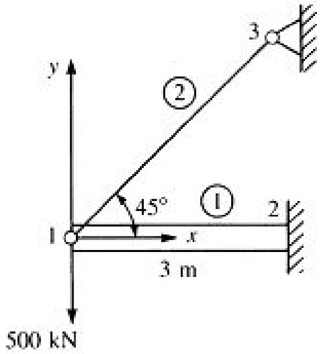


To obtain the actual element local forces, we must subtract the equivalent nodal forces.

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \\ \hat{f}_{3x} \\ \hat{f}_{3y} \\ \hat{m}_3 \end{Bmatrix} = \begin{Bmatrix} 20.63\ k \\ -2.58\ k \\ -832.57\ k\cdot in \\ -20.63\ k \\ 2.58\ k \\ -412.50\ k\cdot in \end{Bmatrix} - \begin{Bmatrix} 0 \\ -20\ k \\ -1600\ k\cdot in \\ 0 \\ -20\ k \\ 1600\ k\cdot in \end{Bmatrix} = \begin{Bmatrix} 20.63\ k \\ 17.42\ k \\ 767.4\ k\cdot in \\ -20.63\ k \\ 22.58\ k \\ -2{,}013\ k\cdot in \end{Bmatrix}$$

Example # 3

The frame shown on the right is fixed at nodes 2 and 3 and subjected to a concentrated load of 500 *kN* applied at node 1. For the bar, $A = 1 \times 10^{-3}$ m^2, for the beam, $A = 2 \times 10^{-3}$ m^2, $I = 5 \times 10^{-5}$ m^4, and $L = 3$ m. Let $E = 210$ GPa for both elements.



Beam Element 1: The angle between x and $\hat{x}$ is 0°

$C = 1 \qquad S = 0$

where

$$\frac{12I}{L^2} = \frac{12(5\times10^{-5})}{(3)^2} = 6.67\times10^{-5}\ m^2 \qquad \frac{6I}{L} = \frac{6(5\times10^{-5})}{3} = 10^{-4}\ m^3$$

$$\frac{E}{L} = \frac{210\times10^6}{3} = 70\times10^6\ kN/m^3$$

Therefore, for element 1:

$$k^{(1)} = 70\times10^3 \begin{array}{c} \begin{matrix} d_{1x} & \quad d_{1y} & \quad \phi_1 \end{matrix} \\ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.067 & 0.10 \\ 0 & 0.10 & 0.20 \end{bmatrix} \end{array} {}^{kN}\!/_{m}$$

Bar Element 2: The angle between x and $\hat{x}$ is 45°

$C = 0.707 \qquad S = 0.707$

where

$$k^{(2)} = \frac{10^{-3}\,m^2\left(210\times10^6\,kN/m^2\right)}{4.24\ m} \begin{array}{c} \begin{matrix} d_{1x} & d_{1y} \end{matrix} \\ \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \end{array} {}^{kN}\!/_{m}$$

$$k^{(2)} = 70\times10^3 \begin{array}{c} \begin{matrix} d_{1x} & \quad d_{1y} \end{matrix} \\ \begin{bmatrix} 0.354 & 0.354 \\ 0.354 & 0.354 \end{bmatrix} \end{array} {}^{kN}\!/_{m}$$

Assembling the elemental stiffness matrices we obtain the global stiffness matrix

$$K = 70\times10^3 \begin{bmatrix} 2.354 & 0.354 & 0 \\ 0.354 & 0.421 & 0.10 \\ 0 & 0.10 & 0.20 \end{bmatrix} {}^{kN}\!/_{m}$$

The global equations are:

$$\begin{Bmatrix} 0 \\ -500\ kN \\ 0 \end{Bmatrix} = 70\times 10^3\ kN/m \begin{bmatrix} 2.354 & 0.354 & 0 \\ 0.354 & 0.421 & 0.10 \\ 0 & 0.10 & 0.20 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ \phi_1 \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} d_{1x} \\ d_{1y} \\ \phi_1 \end{Bmatrix} = \begin{Bmatrix} 0.00388\ m \\ -0.0225\ m \\ 0.0113\ rad \end{Bmatrix}$$

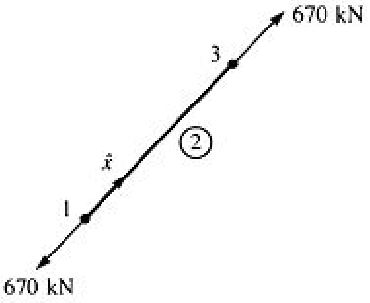
Bar Element: The bar element force-displacement equations can be obtained using $\hat{f} = \hat{k}\overline{T}d$.

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{3x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$

Therefore, the forces in the bar element are:

$$\hat{f}_{1x} = \frac{AE}{L}\left(Cd_{1x} + Sd_{1y}\right) = -670\ kN$$

$$\hat{f}_{3x} = -\frac{AE}{L}\left(Cd_{1x} + Sd_{1y}\right) = 670\ kN$$



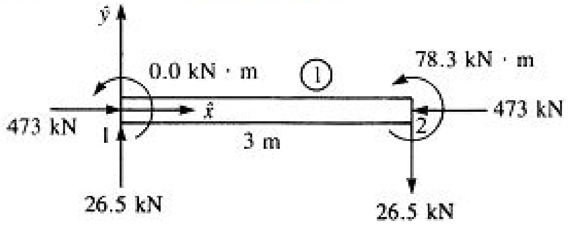
Beam Element: The beam element force-displacement equations can be obtained using $\hat{f} = \hat{k}\hat{d}$. Since the local axis coincides with the global coordinate system, and the displacements at node 2 are zero. Therefore, the local force-displacement equations are:

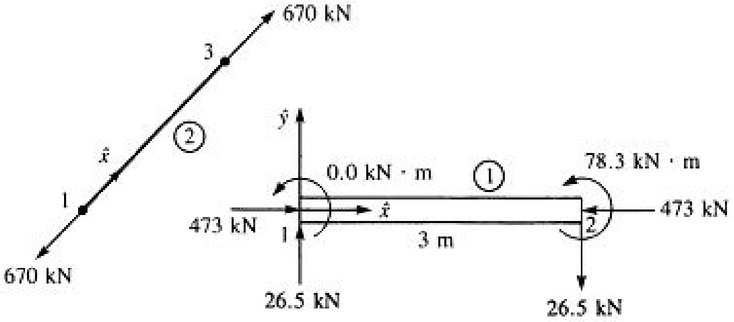
$$\hat{k} = \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{bmatrix} \qquad C_1 = \frac{AE}{L} \qquad C_2 = \frac{EI}{L^3}$$

$$\hat{f}_{(1)} = \hat{k}\hat{d} = 70\times 10^3 \begin{bmatrix} 2 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0.067 & 0.10 & 0 & -0.067 & 0.10 \\ 0 & 0.10 & 0.20 & 0 & -0.10 & 0.10 \\ -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & -0.067 & -0.10 & 0 & 0.067 & -0.10 \\ 0 & 0.10 & 0.10 & 0 & -0.10 & 0.20 \end{bmatrix} \begin{Bmatrix} 0.00388\ m \\ -0.0225\ m \\ 0.0113\ kN\cdot m \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Substituting numerical values into the above equations gives:

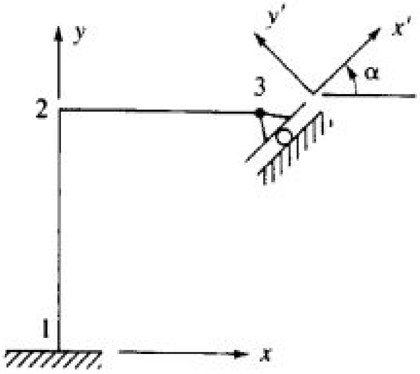
$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \begin{Bmatrix} 473\ kN \\ -26.5\ kN \\ 0.0 \\ -473\ kN \\ 26.5\ kN \\ -78.3\ kN \cdot m \end{Bmatrix}$$





INCLINED OR SKEWED SUPPORTS

If a support is inclined, or skewed, at some angle α for the global x axis, as shown below, the boundary conditions on the displacements are not in the global x-y directions but in the x'-y' directions.



We must transform the local boundary condition of $\mathbf{d'_{3y} = 0}$ (in local coordinates) into the global x-y system. Therefore, the relationship between of the components of the displacement in the local and the global coordinate systems at node 3 is:

$$\begin{Bmatrix} d'_{3x} \\ d'_{3y} \\ \phi'_3 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

We can rewrite the above expression as:

$$\{d'_3\} = [t_3]\{d_3\} \qquad [t_3] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can apply this sort of transformation to the entire displacement vector as:

$$\{d'\} = [T_1]\{d\} \qquad \text{or} \qquad \{d\} = [T_1]^T\{d'\}$$

where the matrix $\mathbf{[T_1]}$ is:

$$[T_1] = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3] \end{bmatrix}$$

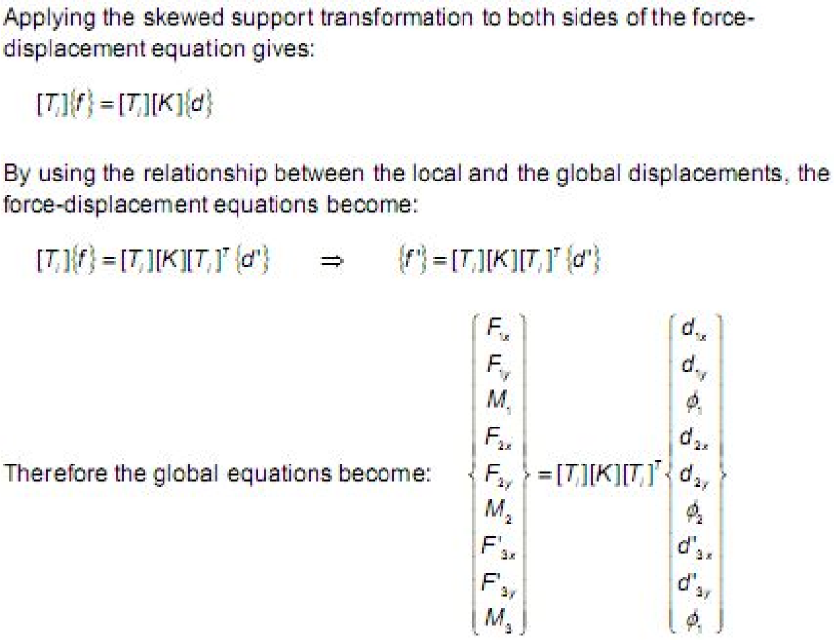
Both the identity matrix $\mathbf{[I]}$ and the matrix $\mathbf{[t_3]}$ are 3 x 3 matrices.

The force vector can be transformed by using the same transformation.

$$\{f'\} = [T_1]\{f\}$$

In global coordinates, the force-displacement equations are:

$$\{f\} = [K]\{d\}$$



Applying the skewed support transformation to both sides of the force-displacement equation gives:

$$[T_i]\{f\} = [T_i][K]\{d\}$$

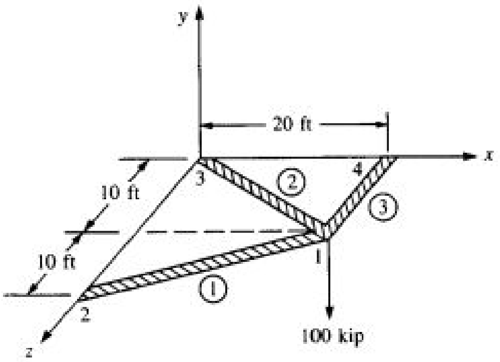
By using the relationship between the local and the global displacements, the force-displacement equations become:

$$[T_i]\{f\} = [T_i][K][T_i]^T\{d'\} \qquad \Rightarrow \qquad \{f'\} = [T_i][K][T_i]^T\{d'\}$$

Therefore the global equations become:
$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ M_1 \\ F_{2x} \\ F_{2y} \\ M_2 \\ F'_{3x} \\ F'_{3y} \\ M_3 \end{Bmatrix} = [T_i][K][T_i]^T \begin{Bmatrix} d_{1x} \\ d_{1y} \\ \phi_1 \\ d_{2x} \\ d_{2y} \\ \phi_2 \\ d'_{3x} \\ d'_{3y} \\ \phi_1 \end{Bmatrix}$$

FEM GRID EXAMPLES
EXAMPLE # 1

Consider the frame shown in the figure below.



The frame is fixed at nodes 2, 3, and 4, and is subjected to a load of 100 *kips* applied at node 1. Assume $I = 400$ in.^{4}, $J = 110$ in.^{4}, $G = 12 \times 10^3$ *ksi*, and $E = 30 \times 10^3$ *ksi* for all elements.

To facilitate a timely solution, the boundary conditions at nodes 2, 3, and 4 are applied to the local stiffness matrices at the beginning of the solution.

$$d_{2y} = \phi_{2x} = \phi_{2z} = 0$$
$$d_{3y} = \phi_{3x} = \phi_{3z} = 0$$
$$d_{4y} = \phi_{4x} = \phi_{4z} = 0$$

Beam Element 1:

$$C = \cos\theta = \frac{x_2 - x_1}{L^{(1)}} = \frac{0-20}{22.36} = -0.894 \qquad S = \sin\theta = \frac{z_2 - z_1}{L^{(1)}} = \frac{20-10}{22.36} = 0.447$$

where

$$\frac{12EI}{L^3} = \frac{12(30 \times 10^3)(400)}{(22.36 \times 12)^3} = 7.45\ {}^{k}\!/_{in} \qquad \frac{6EI}{L^2} = \frac{6(30 \times 10^3)(400)}{(22.36 \times 12)^2} = 1{,}000\ k$$

$$\frac{4EI}{L} = \frac{4(30 \times 10^3)(400)}{(22.36 \times 12)} = 179{,}000\ k \cdot in \qquad \frac{GJ}{L} = \frac{(12 \times 10^3)(110)}{(22.36 \times 12)} = 4{,}920\ k \cdot in$$

The global stiffness matrix for element 1, considering only the parts associated with node 1, and the following relationship:

$$k_g = T_g^T \hat{k}_g T_g$$

$$T_g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & 0.447 \\ 0 & -0.447 & -0.894 \end{bmatrix} \qquad T_g^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & -0.447 \\ 0 & 0.447 & -0.894 \end{bmatrix}$$

$$\hat{k}^{(1)} = \begin{array}{c} \begin{array}{ccc} d_{1y} & \phi_{1x} & \phi_{1z} \end{array} \\ \begin{bmatrix} 7.45 & 0 & 1{,}000 \\ 0 & 4{,}920 & 0 \\ 1{,}000 & 0 & 179{,}000 \end{bmatrix} \end{array} {}^{k}\!/_{in}$$

Therefore, the global stiffness matrix is

$$k^{(1)} = \begin{array}{c} \begin{matrix} d_{1y} & \phi_{1x} & \phi_{1z} \end{matrix} \\ \begin{bmatrix} 7.45 & -447 & -894 \\ -447 & 39{,}700 & 69{,}600 \\ -894 & 69{,}600 & 144{,}000 \end{bmatrix} \end{array} {}^{k}\!/_{in}$$

Beam Element 2:

$$C = \cos\theta = \frac{x_3 - x_1}{L^{(2)}} = \frac{0-20}{22.36} = -0.894 \qquad S = \sin\theta = \frac{z_3 - z_1}{L^{(2)}} = \frac{0-10}{22.36} = -0.447$$

where

$$\frac{12EI}{L^3} = \frac{12(30\times10^3)(400)}{(22.36\times12)^3} = 7.45\ {}^{k}\!/_{in} \qquad \frac{6EI}{L^2} = \frac{6(30\times10^3)(400)}{(22.36\times12)^2} = 1{,}000\ k$$

$$\frac{4EI}{L} = \frac{4(30\times10^3)(400)}{(22.36\times12)} = 179{,}000\ k\cdot in \qquad \frac{GJ}{L} = \frac{(12\times10^3)(110)}{(22.36\times12)} = 4{,}920\ k\cdot in$$

The global stiffness matrix for element 2, considering only the parts associated with node 1, and the following relationship:

$$k_g = T_g{}^T \hat{k}_g T_g$$

$$k^{(2)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & 0.447 \\ 0 & -0.447 & -0.894 \end{bmatrix} \begin{bmatrix} 7.45 & 0 & 1{,}000 \\ 0 & 4{,}920 & 0 \\ 1{,}000 & 0 & 179{,}000 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & -0.447 \\ 0 & 0.447 & -0.894 \end{bmatrix}$$

Therefore, the global stiffness matrix is

$$k^{(2)} = \begin{array}{c} \begin{matrix} d_{1y} & \phi_{1x} & \phi_{1z} \end{matrix} \\ \begin{bmatrix} 7.45 & 447 & -894 \\ 447 & 39{,}700 & -69{,}600 \\ -894 & -69{,}600 & 144{,}000 \end{bmatrix} \end{array} {}^{k}\!/_{in}$$

Beam Element 3:

$$C = \cos\theta = \frac{x_4 - x_1}{L^{(3)}} = \frac{20-20}{10} = 0 \qquad S = \sin\theta = \frac{z_4 - z_1}{L^{(3)}} = \frac{0-10}{10} = -1$$

where

$$\frac{12EI}{L^3} = \frac{12(30\times10^3)(400)}{(10\times12)^3} = 83.3\ k/in \qquad \frac{6EI}{L^2} = \frac{6(30\times10^3)(400)}{(10\times12)^2} = 5{,}000\ k$$

$$\frac{4EI}{L} = \frac{4(30\times10^3)(400)}{(10\times12)} = 400{,}000\ k\cdot in \qquad \frac{GJ}{L} = \frac{(12\times10^3)(110)}{(10\times12)} = 11{,}000\ k\cdot in$$

The global stiffness matrix for element 3, considering only the parts associated with node 1, and the following relationship:

$$k_g = T_g{}^T \hat{k}_g T_g$$

$$k^{(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 83.3 & 0 & 5{,}000 \\ 0 & 11{,}000 & 0 \\ 5{,}000 & 0 & 400{,}000 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Therefore, the global stiffness matrix is

$$k_{(3)} = \begin{array}{c} \begin{array}{ccc} d_{1y} & \phi_{1x} & \phi_{1z} \end{array} \\ \begin{bmatrix} 83.3 & 5{,}000 & 0 \\ 5{,}000 & 400{,}000 & 0 \\ 0 & 0 & 11{,}000 \end{bmatrix} \end{array}$$

Superimposing the three elemental stiffness matrices gives:

$$K = \begin{array}{c} \begin{array}{ccc} d_{1y} & \phi_{1x} & \phi_{1z} \end{array} \\ \begin{bmatrix} 98.2 & 5{,}000 & -1{,}790 \\ 5{,}000 & 479{,}000 & 0 \\ -1{,}790 & 0 & 299{,}000 \end{bmatrix} \end{array}$$

The global equations are:

$$\begin{Bmatrix} F_{1y} = -100\ k \\ M_{1x} = 0 \\ M_{1z} = 0 \end{Bmatrix} = \begin{bmatrix} 98.2 & 5{,}000 & -1{,}790 \\ 5{,}000 & 479{,}000 & 0 \\ -1{,}790 & 0 & 299{,}000 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_{1x} \\ \phi_{1z} \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} d_{1y} \\ \phi_{1x} \\ \phi_{1z} \end{Bmatrix} = \begin{Bmatrix} -2.83\ in \\ 0.0295\ rad \\ -0.0169\ rad \end{Bmatrix}$$

Element 1: The grid element force-displacement equations can be obtained using $\hat{f} = \hat{k}_g \overline{T}_g d$.

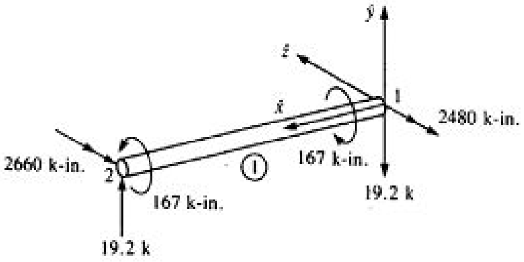
$$\overline{T}_g d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.894 & 0.447 & 0 & 0 & 0 \\ 0 & -0.447 & -0.894 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.894 & 0.447 \\ 0 & 0 & 0 & 0 & -0.447 & -0.894 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ 0.0295\ rad \\ -0.0169\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2.83\ in \\ -0.0339\ rad \\ 0.00192\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Therefore, the local force-displacement equations are:

$$\hat{f}_{(1)} = \hat{k}\overline{T}d = \begin{bmatrix} 7.45 & 0 & 1{,}000 & -7.45 & 0 & 1{,}000 \\ 0 & 4{,}920 & 0 & 0 & -4{,}920 & 0 \\ 1{,}000 & 0 & 179{,}000 & -1{,}000 & 0 & 89{,}500 \\ -7.45 & 0 & -1{,}000 & 7.45 & 0 & -1{,}000 \\ 0 & -4{,}920 & 0 & 0 & 4{,}920 & 0 \\ 1{,}000 & 0 & 89{,}500 & -1{,}000 & 0 & 179{,}000 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ -0.0339\ rad \\ 0.00192\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solving for the forces and moments gives:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_{1x} \\ \hat{m}_{1z} \\ \hat{f}_{2y} \\ \hat{m}_{2x} \\ \hat{m}_{2z} \end{Bmatrix} = \begin{Bmatrix} -19.2\ k \\ -167\ k \cdot in \\ -2{,}480\ k \cdot in \\ 19.2\ k \\ 167\ k \cdot in \\ -2{,}260\ k \cdot in \end{Bmatrix}$$



Element 2: The grid element force-displacement equations can be obtained using $\hat{f} = \hat{k}_G \overline{T}_G d$.

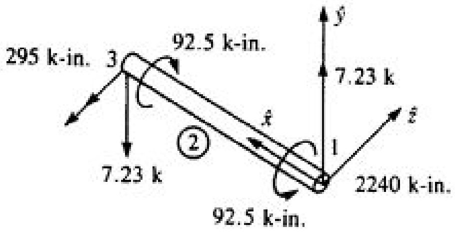
$$\overline{T}_G d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.894 & -0.447 & 0 & 0 & 0 \\ 0 & 0.447 & -0.894 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.894 & -0.447 \\ 0 & 0 & 0 & 0 & 0.447 & -0.894 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ 0.0295\ rad \\ -0.0169\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2.83\ in \\ -0.0188\ rad \\ 0.0283\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Therefore, the local force-displacement equations are:

$$\hat{f}_{(2)} = \hat{k}\overline{T}d = \begin{bmatrix} 7.45 & 0 & 1{,}000 & -7.45 & 0 & 1{,}000 \\ 0 & 4{,}920 & 0 & 0 & -4{,}920 & 0 \\ 1{,}000 & 0 & 179{,}000 & -1{,}000 & 0 & 89{,}500 \\ -7.45 & 0 & -1{,}000 & 7.45 & 0 & -1{,}000 \\ 0 & -4{,}920 & 0 & 0 & 4{,}920 & 0 \\ 1{,}000 & 0 & 89{,}500 & -1{,}000 & 0 & 179{,}000 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ -0.0188\ rad \\ 0.0283\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solving for the forces and moments gives:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_{1x} \\ \hat{m}_{1z} \\ \hat{f}_{3y} \\ \hat{m}_{3x} \\ \hat{m}_{3z} \end{Bmatrix} = \begin{Bmatrix} 7.23\ k \\ -92.5\ k \cdot in \\ -2{,}240\ k \cdot in \\ -7.23\ k \\ 92.5\ k \cdot in \\ -295\ k \cdot in \end{Bmatrix}$$



Element 3: The grid element force-displacement equations can be obtained using $\hat{f} = \hat{k}_G \overline{T}_G d$.

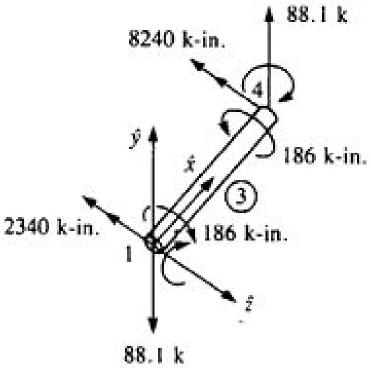
$$\overline{T}_G d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ 0.0295\ rad \\ -0.0169\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2.83\ in \\ 0.0169\ rad \\ 0.0295\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

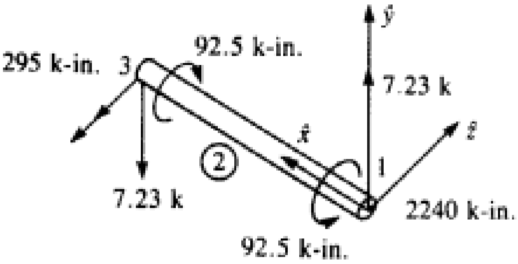
Therefore, the local force-displacement equations are:

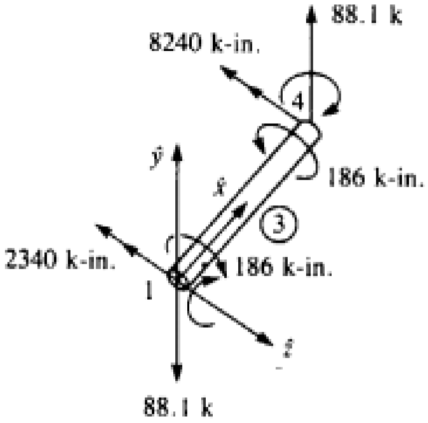
$$\hat{f}_{(3)} = \hat{k}\overline{T}d = \begin{bmatrix} 83.3 & 0 & 5{,}000 & -83.3 & 0 & 5{,}000 \\ 0 & 11{,}000 & 0 & 0 & -11{,}000 & 0 \\ 5{,}000 & 0 & 400{,}000 & -5{,}000 & 0 & 200{,}000 \\ -83.3 & 0 & -5{,}000 & 83.3 & 0 & -5{,}000 \\ 0 & -11{,}000 & 0 & 0 & 11{,}000 & 0 \\ 5{,}000 & 0 & 200{,}000 & -5{,}000 & 0 & 400{,}000 \end{bmatrix} \begin{Bmatrix} -2.83\ in \\ 0.0169\ rad \\ 0.0295\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

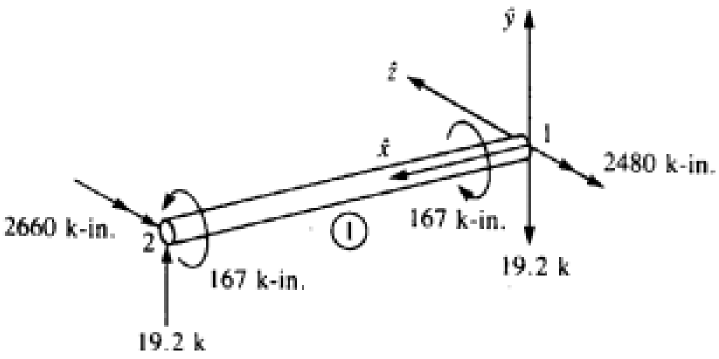
Solving for the forces and moments gives:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_{1x} \\ \hat{m}_{1z} \\ \hat{f}_{4y} \\ \hat{m}_{4x} \\ \hat{m}_{4z} \end{Bmatrix} = \begin{Bmatrix} -88.1\ k \\ 186\ k \cdot in \\ -2{,}340\ k \cdot in \\ 88.1\ k \\ -186\ k \cdot in \\ -8{,}240\ k \cdot in \end{Bmatrix}$$









To check the equilibrium of node 1 the local forces and moments for each element need to be transformed to global coordinates. Recall, that:

$$\hat{f} = Tf \quad \Rightarrow \quad f = T^T \hat{f} \qquad T^T = T^{-1}$$

Since we are only checking the forces and moments at node 1, we need only the upper-left-hand portion of the transformation matrix $\boldsymbol{T_G}$.

Therefore; for **Element 1:**

$$\begin{Bmatrix} f_{1y} \\ m_{1x} \\ m_{1z} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & -0.447 \\ 0 & 0.447 & -0.894 \end{bmatrix} \begin{Bmatrix} -19.2\ k \\ -167\ k \cdot in \\ -2{,}480\ k \cdot in \end{Bmatrix} = \begin{Bmatrix} -19.2\ k \\ 1{,}260\ k \cdot in \\ 2{,}150\ k \cdot in \end{Bmatrix}$$

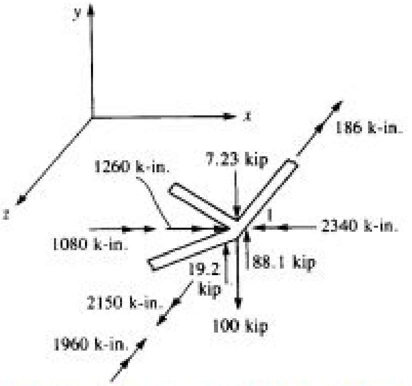
Therefore; for **Element 2:**

$$\begin{Bmatrix} f_{1y} \\ m_{1x} \\ m_{1z} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.894 & 0.447 \\ 0 & -0.447 & -0.894 \end{bmatrix} \begin{Bmatrix} -7.23\ k \\ -92.5\ k \cdot in \\ -2{,}240\ k \cdot in \end{Bmatrix} = \begin{Bmatrix} 7.23\ k \\ 1{,}080\ k \cdot in \\ -1{,}960\ k \cdot in \end{Bmatrix}$$

Therefore; for **Element 3:**

$$\begin{Bmatrix} f_{1y} \\ m_{1x} \\ m_{1z} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} -88.1 k \\ -2{,}340\ k \cdot in \\ -186\ k \cdot in \end{Bmatrix} = \begin{Bmatrix} -88.1 k \\ -2{,}340\ k \cdot in \\ -186\ k \cdot in \end{Bmatrix}$$

The forces and moments that are applied to node 1 by each element are equal in magnitude and opposite direction. Therefore the sum of the forces and moments acting on node 1 are:



$$\sum F_y = -100 - 7.23 + 19.2 + 88.1 = 0.07\ k$$

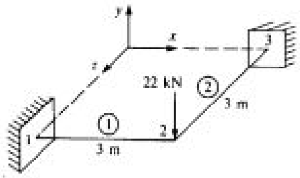
$$\sum M_{1x} = -1{,}260 - 1{,}080 + 2{,}340 = 0.0\ k \cdot in$$

$$\sum M_{1z} = -2{,}150 + 1{,}060 + 186 = -4.0\ k \cdot in$$

The forces and moments accurately satisfy equilibrium considering the amount of truncation error inherent in results of the calculations presented in this example.

EXAMPLE # 2

Consider the frame shown in the figure below.



The frame is fixed at nodes 1 and 3, and is subjected to a load of 22 kN applied at node 2. Assume I = 16.6 x 10^{-5} m^4, J = 4.6 x 10^{-5} m^4, G = 84 GPa, and E = 210 GPa for all elements.

To facilitate a timely solution, the boundary conditions at nodes 1 and 3 are applied to the local stiffness matrices at the beginning of the solution.

$$d_{1y} = \phi_{1x} = \phi_{1z} = 0$$
$$d_{3y} = \phi_{3x} = \phi_{3z} = 0$$

Beam Element 1: the local x axis coincides with the global x axis

$$C = \cos\theta = \frac{x_2 - x_1}{L^{(1)}} = \frac{3}{3} = 1 \qquad S = \sin\theta = \frac{z_2 - z_1}{L^{(1)}} = \frac{0}{3} = 0$$

where

$$\frac{12EI}{L^3} = \frac{12(210\times10^6)(16.6\times10^{-5})}{(3)^3} = 1.55\times10^4\,kN/m$$

$$\frac{6EI}{L^2} = \frac{6(210\times10^6)(16.6\times10^{-5})}{(3)^2} = 2.32\times10^4\,kN$$

$$\frac{4EI}{L} = \frac{4(210\times10^6)(16.6\times10^{-5})}{3} = 4.65\times10^4\,kN{\cdot}m$$

$$\frac{GJ}{L} = \frac{(84\times10^6)(4.6\times10^{-5})}{3} = 0.128\times10^4\,kN{\cdot}m$$

The global stiffness matrix for element 1, considering only the parts associated with node 2, may be obtained from the following relationship:

$$k_G = T_G^T \hat{k}_G T_G$$

$$k^{(1)} = 10^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.55 & 0 & -2.32 \\ 0 & 0.128 & 0 \\ -2.32 & 0 & 4.65 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} kN/m$$

Therefore, the global stiffness matrix is

$$k_{(1)} = 10^4 \begin{array}{c} \begin{array}{ccc} d_{2y} & \phi_{2x} & \phi_{2z} \end{array} \\ \begin{bmatrix} 1.55 & 0 & -2.32 \\ 0 & 0.128 & 0 \\ -2.32 & 0 & 4.65 \end{bmatrix} \end{array} kN/m$$

Beam Element 2: the local $\hat{x}$ axis is located from node 2 to node 3

$$C = \cos\theta = \frac{x_3 - x_2}{L^{(2)}} = \frac{0}{3} = 0 \qquad S = \sin\theta = \frac{z_3 - z_2}{L^{(1)}} = \frac{-3}{3} = -1$$

The global stiffness matrix for element 2, considering only the parts associated with node 2, may be obtained using:

$$k_G = T_G^T \hat{k}_G T_G$$

$$k^{(2)} = 10^4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1.55 & 0 & -2.32 \\ 0 & 0.128 & 0 \\ -2.32 & 0 & 4.65 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} kN/m$$

Therefore, the global stiffness matrix is

$$k_{(2)} = 10^4 \begin{array}{c} \begin{array}{ccc} d_{2y} & \phi_{2x} & \phi_{2z} \end{array} \\ \begin{bmatrix} 1.55 & 2.32 & 0 \\ 2.32 & 4.65 & 0 \\ 0 & 0 & 0.128 \end{bmatrix} \end{array} kN/m$$

Superimposing the two elemental stiffness matrices gives:

$$K = 10^4 \begin{array}{c} \begin{array}{ccc} d_{2y} & \phi_{2x} & \phi_{2z} \end{array} \\ \begin{bmatrix} 3.10 & 2.32 & -2.32 \\ 2.32 & 4.78 & 0 \\ -2.32 & 0 & 4.78 \end{bmatrix} \end{array} kN/m$$

The global equations are:

$$\begin{Bmatrix} F_{2y} = -22\ kN \\ M_{2x} = 0 \\ M_{2z} = 0 \end{Bmatrix} = 10^4 \begin{bmatrix} 3.10 & 2.32 & -2.32 \\ 2.32 & 4.78 & 0 \\ -2.32 & 0 & 4.78 \end{bmatrix} \begin{Bmatrix} d_{2y} \\ \phi_{2x} \\ \phi_{2x} \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} d_{2y} \\ \phi_{2x} \\ \phi_{2z} \end{Bmatrix} = \begin{Bmatrix} -0.00259\ m \\ 0.00126\ rad \\ -0.00126\ rad \end{Bmatrix}$$

Element 1: The grid element force-displacement equations can be obtained using $\hat{f} = \hat{k}_G \overline{T}_G d$.

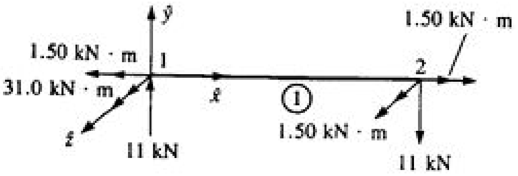
$$\overline{T}_G d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00259\ m \\ 0.00126\ rad \\ -0.00126\ rad \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00259\ m \\ 0.00126\ rad \\ -0.00126\ rad \end{Bmatrix}$$

Therefore, the local force-displacement equations are:

$$\hat{f}_{(1)} = \hat{k}\overline{T}d = 10^4 \begin{bmatrix} 1.55 & 0 & 2.32 & -1.55 & 0 & 2.32 \\ 0 & 0.128 & 0 & 0 & -0.128 & 0 \\ 2.32 & 0 & 4.65 & -2.32 & 0 & 2.33 \\ -1.55 & 0 & -2.32 & 1.55 & 0 & -2.32 \\ 0 & -0.128 & 0 & 0 & 0.128 & 0 \\ 2.32 & 0 & 2.33 & -2.32 & 0 & 4.65 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00259\ m \\ 0.00126\ rad \\ -0.00126\ rad \end{Bmatrix}$$

Solving for the forces and moments gives:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_{1x} \\ \hat{m}_{1z} \\ \hat{f}_{2y} \\ \hat{m}_{2x} \\ \hat{m}_{2z} \end{Bmatrix} = \begin{Bmatrix} 11.0\ kN \\ -1.50\ kN \cdot m \\ 31.0\ kN \cdot m \\ -11.0\ kN \\ 1.50\ kN \cdot m \\ 1.50\ kN \cdot m \end{Bmatrix}$$



Element 2: The grid element force-displacement equations can be obtained using $\hat{f} = \hat{k}_G\overline{T}_G d$.

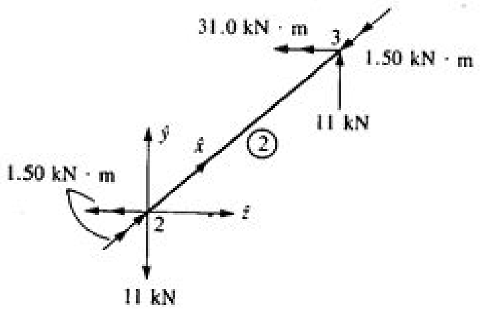
$$\overline{T}_G d = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} -0.00259\ m \\ 0.00126\ rad \\ -0.00126\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -0.00259\ m \\ 0.00126\ rad \\ 0.00126\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Therefore, the local force-displacement equations are:

$$\hat{f}_{(2)} = \hat{k}\overline{T}d = 10^4 \begin{bmatrix} 1.55 & 0 & 2.32 & -1.55 & 0 & 2.32 \\ 0 & 0.128 & 0 & 0 & -0.128 & 0 \\ 2.32 & 0 & 4.65 & -2.32 & 0 & 2.33 \\ -1.55 & 0 & -2.32 & 1.55 & 0 & -2.32 \\ 0 & -0.128 & 0 & 0 & 0.128 & 0 \\ 2.32 & 0 & 2.33 & -2.32 & 0 & 4.65 \end{bmatrix} \begin{Bmatrix} -0.00259\ m \\ 0.00126\ rad \\ 0.00126\ rad \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solving for the forces and moments gives:

$$\begin{Bmatrix} \hat{f}_{2y} \\ \hat{m}_{2x} \\ \hat{m}_{2z} \\ \hat{f}_{3y} \\ \hat{m}_{3x} \\ \hat{m}_{3z} \end{Bmatrix} = \begin{Bmatrix} -11.0\ kN \\ 1.50\ kN \cdot m \\ -1.50\ kN \cdot m \\ 11.0\ kN \\ -1.50\ kN \cdot m \\ -31.0\ kN \cdot m \end{Bmatrix}$$



The resulting free-body diagrams: