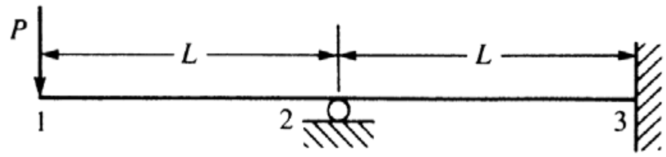
<u>Example # 1</u>

Solve the propped cantilever beam subjected to end load P. The beam is assumed to have constant EI and length 2L. It is supported by a roller at mid length and is built in at the right end.



The beam element stiffness matrices are:

$$k^{(1)} = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} d_{1y} & \phi_1 & d_{2y} & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

$$k^{(2)} = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} d_{2y} & \phi_2 & d_{3y} & \phi_3 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

The total stiffness matrix is:

$$K = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

The beam stiffness equations are:

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix}$$

The boundary conditions are:

$$d_{2y} = d_{3y} = \phi_3 = 0$$

By applying the boundary conditions the beam equations reduce to:

$$\begin{Bmatrix} -P \\ 0 \\ 0 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ \phi_2 \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} d_{1y} \\ \phi_1 \\ \phi_2 \end{Bmatrix} = \frac{PL^2}{4EI} \begin{Bmatrix} -\frac{7L}{3} \\ 3 \\ 1 \end{Bmatrix}$$

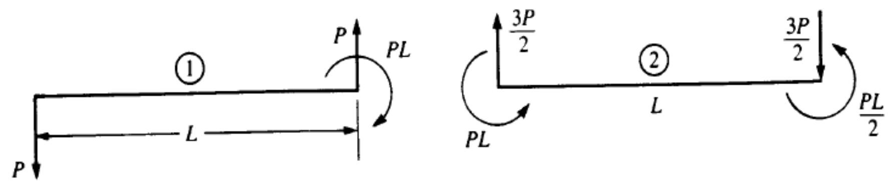
The local nodal forces for element 1:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{P}{4L} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -7L/3 \\ 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \\ P \\ -PL \end{Bmatrix}$$

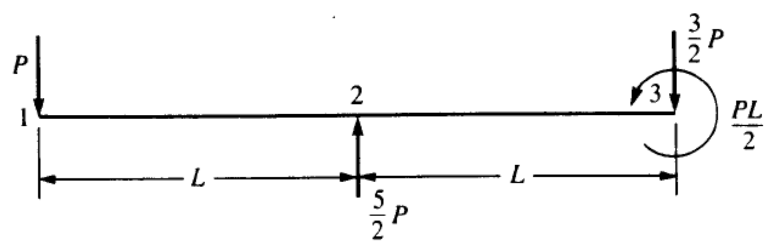
The local nodal forces for element 2:

$$\begin{Bmatrix} \hat{f}_{2y} \\ \hat{m}_2 \\ \hat{f}_{3y} \\ \hat{m}_3 \end{Bmatrix} = \frac{P}{4L}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1.5P \\ PL \\ -1.5P \\ 0.5PL \end{Bmatrix}$$

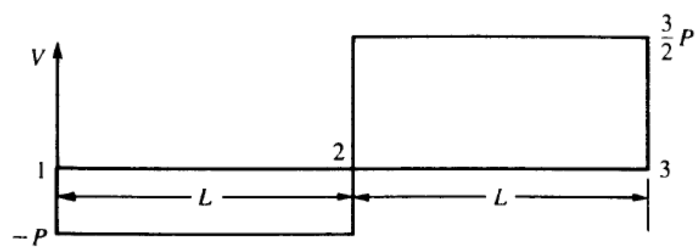
The free-body diagrams for the each element are shown below.

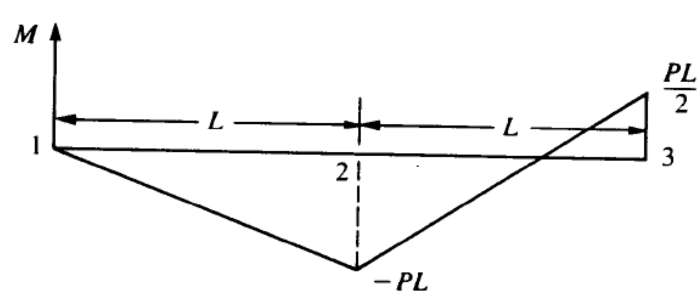


Combining the elements gives the forces and moments for the original beam.



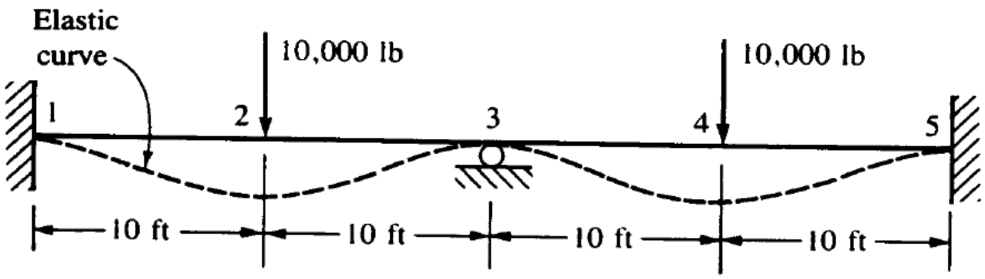
Therefore, the shear force and bending moment diagrams are:





Example # 2

Consider the beam shown below. Assume $\boldsymbol{E} = 30 \times 10^6$ psi and $\boldsymbol{I} = 500$ in.^{4} are constant throughout the beam. Use four elements of equal length to model the beam.



The beam element stiffness matrices are:

$$
\begin{array}{cc}
 & \begin{array}{cccc} d_{iy} & \phi_i & d_{(i+1)y} & \phi_{i+1} \end{array} \\
k^{(i)} = \dfrac{EI}{L^3} & \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}
\end{array}
$$

The beam stiffness equations are:

Element 2

Element 1

$$
\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \\ M_4 \\ F_{5y} \\ M_5 \end{Bmatrix} = \frac{EI}{L^3}
\begin{array}{c}
\begin{array}{cccccccccc} d_{1y} & \phi_1 & d_{2y} & \phi_2 & d_{3y} & \phi_3 & d_{4y} & \phi_4 & d_{5y} & \phi_5 \end{array} \\
\begin{bmatrix}
12 & 6L & -12 & 6L & 0 & 0 & 0 & 0 & 0 & 0 \\
6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-12 & -6L & 12+12 & -6L+6L & -12 & 6L & 0 & 0 & 0 & 0 \\
6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 & 0 & 0 & 0 & 0 \\
0 & 0 & -12 & -6L & 12+12 & -6L+6L & -12 & 6L & 0 & 0 \\
0 & 0 & 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 & 0 & 0 \\
0 & 0 & 0 & 0 & -12 & -6L & 12+12 & -6L+6L & -12 & 6L \\
0 & 0 & 0 & 0 & 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\
0 & 0 & 0 & 0 & 0 & 0 & -12 & -6L & 12 & -6L \\
0 & 0 & 0 & 0 & 0 & 0 & 6L & 2L^2 & -6L & 4L^2
\end{bmatrix}
\end{array}
\begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \\ d_{4y} \\ \phi_4 \\ d_{5y} \\ \phi_5 \end{Bmatrix}
$$

Element 3

Element 4

The boundary conditions are

$$
d_{1y} = \phi_1 = d_{3y} = d_{5y} = \phi_5 = 0
$$

After applying the boundary conditions the global beam equations reduce to:

$$\frac{EI}{L^3}\begin{bmatrix} 24 & 0 & 6L & 0 & 0 \\ 0 & 8L^2 & 2L^2 & 0 & 0 \\ 6L & 2L^2 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -6L & 24 & 0 \\ 0 & 0 & 2L^2 & 0 & 8L^2 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ \phi_2 \\ \phi_3 \\ d_{4y} \\ \phi_4 \end{Bmatrix} = \begin{Bmatrix} -10{,}000\ lb \\ 0 \\ 0 \\ -10{,}000\ lb \\ 0 \end{Bmatrix}$$

Substituting ***L*** = 120 in., ***E*** = 30 x 10^6 psi, and ***I*** = 500 in.^{4} into the above equations and solving for the unknowns gives:

$$d_{2y} = d_{4y} = -0.048\ in \qquad \phi_2 = \phi_3 = \phi_4 = 0$$

The global forces and moments can be determined as:

$$\begin{aligned} F_{1y} &= 5\ kips & M_1 &= 25\ kips{\cdot}ft \\ F_{2y} &= 10\ kips & M_2 &= 0 \\ F_{3y} &= 10\ kips & M_3 &= 0 \\ F_{4y} &= 10\ kips & M_4 &= 0 \\ F_{5y} &= 5\ kips & M_5 &= -25\ kips{\cdot}ft \end{aligned}$$

The local nodal forces for element 1:

$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ -0.048 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 5\ kips \\ 25\ k{\cdot}ft \\ -5\ kips \\ 25\ k{\cdot}ft \end{Bmatrix}$$

The local nodal forces for element 2:

$$\begin{Bmatrix} \hat{f}_{2y} \\ \hat{m}_2 \\ \hat{f}_{3y} \\ \hat{m}_3 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -0.048 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -5\ kips \\ -25\ k{\cdot}ft \\ 5\ kips \\ -25\ k{\cdot}ft \end{Bmatrix}$$

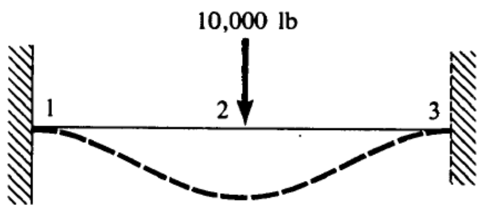
The local nodal forces for element 3:

$$\begin{Bmatrix} \hat{f}_{3y} \\ \hat{m}_3 \\ \hat{f}_{4y} \\ \hat{m}_4 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -0.048 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 5\ kips \\ 25\ k{\cdot}ft \\ -5\ kips \\ 25\ k{\cdot}ft \end{Bmatrix}$$

The local nodal forces for element 4:

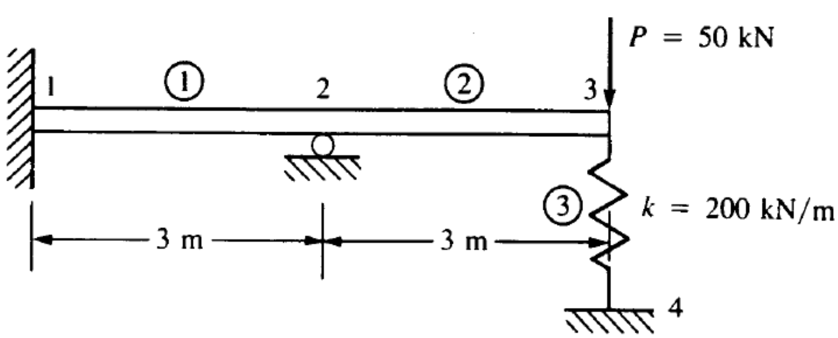
$$\begin{Bmatrix} \hat{f}_{4y} \\ \hat{m}_4 \\ \hat{f}_{5y} \\ \hat{m}_5 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} -0.048 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -5\ kips \\ -25\ k{\cdot}ft \\ 5\ kips \\ -25\ k{\cdot}ft \end{Bmatrix}$$

Note: Due to symmetry about the vertical plane at node 3, we could have worked just half the beam, as shown below.



<u>Example # 3</u>

Consider the beam shown below. Assume ***E*** = 210 GPa and ***I*** = 2 x 10^{-4} m^4 are constant throughout the beam and the spring constant ***k*** = 200 kN/m. Use two beam elements of equal length and one spring element to model the structure.



The beam element stiffness matrices are:

$$k^{(1)} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} d_{1y} & \phi_1 & d_{2y} & \phi_2 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

$$k^{(2)} = \frac{EI}{L^3} \begin{array}{c} \begin{array}{cccc} d_{2y} & \phi_2 & d_{3y} & \phi_3 \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array}$$

The spring element stiffness matrix is:

$$k^{(3)} = \begin{array}{c} \begin{array}{cc} d_{3y} & d_{4y} \end{array} \\ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \end{array} \quad \Rightarrow \quad k^{(3)} = \begin{array}{c} \begin{array}{ccc} d_{3y} & \phi_3 & d_{4y} \end{array} \\ \begin{bmatrix} k & 0 & -k \\ 0 & 0 & 0 \\ -k & 0 & k \end{bmatrix} \end{array}$$

Using the direct stiffness method and superposition gives the global beam equations.

$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L & 0 \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 & 0 \\ 0 & 0 & -12 & -6L & 12+k' & -6L & -k' \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 & 0 \\ 0 & 0 & 0 & 0 & -k' & 0 & k' \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \\ d_{4y} \end{Bmatrix} \qquad k' = \frac{kL^3}{EI}$$

The boundary conditions for this problem are:

$$d_{1y} = \phi_1 = d_{2y} = d_{4y} = 0$$

After applying the boundary conditions the global beam equations reduce to:

$$\begin{Bmatrix} M_2 \\ F_{3y} \\ M_3 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 8L^2 & -6L & 2L^2 \\ -6L & 12+k' & -6L \\ 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

Solving the above equations gives:

$$\begin{Bmatrix} \phi_2 \\ d_{3y} \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{3PL^2}{EI}\left(\dfrac{1}{12+7k'}\right) \\ -\dfrac{7PL^3}{EI}\left(\dfrac{1}{12+7k'}\right) \\ -\dfrac{9PL^2}{EI}\left(\dfrac{1}{12+7k'}\right) \end{Bmatrix} \qquad k' = \frac{kL^3}{EI}$$

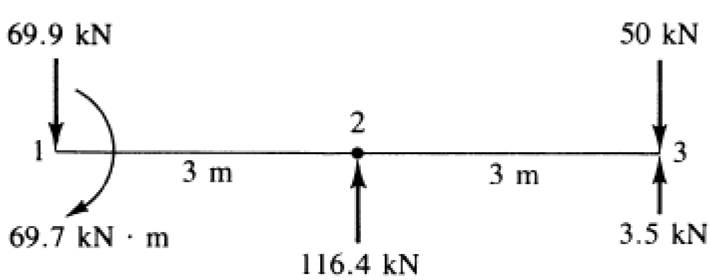
Substituting $\boldsymbol{L}$ = 3 m, $\boldsymbol{E}$ = 210 GPa, $\boldsymbol{I}$ = 2 x 10^{-4} m^4, and $\boldsymbol{k}$ = 200 kN/m in the above equations gives:

$$d_{3y} = -0.0174\ m$$
$$\phi_2 = -0.00249\ rad$$
$$\phi_3 = -0.00747\ rad$$

Substituting the solution back into the global equations gives:

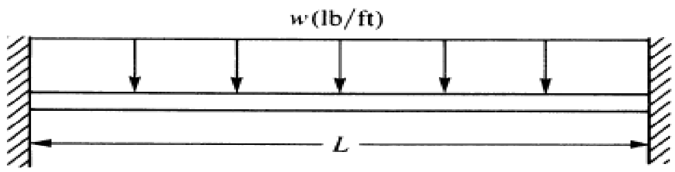
$$\begin{Bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} -69.9\ kN \\ -69.7\ kN \cdot m \\ 116.4\ kN \\ 0 \\ -50\ kN \\ 0 \\ 3.5\ kN \end{Bmatrix}$$

A free-body diagram, including forces and moments acting on the beam is shown below.

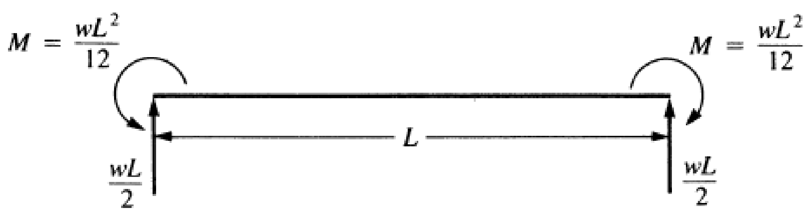


Distributed Loadings

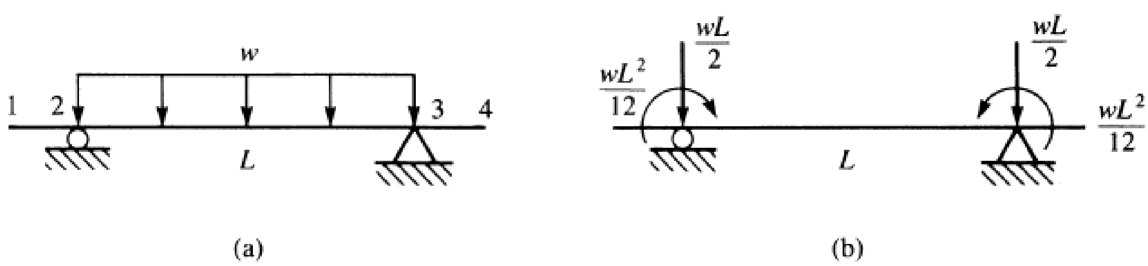
Beam members can support distributed loading as well as concentrated nodal loading. Therefore, we must be able to account for distributed loading. Consider the fixed-fixed beam subjected to a uniformly distributed loading ***w*** shown the figure below. The reactions, determined from structural analysis theory, are called fixed-end reactions. In general, fixed-end reactions are those reactions at the ends of an element if the ends of the element are assumed to be fixed (displacements and rotations are zero). Therefore, guided by the results from structural analysis for the case of a uniformly distributed load, we replace the load by concentrated nodal forces and moments tending to have the same effect on the beam as the actual distributed load.



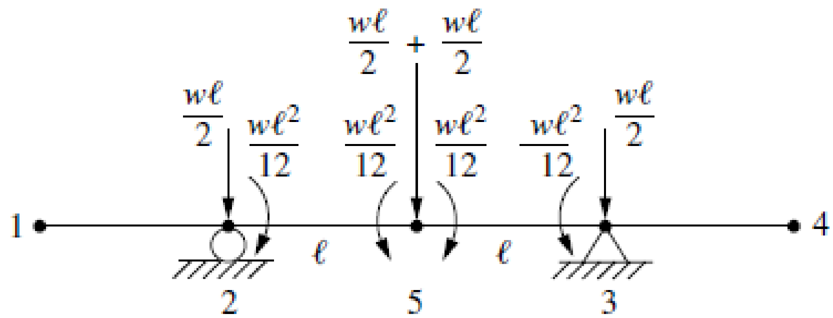
Fixed-fixed beam subjected to a uniformly distributed load



Fixed-end reactions for the above beam



(a) Beam with a distributed load, (b) the equivalent nodal force system,



The enlarged beam (for clarity's sake) with equivalent nodal force system when node 5 is added to the mid span.

Work Equivalence Method

This method is based on the concept that the work done by the distributed load is equal to the work done by the discrete nodal loads. The work done by the distributed load is:

$$W_{distributed} = \int_0 w(\hat{x})\,\hat{v}(\hat{x})\,d\hat{x}$$

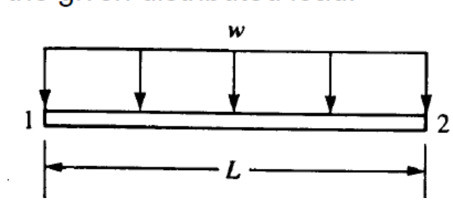
where $\hat{v}(\hat{x})$ is the transverse displacement. The work done by the discrete nodal forces is:

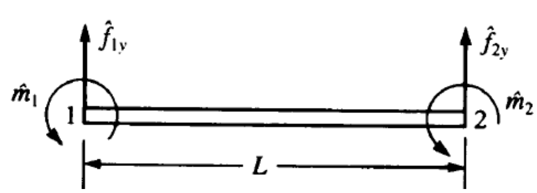
$$W_{nodes} = \hat{m}_1\hat{\phi}_1 + \hat{m}_2\hat{\phi}_2 + \hat{f}_{1y}\hat{d}_{1y} + \hat{f}_{2y}\hat{d}_{2y}$$

The nodal forces can be determined by setting $W_{distributed} = W_{nodes}$ for arbitrary displacements and rotations.

$$W_{distributed} = W_{nodes}$$

Consider the beam, shown below, and determine the equivalent nodal forces for the given distributed load.





Using the work equivalence method or:

$$W_{distributed} = W_{nodes}$$

we get:

$$\int_0^L w(\hat{x})\,\hat{v}(\hat{x})\,d\hat{x} = \hat{m}_1\hat{\phi}_1 + \hat{m}_2\hat{\phi}_2 + \hat{f}_{1y}\hat{d}_{1y} + \hat{f}_{2y}\hat{d}_{2y}$$

Evaluating the left-hand-side of the above expression using $w(\hat{x})$ = -w and $\hat{v}(\hat{x})$ equal to:

$$\hat{v}(\hat{x}) = \left[\frac{2}{L^3}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) + \frac{1}{L^2}\left(\hat{\phi}_1 + \hat{\phi}_2\right)\right]\hat{x}^3 + \left[-\frac{3}{L^2}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) - \frac{1}{L}\left(2\hat{\phi}_1 + \hat{\phi}_2\right)\right]\hat{x}^2 + \hat{\phi}_1\hat{x} + \hat{d}_{1y}$$

gives:

$$\int_0^L w\,\hat{v}(\hat{x})\,d\hat{x} = \frac{Lw}{2}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) - \frac{L^2w}{4}\left(\hat{\phi}_1 + \hat{\phi}_2\right) - Lw\left(\hat{d}_{2y} - \hat{d}_{1y}\right) + \frac{L^2w}{3}\left(2\hat{\phi}_1 + \hat{\phi}_2\right) - \frac{L^2w}{2}\hat{\phi}_1 - wL\hat{d}_{1y}$$

Using a set of arbitrary nodal displacements, such as:

$$d_{1y} = d_{2y} = \phi_2 = 0 \qquad \phi_1 = 1$$

The resulting nodal equivalent force or moment is:

$$\hat{m}_1(1) = -\left(\frac{wL^2}{4} - \frac{2}{3}L^2w + \frac{L^2}{2}w\right) = -\frac{wL^2}{12}$$

Using another set of arbitrary nodal displacements, such as:

$$d_{1y} = d_{2y} = \phi_1 = 0 \qquad \phi_2 = 1$$

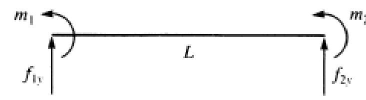
The resulting nodal equivalent force or moment is:

$$\hat{m}_2(1) = -\left(\frac{wL^2}{4} - \frac{wL^2}{3}\right) = \frac{wL^2}{12}$$

Setting the nodal rotations equal zero except for the $\hat{d}_{1y}$ and $\hat{d}_{2y}$ gives:

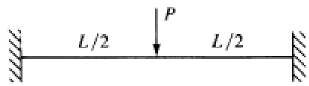
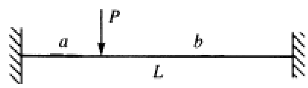
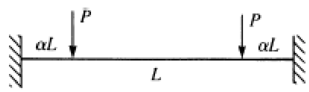
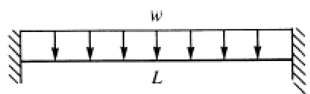
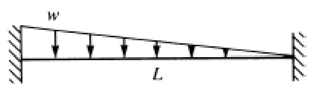
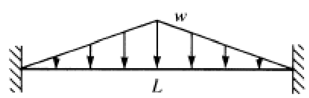
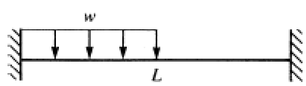
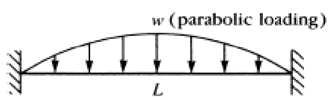
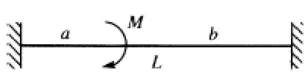
$$\hat{f}_{1y}(1) = -\frac{LW}{2} + Lw - Lw = -\frac{Lw}{2}$$

$$\hat{f}_{2y}(1) = \frac{LW}{2} - Lw = -\frac{Lw}{2}$$



Positive nodal force conventions

Table D–1 Single element equivalent joint forces f_0 for different types of loads

	f_{1y}	m_1	Loading case	f_{2y}	m_2
1.	$\frac{-P}{2}$	$\frac{-PL}{8}$		$\frac{-P}{2}$	$\frac{PL}{8}$
2.	$\frac{-Pb^2(L+2a)}{L^3}$	$\frac{-Pab^2}{L^2}$		$\frac{-Pa^2(L+2b)}{L^3}$	$\frac{Pa^2b}{L^2}$
3.	$-P$	$-\alpha(1-\alpha)PL$		$-P$	$\alpha(1-\alpha)PL$
4.	$\frac{-wL}{2}$	$\frac{-wL^2}{12}$		$\frac{-wL}{2}$	$\frac{wL^2}{12}$
5.	$\frac{-7wL}{20}$	$\frac{-wL^2}{20}$		$\frac{-3wL}{20}$	$\frac{wL^2}{30}$
6.	$\frac{-wL}{4}$	$\frac{-5wL^2}{96}$		$\frac{-wL}{4}$	$\frac{5wL^2}{96}$
7.	$\frac{-13wL}{32}$	$\frac{-11wL^2}{192}$		$\frac{-3wL}{32}$	$\frac{5wL^2}{192}$
8.	$\frac{-wL}{3}$	$\frac{-wL^2}{15}$		$\frac{-wL}{3}$	$\frac{wL^2}{15}$
9.	$\frac{-M(a^2+b^2-4ab-L^2)}{L^3}$	$\frac{Mb(2a-b)}{L^2}$		$\frac{M(a^2+b^2-4ab-L^2)}{L^3}$	$\frac{Ma(2b-a)}{L^2}$