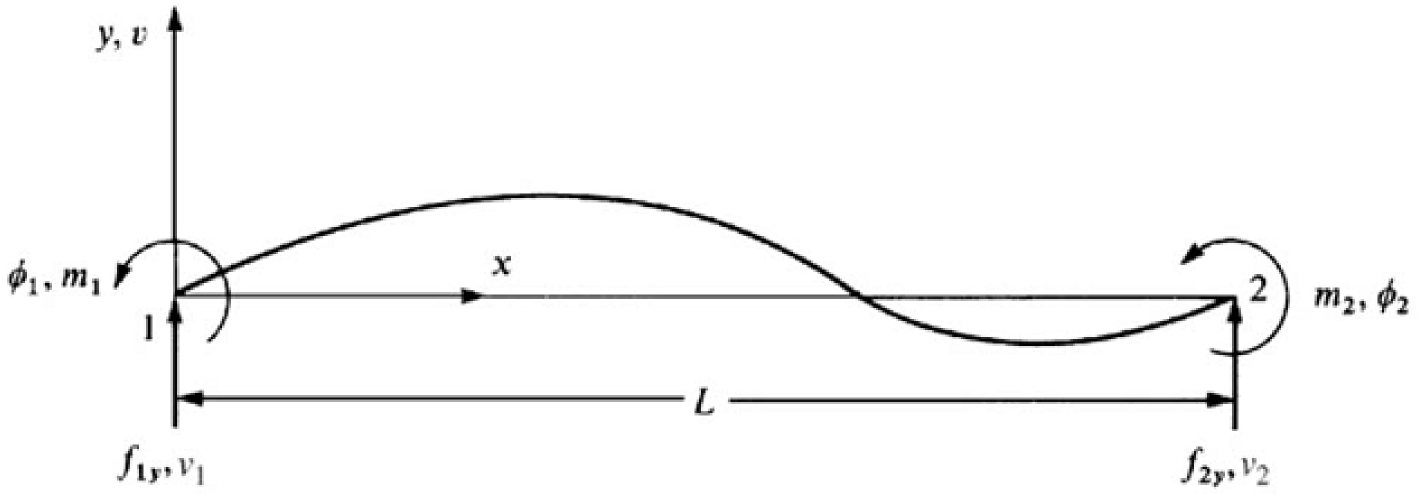
Development of Beam Equations

Beam Stiffness

Consider the beam element shown below.



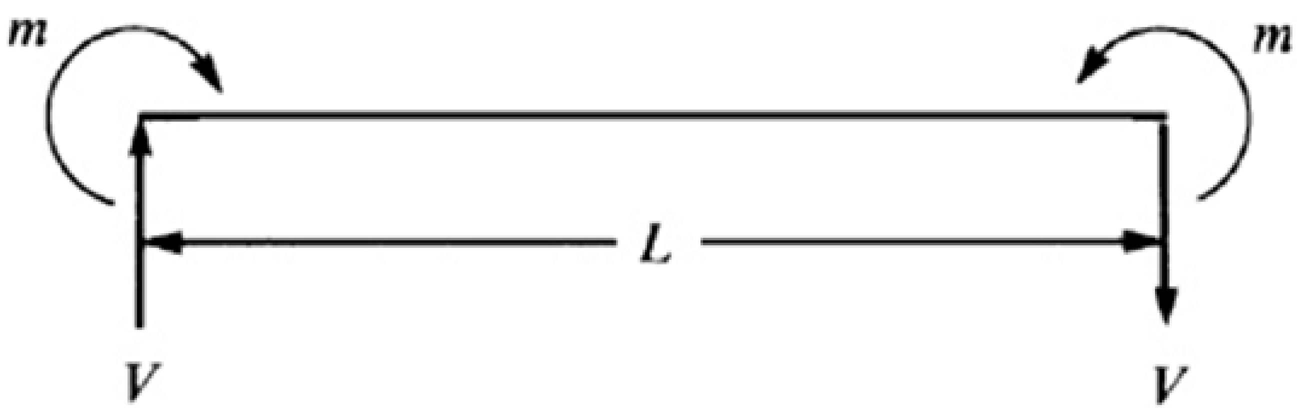
The beam is of length *L* with axial local coordinate *x* and transverse local coordinate *y*.

The local transverse nodal displacements are given by v_i and the rotations by ϕ_i. The local nodal forces are given by f_{iy} and the bending moments by m_i.

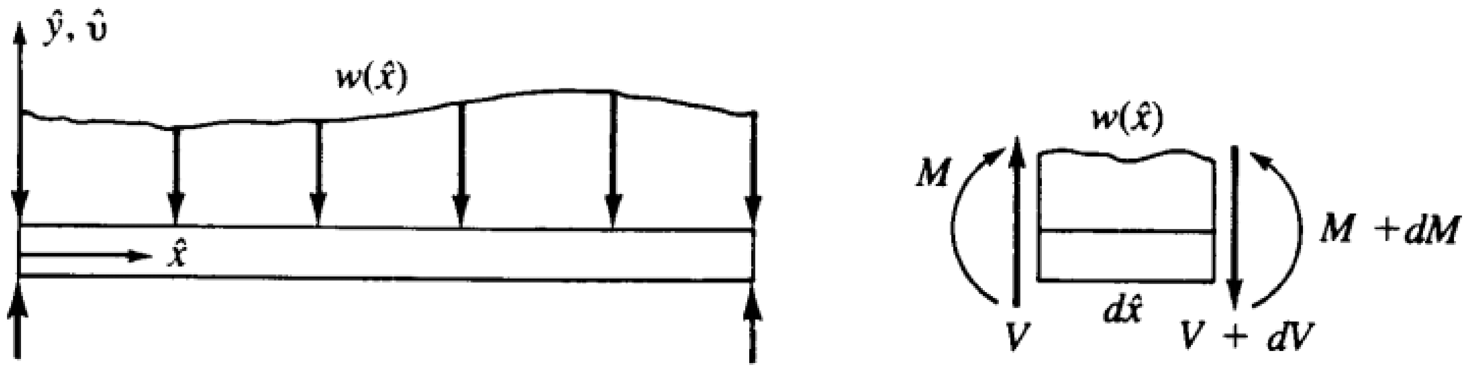
Beam Stiffness

At all nodes, the following sign conventions are used:

1. **Moments** are positive in the counterclockwise direction.
2. **Rotations** are positive in the counterclockwise direction.
3. **Forces** are positive in the positive *y* direction.
4. **Displacements** are positive in the positive *y* direction.



The differential equation governing simple linear-elastic beam behavior can be derived as follows. Consider the beam shown below.



(a) Beam under load $w(\hat{x})$ (b) Differential beam element

Write the equations of equilibrium for the differential element:

$$\sum M_{right-side} = 0 = (M + dM) - M - V d\hat{x} + w(\hat{x}) d\hat{x}\left(\frac{d\hat{x}}{2}\right) \qquad d\hat{x}^2 \approx 0$$

$$\sum F_y = 0 = V - (V + dV) - w(\hat{x})dx$$

From force and moment equilibrium of a differential beam element, we get:

$$\sum M_{right-side} = 0 \quad \Rightarrow \quad -Vdx + dM = 0 \quad \text{or} \quad V = \frac{dM}{dx}$$

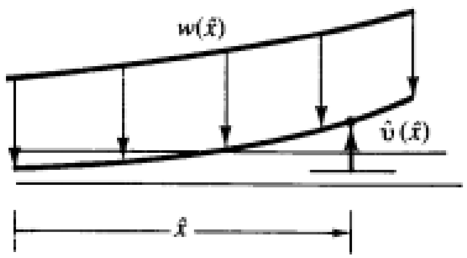
$$\sum F_y = 0 \quad \Rightarrow \quad -wdx - dV = 0 \quad \text{or} \quad w = -\frac{dV}{dx}$$

$$w = -\frac{d}{dx}\left(\frac{dM}{dx}\right)$$

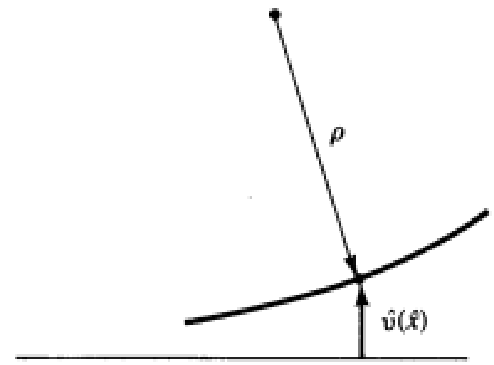
The curvature κ of the beam is related to the moment by:

$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$

where ρ is the radius of the deflected curve, $\hat{v}$ is the transverse displacement function in the $\hat{y}$ direction, ***E*** is the modulus of elasticity, and ***I*** is the principle moment of inertia about $\hat{y}$ direction, as shown below.



(a) Portion of deflected curve of beam



b) Radius of deflected curve at $\hat{v}(\hat{x})$

The curvature for small slopes $\theta = d\hat{v}/d\hat{x}$ is given as:

$$k = \frac{d^2\hat{v}}{d\hat{x}^2}$$

Therefore:

$$\frac{d^2\hat{v}}{d\hat{x}^2} = \frac{M}{EI} \qquad \Rightarrow \qquad M = EI\frac{d^2\hat{v}}{d\hat{x}^2}$$

Substituting the moment expression into the moment-load equations gives:

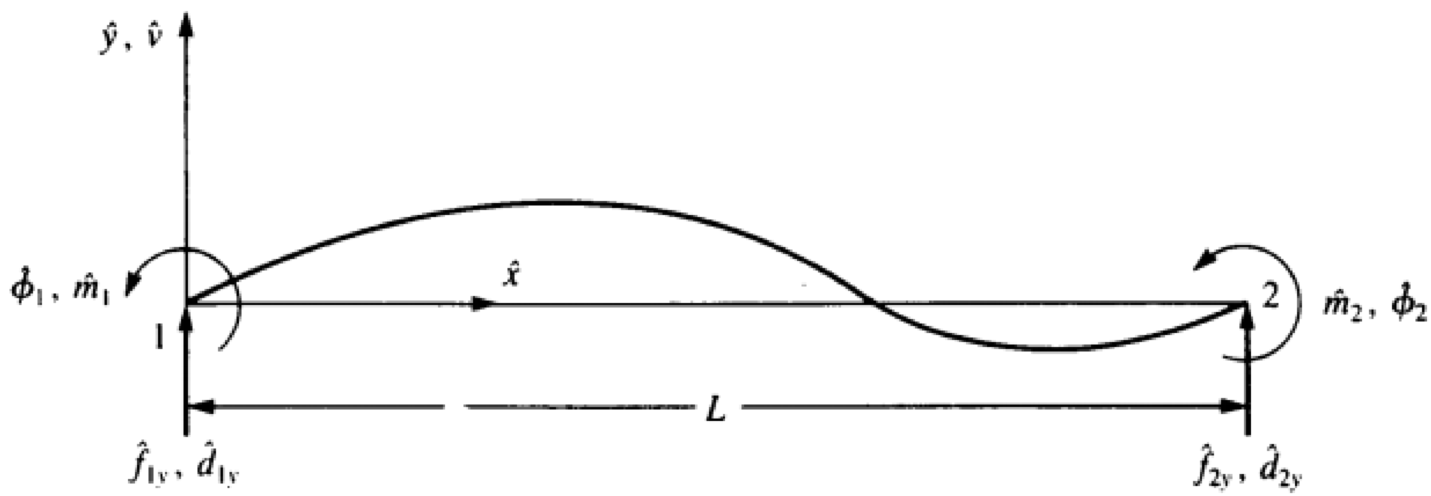
$$\frac{d^2}{d\hat{x}^2}\left(EI\,\frac{d^2\hat{v}}{d\hat{x}^2}\right) = -w(\hat{x})$$

For constant values of ***EI***, the above equation reduces to:

$$EI\left(\frac{d^4\hat{v}}{d\hat{x}^4}\right) = -w(\hat{x})$$

Step 1 - Select Element Type

We will consider the linear-elastic beam element shown below.



Step 2 - Select a Displacement Function

Assume the transverse displacement function v is:

$$v = a_1\hat{x}^3 + a_2\hat{x}^2 + a_3\hat{x} + a_4$$

The number of coefficients in the displacement function, $\boldsymbol{a_i}$, is equal to the total number of degrees of freedom associated with the element (displacement and rotation at each node). The boundary conditions are:

$$\hat{v}(\hat{x} = 0) = \hat{d}_{1y} \qquad \hat{v}(\hat{x} = L) = \hat{d}_{2y}$$

$$\frac{d\hat{v}(\hat{x} = 0)}{d\hat{x}} = \hat{\phi}_1 \qquad \frac{d\hat{v}(\hat{x} = L)}{d\hat{x}} = \hat{\phi}_2$$

Applying the boundary conditions and solving for the unknown coefficients gives:

$$\hat{v}(0) = \hat{d}_{1y} = a_4$$

$$\hat{v}(L) = \hat{d}_{2y} = a_1L^3 + a_2L^2 + a_3L + a_4$$

$$\frac{d\hat{v}(0)}{d\hat{x}} = \hat{\phi}_1 = a_3$$

$$\frac{d\hat{v}(L)}{dx} = \hat{\phi}_2 = 3a_1L^2 + 2a_2L + a_3$$

Solving these equations for a_1, a_2, a_3, and a_4 gives:

$$\hat{v} = \left[\frac{2}{L^3}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) + \frac{1}{L^2}\left(\hat{\phi}_1 - \hat{\phi}_2\right)\right]\hat{x}^3 + \left[-\frac{3}{L^2}\left(\hat{d}_{1y} - \hat{d}_{2y}\right) - \frac{1}{L}\left(2\hat{\phi}_1 + \hat{\phi}_2\right)\right]\hat{x}^2 + \hat{\phi}_1\hat{x} + \hat{d}_{1y}$$

In matrix form the above equations are:

$$\hat{v} = [N]\{\hat{d}\}$$

where

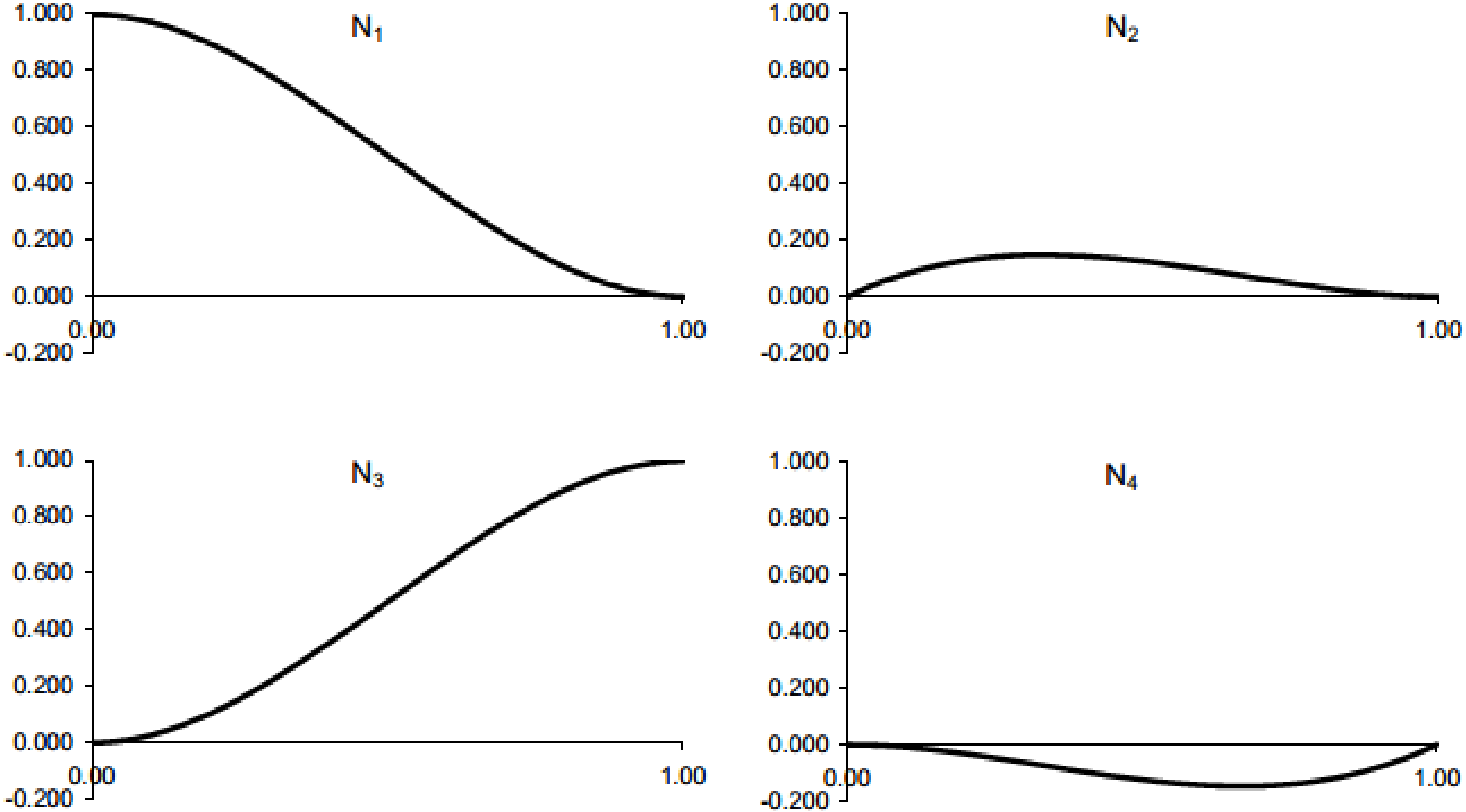
$$\{\hat{d}\} = \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix} \qquad [N] = [N_1\ N_2\ N_3\ N_4]$$

and

$$N_1 = \frac{1}{L^3}\left(2\hat{x}^3 - 3\hat{x}^2 L + L^3\right) \qquad N_2 = \frac{1}{L^3}\left(\hat{x}^3 L - 2\hat{x}^2 L^2 + \hat{x} L^3\right)$$

$$N_3 = \frac{1}{L^3}\left(-2\hat{x}^3 + 3\hat{x}^2 L\right) \qquad N_4 = \frac{1}{L^3}\left(\hat{x}^3 L - \hat{x}^2 L^2\right)$$

N_1, N_2, N_3, and N_4 are called the ***shape functions*** for a beam element.

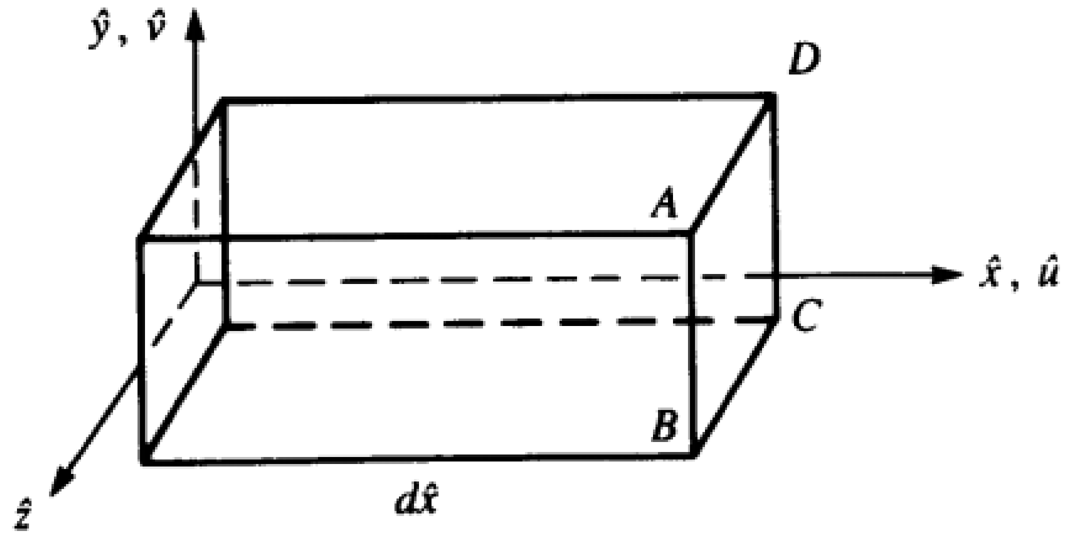


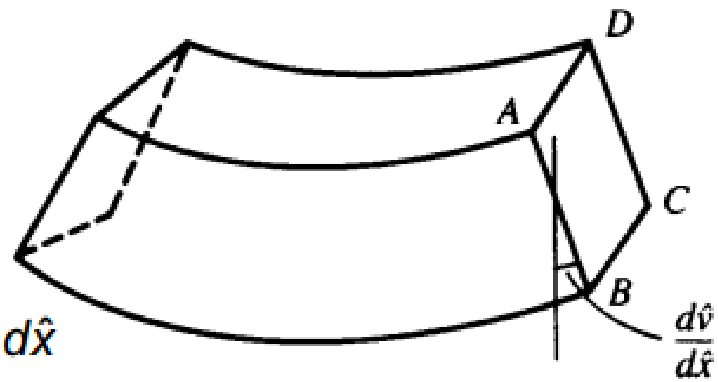
Step 3 - Define the Strain/Displacement and Stress/Strain Relationships

The stress-displacement relationship is:

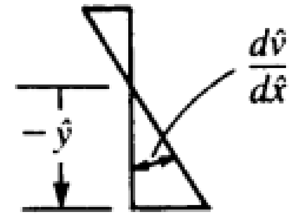
$$\varepsilon_x(\hat{x}, \hat{y}) = \frac{d\hat{u}}{d\hat{x}}$$

where $\hat{u}$ is the axial displacement function. We can relate the axial displacement to the transverse displacement by considering the beam element shown below:





$ds \approx d\hat{x}$



$\theta \approx \frac{d\hat{v}}{d\hat{x}}$

$$\hat{u} = -\hat{y}\,\frac{d\hat{v}}{d\hat{x}}$$

One of the basic assumptions in simple beam theory is that planes remain planar after deformation, therefore:

$$\varepsilon_x(\hat{x},\hat{y}) = -\hat{y}\left(\frac{d^2\hat{v}}{d\hat{x}^2}\right)$$

Moments and shears are related to the transverse displacement as:

$$\hat{m}(\hat{x}) = EI\left(\frac{d^2\hat{v}}{d\hat{x}^2}\right) \qquad \hat{V}(x) = EI\left(\frac{d^3\hat{v}}{d\hat{x}^3}\right)$$

Step 4 - Derive the Element Stiffness Matrix and Equations

Using beam theory sign convention for shear force and bending moment we obtain the following equations:

$$\hat{f}_{1y} = \hat{V} = EI\frac{d^3\hat{v}(0)}{d\hat{x}^3} = \frac{EI}{L^3}\left(12\hat{d}_{1y} + 6L\hat{\phi}_1 - 12\hat{d}_{2y} + 6L\hat{\phi}_2\right)$$

$$\hat{f}_{2y} = -\hat{V} = EI\frac{d^3\hat{v}(L)}{d\hat{x}^3} = \frac{EI}{L^3}\left(-12\hat{d}_{1y} - 6L\hat{\phi}_1 + 12\hat{d}_{2y} - 6L\hat{\phi}_2\right)$$

$$\hat{m}_1 = -\hat{m} = -EI\frac{d^2\hat{v}(0)}{d\hat{x}^2} = \frac{EI}{L^3}\left(6L\hat{d}_{1y} + 4L^2\hat{\phi}_1 - 6L\hat{d}_{2y} + 2L^2\hat{\phi}_2\right)$$

$$\hat{m}_2 = \hat{m} = EI\frac{d^2\hat{v}(L)}{d\hat{x}^2} = \frac{EI}{L^3}\left(6L\hat{d}_{1y} + 2L^2\hat{\phi}_1 - 6L\hat{d}_{2y} + 4L^2\hat{\phi}_2\right)$$

In matrix form the above equations are:

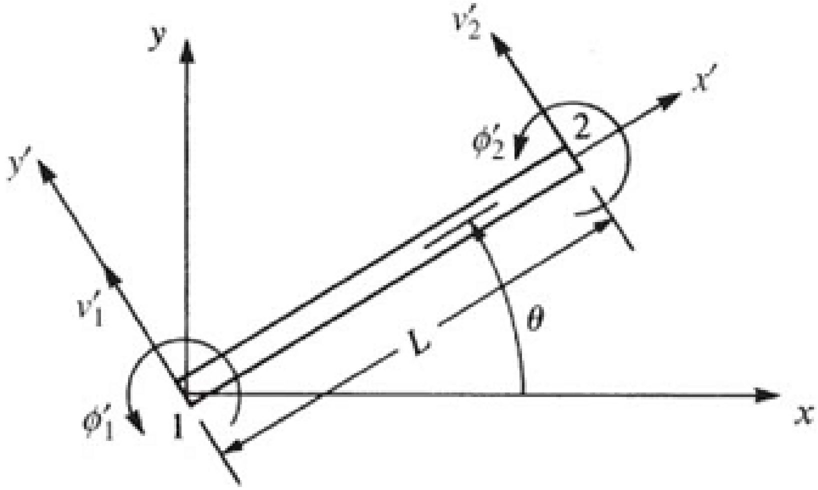
$$\begin{Bmatrix} \hat{f}_{1y} \\ \hat{m}_1 \\ \hat{f}_{2y} \\ \hat{m}_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1y} \\ \hat{\phi}_1 \\ \hat{d}_{2y} \\ \hat{\phi}_2 \end{Bmatrix}$$

where the stiffness matrix is:

$$k = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Two-Dimensional Arbitrarily Oriented Beam Element

We can derive the stiffness matrix for an arbitrarily oriented beam element, in a manner similar to that used for the bar element.



The local axes and are located along the beam element and transverse to the beam element, respectively, and the global axes *x'* and *y'* are located to be convenient for the total structure.

Two-Dimensional Arbitrarily Oriented Beam Element

The transformation from local displacements to global displacements is given in matrix form as:

$$\begin{Bmatrix} u' \\ v' \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} \qquad \begin{matrix} C = \cos\theta \\ S = \sin\theta \end{matrix}$$

Using the second equation for the beam element, we can relate local nodal degrees of freedom to global degree of freedom:

$$\begin{Bmatrix} v'_1 \\ \phi'_1 \\ v'_2 \\ \phi'_2 \end{Bmatrix} = \begin{bmatrix} -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{Bmatrix}$$

$$v'_1 = -Su_1 + Cv_1$$

$$\mathbf{d}' = \bar{\mathbf{T}}\mathbf{d}$$

Two-Dimensional Arbitrarily Oriented Beam Element

For a beam, we will define the following as the ***transformation matrix***:

$$\bar{\mathbf{T}} = \begin{bmatrix} -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Notice that the rotations are not affected by the orientation of the beam.

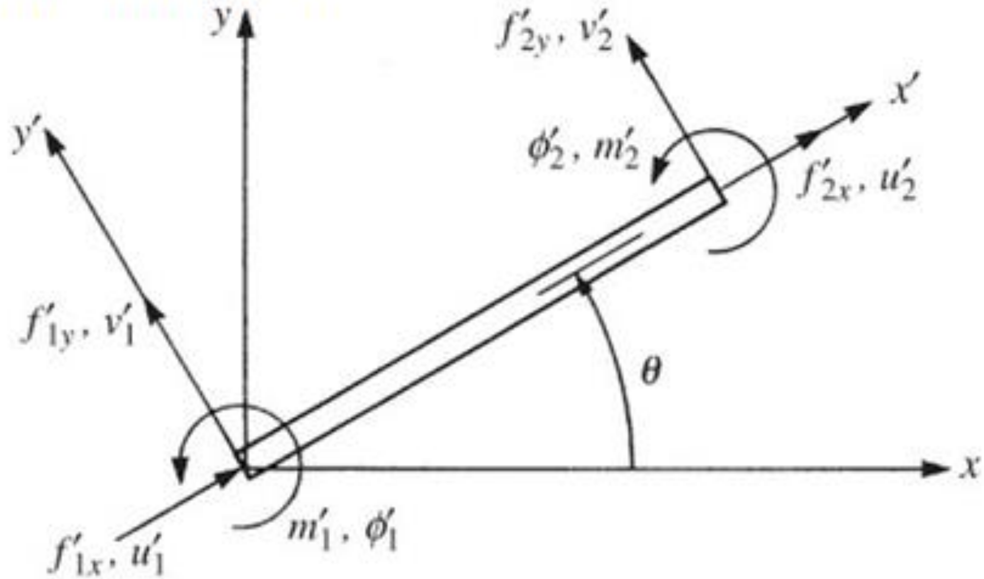
Two-Dimensional Arbitrarily Oriented Beam Element

Substituting the above transformation into the general form of the stiffness matrix $\mathbf{T}^T\mathbf{k}'\mathbf{T}$ gives:

$$\mathbf{k} = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccccc} u_1 & v_1 & \phi_1 & u_2 & v_2 & \phi_2 \end{array} \\ \left[\begin{array}{ccc|ccc} 12S^2 & -12SC & -6LS & -12S^2 & 12SC & -6LS \\ -12SC & 12C^2 & 6LC & 12SC & -12C^2 & 6LC \\ -6LS & 6LC & 4L^2 & 6LS & -6LC & 2L^2 \\ \hline -12S^2 & 12SC & 6LS & 12S^2 & -12SC & 6LS \\ 12SC & -12C^2 & -6LC & -12SC & 12C^2 & -6LC \\ -6LS & 6LC & 2L^2 & 6LS & -6LC & 4L^2 \end{array}\right] \end{array}$$

Two-Dimensional Arbitrarily Oriented Beam Element

Let's now consider the effects of an axial force in the general beam transformation.



Recall the simple axial deformation, define in the spring element:

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix}$$

Two-Dimensional Arbitrarily Oriented Beam Element

Combining the axial effects with the shear force and bending moment effects, in local coordinates gives:

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ m'_1 \\ f'_{2x} \\ f'_{2y} \\ m'_2 \end{Bmatrix} = \left[\begin{array}{ccc|ccc} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\ 0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\ \hline -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\ 0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2 \end{array}\right] \begin{Bmatrix} u'_1 \\ v'_1 \\ \phi'_1 \\ u'_2 \\ v'_2 \\ \phi'_2 \end{Bmatrix}$$

$$C_1 = \frac{AE}{L} \qquad C_2 = \frac{EI}{L^3}$$

Two-Dimensional Arbitrarily Oriented Beam Element

Therefore:

$$
\mathbf{k}' = \left[\begin{array}{ccc|ccc}
C_1 & 0 & 0 & -C_1 & 0 & 0 \\
0 & 12C_2 & 6LC_2 & 0 & -12C_2 & 6LC_2 \\
0 & 6LC_2 & 4C_2L^2 & 0 & -6LC_2 & 2C_2L^2 \\
\hline
-C_1 & 0 & 0 & C_1 & 0 & 0 \\
0 & -12C_2 & -6LC_2 & 0 & 12C_2 & -6LC_2 \\
0 & 6LC_2 & 2C_2L^2 & 0 & -6LC_2 & 4C_2L^2
\end{array}\right]
$$

The above stiffness matrix include the effects of axial force in the *x*' direction, shear force in the *y*' direction, and bending moment about the *z*' axis.

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Two-Dimensional Arbitrarily Oriented Beam Element

The local degrees of freedom may be related to the global degrees of freedom by:

$$\begin{Bmatrix} u_1' \\ v_1' \\ \phi_1' \\ u_2' \\ v_2' \\ \phi_2' \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 & 0 & 0 \\ -S & C & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & C & S & 0 \\ 0 & 0 & 0 & -S & C & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \end{Bmatrix} \qquad \mathbf{d}' = \mathbf{T}\mathbf{d}$$

where **T** has been expanded to include axial effects

Two-Dimensional Arbitrarily Oriented Beam Element

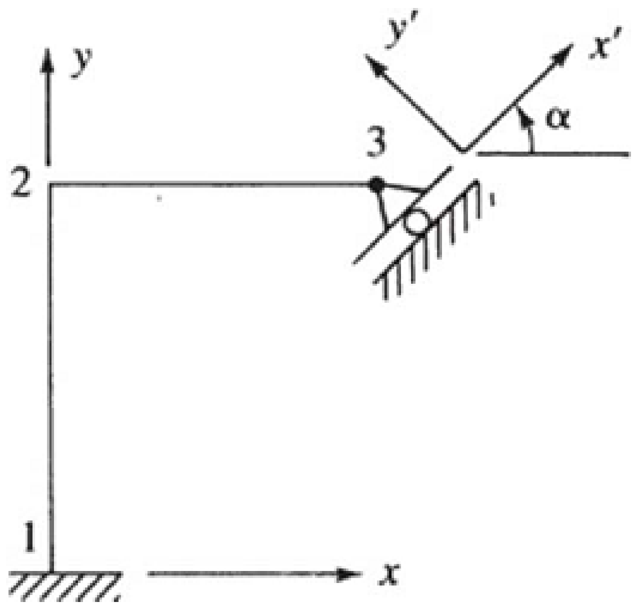
Substituting the above transformation **T** into the general form of the stiffness matrix gives:

$$[k]=\frac{E}{L}\begin{bmatrix} AC^2+\frac{12I}{L^2}S^2 & \left(A-\frac{12I}{L^2}\right)CS & -\frac{6I}{L}S & -\left(AC^2+\frac{12I}{L^2}S^2\right) & -\left(A-\frac{12I}{L^2}\right)CS & -\frac{6I}{L}S \\ & AS^2+\frac{12I}{L^2}C^2 & \frac{6I}{L}C & -\left(A-\frac{12I}{L^2}\right)CS & -\left(AC^2+\frac{12I}{L^2}S^2\right) & \frac{6I}{L}C \\ & & 4I & \frac{6I}{L}S & -\frac{6I}{L}C & 2I \\ \text{symmetric} & & & AC^2+\frac{12I}{L^2}S^2 & \left(A-\frac{12I}{L^2}\right)CS & \frac{6I}{L}S \\ & \text{symmetric} & & & AS^2+\frac{12I}{L^2}C^2 & -\frac{6I}{L}S \\ & & & \text{symmetric} & & 4I \end{bmatrix}$$

Inclined or Skewed Supports

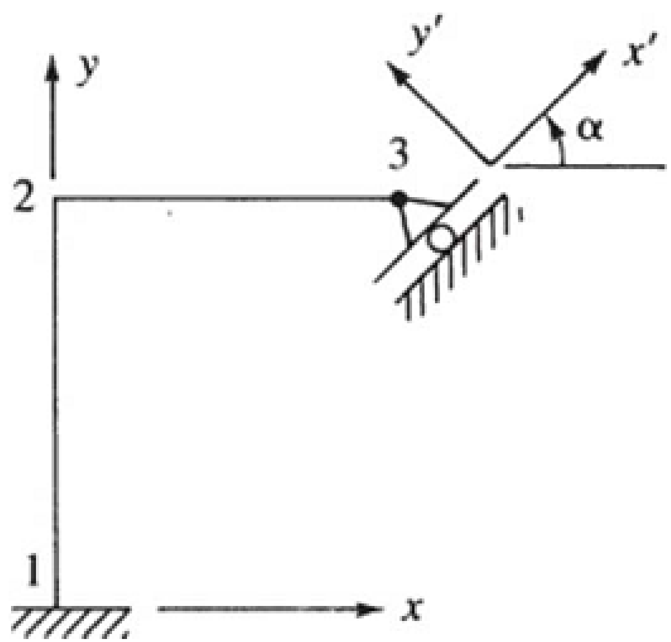
If a support is inclined, or skewed, at some angle α for the global *x* axis, as shown below.

The boundary conditions on the displacements are not in the global *x*-*y* directions but in the *x'*-*y'* directions.



Inclined or Skewed Supports

We must transform the local boundary condition of $\mathbf{v'_3} = 0$ (in local coordinates) into the global *x*-*y* system.



Inclined or Skewed Supports

Therefore, the relationship between of the components of the displacement in the local and the global coordinate systems at node 3 is:

$$\begin{Bmatrix} u'_3 \\ v'_3 \\ \phi'_3 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \\ \phi_3 \end{Bmatrix}$$

We can rewrite the above expression as:

$$\{d'_3\} = [t_3]\{d_3\} \qquad [t_3] = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inclined or Skewed Supports

We can apply this sort of transformation to the entire displacement vector as:

$$\{d'\} = [T_i]\{d\} \qquad \text{or} \qquad \{d\} = [T_i]^T\{d'\}$$

where the matrix **$[T_i]$** is:

$$[T_i] = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3] \end{bmatrix}$$

Both the identity matrix **$[I]$** and the matrix **$[t_3]$** are 3 x 3 matrices.

Inclined or Skewed Supports

The force vector can be transformed by using the same transformation.

$$\{f'\} = [T_i]\{f\}$$

In global coordinates, the force-displacement equations are:

$$\{f\} = [K]\{d\}$$

Applying the skewed support transformation to both sides of the force-displacement equation gives:

$$[T_i]\{f\} = [T_i][K]\{d\}$$

By using the relationship between the local and the global displacements, the force-displacement equations become:

$$[T_i]\{f\} = [T_i][K][T_i]^T\{d'\} \qquad \Rightarrow \qquad \{f'\} = [T_i][K][T_i]^T\{d'\}$$

Inclined or Skewed Supports

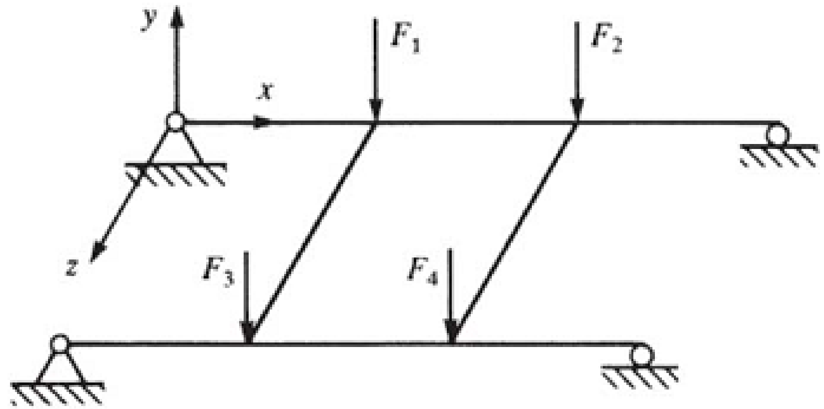
Therefore the global equations become:

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ M_1 \\ F_{2x} \\ F_{2y} \\ M_2 \\ F'_{3x} \\ F'_{3y} \\ M_3 \end{Bmatrix} = [T_i][K][T_i]^T \begin{Bmatrix} u_1 \\ v_1 \\ \phi_1 \\ u_2 \\ v_2 \\ \phi_2 \\ u'_3 \\ v'_3 \\ \phi_3 \end{Bmatrix}$$

Grid Equations

A ***grid*** is: *a structure on which the loads are applied perpendicular to the plane of the structure, as opposed to a plane frame where loads are applied in the plane of the structure.*

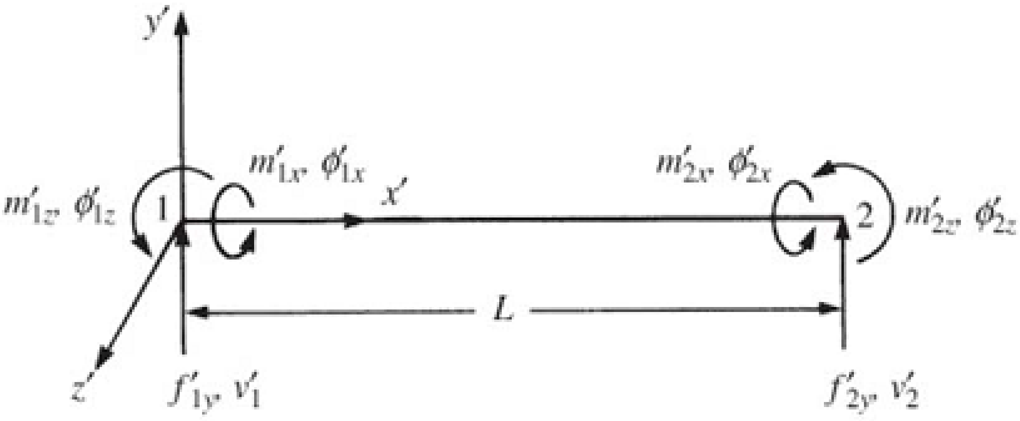
Both torsional and bending moment continuity are maintained at each node in a grid element.



Examples of a grid structure are floors and bridge deck systems.

Grid Equations

A representation of the grid element is shown below:



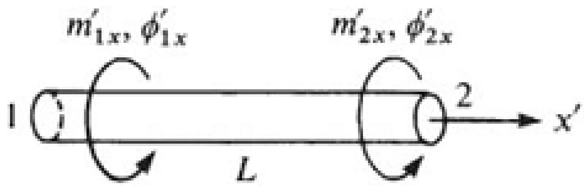
The degrees of freedom for a grid element are: a vertical displacement v'_i (normal to the grid), a torsional rotation ϕ'_{ix} about the *x'* axis, and a bending rotation ϕ'_{iz} about the *z'* axis.

The nodal forces are: a transverse force f'_{iy}, a torsional m'_{ix} moment about the *x'* axis, and a bending moment m'_{iz} about the *z'* axis.

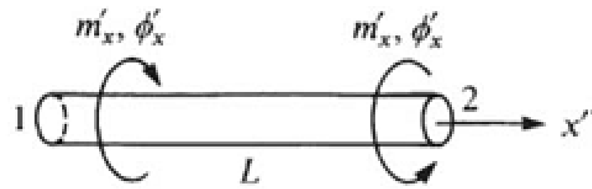
Grid Equations

Let's derive the torsional rotation components of the element stiffness matrix.

Consider the sign convention for nodal torque and angle of twist shown the figure below.



Nodal



Elemental

A linear displacement function is assumed. $\phi = a_1 + a_2 x'$

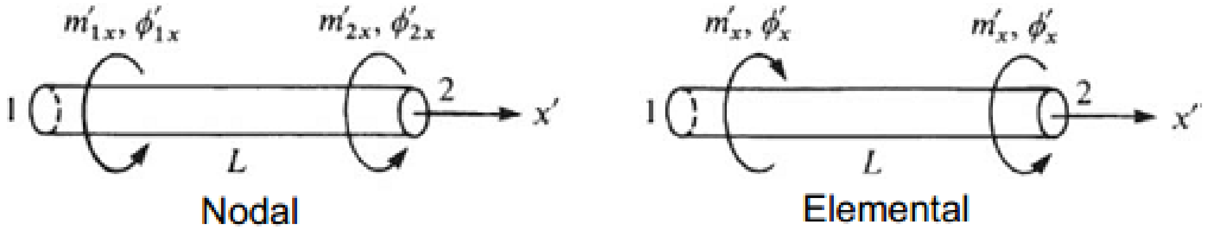
Applying the boundary conditions and solving for the unknown coefficients gives:

$$\phi = \left(\frac{\phi'_{2x} - \phi'_{1x}}{L} \right) x' + \phi'_{1x}$$

Grid Equations

Let's derive the torsional rotation components of the element stiffness matrix.

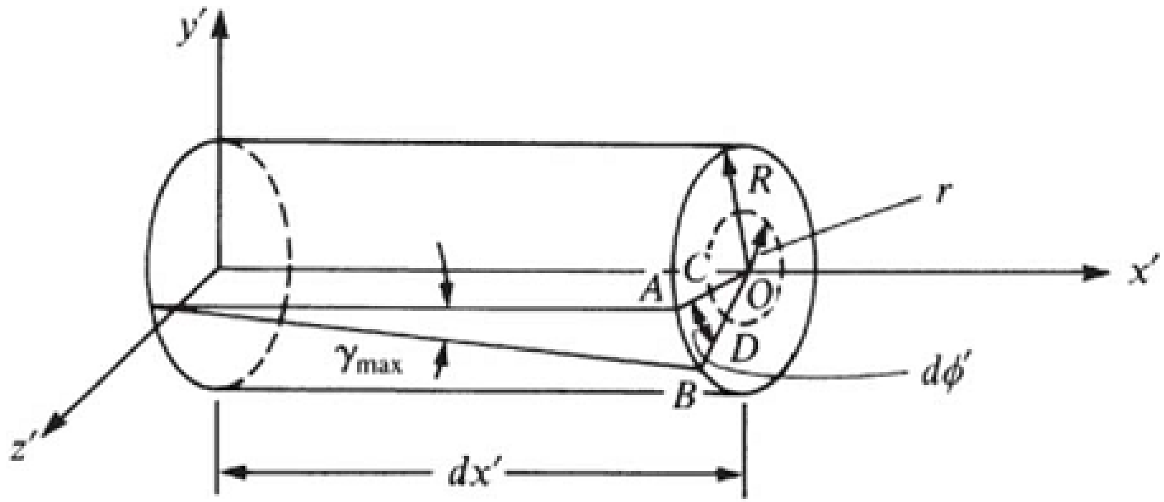
Consider the sign convention for nodal torque and angle of twist shown the figure below.



Or in matrix form: $\phi' = \begin{bmatrix} N_1 & N_2 \end{bmatrix} = \begin{Bmatrix} \phi'_{1x} \\ \phi'_{2x} \end{Bmatrix}$

where: $N_1 = 1 - \dfrac{x'}{L}$ $\qquad N_2 = \dfrac{x'}{L}$

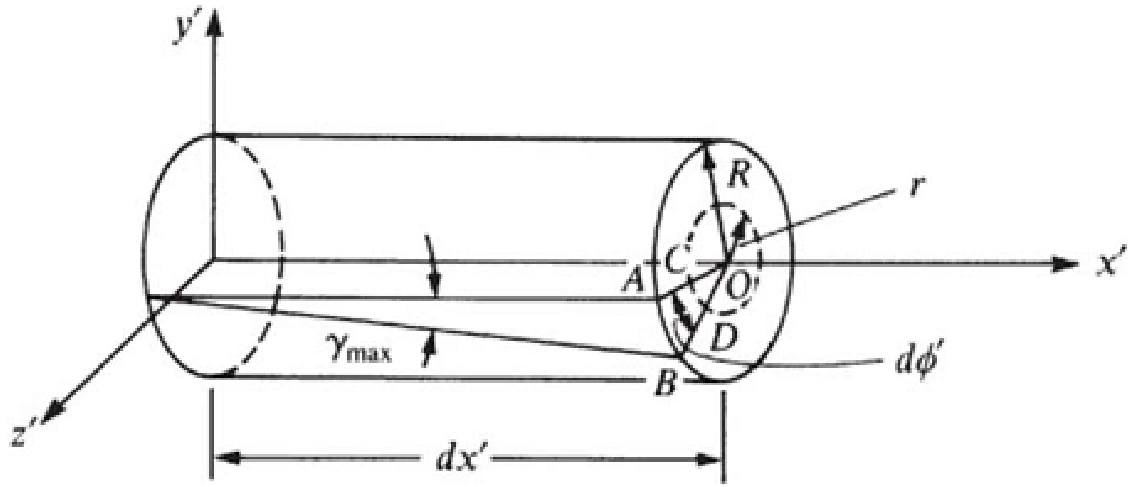
To obtain the relationship between the shear strain γ and the angle of twist ϕ' consider the torsional deformation of the bar as shown below.



If we assume that all radial lines, such as ***OA***, remain straight during twisting or torsional deformation, then the arc length $\overline{AB}$ is:

$$\overline{AB} = \gamma_{max} dx' = R\, d\phi' \qquad \Rightarrow \qquad \gamma_{max} = R \frac{d\phi'}{dx'}$$

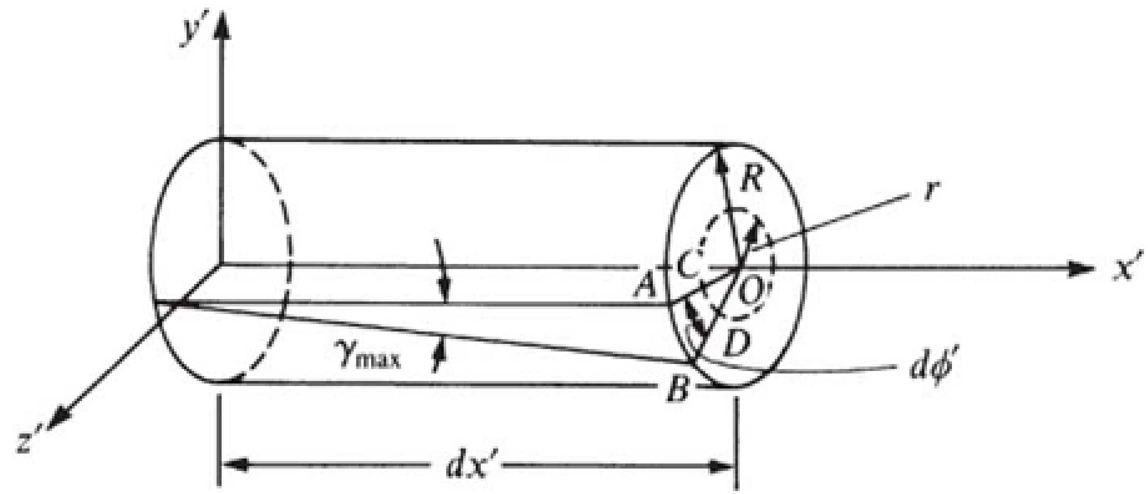
To obtain the relationship between the shear strain γ and the angle of twist ϕ' consider the torsional deformation of the bar as shown below.



At any radial position, *r*, we have, from similar triangles ***OAB*** and ***OCD***:

$$\gamma = r\frac{d\phi'}{dx'} = \frac{r}{L}\left(\phi'_{2x} - \phi'_{1x}\right)$$

To obtain the relationship between the shear strain γ and the angle of twist ϕ' consider the torsional deformation of the bar as shown below.

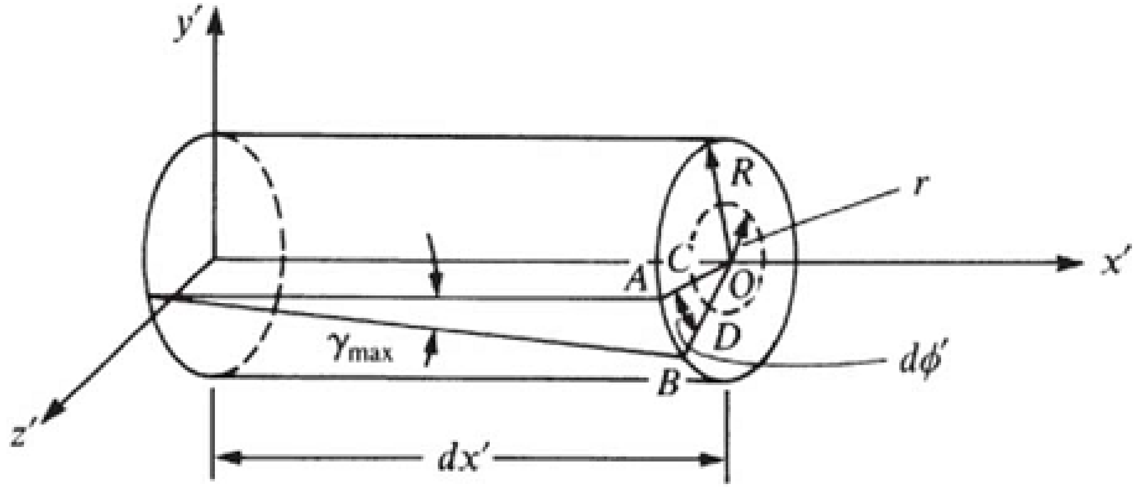


The relationship between shear stress and shear strain is:

$$\tau = G\gamma$$

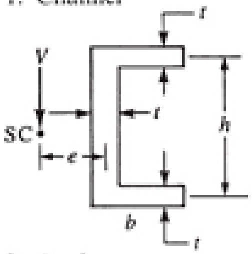
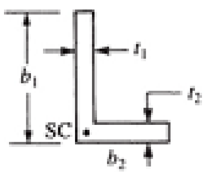
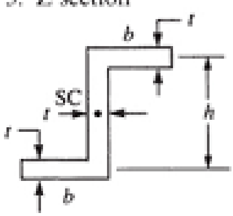
where G is the ***shear modulus*** of the material.

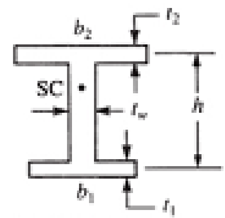
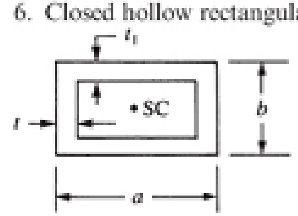
To obtain the relationship between the shear strain γ and the angle of twist ϕ' consider the torsional deformation of the bar as shown below.



From elementary mechanics of materials, we get: $m'_x = \dfrac{\tau J}{R}$

Where J is the ***polar moment of inertia*** for a circular cross section or the ***torsional constant*** for non-circular cross sections.

Cross Section	Torsional Constant
1. Channel 	$J = \frac{t^3}{3}(h + 2b)$ $e = \frac{h^2 b^2 t}{4I}$
2. Angle 	$J = \frac{1}{3}(b_1 t_1^3 + b_2 t_2^3)$
3. Z section 	$J = \frac{t^3}{3}(2b + h)$

Cross Section	Torsional Constant
4. Wide-flanged beam with unequal flanges 	$J = \frac{1}{3}(b_1 t_1^3 + b_2 t_2^3 + h t_w^3)$
5. Solid circular 	$J = \frac{\pi}{2} r^4$
6. Closed hollow rectangular 	$J = \frac{2 t t_1 (a - t)^2 (b - t_1)^2}{a t + b t_1 - t^2 - t_1^2}$

Rewriting the above equation we get: $m'_x = \frac{GJ}{L}(\phi'_{2x} - \phi'_{1x})$

The nodal torque sign convention gives:

$$m'_{1x} = -m'_x$$
$$m'_{2x} = m'_x$$

Therefore: $m'_{1x} = \frac{GJ}{L}(\phi'_{1x} - \phi'_{2x})$ $\quad$ $m'_{2x} = \frac{GJ}{L}(\phi'_{2x} - \phi'_{1x})$

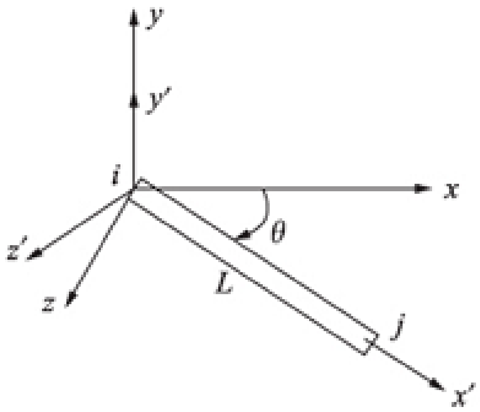
In matrix form the above equations are:

$$\begin{Bmatrix} m'_{1x} \\ m'_{2x} \end{Bmatrix} = \frac{GJ}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \phi'_{1x} \\ \phi'_{2x} \end{Bmatrix}$$

Combining the torsional effects with shear and bending effects, we obtain the local stiffness matrix equations for a grid element.

$$\begin{Bmatrix} f'_{1y} \\ m'_{1x} \\ m'_{1z} \\ f'_{2y} \\ m'_{2x} \\ m'_{2z} \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & 0 & \frac{6EI}{L^2} \\ 0 & \frac{GJ}{L} & 0 & 0 & -\frac{GJ}{L} & 0 \\ \frac{6EI}{L^2} & 0 & \frac{4EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & 0 & -\frac{6EI}{L^2} \\ 0 & -\frac{GJ}{L} & 0 & 0 & \frac{GJ}{L} & 0 \\ \frac{6EI}{L^2} & 0 & \frac{2EI}{L} & -\frac{6EI}{L^2} & 0 & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} v'_1 \\ \phi'_{1x} \\ \phi'_{1z} \\ v'_2 \\ \phi'_{2x} \\ \phi'_{2z} \end{Bmatrix}$$

The ***transformation matrix*** relating local to global degrees of freedom for a grid is:

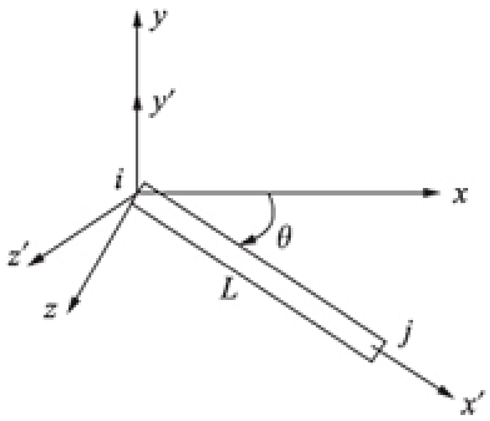


$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C & S & 0 & 0 & 0 \\ 0 & -S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix}$$

where θ is now positive taken counterclockwise from x to x' in the x-z plane: therefore:

$$C = \cos\theta = \frac{x_j - x_i}{L} \qquad S = \sin\theta = \frac{z_j - z_i}{L}$$

The ***transformation matrix*** relating local to global degrees of freedom for a grid is:



$$\mathbf{T}_G = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & C & S & 0 & 0 & 0 \\ 0 & -S & C & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & C & S \\ 0 & 0 & 0 & 0 & -S & C \end{bmatrix}$$

The global stiffness matrix for a grid element arbitrary oriented in the *x*-*z* plane is given by:

$$\mathbf{k}_G = \mathbf{T}_G^{\mathrm{T}} \mathbf{k}'_G \mathbf{T}_G$$