Method of weighted residuals

Engineering problems can be described by ODE (Ordinary differential equation) or PDE (Partial differential equation). It is usually not possible to solve the derived ODE or PDE analytically.

In fields other than structural/solid mechanics, it is quite probable that a variational principle, analogous to the principle of minimum potential energy, for instance, may not be known or even exists. In some flow problems in fluid mechanics and in mass transport problems, we often have only the differential equation and boundary conditions available. However, the finite element method can still be applied.

The methods of weighted residuals applied directly to the differential equation can be used to develop the finite element equations. In this section, we describe Galerkin's residual method in general and then apply it to the bar element. This development provides the basis for later applications of Galerkin's method to the beam element and to the nonstructural heat-transfer element (specifically, the one-dimensional combined conduction, convection, and mass transport element. Because of the mass transport phenomena, the variational formulation is not known (or certainly is difficult to obtain), so Galerkin's method is necessarily applied to develop the finite element equations.

In weighted residual methods, a trial or approximate function is chosen to approximate the independent variable, such as a displacement or a temperature, in a problem defined by a differential equation. This trial function will not, in general, satisfy the governing differential equation. Thus substituting the trial function into the differential equation results in a residual over the whole region of the problem as follows:

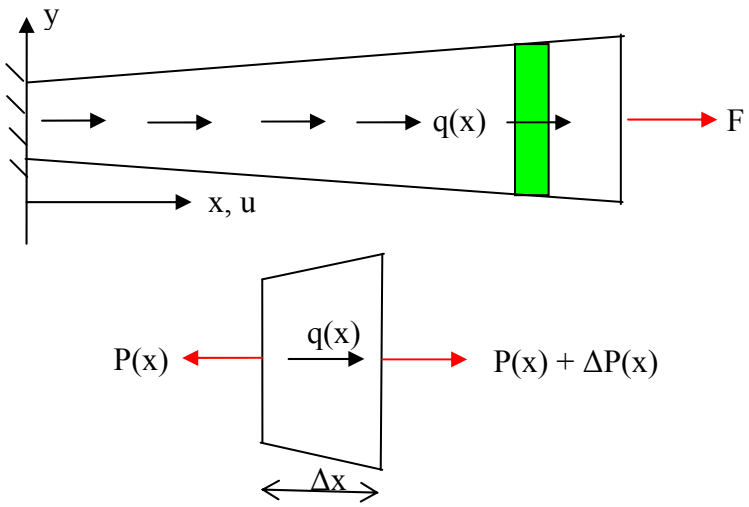
$$\iiint_V R\,dV = \text{minimum}$$

In the residual method, we require that a weighted value of the residual be a minimum over the whole region. The weighting functions allow the weighted integral of residuals to go to zero. If we denote the weighting function by W, the general form of the weighted residual integral is

$$\iiint_V RW\,dV = 0$$

Governing ODE for linear elastic problem of an axially loaded bar

Consider an **infinitesimal** slice of the axial bar



Force equilibrium for the slice:

** Note: variable x are omitted in P(x), A(x), σ(x), E(x), ε(x), q(x), u(x) for clarity.

$$P = q(\Delta x) + (P + \Delta P)$$

$$\Rightarrow \frac{(P + \Delta P) - P}{\Delta x} + q = 0$$

Taking the limit as $\Delta x \to 0$, the LHS function becomes

$$\Rightarrow \lim_{\Delta x \to 0}\left(\frac{(P + \Delta P) - P}{\Delta x}\right) + q = 0$$

$$\Rightarrow \frac{dP}{dx} + q = 0$$

Having

$$P = A\sigma = AE\varepsilon$$

$$\varepsilon = \lim_{\Delta x \to 0}\left(\frac{(u + \Delta u) - u}{\Delta x}\right) = \frac{du}{dx}$$

Hence

$$\frac{dP}{dx} = \frac{d}{dx}\left(AE\frac{du}{dx}\right)$$

The ODE in terms of displacement u(x) is

$$\frac{d}{dx}\left(A(x)E(x)\frac{du}{dx}\right) + q(x) = 0$$

Next step, we identify the boundary conditions:

1. At left end x = 0, u = 0
2. At right end x = L, f = F

The ODE is in terms of u. We need to describe boundary condition (2) in terms of u.

$$F = (A\sigma)\big|_{x=L} = (AE\varepsilon)\big|_{x=L} = \left(AE\frac{du}{dx}\right)\Big|_{x=L}$$

Therefore, the boundary conditions become

1. At left end x = 0, u = 0
2. At right end x = L, $\left.\frac{du}{dx}\right|_{x=L} = \left.\frac{F}{AE}\right|_{x=L}$

The problem can be stated as

Find *u(x)* which satisfies

$$\frac{d}{dx}\left(A(x)\,E(x)\frac{du}{dx}\right) + q(x) = 0$$

u(0) = 0

$$\left.\frac{du}{dx}\right|_{x=L} = \left.\frac{F}{AE}\right|_{x=L}$$

Note: for engineering problem, we usually want to find both

$u(x)$ = displacement at point x

$\sigma(x)$ = stress at point x

To do it, the ODE with boundary conditions is solved to obtain *u(x)*. The stress field $\sigma(x)$ is then obtained from the stress-strain-displacement relationship

$$\sigma(x) = E(x)\,\varepsilon(x) = E(x)\frac{du}{dx}$$

Now we attempt to find the approximated solution for our mathematical problem using a method which is called method of weighted residual

Method of weighted residual

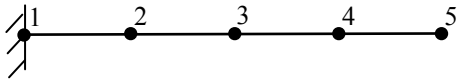
Method outline:

1. Nominate a trial solution $\overline{u(x)}$ which contains n nodal unknown solutions that we need to find.
2. Substitute the trial solution into the ODE. Compute the point-wise residual residual(x)
3. Weight the point-wise residual by n nominated weighting function $w_i(x)$.
4. Integration of each weighted residual over the whole domain is set to zero. A system of n independent equations for n unknown coefficients must be formed from this action.
5. Solve the system of n equations for n unknown coefficients. Use them to recover approximate solution $\overline{u(x)}$.

The method is applied here to solve the ODE for the linear static problem of the axial bar for illustration.

Step 1: Trial solution $\overline{u(x)}$

The problem is described on domain $x \in [0; L]$. We introduce a finite number of nodes n into the domain. For an arbitrary node i^{th}, the location is x_i, the unknown nodal displacement is u_i, ..etc.



An example of the 1D domain with 5 global nodes (n=5).

The trial solution $\overline{u(x)}$ is assumed to be in the form

$$\overline{u(x)} = N_1(x)u_1 + N_2(x)u_2 + N_3(x)u_3 + ... + N_n(x)u_n$$

$$\overline{u(x)} = \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & ... & N_n(x) \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ ... \\ u_n \end{Bmatrix}$$

$$\overline{u(x)} = [N(x)]\{u\}$$

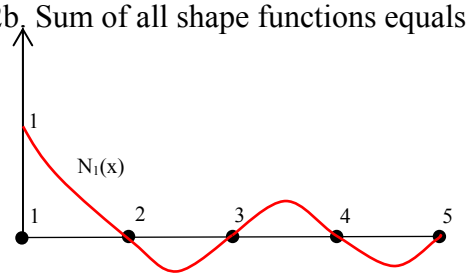
Where

1. n nodal displacements $u_1, u_2, \ldots u_n$ are n unknown that we need to find.

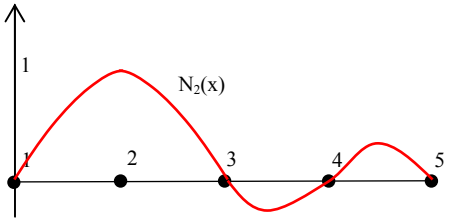
2. Function $N_i(x)$ is called 'shape function'. (Shape function is a key ingredient of the FEM). It is generally defined by continuous polynomial which has following properties:

   2a. The shape function associated with node i has a value of 1 at node i and vanishes at all other nodes.

   2b. Sum of all shape functions equals 1, $\sum_{i=1}^{n} N_i(x) = 1$



$N_1(x)$: High order shape function (in red)



$N_2(x)$: High order shape function (in red)

1st property enforces the approximate solution to be equal the nodal variable at each node, i.e. $\overline{u(x)}\Big|_{x=x_i} = u_i$. Importance of 2nd property reveals throughout the formulation and be addressed in later stage of this lecture.

Extension to 2D and 3D are straightforward

In 2D, $(x) \to (x,y)$, $N(x) \to N(x,y)$, $u(x) \to \begin{cases} u_x(x,y) \\ u_y(x,y) \end{cases}$

In 3D, $(x) \to (x,y,z)$, $N(x) \to N(x,y,z)$, $u(x) \to \begin{cases} u_x(x,y,z) \\ u_y(x,y,z) \\ u_z(x,y,z) \end{cases}$

Step 2: Point-wise residual residual(x)

Mathematical background

For numerical analysis, residual means the error in a result. Suppose we have

$f(x) = b$

Given an approximated solution ~~x_0~~ $f(x_0)$, the residual is the difference

$f(x_o) - b$

Whereas the error is

$x_0 - x$

In our case, we have $\frac{d}{dx}\left(A(x)E(x)\frac{du}{dx} \right) + q(x) = 0$,

By definition, we have b=0, the residual at location x equals

$$residual(x) = \frac{d}{dx}\left(A(x)E(x)\frac{d\overline{u}}{dx} \right) + q(x)$$

If we subtract b=0 from this, it remains same.

As $\overline{u(x)}$ is the approximated solution only, this residual is generally different from 0. If we want $\overline{u(x)}$ to be a good approximation of the exact $u(x)$, we must define the unknown coefficients $u_1, u_2 \ldots u_n$ in such a way that the residual is minimised over the domain $x \in [0; L]$, i.e.

$$\int_0^L residual(x)\, dx = \text{minimum} \quad \text{for } x \in [0; L]$$

Step 3: Weighting function w(x)

To minimize the domain residual, a mathematical method which is called the method of weighted residuals is employed. Note that we have n unknowns to find, therefore a set of n weighting function $w_i(x)$ are selected to yield a system of n linear independent equations. The weighted residuals over the problem domain are then set to zero. We have

$$R_i = \int_0^L w_i(x)\, residual(x)\, dx = 0$$

$$\text{n linear independent equations} \begin{cases} \int_0^L w_1(x)\, residual(x)\, dx = 0 \\ \int_0^L w_2(x)\, residual(x)\, dx = 0 \\ ..... \\ \int_0^L w_n(x)\, residual(x)\, dx = 0 \end{cases}$$

Most popular methods for defining $w_i(x)$ are

1. ***Collocation method:***

The weighting function is the Dirac delta function evaluated at nodal points

$$w_i(x) = \delta(x - x_i)$$

Having $R_i = \int_0^L \delta(x - x_i)\, residual(x) dx = residual(x_i) = 0$

This method enforces the residual to be zero at each nodal point but could be anything in between.

2. ***Least square method:***

The weighting function is defined as the partial derivative of the residual

$$w_i(x) = \frac{\partial(residual)}{\partial u_i}$$

3. ***Galerkin's method:***

The weighting functions are the derivatives of the trial solution $\overline{u(x)}$ with respect to the unknowns

$$w_i(x) = \frac{d\bar{u}}{du_i}$$

$w_i(x) = N_i(x)$, i.e. the shape function

The weighting function defined by Galerkin's method is used in the formulation of the FEM.

Strong form and weak form of the weighted residual R

$$R = \int_0^L w(x) \left[\frac{d}{dx} \left(A(x) E(x) \frac{d\bar{u}}{dx} \right) + q(x) \right] dx$$

When R includes the highest order of derivative term in the differential equation, it is regarded as the strong formulation of the weighted residual method. In this example highest order of differentiation of the variable $\overline{u(x)}$ is 2nd order. To yield a meaningful approximation, the trial solution function $\overline{u(x)}$ must be differentiable

twice and its 2nd derivative must not vanish on the domain $x \in [0; L]$, i.e. the shape functions in $[N(x)]$ have to be at least quadratic functions. To reduce the requirement for $\overline{u(x)}$ in terms of order of differentiability, integration by parts is applied to the strong form

$$R = \int_0^L w(x)\left[\frac{d}{dx}\left(A(x)E(x)\frac{d\bar{u}}{dx}\right) + q(x)\right]dx$$

$$= \left(w(x)A(x)E(x)\frac{d\bar{u}}{dx}\right)\Big|_o^L - \int_0^L A(x)E(x)\frac{d\bar{u}}{dx}\frac{dw}{dx}dx + \int_0^L w(x)\,q(x)dx$$

Mathematics background:

Integration by part states $\int udv = uv - \int vdu$, where u, v are functions u(x), v(x)

The new expression needs 1st order differentiation instead of 2nd order differentiation. As a result, the requirement for $\overline{u(x)}$ and hence $[N(x)]$ is reduced for *R*. This formulation is called the weak formulation of the weighted residual method. *We will use the Galerkin's method and the weak formulation to continue with our FEM formulation.*

Describe weak form of the weighted residual R in matrix form with FEM notations to prepare for systematic FEM formulation

$$w_i(x) = \frac{d\bar{u}}{du_i}$$

Galerkin's method: $w_i(x) = N_i(x)$, i.e. the shape function

$$\frac{dw_i}{dx} = \frac{dN_i}{dx} = N_i'(x)$$

From definition of trial function

(the dash sign ' means differentiation)

$$\frac{d\bar{u}}{dx} = \frac{d}{dx}([N(x)]\{u\}) = \left(\frac{d}{dx}[N(x)]\right)\{u\}$$

$$\frac{d}{dx}[N(x)] = \left[\frac{dN_1}{dx} \quad \frac{dN_2}{dx} \quad \frac{dN_3}{dx} \quad .... \quad \frac{dN_n}{dx}\right]$$

The derivative operator is denoted as matrix [L], the above equation becomes

$$[L][N(x)] = [B(x)]$$

Where $[B(x)]$=Element strain displacement matrix

The approximated strains and stresses are

$$\overline{\varepsilon(x)} = [B(x)]\{u\}$$

$$\overline{\sigma(x)} = E\overline{\varepsilon(x)} = [D][B(x)]\{u\}$$

In above, matrix $[D]$ refers to the stress/strain matrix or constitutive matrix. In this 1D problem, $[D]$=E. Assume E = constant for formulation clarity.

Back to our computation of the weighted residuals, we have

$$R = \left(w(x)A(x)E\frac{d\bar{u}}{dx} \right)\Big|_o^L - \int_0^L A(x)E\frac{d\bar{u}}{dx}\frac{dw}{dx}dx + \int_0^L w(x)\,q(x)dx = 0$$

n linear independent equations are

$$\begin{cases} R_1 = \left(w_1(x)A(x)E\frac{d\bar{u}}{dx} \right)\Big|_o^L - \int_0^L A(x)E\frac{d\bar{u}}{dx}\frac{dw_1}{dx}dx + \int_0^L w_1(x)\,q(x)dx = 0 \\ R_2 = \left(w_2(x)A(x)E\frac{d\bar{u}}{dx} \right)\Big|_o^L - \int_0^L A(x)E\frac{d\bar{u}}{dx}\frac{dw_2}{dx}dx + \int_0^L w_2(x)\,q(x)dx = 0 \\ R_3 = \left(w_3(x)A(x)E\frac{d\bar{u}}{dx} \right)\Big|_o^L - \int_0^L A(x)E\frac{d\bar{u}}{dx}\frac{dw_3}{dx}dx + \int_0^L w_3(x)\,q(x)dx = 0 \\ .... \\ R_n = \left(w_n(x)A(x)E\frac{d\bar{u}}{dx} \right)\Big|_o^L - \int_0^L A(x)E\frac{d\bar{u}}{dx}\frac{dw_n}{dx}dx + \int_0^L w_n(x)\,q(x)dx = 0 \end{cases}$$

By substituting the terms accordingly, the system becomes

$$\begin{cases} R_1 = \left(N_1(x)A(x)E\frac{d\bar{u}}{dx} \right)\Big|_{x=L} - \left(N_1(x)A(x)E\frac{d\bar{u}}{dx} \right)\Big|_{x=0} - \left(\int_0^L N_1'(x)[D][B(x)]A(x)dx \right)\{u\} + \int_0^L N_1(x)\,q(x)dx = 0 \\ R_2 = \left(N_2(x)A(x)E\frac{d\bar{u}}{dx} \right)\Big|_{x=L} - \left(N_2(x)A(x)E\frac{d\bar{u}}{dx} \right)\Big|_{x=0} - \left(\int_0^L N_2'(x)[D][B(x)]A(x)dx \right)\{u\} + \int_0^L N_2(x)\,q(x)dx = 0 \\ R_3 = \left(N_3(x)A(x)E\frac{d\bar{u}}{dx} \right)\Big|_{x=L} - \left(N_3(x)A(x)E\frac{d\bar{u}}{dx} \right)\Big|_{x=0} - \left(\int_0^L N_3'(x)[D][B(x)]A(x)dx \right)\{u\} + \int_0^L N_3(x)\,q(x)dx = 0 \\ .... \\ R_n = \left(N_n(x)A(x)E\frac{d\bar{u}}{dx} \right)\Big|_{x=L} - \left(N_n(x)A(x)E\frac{d\bar{u}}{dx} \right)\Big|_{x=0} - \left(\int_0^L N_n'(x)[D][B(x)]A(x)dx \right)\{u\} + \int_0^L N_n(x)\,q(x)dx = 0 \end{cases}$$

With $\begin{matrix} N_i(x) = 1 \text{ if } x = x_i \\ N_i(x) = 0 \text{ if } x = x_j, j \neq i \end{matrix}$, the first two terms in above equations equals zero except for 1st equation ($x = x_1 = 0$) and last equation ($x = x_n = L$) . Hence,

$$
\begin{cases}
R_1 = \left(-A(x)\,E\,\dfrac{d\bar{u}}{dx} \right)\Big|_{x=0} - \left(\displaystyle\int_0^L N_1'(x)[D][B(x)]A(x)dx \right)\{u\} + \displaystyle\int_0^L N_1(x)\,q(x)dx = 0 \\
R_2 = \quad 0 \quad - \left(\displaystyle\int_0^L N_2'(x)[D][B(x)]A(x)dx \right)\{u\} + \displaystyle\int_0^L N_2(x)\,q(x)dx = 0 \\
R_3 = \quad 0 \quad - \left(\displaystyle\int_0^L N_3'(x)[D][B(x)]A(x)dx \right)\{u\} + \displaystyle\int_0^L N_3(x)\,q(x)dx = 0 \\
\ldots. \\
R_n = \left(A(x)\,E\,\dfrac{d\bar{u}}{dx} \right)\Big|_{x=L} - \left(\displaystyle\int_0^L N_n'(x)[D][B(x)]A(x)dx \right)\{u\} + \displaystyle\int_0^L N_n(x)\,q(x)dx = 0
\end{cases}
$$

$\left(-A(x)\,E\,\dfrac{d\bar{u}}{dx} \right)\Big|_{x=0}$ and $\left(A(x)\,E\,\dfrac{d\bar{u}}{dx} \right)\Big|_{x=L}$ are the external nodal forces F_0 and F_L at the start and the end of the axial bar. All first terms form a vector {F} of size n x 1

$$\{F\} = [F_0 \quad 0 \ldots \quad 0 \quad F_L]^T$$

Write all 2nd terms in matrix form. They form negative of the multiplication of size n x n stiffness matrix [K] and size n x 1 nodal displacement vector {u}.

$$[K] = \int_0^L [B(x)]^T [D][B(x)]\,A(x)dx$$

$$[K] = AE \int_0^L \begin{bmatrix} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{bmatrix} \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix} dx$$

$$[K] = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

When the Galerkin's method and the weak formulation are used, combination of these two results in symmetric element stiffness matrix and banded symmetric global stiffness matrix for unknown nodal displacements in the trial solution. This advantage reduces significantly the computational cost for solving the ODE.

Consider the term $\int_0^L N_i(x)\,q(x)dx$, the weighting function $N_i(x)$ weights and lumps the traction $q(x)$ to node i. The integral $\int_0^L N_i(x)\,q(x)dx$ hence computes the equivalent nodal force at node i. Write all 3rd terms in matrix form. They form a nodal force vector {f} of size n x 1.

$$f = \int_0^L \begin{bmatrix} N_1(x) & N_2(x) & .... & N_n(x) \end{bmatrix}^T q(x)dx$$

We combine $\{F\}$ into $\{f\}$ and modify [K] to include the boundary conditions

The system of n linear independent equations becomes

$$[K]\{u\} = \{f\}$$

We solve for the nodal displacement vector $\{u\}$. Then we compute our approximated solution

$$\overline{u(x)} = [N(x)]\{u\}$$

$$\overline{\sigma(x)} = [D][B(x)]\{u\}$$

2nd property of shape functions states that the sum of all shape functions equals 1, i.e. $\sum_{i=1}^{n} N_i(x) = 1$. In Galerkin's method, the shape functions are also the weighting function. This property ensures that shape functions can act as weighting factors to the integrals, i.e. summation of the weighted values recover the domain value.

For example,

If we sum the weighted residuals, we have domain residual

$$\int_0^L \left[\frac{d}{dx}\left(A(x)\,E(x)\frac{d\bar{u}}{dx} \right) + q(x) \right] dx = \int_0^L \left(\sum_{i=1}^{n} N_i(x) \right) \left[\frac{d}{dx}\left(A(x)\,E(x)\frac{d\bar{u}}{dx} \right) + q(x) \right] dx$$

$$= R_1 + R_2 + ... + R_n$$

When weighting the traction over the domain, we have

$$f = \int_0^L \begin{bmatrix} N_1(x) & N_2(x) & .... & N_n(x) \end{bmatrix}^T q(x)dx$$

Summation of all equivalent nodal force terms in the nodal force vector recovers full traction over the domain

$$\int_0^L N_1(x)q(x)dx + \int_0^L N_2(x)q(x)dx + ... + \int_0^L N_n(x)q(x)dx = \int_0^L \left(\sum_{i=1}^{n} N_i(x) \right) q(x)dx$$