Potential Energy Approach to Derive Bar Element Equations

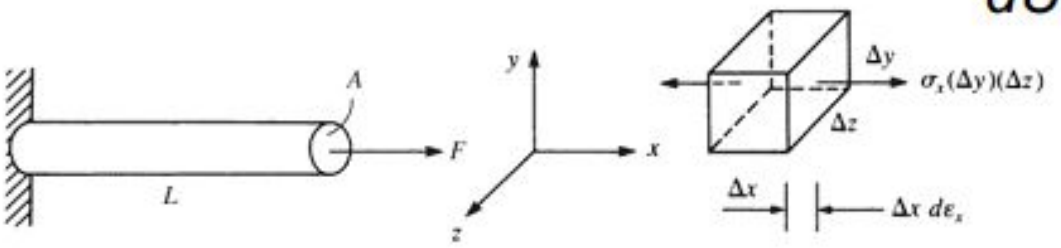
Let's derive the equations for a bar element using the principle of minimum potential energy.

The total potential energy, π_p, is defined as the sum of the internal strain energy U and the potential energy of the external forces Ω:

$$\pi_p = U + \Omega$$

The differential internal work (strain energy) dU in a one-dimensional bar element is:

$$dU = \sigma_x(\Delta y)(\Delta z)(\Delta x)d\varepsilon_x$$



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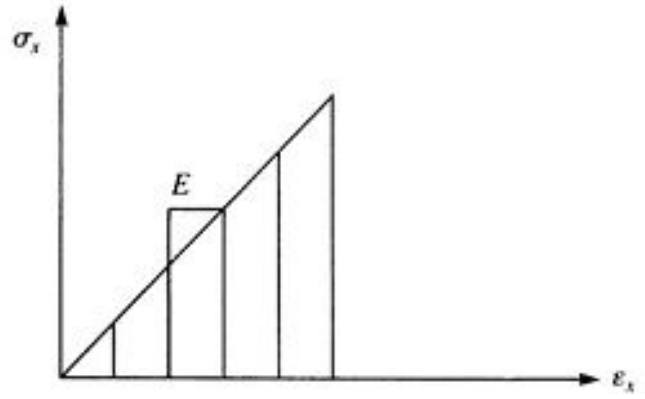
If we let the volume of the element approach zero, then:

$$dU = \sigma_x d\varepsilon_x dV$$

Summing the differential energy over the whole bar gives:

$$U = \int_V \left\{ \int_0^{\varepsilon_x} \sigma_x d\varepsilon_x \right\} dV$$

For a linear-elastic material (Hooke's law) as shown below:



$$\sigma_x = E\varepsilon_x$$

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The internal strain energy statement becomes

$$U = \frac{1}{2}\int_V \sigma_x \varepsilon_x dV$$

The potential energy of the external forces is:

$$\Omega = -\int_V X_b u \, dV - \int_S T_x u_s \, dS - \sum_{i=1}^{M} f_{ix} u_i$$

where X_b is the body force (force per unit volume), T_x is the traction (force per unit area), and f_{ix} is the nodal concentrated force. All of these forces are considered to act in the local x direction.

Potential Energy Approach to Derive Bar Element Equations

Apply the following steps when using the principle of minimum potential energy to derive the finite element equations.

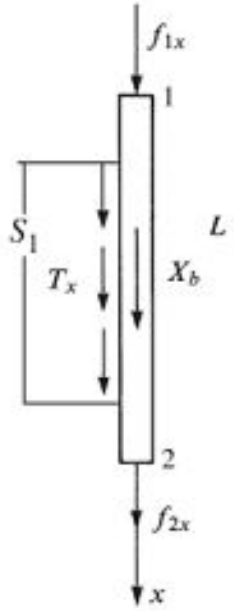
1. Formulate an expression for the total potential energy.

2. Assume a displacement pattern.

3. Obtain a set of simultaneous equations minimizing the total potential energy with respect to the displacement parameters.

Potential Energy Approach to Derive Bar Element Equations

Consider the following bar element, as shown below:



$$\pi_p = \frac{A}{2}\int_0^L \sigma_x \varepsilon_x \, dx - f_{1x}u_1 - f_{2x}u_2 - \int_V X_b u \, dV - \int_S T_x u_s \, dS$$

We can approximate the axial displacement as:

$$u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad N_1 = 1 - \frac{x}{L} \qquad N_2 = \frac{x}{L}$$

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Using the stress-strain relationships, the axial strain is:

$$\varepsilon_x = \frac{du}{dx} = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

where N_1 and N_2 are the interpolation functions gives as:

$$\varepsilon_x = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \qquad \{\varepsilon_x\} = [B]\{d\}$$

$$B = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

The axial stress-strain relationship is: $\{\sigma_x\} = [D]\{\varepsilon_x\}$

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Where $[D] = [E]$ for the one-dimensional stress-strain relationship and E is the modulus of elasticity.

Therefore, stress can be related to nodal displacements as:

$$\{\sigma_x\} = [D][B]\{d\}$$

The total potential energy expressed in matrix form is:

$$\pi_p = \frac{A}{2}\int_0^L \{\sigma_x\}^T \{\varepsilon_x\}\, dx - \{d\}^T \{P\} - \int_V \{u\}^T \{X_b\}\, dV - \int_S \{u\}^T \{T_x\}\, dS$$

where $\{P\}$ represented the concentrated nodal loads.

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If we substitute the relationship between $\hat{u}$ and $\hat{d}$ into the energy equations we get:

$$\pi_p = \frac{A}{2}\int_0^L \{d\}^T [B]^T [D]^T [B]\{d\}\, dx - \{d\}^T \{P\}$$
$$- \int_V \{d\}^T [N]^T \{X_b\}\, dV - \int_S \{d\}^T [N_s]^T \{T_x\}\, dS$$

In the above expression for potential energy π_p is a function of the **d**, that is: $\pi_p = \pi_p(u_1, u_2)$.

However, [*B*] and [*D*] and the nodal displacements *u* are not a function of *x*.

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Integration the energy expression with respect to *x* gives:

$$\pi_p = \frac{AL}{2}\{d\}^T [B]^T [D]^T [B]\{d\} - \{d\}^T \{f\}$$

where

$$\{f\} = \{P\} + \int_V [N]^T \{X_b\}\, dV + \int_S [N]^T \{X_b\}\, dS$$

We can define the surface tractions and body-force matrices as:

$$\{f_s\} = \int_S [N]^T \{T_x\}\, dS \qquad \{f_b\} = \int_V [N]^T \{X_b\}\, dV$$

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Minimization of π_p with respect to each nodal displacement requires that:

$$\frac{\partial \pi_p}{\partial u_1} = 0 \qquad \frac{\partial \pi_p}{\partial u_2} = 0$$

For convenience, let's define the following

$$\{U^*\} = \{d\}^T [B]^T [D]^T [B] \{d\}$$

$$\{U^*\} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} [E] \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Potential Energy Approach to Derive Bar Element Equations

Simplifying the above expression gives:

$$U^* = \frac{E}{L^2}\left(u_1^{\,2} - 2u_1u_2 + u_2^{\,2}\right)$$

The loading on a bar element is given as:

$$\{d\}^T \{f\} = u_1 f_{1x} + u_2 f_{2x}$$

Therefore, the minimum potential energy is:

$$\frac{\partial \pi_p}{\partial u_1} = \frac{AE}{2L}\left(2u_1 - 2u_2\right) - f_{1x} = 0$$

$$\frac{\partial \pi_p}{\partial u_2} = \frac{AE}{2L}\left(-2u_1 + 2u_2\right) - f_{2x} = 0$$

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The above equations can be written in matrix form as:

$$\frac{\partial \pi_p}{\partial (d)} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} f_{1x} \\ f_{2x} \end{pmatrix} = 0$$

The stiffness matrix for a bar element is: $[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

This form of the stiffness matrix obtained from the principle of minimum potential energy is identical to the stiffness matrix derived from the equilibrium equations.