Chapter # 3

Development of Truss Equations

Development of Truss Equations

Having set forth the foundation on which the direct stiffness method is based, we will now derive the stiffness matrix for a linear-elastic bar (or truss) element using the general steps outlined in Chapter 1.

We will include the introduction of both a local coordinate system, chosen with the element in mind, and a global or reference coordinate system, chosen to be convenient (for numerical purposes) with respect to the overall structure.

We will also discuss the transformation of a vector from the local coordinate system to the global coordinate system, using the concept of transformation matrices to express the stiffness matrix of an arbitrarily oriented bar element in terms of the global system.

Development of Truss Equations

Next we will describe how to handle inclined, or skewed, supports.

We will then extend the stiffness method to include space trusses.

We will develop the transformation matrix in three-dimensional space and analyze a space truss.

We will then use the principle of minimum potential energy and apply it to the bar element equations.

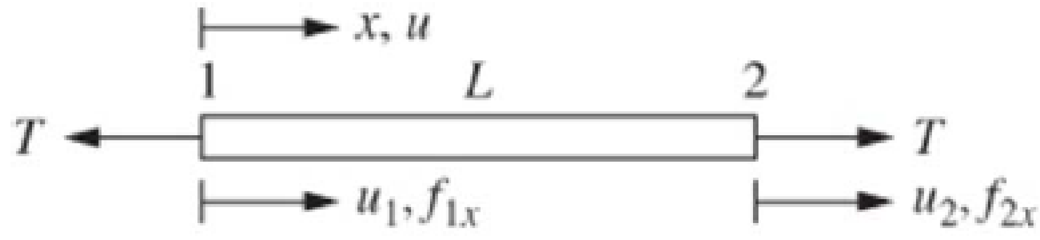
Finally, we will apply Galerkin's residual method to derive the bar element equations.

Development of Truss Equations

Development of Truss Equations

Stiffness Matrix for a Bar Element

Consider the derivation of the stiffness matrix for the linear-elastic, constant cross-sectional area (prismatic) bar element show below.



The bar element has a constant cross-section *A*, an initial length *L*, and modulus of elasticity *E*.

The nodal degrees of freedom are the local axial displacements u_1 and u_2 at the ends of the bar.

Stiffness Matrix for a Bar Element

The strain-displacement relationship is: $\sigma = E\varepsilon \qquad \varepsilon = \dfrac{du}{dx}$

From equilibrium of forces, assuming no distributed loads acting on the bar, we get:

$$A\sigma_x = T = \text{constant}$$

Combining the above equations gives:

$$AE\frac{du}{dx} = T = \text{constant}$$

Taking the derivative of the above equation with respect to the local coordinate x gives:

$$\frac{d}{dx}\left(AE\frac{du}{dx}\right) = 0$$

Stiffness Matrix for a Bar Element

The following assumptions are considered in deriving the bar element stiffness matrix:

1. The bar cannot sustain shear force: $f_{1y} = f_{2y} = 0$

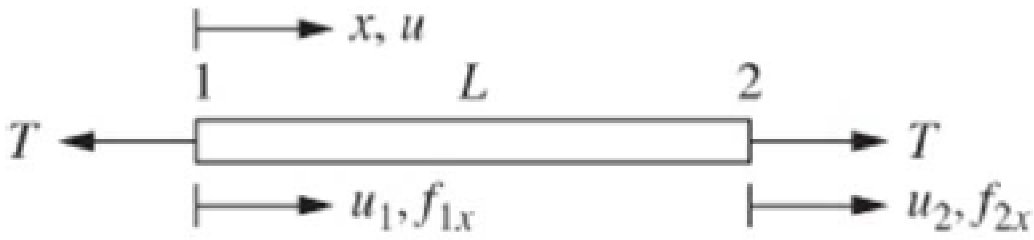
2. Any effect of transverse displacement is ignored.

3. Hooke's law applies; stress is related to strain: $\sigma_x = E\varepsilon_x$

Stiffness Matrix for a Bar Element

Step 1 - Select Element Type

We will consider the linear bar element shown below.



Stiffness Matrix for a Bar Element

Step 2 - Select a Displacement Function

A linear displacement function u is assumed: $u = a_1 + a_2 x$

The number of coefficients in the displacement function, $\boldsymbol{a}_i$, is equal to the total number of degrees of freedom associated with the element.

Applying the boundary conditions and solving for the unknown coefficients gives:

$$u = \left(\frac{u_2 - u_1}{L} \right) x + u_1 \qquad u = \begin{bmatrix} \left(1 - \frac{x}{L} \right) & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

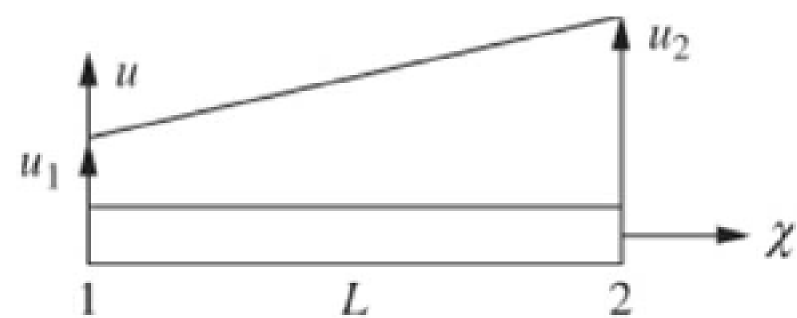
Step 2 - Select a Displacement Function

Or in another form: $u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

where $\boldsymbol{N_1}$ and $\boldsymbol{N_2}$ are the interpolation functions gives as:

$$N_1 = 1 - \frac{x}{L} \qquad N_2 = \frac{x}{L}$$

The linear displacement function $\hat{u}$ plotted over the length of the bar element is shown below.



Stiffness Matrix for a Bar Element

Step 3 - Define the Strain/Displacement and Stress/Strain Relationships

The stress-displacement relationship is: $\varepsilon_x = \frac{du}{dx} = \frac{u_2 - u_1}{L}$

Step 4 - Derive the Element Stiffness Matrix and Equations

We can now derive the element stiffness matrix as follows:

$$T = A\sigma_x$$

Substituting the stress-displacement relationship into the above equation gives:

$$T = AE\left(\frac{u_2 - u_1}{L}\right)$$

Stiffness Matrix for a Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

The nodal force sign convention, defined in element figure, is:

$$f_{1x} = -T \qquad f_{2x} = T$$

therefore, $$f_{1x} = AE\left(\frac{u_1 - u_2}{L}\right) \qquad f_{2x} = AE\left(\frac{u_2 - u_1}{L}\right)$$

Writing the above equations in matrix form gives:

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Notice that ***AE/L*** for a bar element is analogous to the spring constant ***k*** for a spring element.

Stiffness Matrix for a Bar Element

Step 5 - Assemble the Element Equations and Introduce Boundary Conditions

The ***global stiffness matrix*** and the ***global force vector*** are assembled using the nodal force equilibrium equations, and force/deformation and compatibility equations.

$$\mathbf{K} = [K] = \sum_{e=1}^{n} \mathbf{k}^{(e)} \qquad \mathbf{F} = \{F\} = \sum_{e=1}^{n} \mathbf{f}^{(e)}$$

Where **k** and **f** are the element stiffness and force matrices expressed in global coordinates.

Stiffness Matrix for a Bar Element

Step 6 - Solve for the Nodal Displacements

Solve the displacements by imposing the boundary conditions and solving the following set of equations:

$$\mathbf{F} = \mathbf{K}\mathbf{u}$$

Step 7 - Solve for the Element Forces

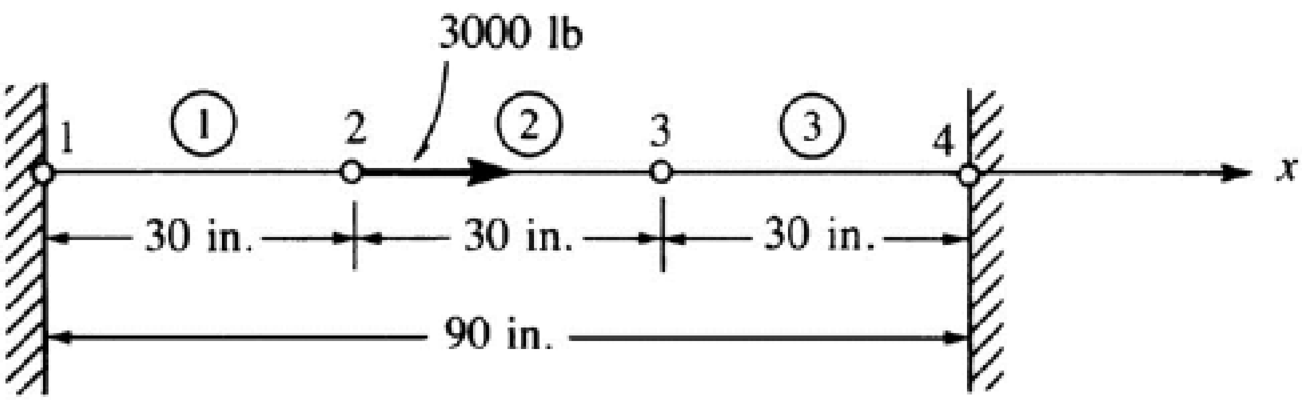
Once the displacements are found, the stress and strain in each element may be calculated from:

$$\varepsilon_x = \frac{du}{dx} = \frac{u_2 - u_1}{L} \qquad \sigma_x = E\varepsilon_x$$

Stiffness Matrix for a Bar Element

Example 1 - Bar Problem

Consider the following three-bar system shown below. Assume for elements 1 and 2: $A = 1\ in^2$ and $E = 30\ (10^6)\ psi$ and for element 3: $A = 2\ in^2$ and $E = 15\ (10^6)\ psi$.



Determine: (a) the global stiffness matrix, (b) the displacement of nodes 2 and 3, and (c) the reactions at nodes 1 and 4.

Stiffness Matrix for a Bar Element

Example 1 - Bar Problem

For elements 1 and 2:

$$\mathbf{k}^{(1)} = \mathbf{k}^{(2)} = \frac{(1)\left(30\times10^{6}\right)}{30}\begin{array}{ccl} 1 & 2 & \text{node numbers for element 1} \\ 2 & 3 & \text{node numbers for element 2} \\ \multicolumn{2}{c}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} & \end{array} {}^{lb}\!/_{in} = 10^{6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} {}^{lb}\!/_{in}$$

For element 3:

$$\mathbf{k}^{(3)} = \frac{(2)\left(15\times10^{6}\right)}{30}\begin{array}{ccl} 3 & 4 & \text{node numbers for element 3} \\ \multicolumn{2}{c}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} & \end{array} {}^{lb}\!/_{in} = 10^{6}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} {}^{lb}\!/_{in}$$

As before, the numbers above the matrices indicate the displacements associated with the matrix.

Stiffness Matrix for a Bar Element

Example 1 - Bar Problem

Assembling the global stiffness matrix by the direct stiffness methods gives:

E1 E 2 E 3

$$\mathbf{K} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Relating global nodal forces related to global nodal displacements gives:

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 1 – Bar Problem

The boundary conditions are: $u_1 = u_4 = 0$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^6 \left[\begin{array}{c|cc|c} 1 & -1 & 0 & 0 \\ \hline -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ \hline 0 & 0 & -1 & 1 \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Applying the boundary conditions and the known forces ($\boldsymbol{F}_{2x}$ = 3000 lb.) gives:

$$\begin{Bmatrix} 3000 \\ 0 \end{Bmatrix} = 10^6 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$

Stiffness Matrix for a Bar Element

Example 1 – Bar Problem

Solving for $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ gives:

$$u_2 = 0.002\,in$$
$$u_3 = 0.001\,in$$

The global nodal forces are calculated as:

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = 10^6 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.002 \\ 0.001 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2000 \\ 3000 \\ 0 \\ -1000 \end{Bmatrix} lbs$$

Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

In many problems it is convenient to introduce both ***local*** and ***global*** (or reference) coordinates.

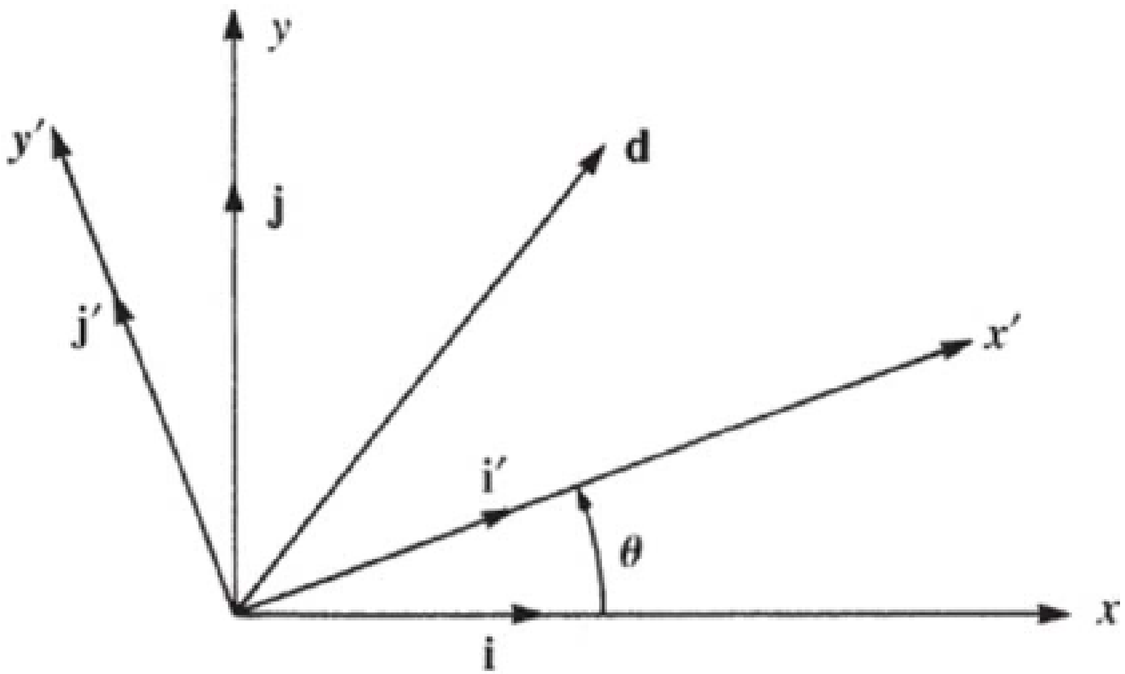
Local coordinates are always chosen to conveniently represent the individual element.

Global coordinates are chosen to be convenient for the whole structure.

Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

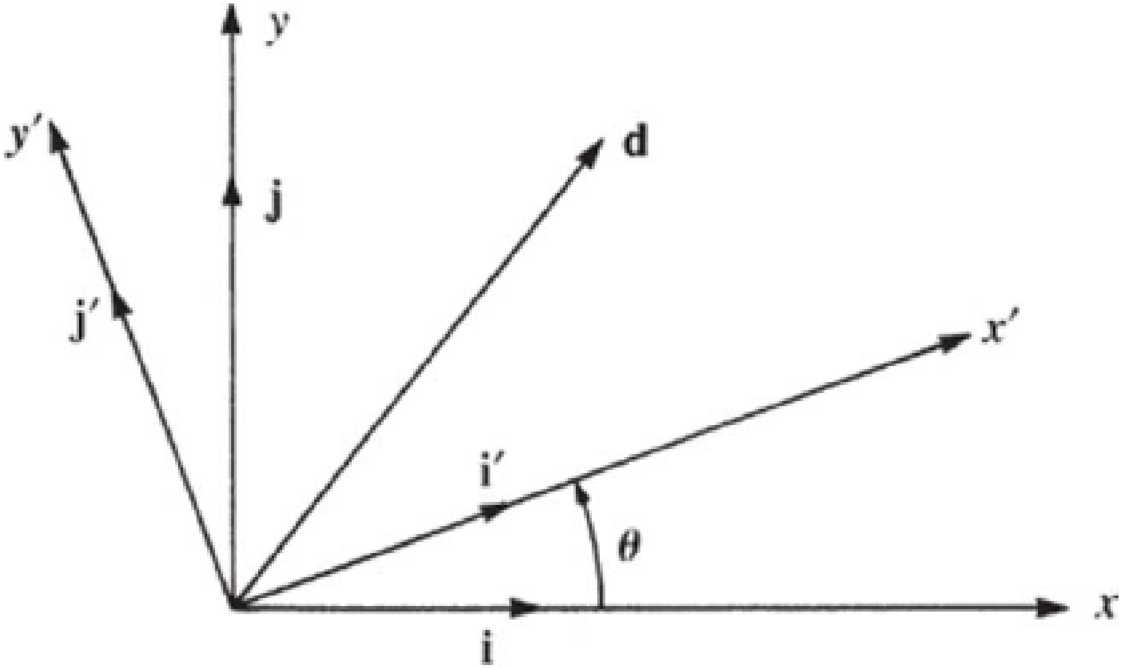
Given the nodal displacement of an element, represented by the vector **d** in the figure below, we want to relate the components of this vector in one coordinate system to components in another.



Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

Let's consider that **d** does not coincident with either the local or global axes. In this case, we want to relate global displacement components to local ones. In so doing, we will develop a ***transformation matrix*** that will subsequently be used to develop the global stiffness matrix for a bar element.

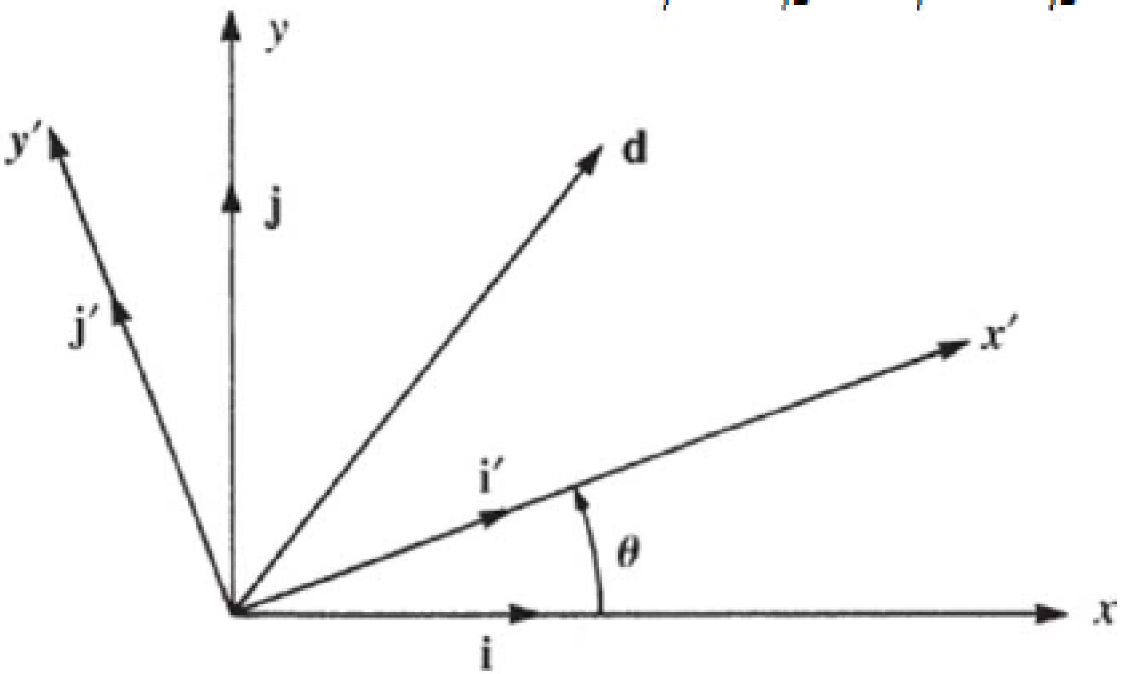


Stiffness Matrix for a Bar Element

Transformation of Vectors in Two Dimensions

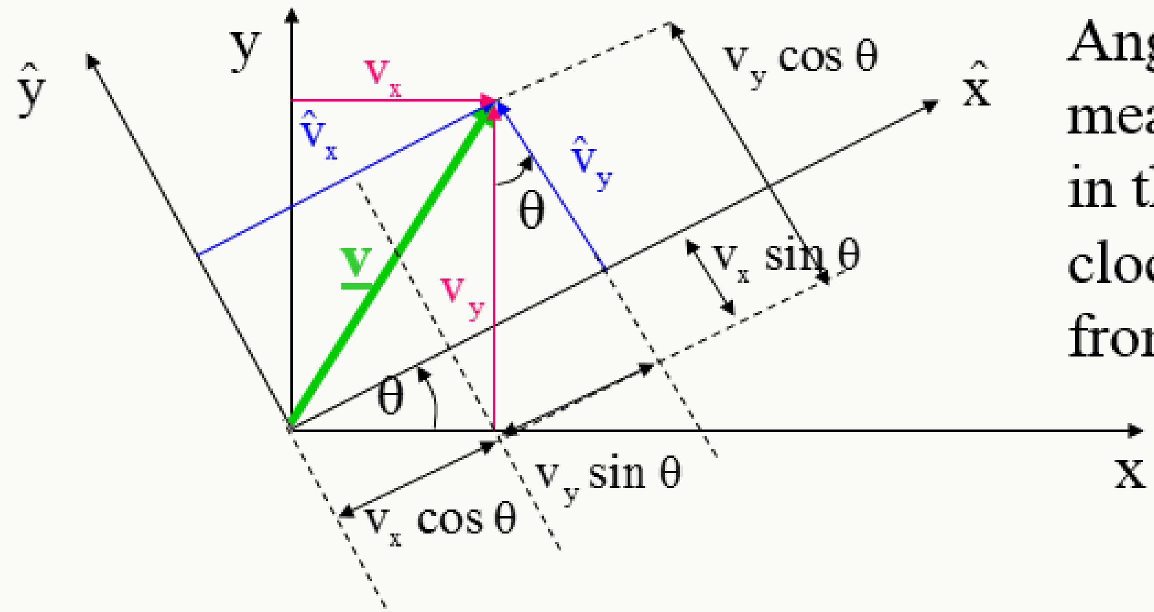
We define the angle θ to be positive when measured counterclockwise from *x* to *x*'. We can express vector displacement **d** in both global and local coordinates by:

$$\mathbf{d} = u_1\mathbf{i} + v_1\mathbf{j} = u_1'\mathbf{i}' + v_1'\mathbf{j}'$$



Stiffness Matrix for a Bar Element

Transformation of a vector in two dimensions



Angle θ is measured positive in the counter clockwise direction from the +x axis)

The vector $\underline{v}$ has components (v_x, v_y) in the global coordinate system and ($\hat{v}_x$, $\hat{v}_y$) in the local coordinate system. From geometry

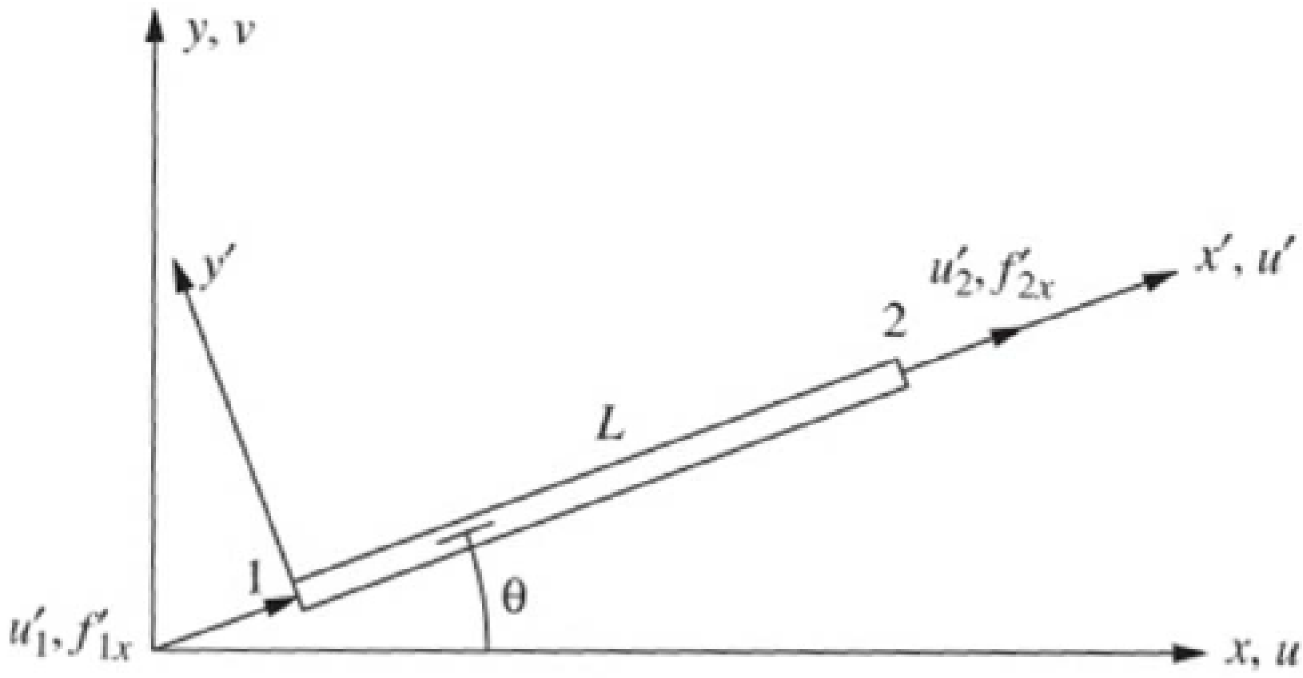
$$\hat{v}_x = v_x \cos\theta + v_y \sin\theta$$

$$\hat{v}_y = -v_x \sin\theta + v_y \cos\theta$$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

We will now use the transformation relationship developed above to obtain the global stiffness matrix for a bar element.



Stiffness Matrix for a Bar Element

Global Stiffness Matrix

We known that for a bar element in local coordinates we have:

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} \qquad \mathbf{f}' = \mathbf{k}'\mathbf{d}'$$

We want to relate the global element forces **f** to the global displacements **d** for a bar element with an arbitrary orientation.

$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = k \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \qquad \mathbf{f} = \mathbf{k}\mathbf{d}$$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

Using the relationship between local and global components, we can develop the global stiffness matrix.

We already know the transformation relationships:

$$u'_1 = u_1\cos\theta + v_1\sin\theta \qquad\qquad u'_2 = u_2\cos\theta + v_2\sin\theta$$

Combining both expressions for the two local degrees-of-freedom, in matrix form, we get:

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\mathbf{d}' = \mathbf{T}^{*}\mathbf{d}$$

$$\mathbf{T}^{*} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

A similar expression for the force transformation can be developed.

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}$$

$$\boxed{\mathbf{f}' = \mathbf{T}^{*}\mathbf{f}}$$

Substituting the global force expression into element force equation gives:

$$\mathbf{f}' = \mathbf{k}'\mathbf{d}' \quad \Rightarrow \quad \mathbf{T}^{*}\mathbf{f} = \mathbf{k}'\mathbf{d}'$$

Substituting the transformation between local and global displacements gives:

$$\boxed{\mathbf{d}' = \mathbf{T}^{*}\mathbf{d}} \quad \Rightarrow \quad \boxed{\mathbf{T}^{*}\mathbf{f} = \mathbf{k}'\mathbf{T}^{*}\mathbf{d}}$$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

The matrix **T*** is not a square matrix so we cannot invert it.

Let's expand the relationship between local and global displacement.

$$\begin{Bmatrix} u_1' \\ v_1' \\ u_2' \\ v_2' \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \qquad \mathbf{d}' = \mathbf{T}\mathbf{d}$$

where **T** is:

$$\mathbf{T} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

We can write a similar expression for the relationship between local and global forces.

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix} \qquad \mathbf{f'} = \mathbf{Tf}$$

Therefore our original local coordinate force-displacement expression

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} \qquad \mathbf{f'} = \mathbf{k'd'}$$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

May be expanded:

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix}$$

The global force-displacement equations are:

$$\mathbf{f}' = \mathbf{k}'\mathbf{d}' \quad \Rightarrow \quad \mathbf{T}\mathbf{f} = \mathbf{k}'\mathbf{T}\mathbf{d}$$

Multiply both side by $\mathbf{T}^{-1}$ we get: $\mathbf{f} = \mathbf{T}^{-1}\mathbf{k}'\mathbf{T}\mathbf{d}$

where $\mathbf{T}^{-1}$ is the ***inverse*** of **T**. It can be shown that: $\mathbf{T}^{-1} = \mathbf{T}^{T}$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

The global force-displacement equations become: $\mathbf{f} = \mathbf{T}^T \mathbf{k}' \mathbf{T} \mathbf{d}$

Where the global stiffness matrix $\mathbf{k}$ is: $\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$

Expanding the above transformation gives:

$$\mathbf{k} = \frac{AE}{L} \begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

We can assemble the total stiffness matrix by using the above element stiffness matrix and the direct stiffness method.

$$\mathbf{K} = [K] = \sum_{e=1}^{n} \mathbf{k}^{(e)} \qquad \mathbf{F} = \{F\} = \sum_{e=1}^{n} \mathbf{f}^{(e)} \qquad \mathbf{F} = \mathbf{K}\mathbf{d}$$

Stiffness Matrix for a Bar Element

Global Stiffness Matrix

Local forces can be computed as:

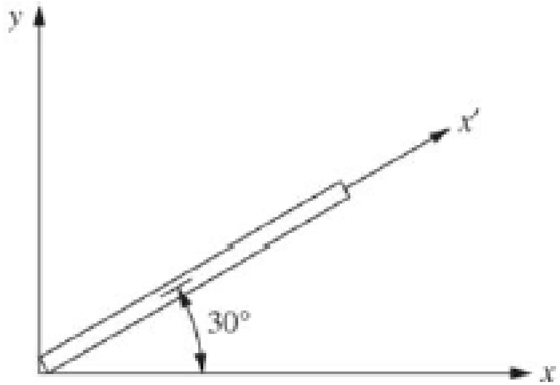
$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u'_1 \\ v'_1 \\ u'_2 \\ v'_2 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} Cu_1 + Sv_1 - Cu_2 - Sv_2 \\ 0 \\ -Cu_1 - Sv_1 + Cu_2 + Sv_2 \\ 0 \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Example 3 - Bar Element Problem

For the bar element shown below, evaluate the global stiffness matrix. Assume the cross-sectional area is 2 *in*^{2}, the length is 60 *in*, and the *E* is 30 x 10^6 *psi*.



$$\mathbf{k} = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

Therefore: $C = \cos 30^o = \frac{\sqrt{3}}{2}$ $S = \sin 30^o = \frac{1}{2}$

Stiffness Matrix for a Bar Element

Example 3 - Bar Element Problem

The global elemental stiffness matrix is:

$$\mathbf{k} = \frac{(2\,in^2)(30\times10^6\,psi)}{60\,in}\begin{bmatrix} 3/4 & \sqrt{3}/4 & -3/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 & -\sqrt{3}/4 & -1/4 \\ -3/4 & -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 \\ -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & 1/4 \end{bmatrix} {lb}/{in}$$

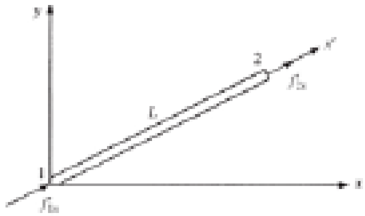
Simplifying the global elemental stiffness matrix is:

$$\mathbf{k} = 10^6\begin{bmatrix} 0.750 & 0.433 & -0.750 & -0.433 \\ 0.433 & 0.250 & -0.433 & -0.250 \\ -0.750 & -0.433 & 0.750 & 0.433 \\ -0.433 & -0.250 & 0.433 & 0.250 \end{bmatrix} {lb}/{in}$$

Stiffness Matrix for a Bar Element

Computation of Stress for a Bar in the x-y Plane

For a bar element the local forces are related to the local displacements by:



$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix}$$

The force-displacement equation for f'_{2x} is:

$$f'_{2x} = \frac{AE}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix}$$

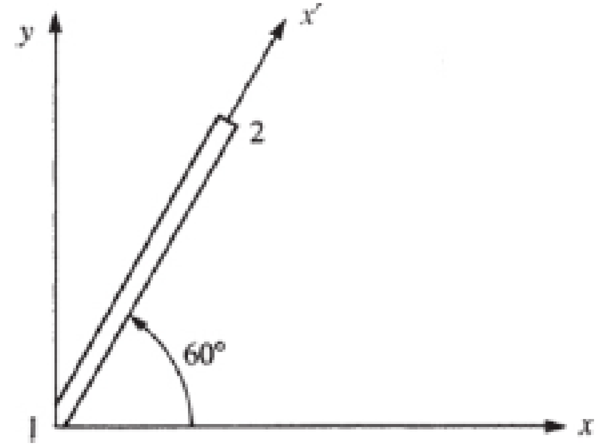
The stress in terms of global displacement is:

$$\sigma = \frac{E}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} C & S & 0 & 0 \\ 0 & 0 & C & S \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \frac{E}{L}\begin{bmatrix} -Cu_1 - Sv_1 + Cu_2 + Sv_2 \end{bmatrix}$$

Stiffness Matrix for a Bar Element

Example 4 - Bar Element Problem

For the bar element shown below, determine the axial stress. Assume the cross-sectional area is 4 x 10^{-4} m^2, the length is 2 *m*, and the ***E*** is 210 *GPa*.



The global displacements are known as $\boldsymbol{u}_1$ = 0.25 *mm*, $\boldsymbol{v}_1$ = 0, $\boldsymbol{u}_2$ = 0.5 *mm*, and $\boldsymbol{v}_2$ = 0.75 *mm*.

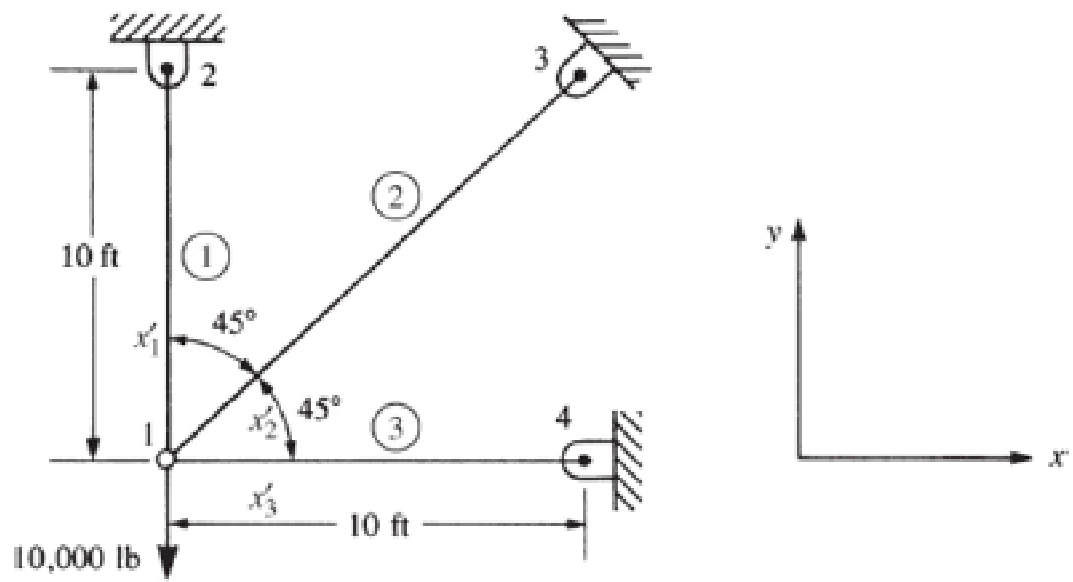
$$\sigma = \frac{210 \times 10^6}{2}\left[-\frac{1}{2}(0.25) - \frac{\sqrt{3}}{4}(0) + \frac{1}{2}(0.5) + \frac{\sqrt{3}}{4}(0.75)\right] {}^{KN}\!/\!_{m}$$

$$\sigma = 81.32 \times 10^3 \, {}^{kN}\!/\!_{m^2} = 81.32 \, MPa$$

Example 5 - Plane Truss Problem

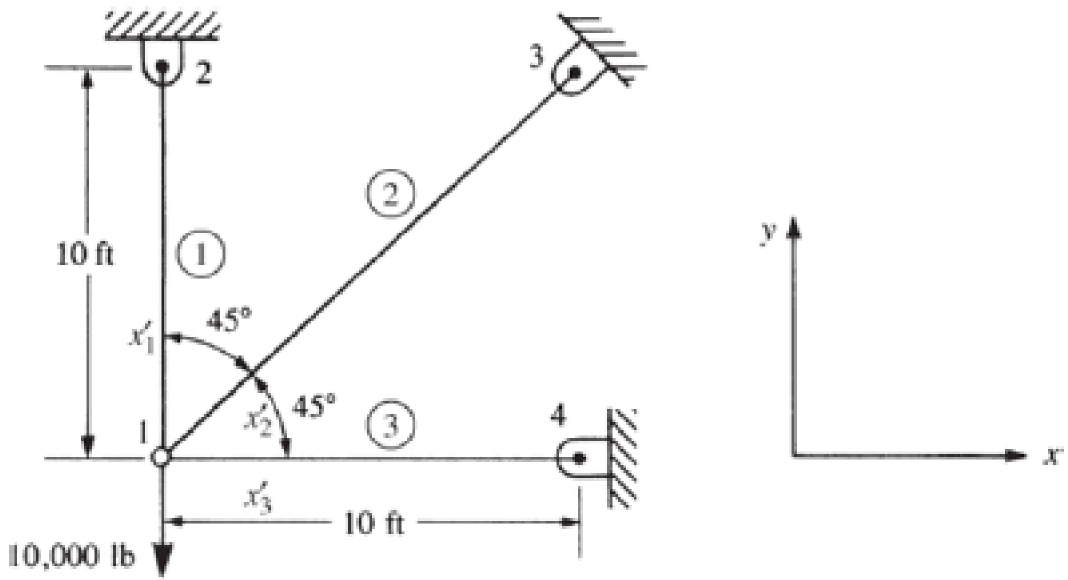
The plane truss shown below is composed of three bars subjected to a downward force of 10 *kips* at node 1. Assume the cross-sectional area $A = 2\ in^2$ and E is $30 \times 10^6\ psi$ for all elements.

Determine the x and y displacement at node 1 and stresses in each element.



Example 5 - Plane Truss Problem

Element	Node 1	Node 2	θ	C	S
1	1	2	90°	0	1
2	1	3	45°	0.707	0.707
3	1	4	0°	1	0



Example 5 - Plane Truss Problem

The global elemental stiffness matrix are:

$$\mathbf{k} = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

element 1:

$$C = 0 \quad S = 1 \quad \Rightarrow \quad \mathbf{k}^{(1)} = \frac{(2\,in^2)(30\times 10^6\,psi)}{120\,in} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} {}^{lb}\!/_{in}$$

element 2:

$$C = \tfrac{\sqrt{2}}{2} \quad S = \tfrac{\sqrt{2}}{2} \quad \Rightarrow \quad \mathbf{k}^{(2)} = \frac{(2\,in^2)(30\times 10^6\,psi)}{240\sqrt{2}\,in} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \end{array} {}^{lb}\!/_{in}$$

element 3:

$$C = 1 \quad S = 0 \quad \Rightarrow \quad \mathbf{k}^{(3)} = \frac{(2\,in^2)(30\times 10^6\,psi)}{120\,in} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} {}^{lb}\!/_{in}$$

Example 5 - Plane Truss Problem

The total global stiffness matrix is:

$$
\mathbf{K} = 5\times10^5 \begin{bmatrix}
1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\
0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\
-0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \, {}^{lb}\!/_{in}
$$

element 1

element 2

element 3

The total global force-displacement equations are:

$$
\begin{Bmatrix} 0 \\ -10,000 \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = 5\times10^5 \begin{bmatrix}
1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\
0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
-0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\
-0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

Example 5 - Plane Truss Problem

Applying the boundary conditions for the truss, the above equations reduce to:

$$\begin{Bmatrix} 0 \\ -10,000 \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{Bmatrix} = 5\times 10^{5} \begin{bmatrix} 1.354 & 0.354 & 0 & 0 & -0.354 & -0.354 & -1 & 0 \\ 0.354 & 1.354 & 0 & -1 & -0.354 & -0.354 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -0.354 & -0.354 & 0 & 0 & 0.354 & 0.354 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Example 5 - Plane Truss Problem

Applying the boundary conditions for the truss, the above equations reduce to:

$$\begin{Bmatrix} 0 \\ -10{,}000 \end{Bmatrix} = 5\times 10^{5}\begin{bmatrix} 1.354 & 0.354 \\ 0.354 & 1.354 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

Solving the equations gives:

$$u_1 = 0.414\times 10^{-2}\, in$$

$$v_1 = -1.59\times 10^{-2}\, in$$

The stress in an element is:

$$\sigma = \frac{E}{L}\left[-Cu_1 - Sv_1 + Cu_2 + Sv_2\right]$$

Example 5 - Plane Truss Problem

Element	Node 1	Node 2	θ	C	S
1	1	2	90°	0	1
2	1	3	45°	0.707	0.707
3	1	4	0°	1	0

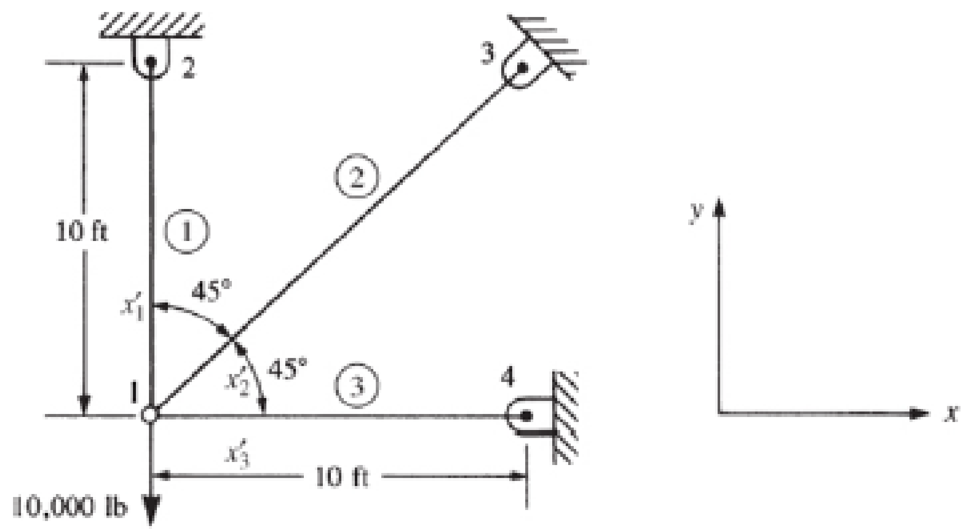
element 1 $\sigma^{(1)} = \dfrac{30\times10^{6}}{120}\left[-v_1\right] = 3{,}965\ psi$

element 2 $\sigma^{(2)} = -\dfrac{30\times10^{6}}{120}\left[(0.707)u_1 + (0.707)v_1\right] = 1{,}471\ psi$

element 3 $\sigma^{(3)} = \dfrac{30\times10^{6}}{120}\left[-u_1\right] = -1{,}035\ psi$

Example 5 - Plane Truss Problem

Let's check equilibrium at node 1:

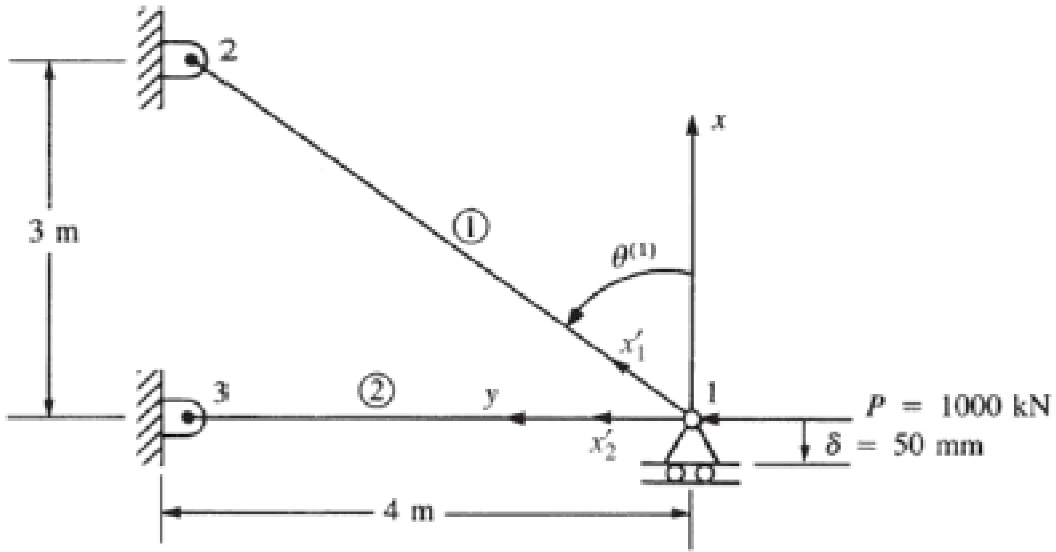


$$\sum F_x = (1{,}471\ psi)(2\ in^2)(0.707) - (1{,}035\ psi)(2\ in^2) = 0$$

$$\sum F_y = (3{,}965\ psi)(2\ in^2) + (1{,}471\ psi)(2\ in^2)(0.707) - 10{,}000 = 0$$

Example 7 - Plane Truss Problem

Consider the two bar truss shown below.



Determine the displacement in the *y* direction of node 1 and the axial force in each element.

Assume E = 210 GPa and $A = 6 \times 10^{-4}\ m^2$

Example 7 - Plane Truss Problem

The global elemental stiffness matrix for ***element 1*** is:

$$\cos\theta^{(1)} = \frac{3}{5} = 0.6 \qquad \sin\theta^{(1)} = \frac{4}{5} = 0.8$$

$$\mathbf{k}^{(1)} = \frac{210\times10^{6}(6\times10^{-4})}{5}\begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -0.48 & -0.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix}$$

Simplifying the above expression gives:

$$\mathbf{k}^{(1)} = 25{,}200\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 \\ 0.48 & 0.64 & -0.48 & -0.64 \\ -0.36 & -0.48 & 0.36 & 0.48 \\ -0.48 & -0.64 & 0.48 & 0.64 \end{bmatrix} \end{array}$$

Example 7 - Plane Truss Problem

The global elemental stiffness matrix for ***element 2*** is:

$$\cos\theta^{(2)} = 0 \qquad \sin\theta^{(2)} = 1$$

$$\mathbf{k}^{(2)} = \frac{(210\times 10^{6})(6\times 10^{-4})}{4}\begin{bmatrix} 0 & 0 & 0 & -0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Simplifying the above expression gives:

$$\mathbf{k}^{(2)} = 25,200\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.25 & 0 & -1.25 \\ 0 & 0 & 0 & 0 \\ 0 & -1.25 & 0 & 1.25 \end{bmatrix} \end{array}$$

Example 7 - Plane Truss Problem

The total global equations are:

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = 25{,}200 \begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 & 0 & 0 \\ 0.48 & 1.89 & -0.48 & -0.64 & 0 & -1.25 \\ -0.36 & -0.48 & 0.36 & 0.48 & 0 & 0 \\ -0.48 & -0.64 & 0.48 & 0.64 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.25 & 0 & 0 & 0 & 1.25 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

The displacement boundary conditions are:

$$u_1 = \delta \qquad u_2 = v_2 = u_3 = v_3 = 0$$

Example 7 - Plane Truss Problem

The total global equations are:

$$\begin{Bmatrix} F_{1x} \\ P \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = 25{,}200 \begin{bmatrix} 0.36 & 0.48 & -0.36 & -0.48 & 0 & 0 \\ 0.48 & 1.89 & -0.48 & -0.64 & 0 & -1.25 \\ -0.36 & -0.48 & 0.36 & 0.48 & 0 & 0 \\ -0.48 & -0.64 & 0.48 & 0.64 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.25 & 0 & 0 & 0 & 1.25 \end{bmatrix} \begin{Bmatrix} \delta \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

By applying the boundary conditions the force-displacement equations reduce to:

$$P = 25{,}200(0.48\delta + 1.89v_1)$$

Example 7 - Plane Truss Problem

Solving the equation gives: $v_1 = (2.1 \times 10^{-5})P - 0.25\delta$

By substituting P = 1,000 kN and δ = -0.05 m in the above equation gives:

$$v_1 = 0.0337\,m$$

The local element forces for element 1 are:

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = 25{,}200 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \end{bmatrix} \begin{Bmatrix} u_1 = -0.05 \\ v_1 = 0.0337 \\ u_2 \\ v_2 \end{Bmatrix}$$

The element forces are: $f_{1x} = -76.6\,kN$ $\qquad f_{2x} = 76.7\,kN$

Example 7 - Plane Truss Problem

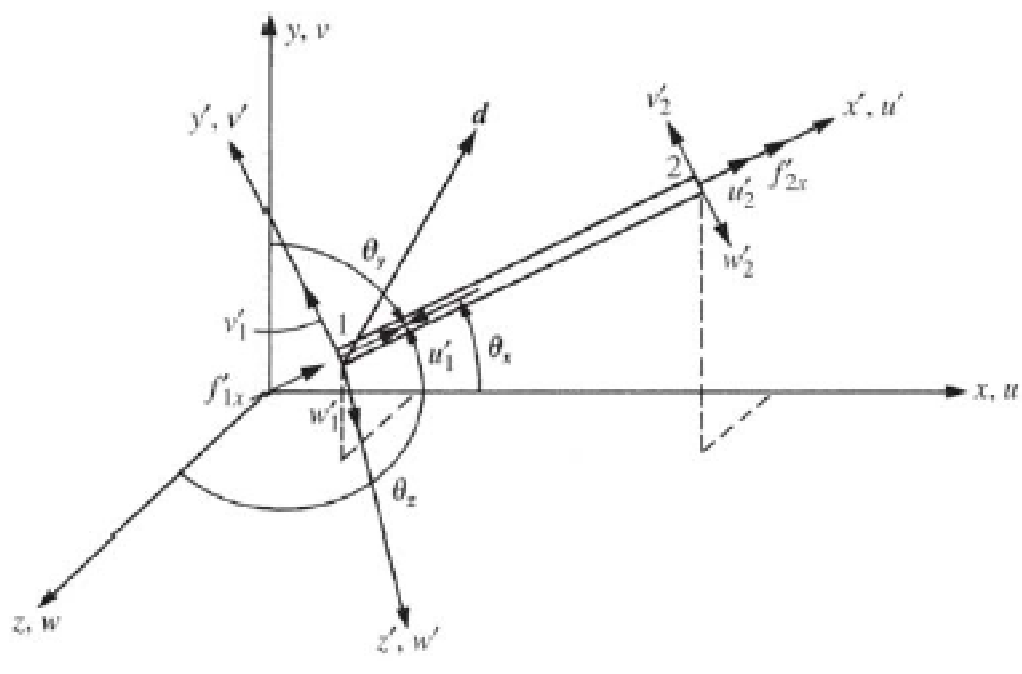
The local element forces for element 2 are:

$$\begin{Bmatrix} f_{1x} \\ f_{3x} \end{Bmatrix} = 31{,}500 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 = -0.05 \\ v_1 = 0.0337 \\ u_3 \\ v_3 \end{Bmatrix}$$

The element forces are: $f_{1x} = 1{,}061\,kN$ $\qquad f_{3x} = -1{,}061\,kN$

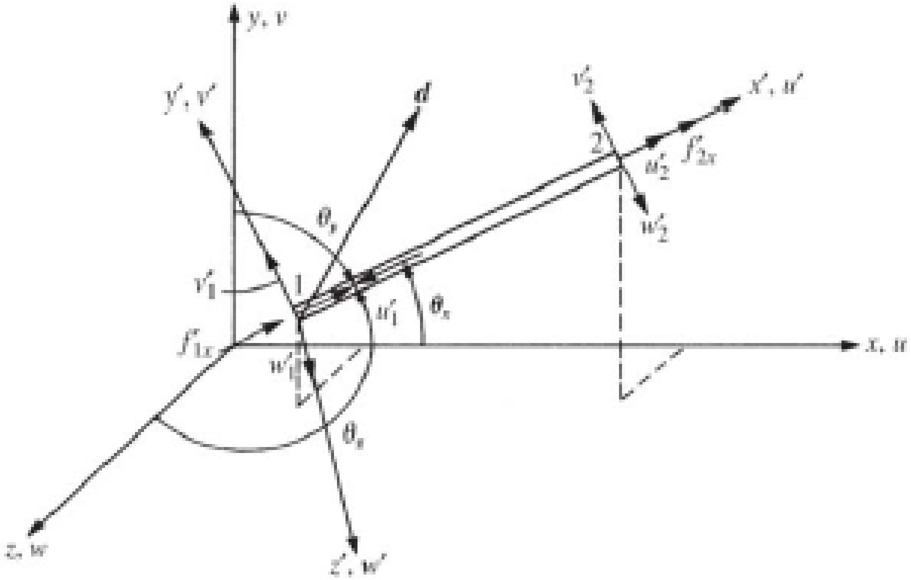
Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

Let's derive the transformation matrix for the stiffness matrix for a bar element in three-dimensional space as shown below:



Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

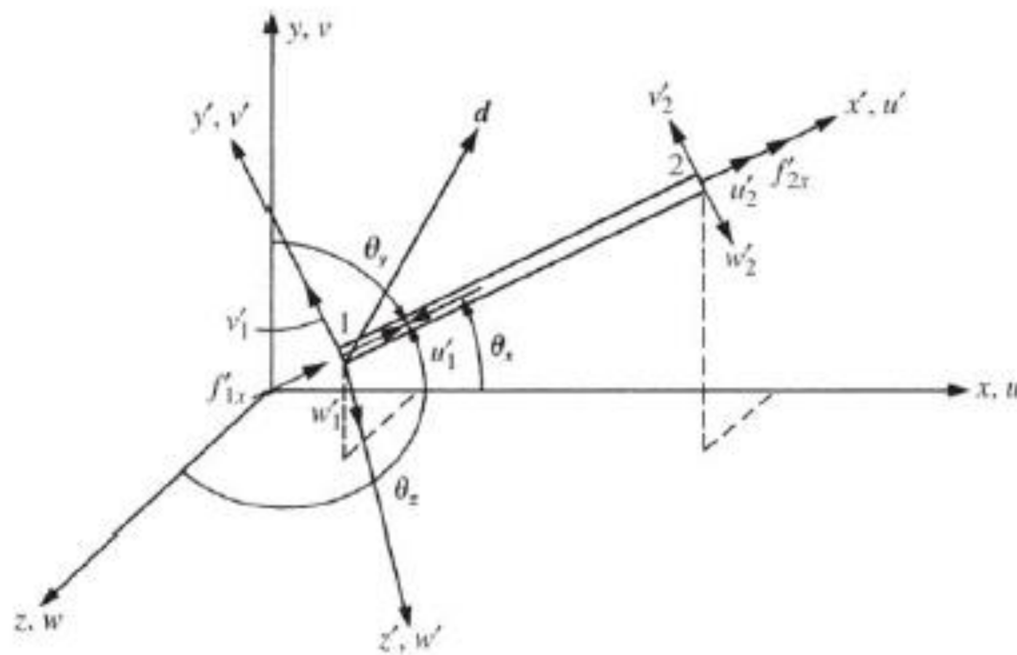
The coordinates at node 1 are x_1, y_1, and z_1, and the coordinates of node 2 are x_2, y_2, and z_2. Also, let θ_x, θ_y, and θ_z be the angles measured from the global x, y, and z axes, respectively, to the local axis.



Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

The three-dimensional vector representing the bar element is gives as:

$$\mathbf{d} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = u'\mathbf{i}' + v'\mathbf{j}' + w'\mathbf{k}'$$



Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

Taking the dot product of the above equation with $\mathbf{i}'$ gives:

$$u(\mathbf{i} \cdot \mathbf{i}') + v(\mathbf{j} \cdot \mathbf{i}') + w(\mathbf{k} \cdot \mathbf{i}') = u'$$

By the definition of the dot product we get:

$$\mathbf{i} \cdot \mathbf{i}' = \frac{x_2 - x_1}{L} = C_x \qquad \mathbf{j} \cdot \mathbf{i}' = \frac{y_2 - y_1}{L} = C_y \qquad \mathbf{k} \cdot \mathbf{i}' = \frac{z_2 - z_1}{L} = C_z$$

where $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$$C_x = \cos\theta_x \qquad C_y = \cos\theta_y \qquad C_z = \cos\theta_z$$

where $\boldsymbol{C_x}$, $\boldsymbol{C_y}$, and $\boldsymbol{C_z}$ are projections of $\mathbf{i}'$ on to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, respectively.

Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

Therefore: $u' = C_x u + C_y v + C_z w$

The transformation between local and global displacements is:

$$\begin{Bmatrix} u'_1 \\ u'_2 \end{Bmatrix} = \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

$$\mathbf{d}' = \mathbf{T}^* \mathbf{d}$$

$$\mathbf{T}^* = \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}$$

Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

The transformation from the local to the global stiffness matrix is:

$$\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$$

$$\mathbf{k} = \begin{bmatrix} C_x & 0 \\ C_y & 0 \\ C_z & 0 \\ 0 & C_x \\ 0 & C_y \\ 0 & C_z \end{bmatrix} \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} C_x & C_y & C_z & 0 & 0 & 0 \\ 0 & 0 & 0 & C_x & C_y & C_z \end{bmatrix}$$

$$\mathbf{k} = \frac{AE}{L} \left[\begin{array}{ccc|ccc} C_x^2 & C_xC_y & C_xC_z & -C_x^2 & -C_xC_y & -C_xC_z \\ C_xC_y & C_y^2 & C_yC_z & -C_xC_y & -C_y^2 & -C_yC_z \\ C_xC_z & C_yC_z & C_z^2 & -C_xC_z & -C_yC_z & -C_z^2 \\ \hline -C_x^2 & -C_xC_y & -C_xC_z & C_x^2 & C_xC_y & C_xC_z \\ -C_xC_y & -C_y^2 & -C_yC_z & C_xC_y & C_y^2 & C_yC_z \\ -C_xC_z & -C_yC_z & -C_z^2 & C_xC_z & C_yC_z & C_z^2 \end{array}\right]$$

Transformation Matrix and Stiffness Matrix for a Bar in Three-Dimensional Space

The global stiffness matrix can be written in a more convenient form as:

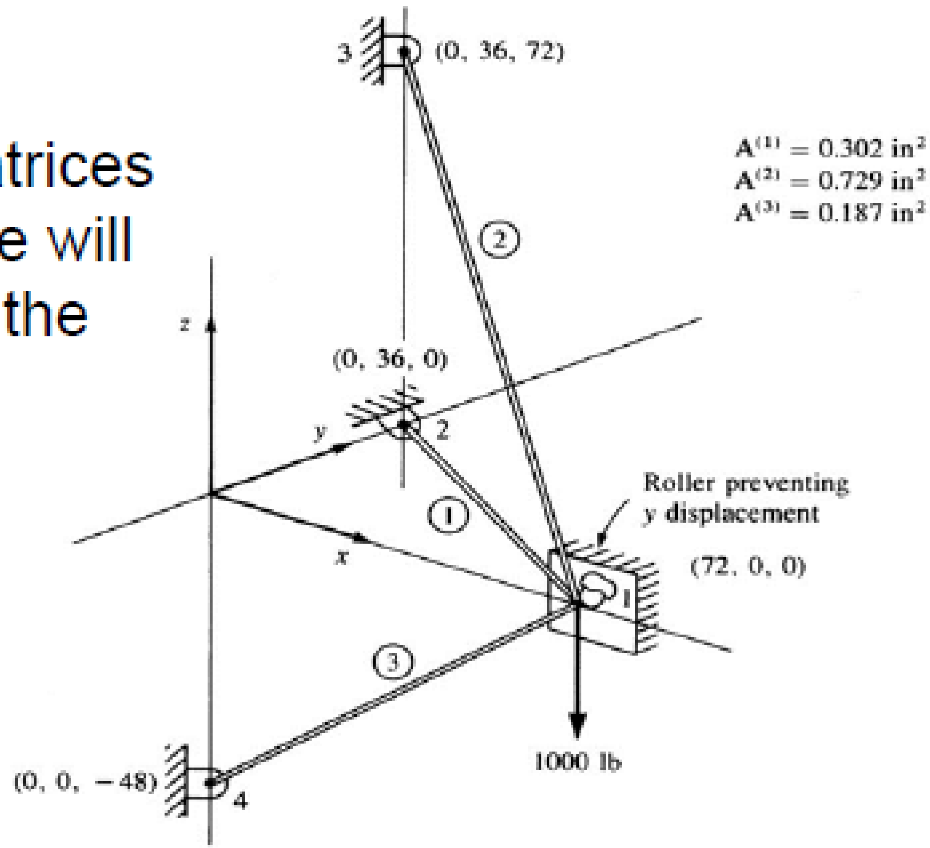
$$\mathbf{k} = \frac{AE}{L}\begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix} \qquad \lambda = \begin{bmatrix} C_x{}^2 & C_x C_y & C_x C_z \\ C_x C_y & C_y{}^2 & C_y C_z \\ C_x C_z & C_y C_z & C_z{}^2 \end{bmatrix}$$

Example 8 – Space Truss Problem

Consider the space truss shown below. The modulus of elasticity, E = 1.2 x 10^6 *psi* for all elements. Node 1 is constrained from movement in the *y* direction.

To simplify the stiffness matrices for the three elements, we will express each element in the following form:

$$\mathbf{k} = \frac{AE}{L}\begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix}$$



Consider element 1:

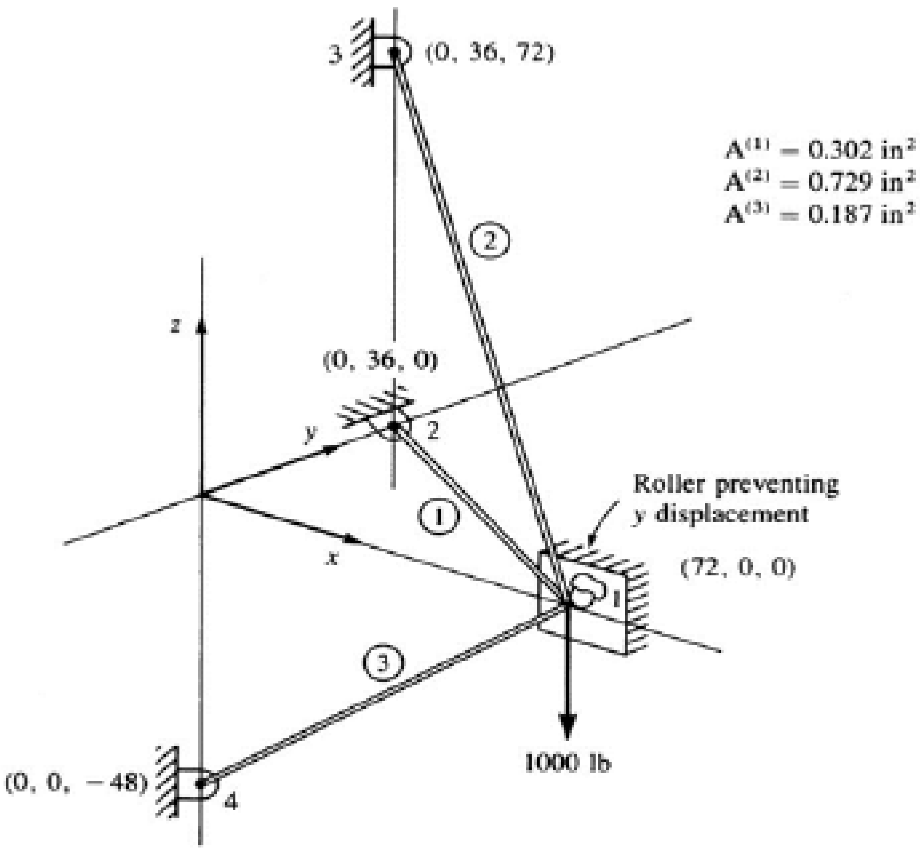
$$L^{(1)} = \sqrt{(-72)^2 + (36)^2} = 80.5\,in$$

$$C_x = \frac{-72}{80.5} = -0.89$$

$$C_y = \frac{36}{80.5} = 0.45$$

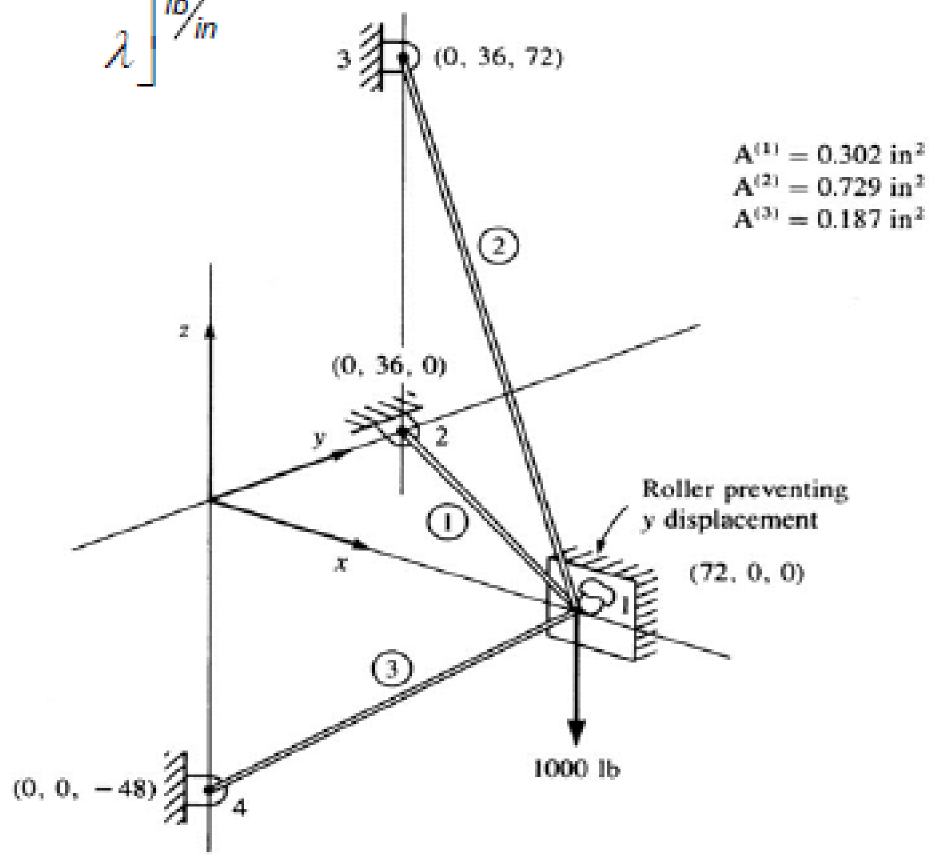
$$C_z = 0$$

$$\lambda = \begin{bmatrix} 0.79 & -0.40 & 0 \\ -0.40 & 0.20 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Consider element 1:

$$\mathbf{k} = \frac{(0.302\,in^2)(1.2\times10^6\,psi)}{80.5\,in}\begin{array}{c} \begin{array}{cc} u_1\;v_1\;w_1 & \qquad u_2\;v_2\;w_2 \end{array} \\ \begin{bmatrix}\lambda & -\lambda \\ -\lambda & \lambda\end{bmatrix}\end{array} {}^{lb}\!/_{in}$$



Consider element 2:

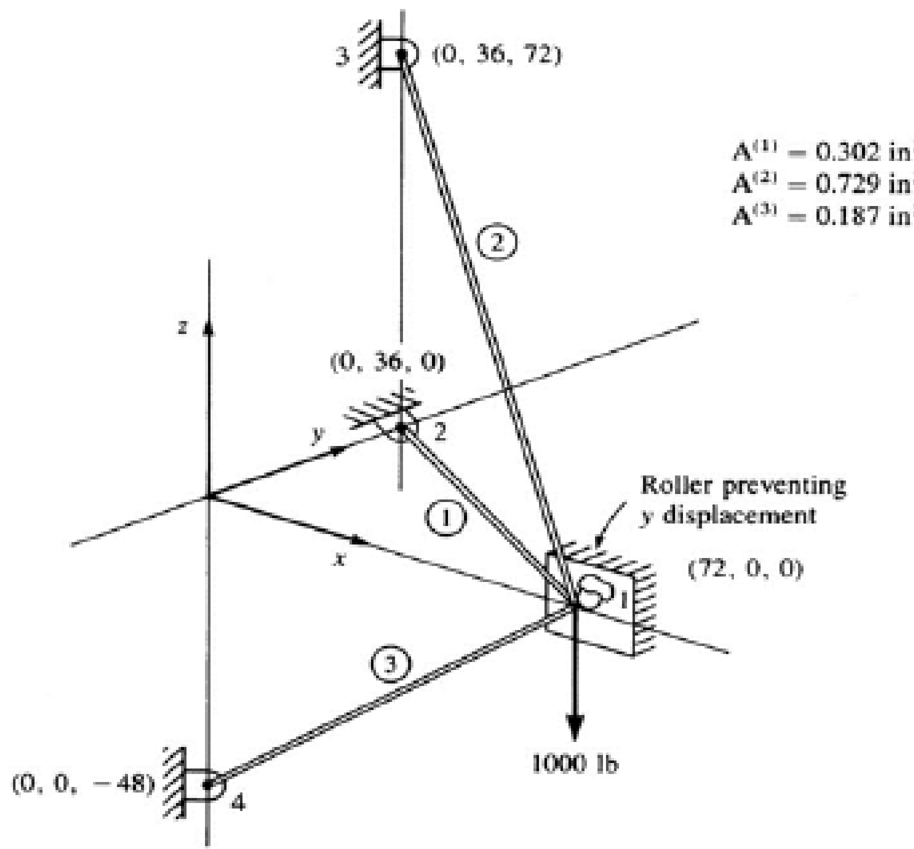
$$L^{(2)} = \sqrt{(-72)^2 + (36)^2 + (72)^2} = 108\,in$$

$$C_x = \frac{-72}{108} = -0.667$$

$$C_y = \frac{36}{108} = 0.33$$

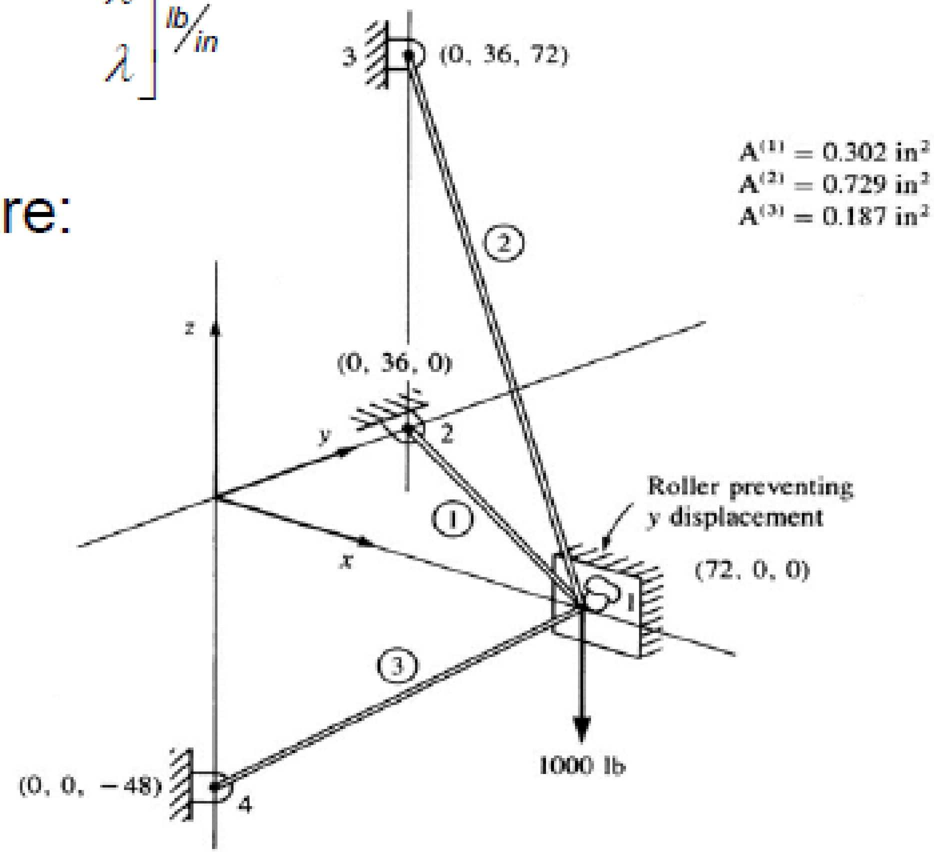
$$C_z = \frac{72}{108} = 0.667$$

$$\lambda = \begin{bmatrix} 0.45 & -0.22 & -0.45 \\ -0.22 & 0.11 & 0.45 \\ -0.45 & 0.45 & 0.45 \end{bmatrix}$$



Consider element 2:

$$
\mathbf{k} = \frac{(0.729\,in^2)(1.2\times10^6\,psi)}{108\,in}
\begin{array}{cc}
u_1\; v_1\; w_1 & u_3\; v_3\; w_3 \\
\left[\begin{array}{c} \lambda \\ -\lambda \end{array}\right. & \left.\begin{array}{c} -\lambda \\ \lambda \end{array}\right]
\end{array}
\,{}^{lb}\!/_{in}
$$



The boundary conditions are:

$$u_2 = v_2 = w_2 = 0$$

$$u_3 = v_3 = w_3 = 0$$

$$u_4 = v_4 = w_4 = 0$$

$$v_1 = 0$$

Consider element 3: $L^{(3)} = \sqrt{(x_4 - x_1)^2 + (y_4 - y_1)^2 + (z_4 - z_1)^2}$

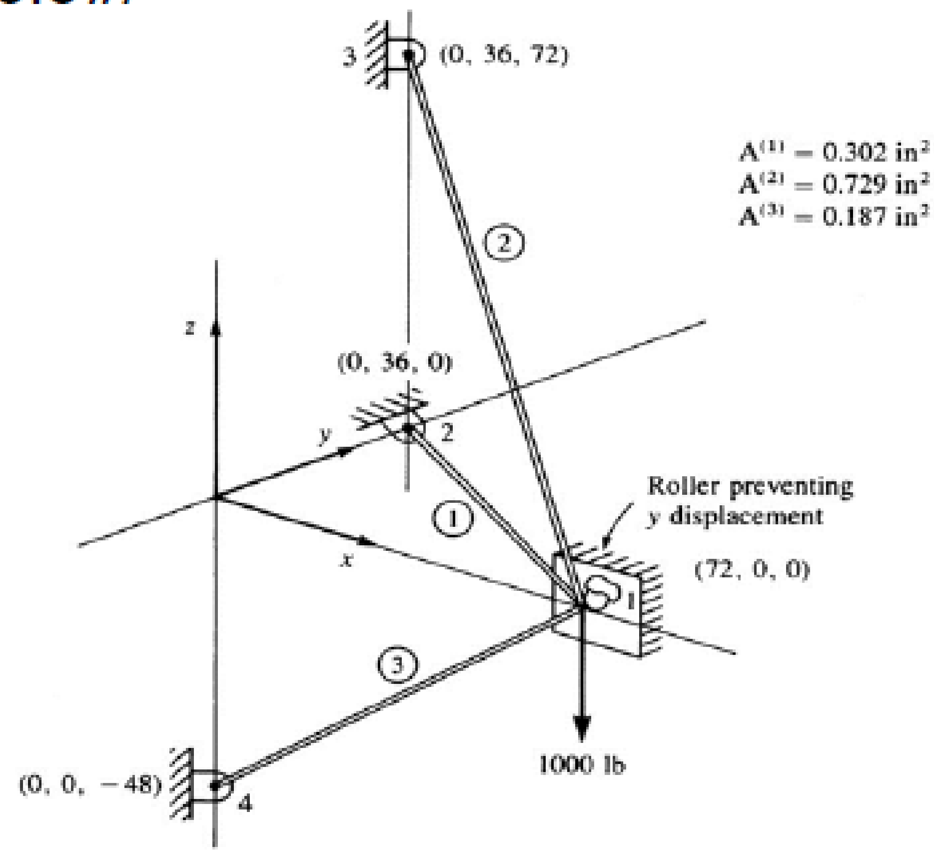
$$L^{(3)} = \sqrt{(-72)^2 + (-48)^2} = 86.5\,in$$

$$C_x = \frac{-72}{86.5} = -0.833$$

$$C_y = 0$$

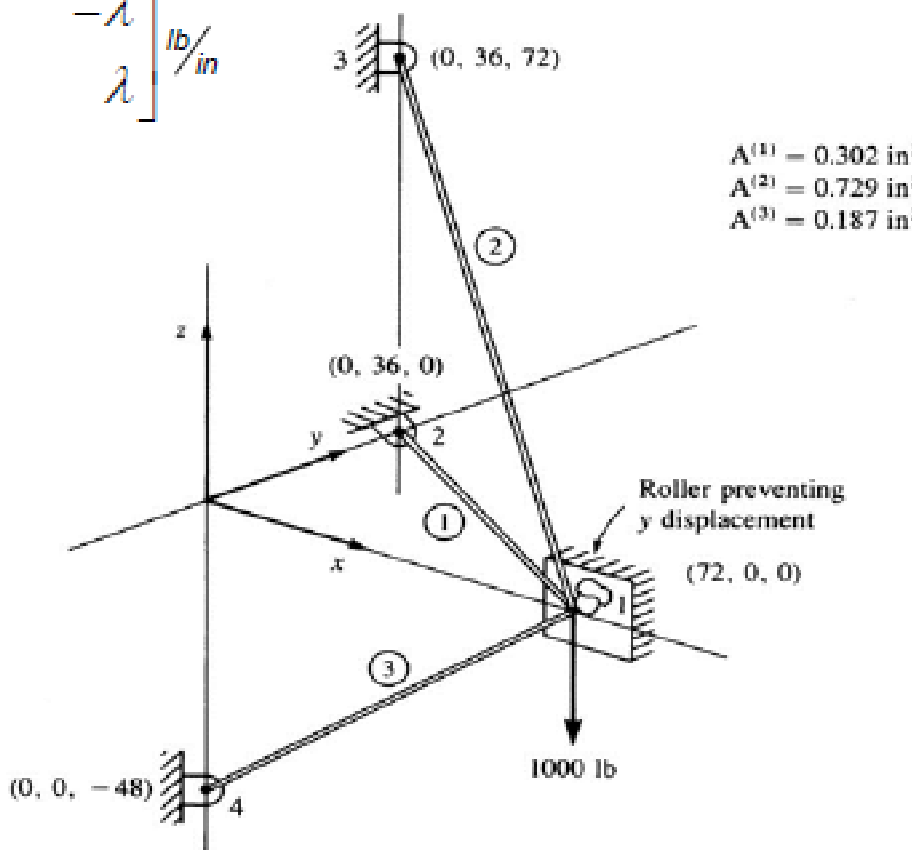
$$C_z = \frac{-48}{86.5} = -0.550$$

$$\lambda = \begin{bmatrix} 0.69 & 0 & 0.46 \\ 0 & 0 & 0 \\ 0.46 & 0 & 0.30 \end{bmatrix}$$



Consider element 3:

$$
\mathbf{k} = \frac{(0.187\,in^2)(1.2\times10^6\,psi)}{86.5\,in}
\begin{array}{c}
\begin{array}{cc} u_1\ v_1\ w_1 & u_4\ v_4\ w_4 \end{array} \\
\begin{bmatrix} \lambda & -\lambda \\ -\lambda & \lambda \end{bmatrix}
\end{array}
\ {}^{lb}/_{in}
$$



Canceling the rows and the columns associated with the boundary conditions reduces the global stiffness matrix to:

$$\begin{array}{cc} & \begin{array}{cc} u_1 & w_1 \end{array} \\ \mathbf{K} = & \begin{bmatrix} 9{,}000 & -2{,}450 \\ -2{,}450 & 4{,}450 \end{bmatrix} \end{array}$$

The global force-displacement equations are:

$$\begin{bmatrix} 9{,}000 & -2{,}450 \\ -2{,}450 & 4{,}450 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1{,}000 \end{Bmatrix}$$

Solving the equation gives:

$$u_1 = -0.072\ in \qquad w_1 = -0.264\ in$$

It can be shown, that the local forces in an element are:

$$\begin{Bmatrix} f'_{ix} \\ f'_{jx} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} -C_x & -C_y & -C_z & C_x & C_y & C_z \\ C_x & C_y & C_z & -C_x & -C_y & -C_z \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ w_i \\ u_j \\ v_j \\ w_j \end{Bmatrix}$$

The stress in an element is:

$$\sigma = \frac{E}{L}\begin{bmatrix} -C_x & -C_y & -C_z & C_x & C_y & C_z \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ w_i \\ u_j \\ v_j \\ w_j \end{Bmatrix}$$

The stress in element 1 is:

$$\sigma^{(1)} = \frac{1.2\times10^6}{80.5}\begin{bmatrix}0.89 & 0.45 & 0 & -0.89 & 0.45 & 0\end{bmatrix}\begin{Bmatrix}-0.072\\0\\-0.264\\0\\0\\0\end{Bmatrix}$$

$$\sigma^{(1)} = -955\ psi$$

The stress in element 2 is:

$$\sigma^{(2)} = \frac{1.2\times10^6}{108}\begin{bmatrix}0.667 & -0.33 & -0.667 & -0.667 & 0.33 & 0.667\end{bmatrix}\begin{Bmatrix}-0.072\\0\\-0.264\\0\\0\\0\end{Bmatrix}$$

$$\sigma^{(2)} = 1{,}423\ psi$$

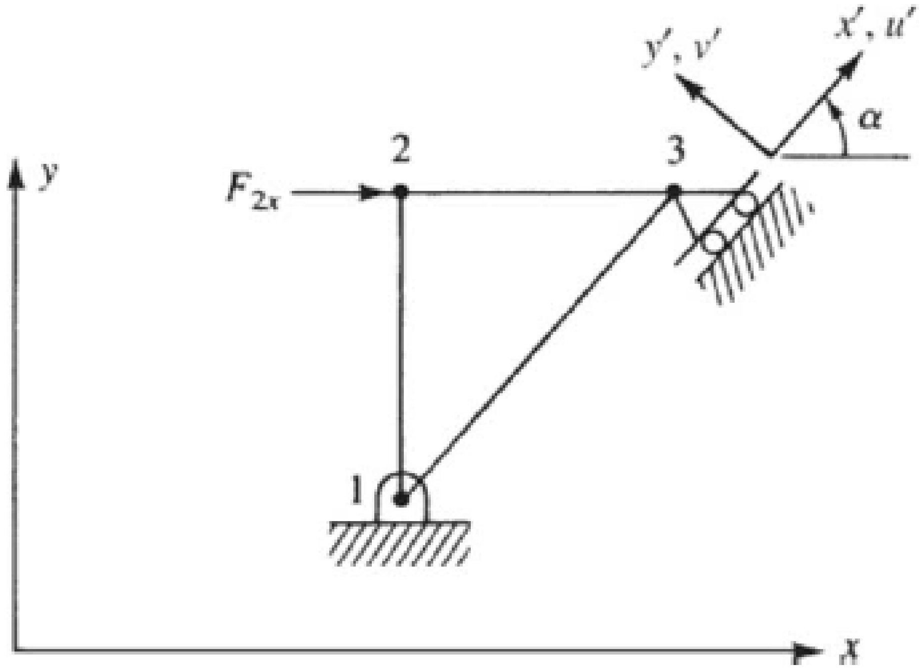
The stress in element 3 is:

$$\sigma^{(3)} = \frac{1.2\times10^6}{86.5}\begin{bmatrix}0.83 & 0 & 0.55 & -0.83 & 0 & -0.55\end{bmatrix}\begin{Bmatrix}-0.072 \\ 0 \\ -0.264 \\ 0 \\ 0 \\ 0\end{Bmatrix}$$

$$\sigma^{(3)} = 2{,}843\ psi$$

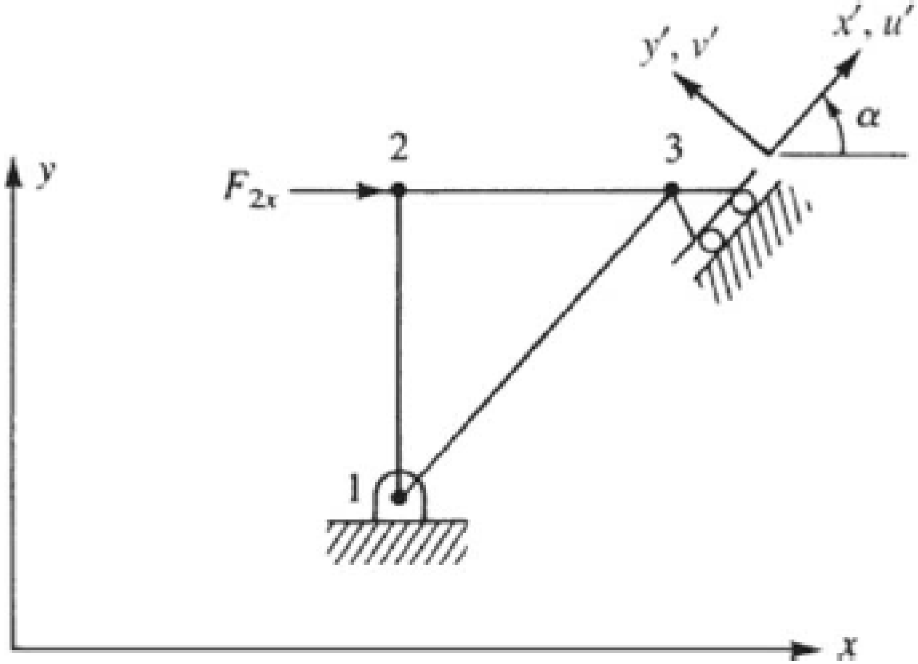
Inclined, or Skewed, Supports

If a support is inclined, or skewed, at some angle α for the global *x* axis, as shown below, the boundary conditions on the displacements are not in the global *x*-*y* directions but in the *x'*-*y'* directions.



Inclined, or Skewed, Supports

We must transform the local boundary condition of $\boldsymbol{v'_3} = 0$ (in local coordinates) into the global x-y system.



Inclined, or Skewed, Supports

Therefore, the relationship between of the components of the displacement in the local and the global coordinate systems at node 3 is:

$$\begin{Bmatrix} u'_3 \\ v'_3 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix}$$

We can rewrite the above expression as:

$$\{d'_3\} = [t_3]\{d_3\} \qquad [t_3] = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

We can apply this sort of transformation to the entire displacement vector as:

$$\{d'\} = [T_1]\{d\} \quad or \quad \{d\} = [T_1]^T\{d'\}$$

Inclined, or Skewed, Supports

Where the matrix $[T_1]^T$ is:

$$[T_1]^T = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3] \end{bmatrix}$$

Both the identity matrix $[I]$ and the matrix $[t_3]$ are 2 x 2 matrices.

The force vector can be transformed by using the same transformation.

$$\{f'\} = [T_1]\{f\}$$

In global coordinates, the force-displacement equations are:

$$\{f\} = [K]\{d\}$$

Inclined, or Skewed, Supports

Applying the skewed support transformation to both sides of the equation gives:

$$[T_1]\{f\} = [T_1][K]\{d\}$$

By using the relationship between the local and the global displacements, the force-displacement equations become:

$$\{f'\} = [T_1][K][T_1]^T\{d'\}$$

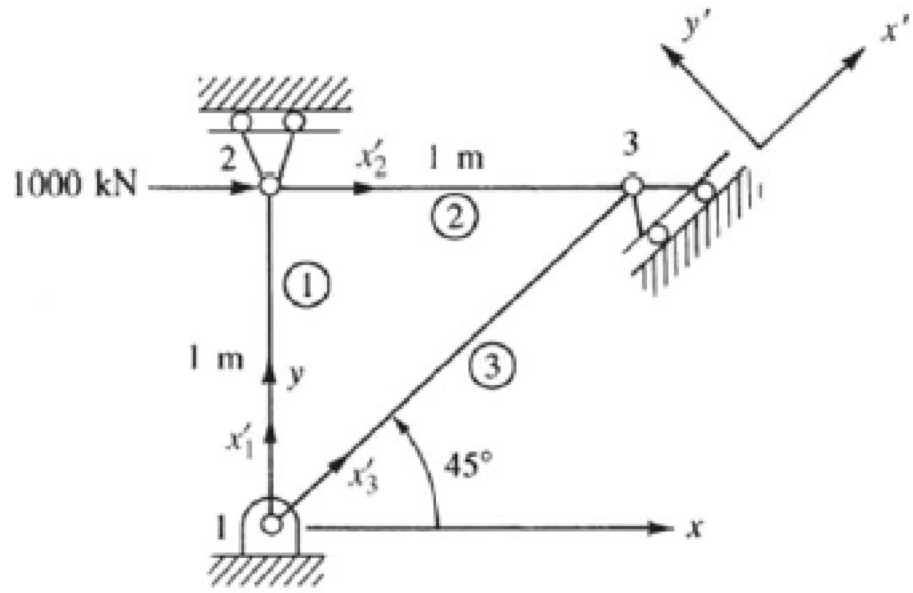
Therefore the global equations become:

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F'_{3x} \\ F'_{3y} \end{Bmatrix} = [T_1][K][T_1]^T \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u'_3 \\ v'_3 \end{Bmatrix}$$

Example 9 – Space Truss Problem

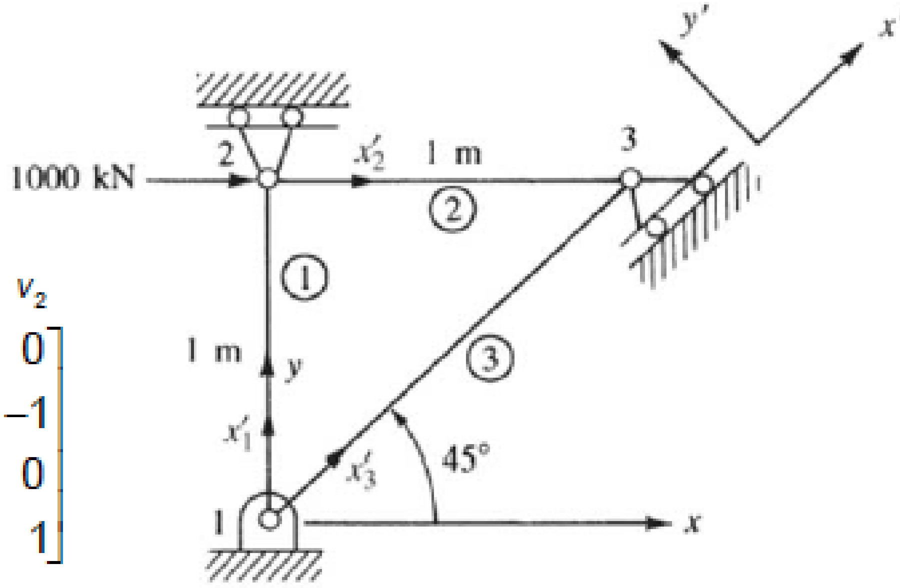
Consider the plane truss shown below. Assume E = 210 GPa, A = 6 x 10^{-4} m^2 for element 1 and 2, and $A = \sqrt{2}(6 \times 10^{-4}) m^2$ for element 3.

Determine the stiffness matrix for each element.



The global elemental stiffness matrix for ***element 1*** is:

$\cos\theta^{(1)} = 0 \qquad \sin\theta^{(1)} = 1$

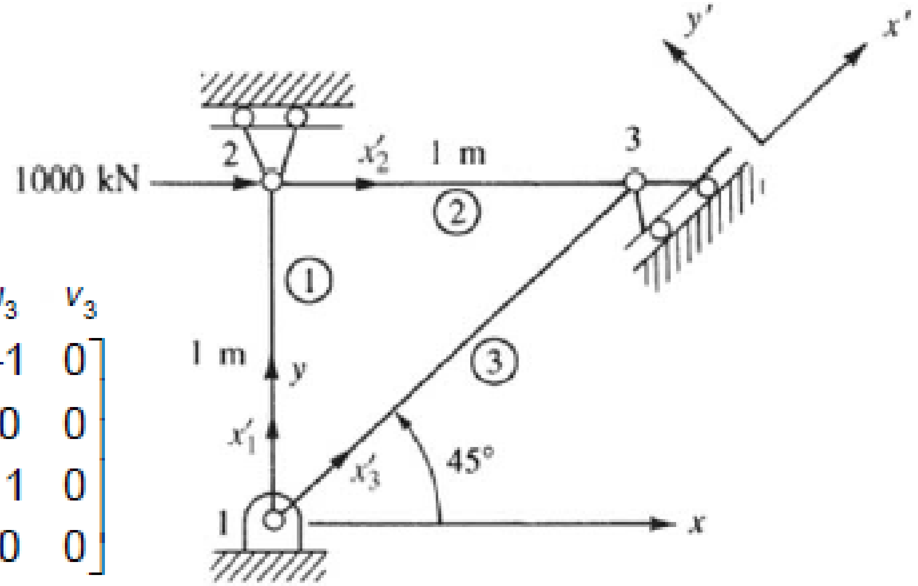


$$\mathbf{k}^{(1)} = \frac{(210\times10^{6}\,kN/m^{2})(6\times10^{-4}\,m^{-2})}{1\,m}\begin{array}{c} \begin{matrix} u_1 & v_1 & u_2 & v_2 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Example 9 – Space Truss Problem

The global elemental stiffness matrix for ***element 2*** is:

$\cos\theta^{(2)} = 1 \qquad \sin\theta^{(2)} = 0$

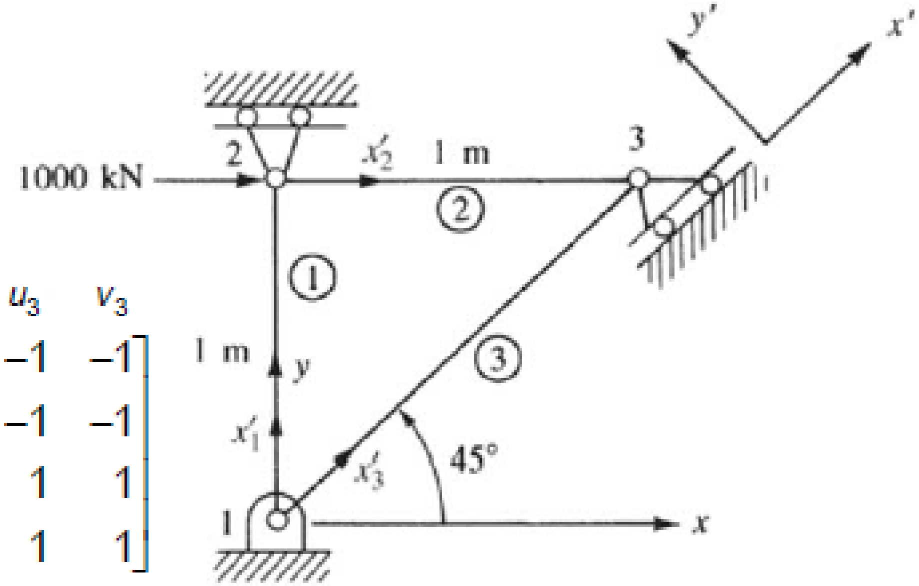


$$\mathbf{k}^{(2)} = \frac{(210\times10^{6}\,kN/m^{2})(6\times10^{-4}\,m^{2})}{1\,m}\begin{array}{c} \begin{matrix} u_2 & v_2 & u_3 & v_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The global elemental stiffness matrix for ***element 3*** is:

$$\cos\theta^{(3)} = \frac{\sqrt{2}}{2} \qquad \sin\theta^{(3)} = \frac{\sqrt{2}}{2}$$

$$\mathbf{k}^{(3)} = \frac{(210\times 10^6\,kN/m^2)(6\sqrt{2}\times 10^{-4}\,m^2)}{2\sqrt{2}\,m} \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix} \end{array}$$



Using the direct stiffness method, the global stiffness matrix is:

$$\mathbf{K} = 1{,}260 \times 10^5 \, {}^{N}\!/\!_{m} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ 0.5 & 1.5 & 0 & -1 & -0.5 & -0.5 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.5 & -0.5 & -1 & 0 & 1.5 & 0.5 \\ -0.5 & -0.5 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

We must transform the global displacements into local coordinates. Therefore the transformation $[T_1]$ is:

$$[T_1] = \begin{bmatrix} [I] & [0] & [0] \\ [0] & [I] & [0] \\ [0] & [0] & [t_3] \end{bmatrix} = \left[\begin{array}{cc:cc:cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & 0 & 0 & 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{array}\right]$$

The first step in the matrix transformation to find the product of $[T_1][K]$.

$$[T_1][K] = 1{,}260 \times 10^5 \, {}^{N}\!/\!_{m} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ 0.5 & 1.5 & 0 & -1 & -0.5 & -0.5 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.414 & 0.707 \\ 0 & 0 & 0.707 & 0 & -0.70 & 0 \end{bmatrix}$$

The next step in the matrix transformation to find the product of $[T_1][K][T_1]^T$.

$$[T_1][K][T_1]^T = 1{,}260 \times 10^5 \, {}^{N}\!/\!_{m} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.707 & 0 \\ 0.5 & 1.5 & 0 & -1 & -0.707 & 0 \\ 0 & 0 & 1 & 0 & -0.707 & 0.707 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.5 & -0.5 \\ 0 & 0 & 0.707 & 0 & -0.5 & 0.5 \end{bmatrix}$$

The displacement boundary conditions are: $u_1 = v_1 = v_2 = v'_3 = 0$

$$
\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F'_{3x} \\ F'_{3y} \end{Bmatrix} = 1{,}260 \times 10^5 \, \tfrac{N}{m} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.707 & 0 \\ 0.5 & 1.5 & 0 & -1 & -0.707 & 0 \\ 0 & 0 & 1 & 0 & -0.707 & 0.707 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.5 & -0.5 \\ 0 & 0 & 0.707 & 0 & -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u'_3 \\ v'_3 \end{Bmatrix}
$$

By applying the boundary conditions the global force-displacement equations are:

$$1{,}260 \times 10^{5}\, {}^{N}\!/_{m} \begin{bmatrix} 1 & -0.707 \\ -0.707 & 1.5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u'_3 \end{Bmatrix} = \begin{Bmatrix} F_{2x} = 1{,}000\ kN \\ F'_{3x} = 0 \end{Bmatrix}$$

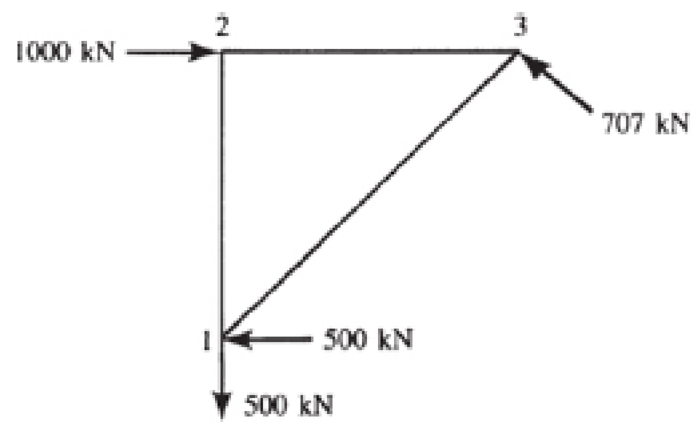
Solving the equation gives: $u_2 = 11.91\, mm$ $u'_3 = 5.61\, mm$

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F'_{3x} \\ F'_{3y} \end{Bmatrix} = 1{,}260 \times 10^2 \, {}^{N}\!/\!_{mm} \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.707 & 0 \\ 0.5 & 1.5 & 0 & -1 & -0.707 & 0 \\ 0 & 0 & 1 & 0 & -0.707 & 0.707 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & -0.707 & 0 & 1.5 & -0.5 \\ 0 & 0 & 0.707 & 0 & -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 11.91 \\ 0 \\ 5.61 \\ 0 \end{Bmatrix}$$

Therefore:

$$F_{1x} = -500\ kN \qquad F_{1y} = -500\ kN$$

$$F_{2y} = 0 \qquad F'_{3y} = 707\ kN$$



Development of Truss Equations

3.8 Use of Symmetry in Structure

Reflective symmetry

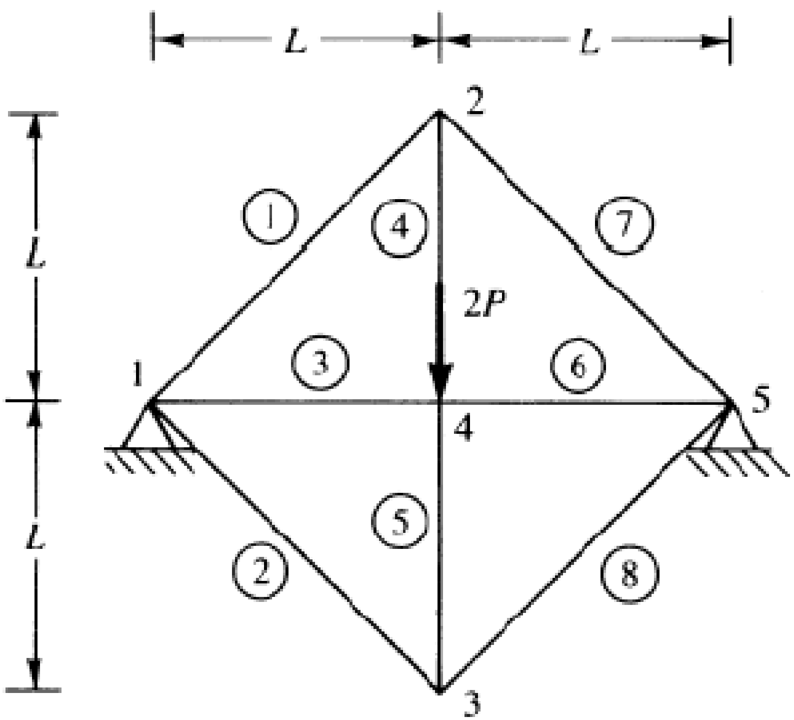


Figure 3–20 Plane truss

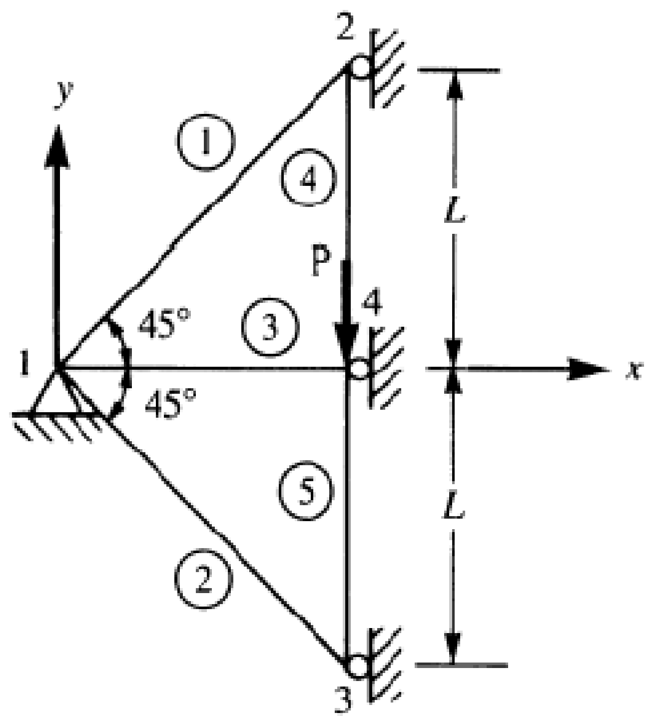


Figure 3–21 Truss of Figure 3–20 reduced by symmetry

3.8 Use of Symmetrv in Structure

Example 3.10

Table 3–2 Data for the truss of Figure 3–21

Element	θ°	C	S	C^2	S^2	CS
1	45°	$\sqrt{2}/2$	$\sqrt{2}/2$	$1/2$	$1/2$	$1/2$
2	315°	$\sqrt{2}/2$	$-\sqrt{2}/2$	$1/2$	$1/2$	$-1/2$
3	0°	1	0	1	0	0
4	90°	0	1	0	1	0
5	90°	0	1	0	1	0

using Eq. (3.4.23) along with Table 3–2 for the direction cosines, we obtain

$$\underline{k}^{(1)} = \frac{\sqrt{2}AE}{\sqrt{2}L} \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{1y} & d_{2x} & d_{2y} \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

Similarly, for elements 2–5, we obtain

$$\underline{k}^{(2)} = \frac{\sqrt{2}AE}{\sqrt{2}L} \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{1y} & d_{3x} & d_{3y} \end{array} \\ \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$$

3.8 Use of Symmetry in Structure

Example 3.10

$$\underline{k}^{(3)} = \frac{AE}{L} \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{1y} & d_{4x} & d_{4y} \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\underline{k}^{(4)} = \frac{AE}{L} \begin{array}{c} \begin{array}{cccc} d_{4x} & d_{4y} & d_{2x} & d_{2y} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

$$\underline{k}^{(5)} = \frac{AE}{L} \begin{array}{c} \begin{array}{cccc} d_{3x} & d_{3y} & d_{4x} & d_{4y} \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

3.8 Use of Symmetry in Structure

Example 3.10

$$\frac{AE}{L}\begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}\begin{Bmatrix} d_{2y} \\ d_{3y} \\ d_{4y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -P \end{Bmatrix}$$

On solving Eq. (3.8.6) for the displacements, we obtain

$$d_{2y} = \frac{-PL}{AE} \qquad d_{3y} = \frac{-PL}{AE} \qquad d_{4y} = \frac{-2PL}{AE}$$