INTRODUCTION TO THE STIFFNESS METHOD

Potential Energy Approach to Derive Spring Element Equations

Potential Energy Approach to Derive Spring Element Equations

One of the alternative methods often used to derive the element equations and the stiffness matrix for an element is based on the ***principle of minimum potential energy***.

This method has the advantage of being more general than the methods involving nodal and element equilibrium equations, along with the stress/strain law for the element.

The principle of minimum potential energy is more adaptable for the determination of element equations for complicated elements (those with large numbers of degrees of freedom) such as the plane stress/strain element, the axisymmetric stress element, the plate bending element, and the three-dimensional solid stress element.

Total Potential Energy

The total potential energy is defined as the sum of the internal strain energy ***U*** and the potential energy of the external forces **Ω**:

$$\pi_p = U + \Omega$$

Strain energy is the capacity of the internal forces (or stresses) to do work through deformations (strains) in the structure.

The potential energy of the external forces **Ω** is the capacity of forces such as body forces, surface traction forces, and applied nodal forces to do work through the deformation of the structure.

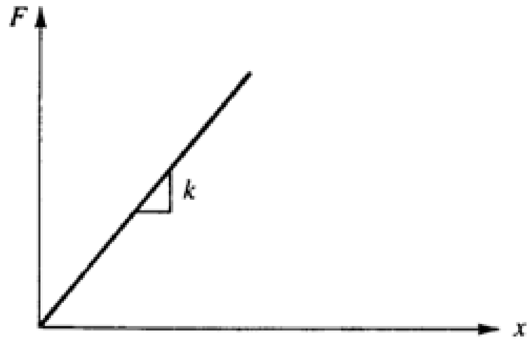
Total Potential Energy

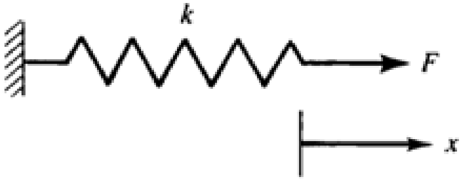
Recall the force-displacement relationship for a linear spring:

$$F = kx$$

The differential internal work (or strain energy) *dU* in the spring is the internal force multiplied by the change in displacement which the force moves through:

$$dU = F dx = (kx) dx$$



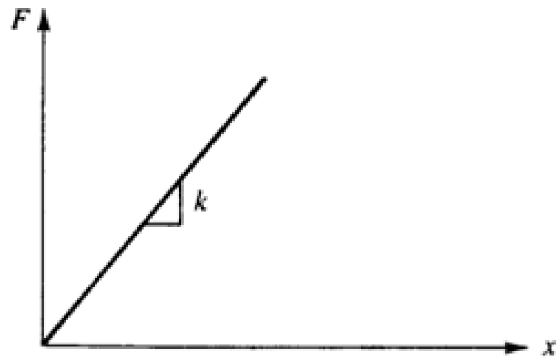


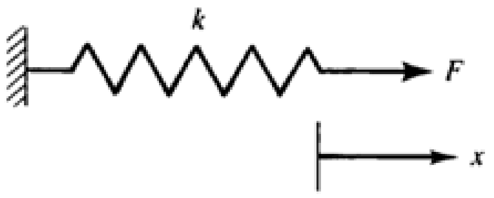
Total Potential Energy

The total strain energy is: $$U = \int_L dU = \int_0^x (kx)\, dx = \frac{1}{2}kx^2$$

The strain energy is the area under the force-displacement curve. The potential energy of the external forces is the work done by the external forces:

$$\Omega = -Fx$$

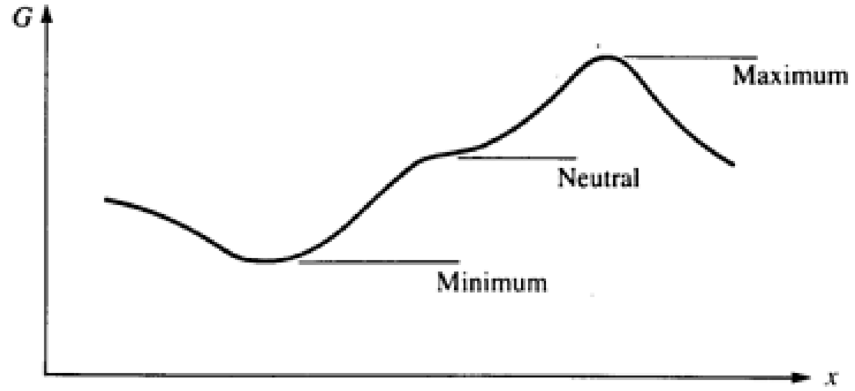




Total Potential Energy

Therefore, the total potential energy is: $\pi_p = \frac{1}{2}kx^2 - Fx$

The concept of a ***stationary value*** of a function *G* is shown below:



$$\frac{dG}{dx} = 0$$

The function *G* is expressed in terms of *x*. To find a value of *x* yielding a stationary value of *G*(*x*), we use differential calculus to differentiate *G* with respect to *x* and set the expression equal to zero.

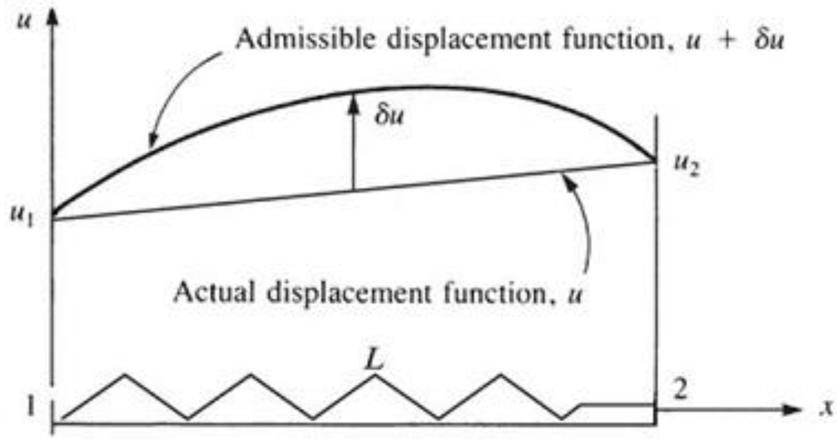
Total Potential Energy

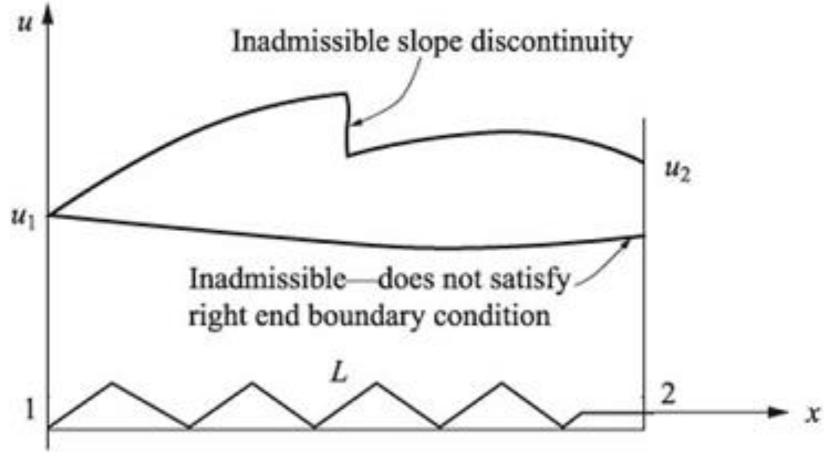
We can replace G with the total potential energy π_p and the coordinate x with a discrete value d_i. To minimize π_p we first take the ***variation*** of π_p (we will not cover the details of variational calculus):

$$\delta\pi_p = \frac{\partial \pi_p}{\partial d_1}\delta d_1 + \frac{\partial \pi_p}{\partial d_2}\delta d_2 + \ldots + \frac{\partial \pi_p}{\partial d_n}\delta d_n$$

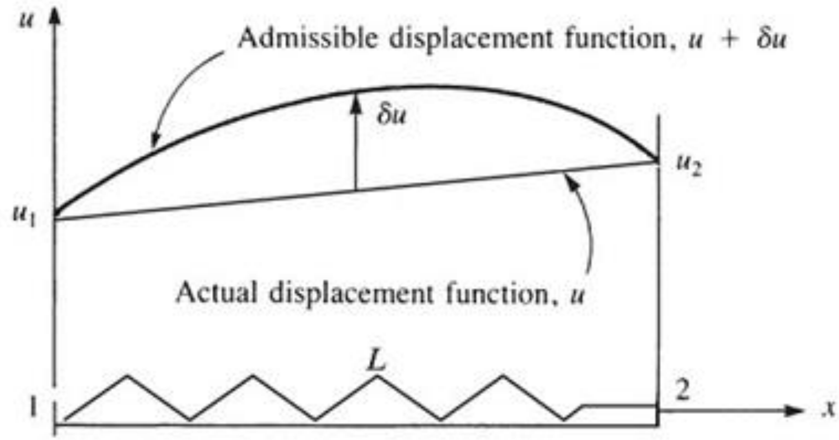
The principle states that equilibrium exist when the d_i define a structure state such that $\delta\pi_p = 0$ for arbitrary admissible variations δd_i from the equilibrium state.

Total Potential Energy





Total Potential Energy



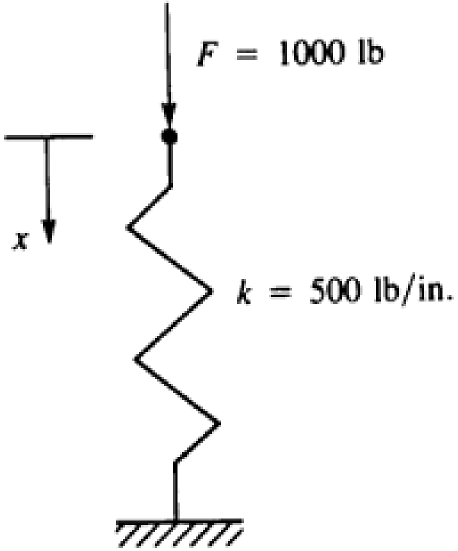
To satisfy $\delta\pi_p = 0$, all coefficients associated with δd_i must be zero independently, therefore:

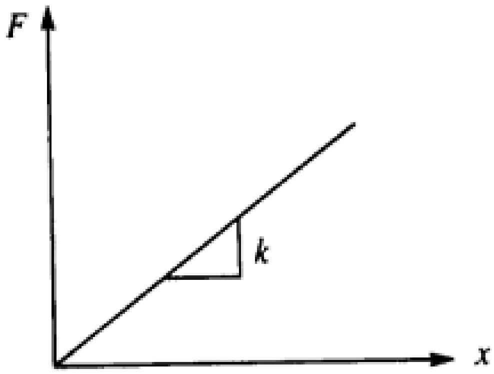
$$\frac{\partial \pi_p}{\partial d_i} = 0 \quad i = 1, 2, \cdots, n \qquad or \qquad \frac{\partial \pi_p}{\partial \{d\}} = 0$$

Total Potential Energy – Example 5

Consider the following linear-elastic spring system subjected to a force of 1,000 *lb*.

Evaluate the potential energy for various displacement values and show that the minimum potential energy also corresponds to the equilibrium position of the spring.





Total Potential Energy – Example 5

The total potential energy is defined as the sum of the internal strain energy *U* and the potential energy of the external forces Ω:

$$\pi_p = U + \Omega \qquad U = \frac{1}{2}kx^2 \qquad \Omega = -Fx$$

The variation of π_p with respect to *x* is: $\delta\pi_p = \frac{\partial \pi_p}{\partial x}\delta x = 0$

Since δx is arbitrary and might not be zero, then: $\frac{\partial \pi_p}{\partial x} = 0$

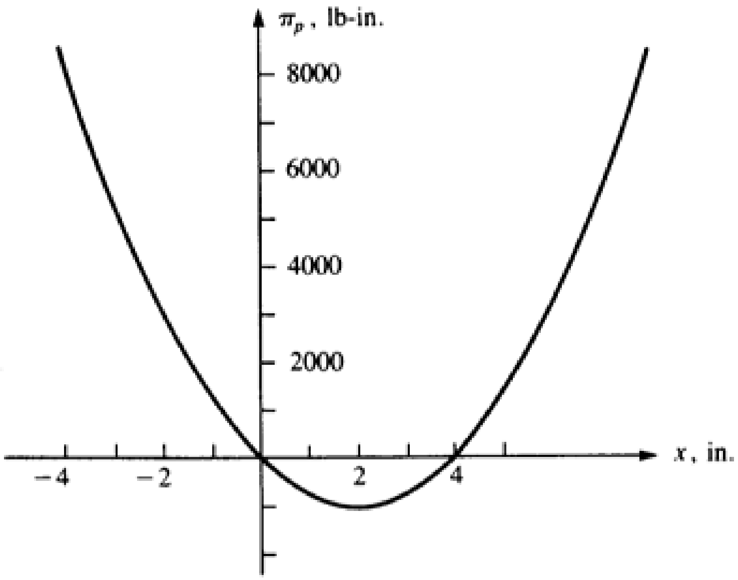
Total Potential Energy – Example 5

Using our express for π_p, we get:

$$\pi_p = \frac{1}{2}kx^2 - Fx = \frac{1}{2}500(^{lb}/_{in})\,x^2 - (1{,}000\,lb)\,x$$

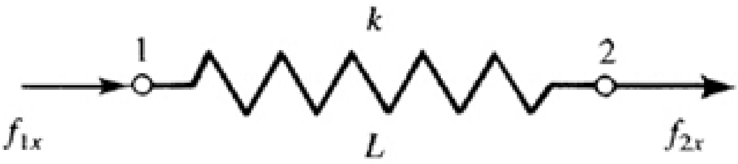
$$\frac{\partial \pi_p}{\partial x} = 0 = 500x - 1{,}000 \qquad x = 2.0\,in$$

If we had plotted the total potential energy function π_p for various values of deformation we would get:



Total Potential Energy

Let's derive the spring element equations and stiffness matrix using the principal of minimum potential energy. Consider the linear spring subjected to nodal forces shown below:



The total potential energy π_p

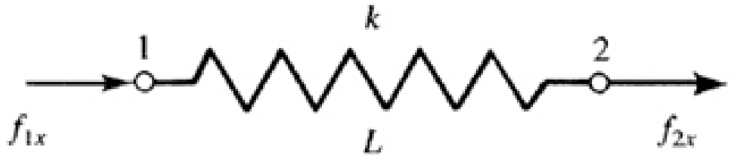
$$\pi_p = \frac{1}{2}k\left(u_2 - u_1\right)^2 - f_{1x}u_1 - f_{2x}u_2$$

Expanding the above express gives:

$$\pi_p = \frac{1}{2}k\left(u_2^2 - 2u_1u_2 + u_1^2\right) - f_{1x}u_1 - f_{2x}u_2$$

Total Potential Energy

Let's derive the spring element equations and stiffness matrix using the principal of minimum potential energy. Consider the linear spring subjected to nodal forces shown below:



Recall: $\dfrac{\partial \pi_p}{\partial d_i} = 0 \quad i = 1, 2, \cdots, n \quad or \quad \dfrac{\partial \pi_p}{\partial \{d\}} = 0$

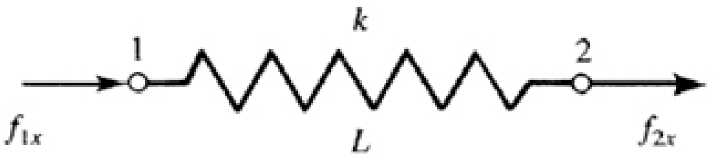
Therefore:

$$\frac{\partial \pi_p}{\partial u_1} = \frac{k}{2}\left(-2u_2 + 2u_1\right) - f_{1x} = 0$$

$$\frac{\partial \pi_p}{\partial u_2} = \frac{k}{2}\left(2u_2 - 2u_1\right) - f_{2x} = 0$$

Total Potential Energy

Let's derive the spring element equations and stiffness matrix using the principal of minimum potential energy. Consider the linear spring subjected to nodal forces shown below:



Therefore: $k\left(u_1 - u_2\right) = f_{1x}$

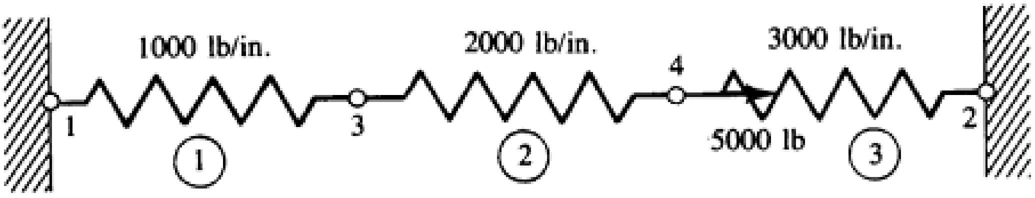
$$k\left(-u_1 + u_2\right) = f_{2x}$$

In matrix form the equations are:

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Total Potential Energy – Example 6

Obtain the total potential energy of the spring system shown below and find its minimum value.



The potential energy π_p for element 1 is:

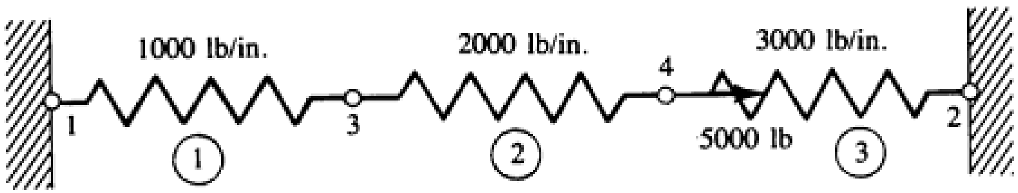
$$\pi_p^{(1)} = \frac{1}{2}k_1\left(u_3 - u_1\right)^2 - f_{1x}u_1 - f_{3x}u_3$$

The potential energy π_p for element 2 is:

$$\pi_p^{(2)} = \frac{1}{2}k_2\left(u_4 - u_3\right)^2 - f_{3x}u_3 - f_{4x}u_4$$

Total Potential Energy – Example 6

Obtain the total potential energy of the spring system shown below and find its minimum value.



The potential energy π_p for element 3 is:

$$\pi_p^{(3)} = \frac{1}{2}k_3\left(u_2 - u_4\right)^2 - f_{2x}u_2 - f_{4x}u_4$$

The total potential energy π_p for the spring system is:

$$\pi_p = \sum_{e=1}^{3} \pi_p^{(e)}$$

Total Potential Energy – Example 6

Minimizing the total potential energy π_p:

$$\frac{\partial \pi_p}{\partial u_1} = 0 = -k_1 u_3 + k_1 u_1 - f_{1x}^{(1)}$$

$$\frac{\partial \pi_p}{\partial u_2} = 0 = k_3 u_2 - k_3 u_4 - f_{2x}^{(3)}$$

$$\frac{\partial \pi_p}{\partial u_3} = 0 = k_1 u_3 - k_1 u_1 - k_2 u_4 + k_2 u_3 - f_{3x}^{(1)} - f_{3x}^{(2)}$$

$$\frac{\partial \pi_p}{\partial u_4} = 0 = k_2 u_4 - k_2 u_3 - k_3 u_2 + k_3 u_4 - f_{4x}^{(2)} - f_{4x}^{(3)}$$

Total Potential Energy – Example 6

In matrix form:

$$\begin{bmatrix} k_1 & 0 & k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1 + k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ f_{2x} \\ f_{3x}^{(1)} + f_{3x}^{(2)} \\ f_{4x}^{(2)} + f_{4x}^{(3)} \end{Bmatrix}$$

Using the following force equilibrium equations:

$$f_{1x}^{(1)} = F_{1x}$$

$$f_{2x}^{(3)} = F_{2x}$$

$$f_{3x}^{(1)} + f_{3x}^{(2)} = F_{3x}$$

$$f_{4x}^{(2)} + f_{4x}^{(3)} = F_{4x}$$

Total Potential Energy – Example 6

The global force-displacement equations are:

$$\begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix}$$

The above equations are identical to those we obtained through the direct stiffness method.