Introduction to the Stiffness (Displacement) Method

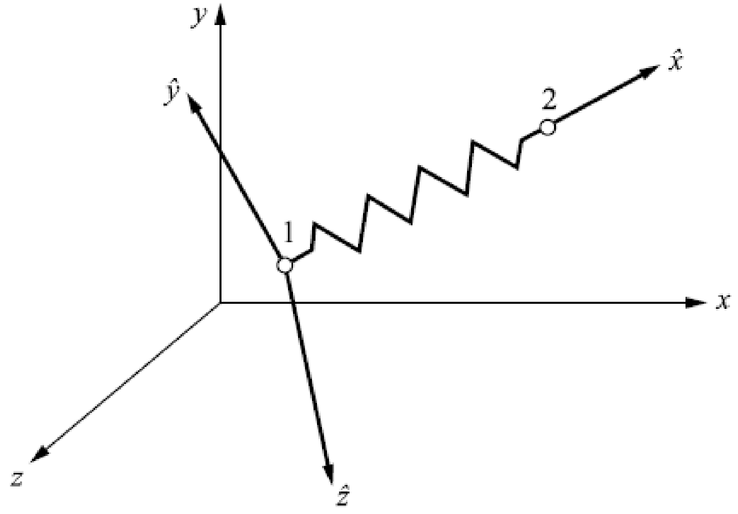
Definition of the Stiffness Matrix

$$\boxed{\underline{\hat{f}} = \underline{\hat{k}}\,\underline{\hat{d}}}$$

Element stiffness equation in local coordinate system

$$\boxed{\underline{F} = \underline{K}\,\underline{d}}$$

Global stiffness equation in global coordinate system



Local $(\hat{x}, \hat{y}, \hat{z})$ and global (x, y, z) coordinate systems

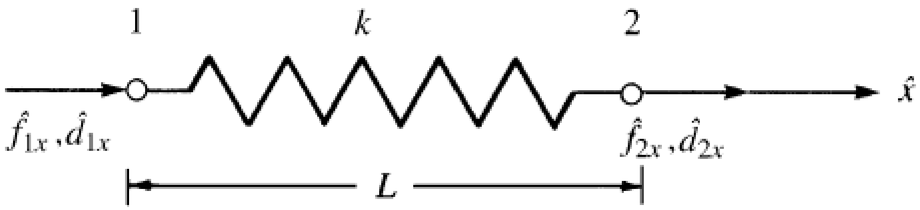
Introduction to the Stiffness (Displacement) Method

Derivation of the Stiffness Matrix for a Spring Element

Degrees of freedom:

$\hat{d}_{1x}$ and $\hat{d}_{2x}$

k : spring constant or stiffness of the spring



Linear spring element with positive nodal displacement and force conventions

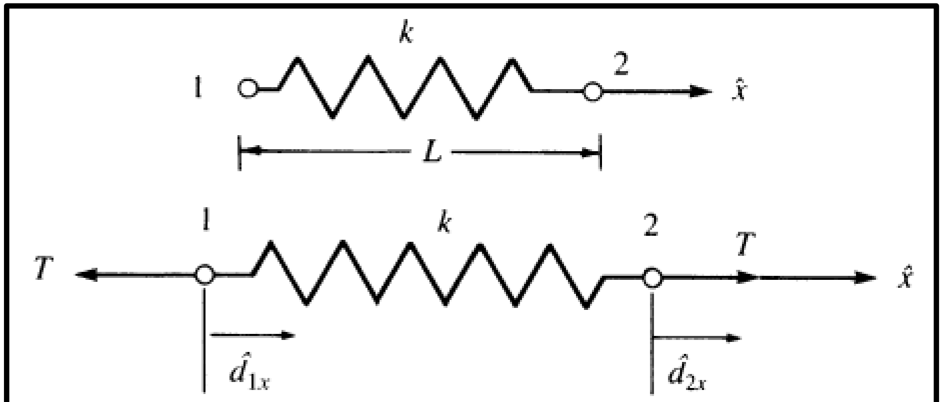
$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

We now use the general steps to derive the stiffness matrix for the spring element

Introduction to the Stiffness (Displacement) Method

Derivation of the Stiffness Matrix for a Spring Element

Step 1 Select the Element Type



Linear spring subjected to tensile forces

Consider the ***linear spring element*** subjected to resulting nodal tensile forces directed along the spring axial direction, so as to be in equilibrium.

Introduction to the Stiffness (Displacement) Method

Derivation of the Stiffness Matrix for a Spring Element

Step 2 Select a Displacement Function

$$\boxed{\hat{u} = a_1 + a_2\hat{x}}$$

$$\boxed{\hat{u} = \begin{bmatrix} 1 & \hat{x} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}}$$

$$\hat{u}(0) = \hat{d}_{1x} = a_1$$

$$\hat{u}(L) = \hat{d}_{2x} = a_2 L + \hat{d}_{1x}$$

$$a_2 = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

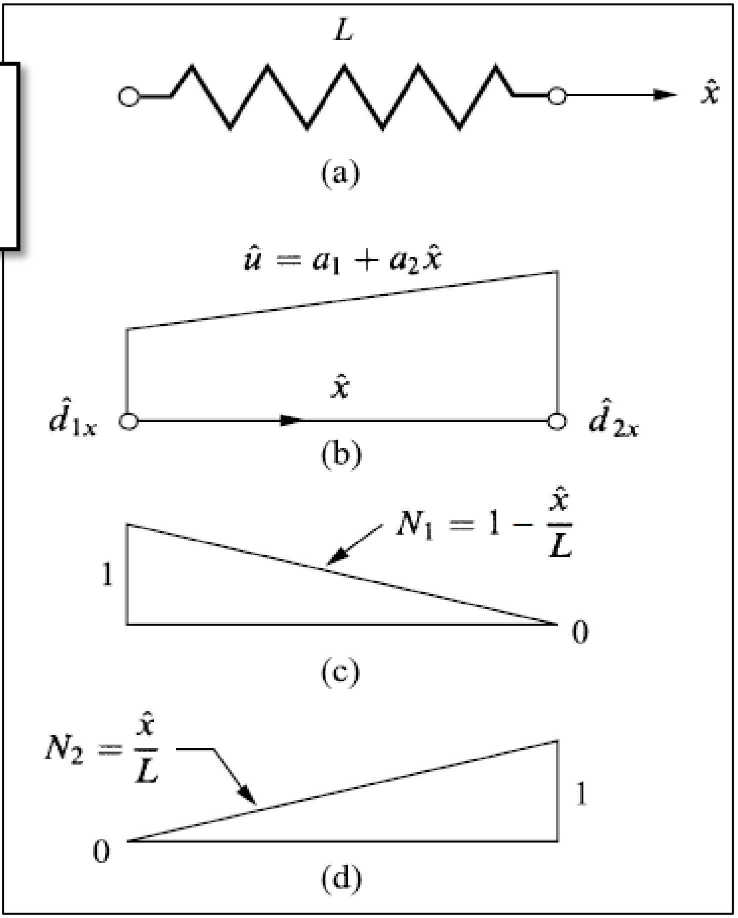
$$\hat{u} = \left(\frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}\right)\hat{x} + \hat{d}_{1x}$$

$$\hat{u} = \begin{bmatrix} 1 - \dfrac{\hat{x}}{L} & \dfrac{\hat{x}}{L} \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$\hat{u} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$N_1 = 1 - \frac{\hat{x}}{L} \quad \text{and} \quad N_2 = \frac{\hat{x}}{L}$$

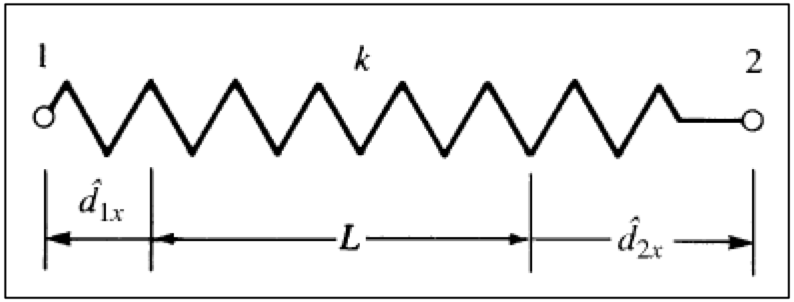
shape functions (or) Interpolation functions



Introduction to the Stiffness (Displacement) Method

Derivation of the Stiffness Matrix for a Spring Element

Step 3 Define the Strain/Displacement and Stress/Strain Relationships



Deformed spring

- Due to the tensile force T, the deformation (total elongation) of the spring:

$$\delta = \hat{u}(L) - \hat{u}(0) = \hat{d}_{2x} - \hat{d}_{1x}$$

- Force/deformation relationship:

$$T = k\delta$$

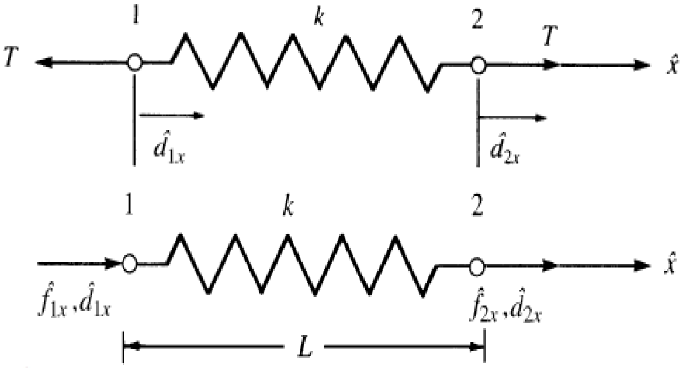
$$T = k(\hat{d}_{2x} - \hat{d}_{1x})$$

Introduction to the Stiffness (Displacement) Method

Derivation of the Stiffness Matrix for a Spring Element

Step 4 Derive the Element Stiffness Matrix and Equations

- By the sign convention for nodal forces and equilibrium,



$$\hat{f}_{1x} = -T \qquad \hat{f}_{2x} = T$$

$$T = -\hat{f}_{1x} = k(\hat{d}_{2x} - \hat{d}_{1x})$$

$$T = \hat{f}_{2x} = k(\hat{d}_{2x} - \hat{d}_{1x})$$

$$\hat{f}_{1x} = k(\hat{d}_{1x} - \hat{d}_{2x})$$

$$\hat{f}_{2x} = k(\hat{d}_{2x} - \hat{d}_{1x})$$

$$\boxed{\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}}$$

Local stiffness matrix

$$\boxed{\underline{\hat{k}} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}$$

Introduction to the Stiffness (Displacement) Method

Derivation of the Stiffness Matrix for a Spring Element

Step 5 Assemble the Element Equations to Obtain the Global Equations and Introduce Boundary Conditions

Global stiffness matrix

$$\boxed{\underline{K} = [K] = \sum_{e=1}^{N} \underline{k}^{(e)}}$$

Global force matrix

$$\boxed{\underline{F} = \{F\} = \sum_{e=1}^{N} \underline{f}^{(e)}}$$

where $\underline{k}$ and $\underline{f}$ are now element stiffness and force matrices expressed in a global reference frame.

Note: **The element matrices must be assembled properly according to the direct stiffness method**

Introduction to the Stiffness (Displacement) Method

Derivation of the Stiffness Matrix for a Spring Element

Step 6 Solve for the Nodal Displacements

- The displacements are then determined by imposing boundary conditions, such as support conditions, and solving a system of equations, $\underline{F} = \underline{K}\underline{d}$, simultaneously.

Step 7 Solve for the Element Forces

- Finally, the element forces are determined by back-substitution, applied to each element, into equations similar to,

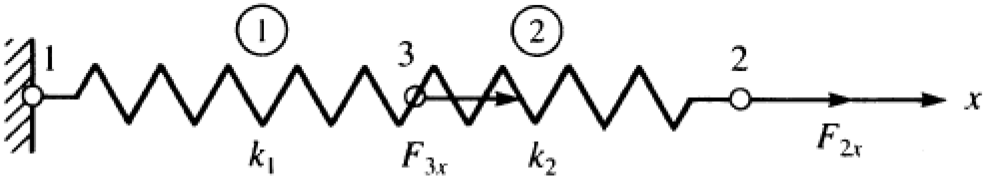
$$\hat{f}_{1x} = k(\hat{d}_{1x} - \hat{d}_{2x})$$
$$\hat{f}_{2x} = k(\hat{d}_{2x} - \hat{d}_{1x})$$

Introduction to the Stiffness (Displacement) Method

Example of a Spring Assemblage

DIRECT EQUILIBRIUM APPROACH

TOTAL STIFFNESS MATRIX



Element 1

$$\begin{Bmatrix} f_{1x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} d_{1x}^{(1)} \\ d_{3x}^{(1)} \end{Bmatrix}$$

Element 2

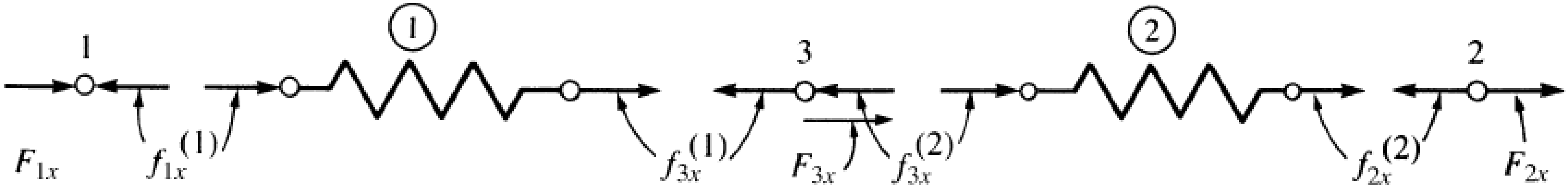
$$\begin{Bmatrix} f_{3x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{3x}^{(2)} \\ d_{2x}^{(2)} \end{Bmatrix}$$

Continuity Or Compatibility Requirement →

$$d_{3x}^{(1)} = d_{3x}^{(2)} = d_{3x}$$

Introduction to the Stiffness (Displacement) Method

Example of a Spring Assemblage



Free-body diagrams of each element and node

Note: nodal forces consistent with element force sign convention

Nodal Equilibrium Equations

$$F_{3x} = f_{3x}^{(1)} + f_{3x}^{(2)}$$

$$F_{2x} = f_{2x}^{(2)}$$

$$F_{1x} = f_{1x}^{(1)}$$

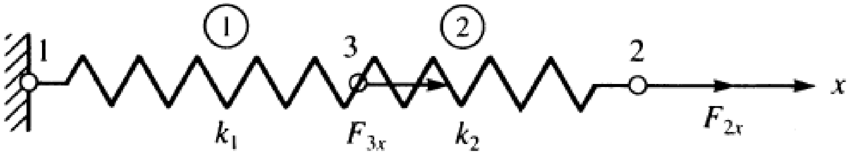
$$F_{3x} = (-k_1 d_{1x} + k_1 d_{3x}) + (k_2 d_{3x} - k_2 d_{2x})$$

$$F_{2x} = -k_2 d_{3x} + k_2 d_{2x}$$

$$F_{1x} = k_1 d_{1x} - k_1 d_{3x}$$

Introduction to the Stiffness (Displacement) Method

Example of a Spring Assemblage



Global Stiffness Equation:

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix}$$

$$\boxed{\underline{F} = \underline{K}\underline{d}}$$

$$\underline{F} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} \qquad \underline{K} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \qquad \underline{d} = \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix}$$

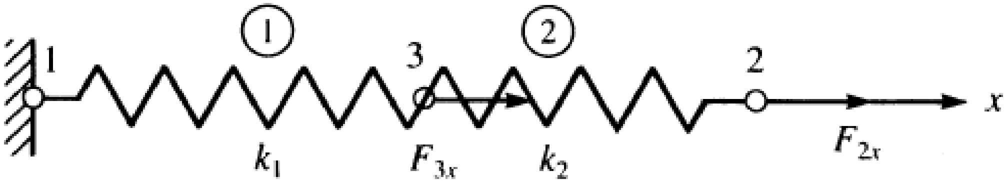
Global Nodal Force Matrix

Total or Global or System Stiffness Matrix

Global Nodal Displacement Matrix

Introduction to the Stiffness (Displacement) Method

Assembling the Total Stiffness Matrix by Superposition (Direct Stiffness Method)



$$\underline{k}^{(1)} = \begin{array}{cc|c} d_{1x} & d_{3x} & \\ k_1 & -k_1 & d_{1x} \\ -k_1 & k_1 & d_{3x} \end{array}$$

$$k_1 \begin{array}{ccc} d_{1x} & d_{2x} & d_{3x} \end{array} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x}^{(1)} \\ d_{2x}^{(1)} \\ d_{3x}^{(1)} \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(1)} \\ f_{3x}^{(1)} \end{Bmatrix}$$

$$\underline{k}^{(2)} = \begin{array}{cc|c} d_{3x} & d_{2x} & \\ k_2 & -k_2 & d_{3x} \\ -k_2 & k_2 & d_{2x} \end{array}$$

$$k_2 \begin{array}{ccc} d_{1x} & d_{2x} & d_{3x} \end{array} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x}^{(2)} \\ d_{2x}^{(2)} \\ d_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} f_{1x}^{(2)} \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix}$$

Introduction to the Stiffness (Displacement) Method

Assembling the Total Stiffness Matrix by Superposition (Direct Stiffness Method)

$$\begin{Bmatrix} f_{1x}^{(1)} \\ 0 \\ f_{3x}^{(1)} \end{Bmatrix} + \begin{Bmatrix} 0 \\ f_{2x}^{(2)} \\ f_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

Force Equilibrium at Each Node

$$k_1 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x}^{(1)} \\ d_{2x}^{(1)} \\ d_{3x}^{(1)} \end{Bmatrix} + k_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x}^{(2)} \\ d_{2x}^{(2)} \\ d_{3x}^{(2)} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

$$\boxed{\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}}$$

Introduction to the Stiffness (Displacement) Method

Assembling the Total Stiffness Matrix by Superposition (Direct Stiffness Method)

Direct, or Short-cut, Form of the Direct Stiffness Method

$$\underline{k}^{(1)} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} d_{1x} & d_{3x} \end{array}$$

$$\underline{k}^{(1)} = \begin{array}{cc|c} d_{1x} & d_{3x} & \\ k_1 & -k_1 & d_{1x} \\ -k_1 & k_1 & d_{3x} \end{array}$$

$$\underline{k}^{(2)} = \begin{array}{cc|c} d_{3x} & d_{2x} & \\ k_2 & -k_2 & d_{3x} \\ -k_2 & k_2 & d_{2x} \end{array}$$

$$\underline{K} = \begin{array}{ccc|c} d_{1x} & d_{2x} & d_{3x} & \\ k_1 & 0 & -k_1 & d_{1x} \\ 0 & k_2 & -k_2 & d_{2x} \\ -k_1 & -k_2 & k_1+k_2 & d_{3x} \end{array}$$

Introduction to the Stiffness (Displacement) Method

Boundary Conditions (BCs)

- Without specifying boundary conditions:
  - Mathematically: $\underline{K}$ is singular, its determinant will be zero, and its inverse will not exist
  - Physically: the structural system is unstable, free to move as a rigid body and not resist any applied loads
- Boundary conditions are of two general types:
  - Homogeneous BCs,occur at locations that are completely prevented from movement
  - Nonhomogeneous BCs, occur where finite nonzero values of displacement are specified, such as the settlement of a support.

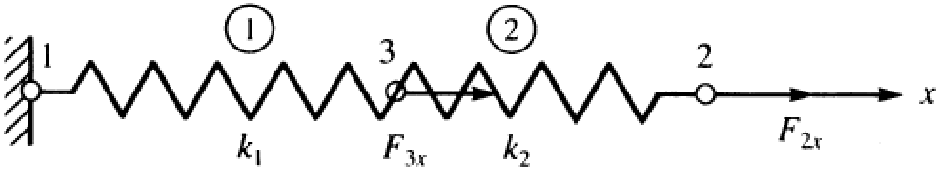
Introduction to the Stiffness (Displacement) Method

Boundary Conditions

Homogeneous Boundary Conditions

$$\boxed{d_{1x} = 0}$$

Node 1 is fixed



$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1+k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

$$k_1(0) + (0)d_{2x} - k_1 d_{3x} = F_{1x}$$

→ **Unknown Reaction**

$$0(0) + k_2 d_{2x} - k_2 d_{3x} = F_{2x}$$

$$-k_1(0) - k_2 d_{2x} + (k_1 + k_2) d_{3x} = F_{3x}$$

→ **Known Applied Loads**

Introduction to the Stiffness (Displacement) Method

Boundary Conditions

- Delete the rows and columns corresponding to the zero-displacement degrees of freedom from the original set of equations,

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} F_{2x} \\ F_{3x} \end{Bmatrix}$$

- Then solve for the unknown displacements and the reaction,

$$\begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}^{-1} \begin{Bmatrix} F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{k_2} + \dfrac{1}{k_1} & \dfrac{1}{k_1} \\ \dfrac{1}{k_1} & \dfrac{1}{k_1} \end{bmatrix} \begin{Bmatrix} F_{2x} \\ F_{3x} \end{Bmatrix}$$

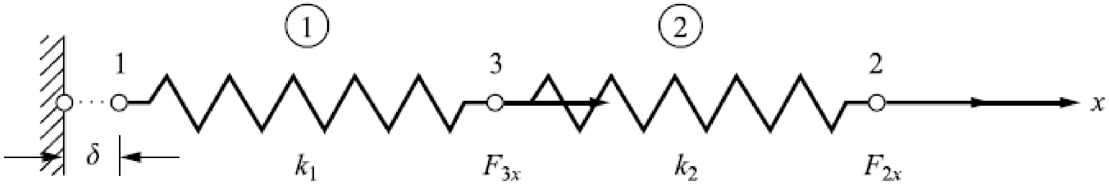
$$F_{1x} = -k_1 d_{3x} = -F_{2x} - F_{3x}$$

Note: for matrix inverse, review Appendix A, Page 716

Introduction to the Stiffness (Displacement) Method

Boundary Conditions

Nonhomogeneous Boundary Conditions



$$\boxed{d_{1x} = \delta}$$

Known displacement at node 1

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} \delta \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

$$k_1\delta + 0d_{2x} - k_1 d_{3x} = F_{1x}$$

→ **Unknown Reaction**

$$0\delta + k_2 d_{2x} - k_2 d_{3x} = F_{2x}$$

$$-k_1\delta - k_2 d_{2x} + (k_1 + k_2) d_{3x} = F_{3x}$$

→ **Known Applied Loads**

Introduction to the Stiffness (Displacement) Method

Boundary Conditions

- Transform the terms associated with the known displacements to the right-side force matrix before solving for the unknown nodal displacements.

$$k_2 d_{2x} - k_2 d_{3x} = F_{2x}$$

$$-k_2 d_{2x} + (k_1 + k_2) d_{3x} = +k_1 \delta + F_{3x}$$

- Then solve for the unknown displacements and the reaction,

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} F_{2x} \\ k_1 \delta + F_{3x} \end{Bmatrix}$$

$$F_{1x} = k_1 \delta - k_1 d_{3x}$$

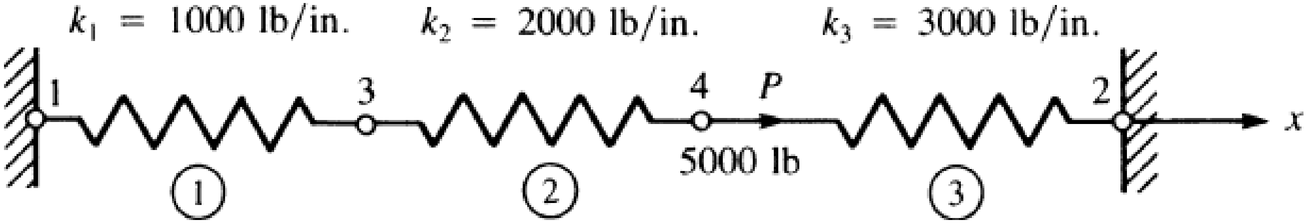
Introduction to the Stiffness (Displacement) Method

Boundary Conditions

- Properties of the stiffness matrix:
  - $\underline{K}$ is symmetric, as is each of the element stiffness matrices
  - $\underline{K}$ is singular, and thus no inverse exists until sufficient BCs are imposed to remove the singularity and prevent rigid body motion.
  - the main diagonal terms of $\underline{K}$ are always positive.

Introduction to the Stiffness (Displacement) Method

Example 1



- Nodes 1 and 2 are fixed
- A force of 5000 lb is applied at node 4 in the x direction.

Obtain:

(a) The global stiffness matrix
(b) The displacements of nodes 3 and 4
(c) The reaction forces at nodes 1 and 2
(d) The forces in each spring

Introduction to the Stiffness (Displacement) Method

Example 1

(a) The global stiffness matrix

Element Stiffness Matrix

$$\underline{k}^{(1)} = \begin{array}{cc} \begin{array}{cc} 1 & 3 \end{array} & \\ \left[\begin{array}{cc} 1000 & -1000 \\ -1000 & 1000 \end{array}\right] & \begin{array}{c} 1 \\ 3 \end{array} \end{array} \qquad \underline{k}^{(2)} = \begin{array}{cc} \begin{array}{cc} 3 & 4 \end{array} & \\ \left[\begin{array}{cc} 2000 & -2000 \\ -2000 & 2000 \end{array}\right] & \begin{array}{c} 3 \\ 4 \end{array} \end{array} \qquad \underline{k}^{(3)} = \begin{array}{cc} \begin{array}{cc} 4 & 2 \end{array} & \\ \left[\begin{array}{cc} 3000 & -3000 \\ -3000 & 3000 \end{array}\right] & \begin{array}{c} 4 \\ 2 \end{array} \end{array}$$

- Using the concept of superposition (the direct stiffness method), we obtain the global stiffness matrix as

$$\underline{K} = \underline{k}^{(1)} + \underline{k}^{(2)} + \underline{k}^{(3)}$$

$$\underline{K} = \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \left[\begin{array}{cccc} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 1000+2000 & -2000 \\ 0 & -3000 & -2000 & 2000+3000 \end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

Introduction to the Stiffness (Displacement) Method

Example 1

<u>(b) The displacements of nodes 3 and 4</u>

Global Stiffness Equation

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} \cancel{1000} & \cancel{0} & \cancel{-1000} & \cancel{0} \\ \cancel{0} & \cancel{3000} & \cancel{0} & \cancel{-3000} \\ \cancel{-1000} & \cancel{0} & 3000 & -2000 \\ \cancel{0} & \cancel{-3000} & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \end{Bmatrix}$$

Homogeneous Boundary Conditions: $d_{1x} = 0 \text{ and } d_{2x} = 0$

$$\begin{array}{c} F_{3x} \longleftarrow \\ F_{4x} \longleftarrow \end{array} \begin{Bmatrix} 0 \\ 5000 \end{Bmatrix} = \begin{bmatrix} 3000 & -2000 \\ -2000 & 5000 \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{4x} \end{Bmatrix}$$

Global nodal displacements: $d_{3x} = \dfrac{10}{11} \text{ in.} \qquad d_{4x} = \dfrac{15}{11} \text{ in.}$

Introduction to the Stiffness (Displacement) Method

Example 1

(c) The reaction forces at nodes 1 and 2

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \frac{10}{11} \\ \frac{15}{11} \end{Bmatrix}$$

$$F_{1x} = \frac{-10{,}000}{11} \text{ lb}$$

$$F_{2x} = \frac{-45{,}000}{11} \text{ lb}$$

$$F_{3x} = 0$$

$$F_{4x} = 5000 \text{ lb}$$

The sum of the reactions F_{1x} and F_{2x} is equal in magnitude but opposite in direction to the applied force F_{4x}. This result verifies equilibrium of the whole spring assemblage.

Introduction to the Stiffness (Displacement) Method

Example 1

(d) The forces in each spring

Element 1

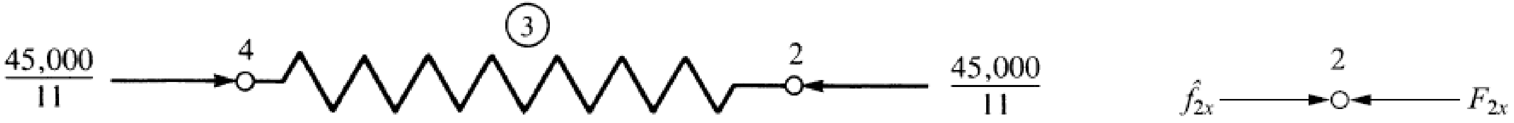
$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{10}{11} \end{Bmatrix} \qquad \hat{f}_{1x} = \frac{-10{,}000}{11}\ \text{lb} \qquad \hat{f}_{3x} = \frac{10{,}000}{11}\ \text{lb}$$

Element 2

$$\begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{4x} \end{Bmatrix} = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \begin{Bmatrix} \frac{10}{11} \\ \frac{15}{11} \end{Bmatrix} \qquad \hat{f}_{3x} = \frac{-10{,}000}{11}\ \text{lb} \qquad \hat{f}_{4x} = \frac{10{,}000}{11}\ \text{lb}$$

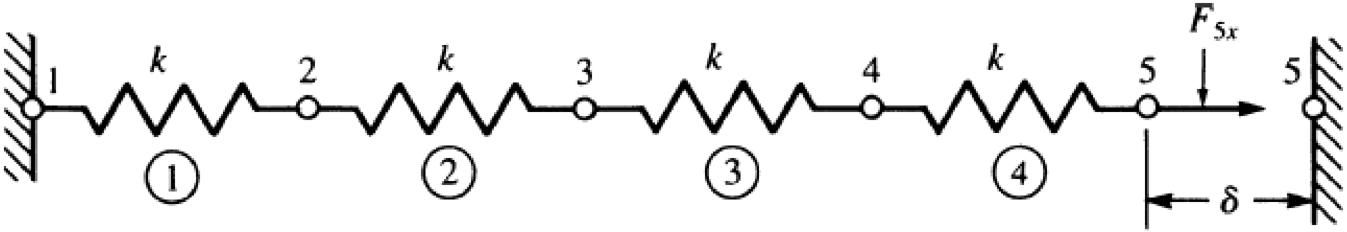
Element 3

$$\begin{Bmatrix} \hat{f}_{4x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \begin{Bmatrix} \frac{15}{11} \\ 0 \end{Bmatrix} \qquad \hat{f}_{4x} = \frac{45{,}000}{11}\ \text{lb} \qquad \hat{f}_{2x} = \frac{-45{,}000}{11}\ \text{lb}$$



Introduction to the Stiffness (Displacement) Method

Example 2



- Node 1 is fixed
- Node 5 is given a fixed, known displacement d δ = 20 mm.
- The spring constants are all equal to k = 200 kN/m.
- **Obtain:**

(a) The global stiffness matrix
(b) The displacements of nodes 2 and 4
(c) The global nodal forces
(d) The local element forces

Introduction to the Stiffness (Displacement) Method

Example 2

(a) The global stiffness matrix

Element Stiffness Matrix

$$\underline{k}^{(1)} = \underline{k}^{(2)} = \underline{k}^{(3)} = \underline{k}^{(4)} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix}$$

Global Stiffness Matrix

$$\underline{K} = \begin{bmatrix} 200 & -200 & 0 & 0 & 0 \\ -200 & 400 & -200 & 0 & 0 \\ 0 & -200 & 400 & -200 & 0 \\ 0 & 0 & -200 & 400 & -200 \\ 0 & 0 & 0 & -200 & 200 \end{bmatrix} \frac{\text{kN}}{\text{m}}$$

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(b) The displacements of nodes 2 and 4

Example 2

$$\begin{Bmatrix} \cancel{F_{1x}} \\ F_{2x} \\ F_{3x} \\ F_{4x} \\ \cancel{F_{5x}} \end{Bmatrix} = \begin{bmatrix} \cancel{200} & \cancel{-200} & \cancel{0} & \cancel{0} & \cancel{0} \\ -200 & 400 & -200 & 0 & 0 \\ 0 & -200 & 400 & -200 & 0 \\ 0 & 0 & -200 & 400 & -200 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{-200} & \cancel{200} \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \\ d_{5x} \end{Bmatrix}$$

$$d_{1x} = 0 \text{ and } d_{5x} = 20 \text{ mm}$$

$$F_{2x} = 0, \ F_{3x} = 0, \text{ and } F_{4x} = 0$$

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} -200 & 400 & -200 & 0 & 0 \\ 0 & -200 & 400 & -200 & 0 \\ 0 & 0 & -200 & 400 & \boxed{-200} \end{bmatrix} \begin{Bmatrix} 0 \\ d_{2x} \\ d_{3x} \\ d_{4x} \\ \boxed{0.02 \text{ m}} \end{Bmatrix}$$

Transposing the product of the appropriate stiffness coefficient (-200) multiplied by the known displacement (0.02m) to the left side.

$$\begin{Bmatrix} 0 \\ 0 \\ 4 \text{ kN} \end{Bmatrix} = \begin{bmatrix} 400 & -200 & 0 \\ -200 & 400 & -200 \\ 0 & -200 & 400 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \\ d_{4x} \end{Bmatrix} \longrightarrow$$

$$d_{2x} = 0.005 \text{ m}$$

$$d_{3x} = 0.01 \text{ m}$$

$$d_{4x} = 0.015 \text{ m}$$

Introduction to the Stiffness (Displacement) Method

<u>(c) The global nodal forces</u>

Example 2

$$F_{1x} = (-200)(0.005) = -1.0 \text{ kN}$$

$$F_{2x} = (400)(0.005) - (200)(0.01) = 0$$

$$F_{3x} = (-200)(0.005) + (400)(0.01) - (200)(0.015) = 0$$

$$F_{4x} = (-200)(0.01) + (400)(0.015) - (200)(0.02) = 0$$

$$F_{5x} = (-200)(0.015) + (200)(0.02) = 1.0 \text{ kN}$$

Introduction to the Stiffness (Displacement) Method

Example 2

(d) The local element forces

Element 1

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.005 \end{Bmatrix}$$

$\hat{f}_{1x} = -1.0 \text{ kN} \qquad \hat{f}_{2x} = 1.0 \text{ kN}$

Element 3

$$\begin{Bmatrix} \hat{f}_{3x} \\ \hat{f}_{4x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.01 \\ 0.015 \end{Bmatrix}$$

$\hat{f}_{3x} = -1 \text{ kN} \qquad \hat{f}_{4x} = 1 \text{ kN}$

Element 2

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{3x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.005 \\ 0.01 \end{Bmatrix}$$

$\hat{f}_{2x} = -1 \text{ kN} \qquad \hat{f}_{3x} = 1 \text{ kN}$

Element 4

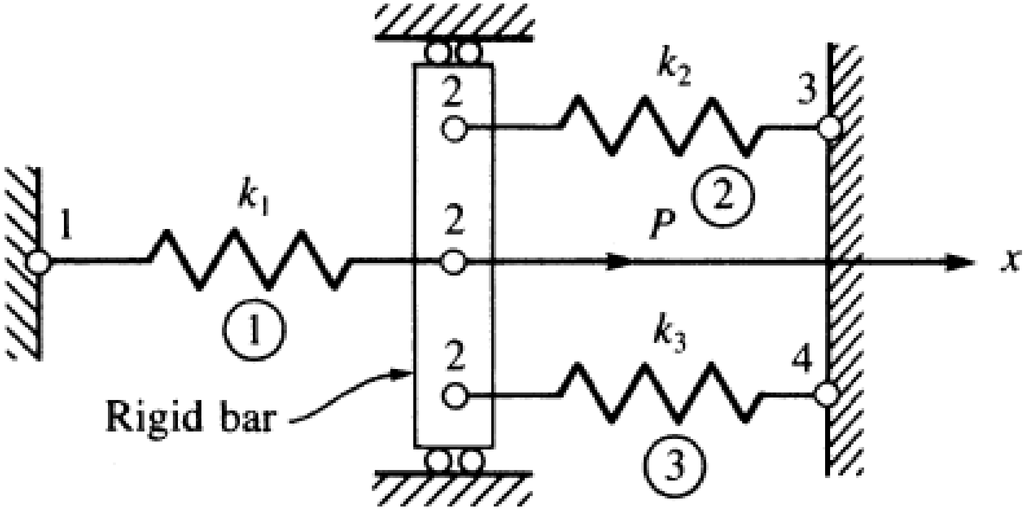
$$\begin{Bmatrix} \hat{f}_{4x} \\ \hat{f}_{5x} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0.015 \\ 0.02 \end{Bmatrix}$$

$\hat{f}_{4x} = -1 \text{ kN} \qquad \hat{f}_{5x} = 1 \text{ kN}$

Introduction to the Stiffness (Displacement) Method

Example 3

P is an applied force at node 2



- Formulate the global stiffness matrix and equations for solution of the unknown global displacement and forces.
  a) Using the direct equilibrium approach
  b) Using the direct stiffness method

Introduction to the Stiffness (Displacement) Method

Example 3

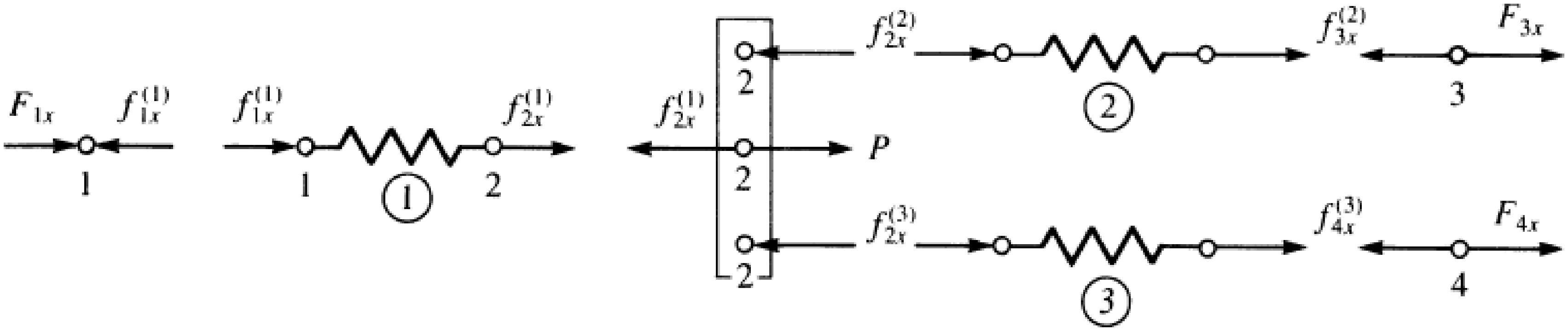
a) Using the direct equilibrium approach

- The boundary conditions: $d_{1x} = 0 \qquad d_{3x} = 0 \qquad d_{4x} = 0$
- The compatibility condition at node 2 is, $d_{2x}^{(1)} = d_{2x}^{(2)} = d_{2x}^{(3)} = d_{2x}$
- The nodal equilibrium conditions are,

$$F_{1x} = f_{1x}^{(1)} \qquad F_{3x} = f_{3x}^{(2)}$$

$$P = f_{2x}^{(1)} + f_{2x}^{(2)} + f_{2x}^{(3)} \qquad F_{4x} = f_{4x}^{(3)}$$

- The element and nodal force free-body diagrams,



Introduction to the Stiffness (Displacement) Method

Example 3

a) Using the direct equilibrium approach

- The total or global equilibrium equations,

$$F_{1x} = k_1 d_{1x} - k_1 d_{2x}$$

$$P = -k_1 d_{1x} + k_1 d_{2x} + k_2 d_{2x} - k_2 d_{3x} + k_3 d_{2x} - k_3 d_{4x}$$

$$F_{3x} = -k_2 d_{2x} + k_2 d_{3x}$$

$$F_{4x} = -k_3 d_{2x} + k_3 d_{4x}$$

Global Stiffness Matrix

$$\begin{Bmatrix} F_{1x} \\ P \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \end{Bmatrix}$$

$$d_{2x} = \frac{P}{k_1 + k_2 + k_3} \qquad F_{1x} = -k_1 d_{2x} \qquad F_{3x} = -k_2 d_{2x} \qquad F_{4x} = -k_3 d_{2x}$$

Introduction to the Stiffness (Displacement) Method

Example 3

a) Using the direct stiffness method

$$\underline{k}^{(1)} = \begin{array}{c} \begin{array}{cc} d_{1x} & d_{2x} \end{array} \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \end{array} \quad \underline{k}^{(2)} = \begin{array}{c} \begin{array}{cc} d_{2x} & d_{3x} \end{array} \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \end{array} \quad \underline{k}^{(3)} = \begin{array}{c} \begin{array}{cc} d_{2x} & d_{4x} \end{array} \\ \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix} \end{array}$$

Global Stiffness Matrix

$$\underline{K} = \begin{array}{c} \begin{array}{cccc} d_{1x} & d_{2x} & d_{3x} & d_{4x} \end{array} \\ \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_2 & -k_3 \\ 0 & -k_2 & k_2 & 0 \\ 0 & -k_3 & 0 & k_3 \end{bmatrix} \end{array}$$

Then write the global equilibrium equations and solve for the global forces.