INTERPOLATION FUNCTIONS

The basic idea of the finite element method is piecewise approximation i.e., the solution of a complicated problem is obtained by dividing the region of interest into small regions (finite elements) and approximating the solution over each subregion by a simple function. Thus, a necessary and important step is that of choosing a simple function for the solution in each element.

The functions used to represent the behavior of the solution within an element are called interpolation functions or approximating functions or interpolation models.

Polynomial Form of Interpolation Functions

Polynomial-type interpolation functions have been most widely used in the literature due to the following reasons:

a) It is easier to formulate and computerize the finite element equations with polynomial-type interpolation functions. Specifically, it is easier to perform differentiation or integration with polynomials.

b) It is possible to improve the accuracy of the results by increasing the order of the polynomial as shown in the figure below. Theoretically a polynomial of infinite order corresponds to the exact solution. But in practice we use polynomials of finite order only as an approximation.

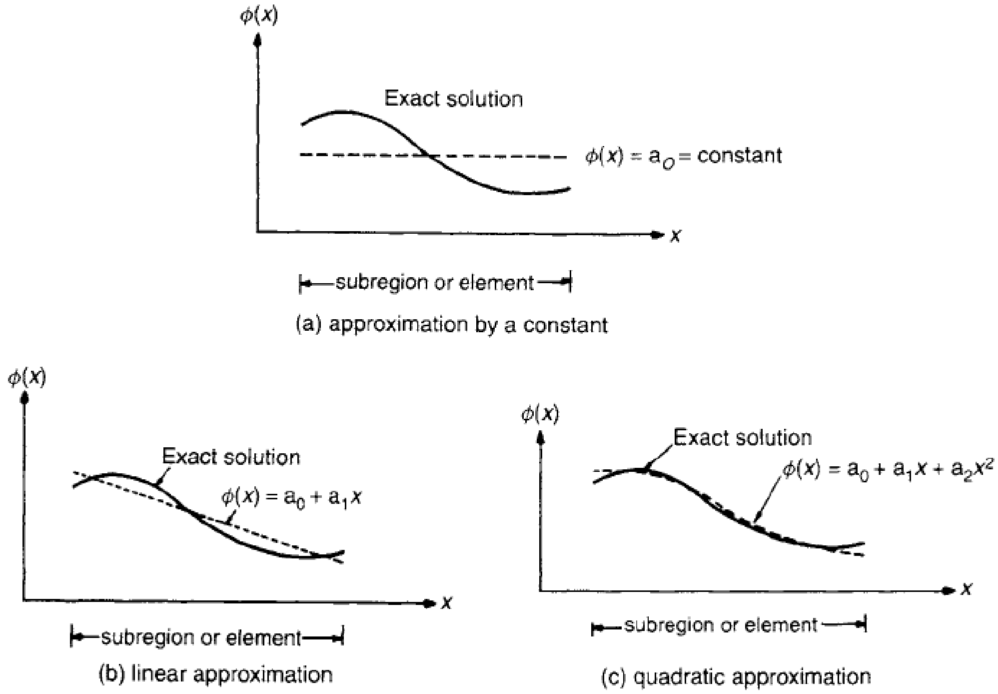


Figure: Polynomial Approximation in One Dimension**.**

Although trigonometric functions also possess some of these properties, they are seldom used in the finite element analysis.

(i.) If a polynomial type of variation is assumed for the field variable $\phi(x)$ in a one-dimensional element, $\phi(x)$ can be expressed as:

$$\phi(x) = a_1 + a_2 x + a_3 x^2 + ... + a_m x^n$$

Where $a_1, a_2, a_3............., a_n$ are the coefficients of the polynomial, also known as generalized coordinates, n is the degree of the polynomial and m is the number of polynomial coefficients.

(ii.) For two dimensional finite elements the polynomial form of interpolation functions can be expressed as:

$$\phi(x, y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy + ... + a_m y^n$$

(iii.) For three dimensional finite elements the polynomial form of interpolation functions can be expressed as:

$$\phi(x, y, z) = a_1 + a_2 x + a_3 y + a_4 z + a_5 x^2 + a_6 y^2 + a_7 z^2 + a_8 xy + a_9 yz + a_{10} zx............ + a_m z^n$$

Finite elements can be classified into three categories as **simplex, complex, and multiplex** elements depending on the geometry of the element and the order of the polynomial used in the interpolation function.

- **Simplex elements** are those for which the approximating polynomial consists of constant and linear terms. For example, the simplex element in two dimensions is a triangle with three nodes (corners).
- **Complex elements** are those for which the approximating polynomial consists of quadratic, cubic, and higher order terms, according to the need in addition to the constant and linear terms. For example, a triangular element with three corner nodes and three midside nodes satisfies this requirement.
- **Multiplex elements** are those whose boundaries are parallel to the coordinate axes to achieve inter-element continuity and whose approximating polynomials contain higher order terms. The rectangular element is an example of a multiplex element in two dimensions. **Note:** The boundaries of the simplex and complex elements need not be parallel to the coordinate axes.)

- If the interpolation polynomial is of order two or more, the element is known as a higher order element. In higher order elements, some secondary (midside and/or interior) nodes are introduced in addition to the primary (corner) nodes to match the number of nodal degrees of freedom with the number of constants (generalized coordinates) in the interpolation polynomial.
- In general, fewer higher order elements are needed to achieve the same degree of accuracy in the final results. Although it does not reduce the computational time, the reduction in the number of elements generally reduces the effort needed in the preparation of data and hence the chances of errors in the input data.
- The higher order elements are especially useful in cases in which tile gradient of the field variable is expected to vary rapidly. In these cases, the simplex elements, which approximate the gradient by a set of constant values, do not yield good results. The combination of greater accuracy and a reduction in the data preparation effort has resulted in the widespread use of higher order elements in several practical applications.

SELECTION OF THE ORDER OF THE INTERPOLATION POLYNOMIAL

While choosing the order of the polynomial in a polynomial-type interpolation function, the following considerations have to be taken into account:

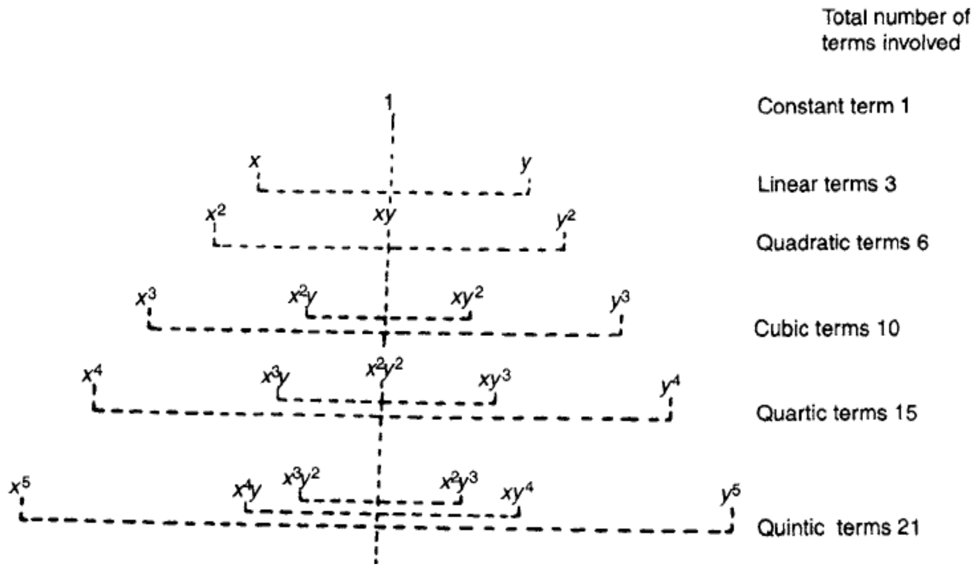
a. The interpolation polynomial should satisfy, as far as possible, the **convergence requirements.**

b. The pattern of variation of the field variable resulting from the polynomial model should be **independent of the local coordinate system.**

c. The number of **generalized coordinates (a_i) should be equal to the number of nodal degrees of freedom of the element.**

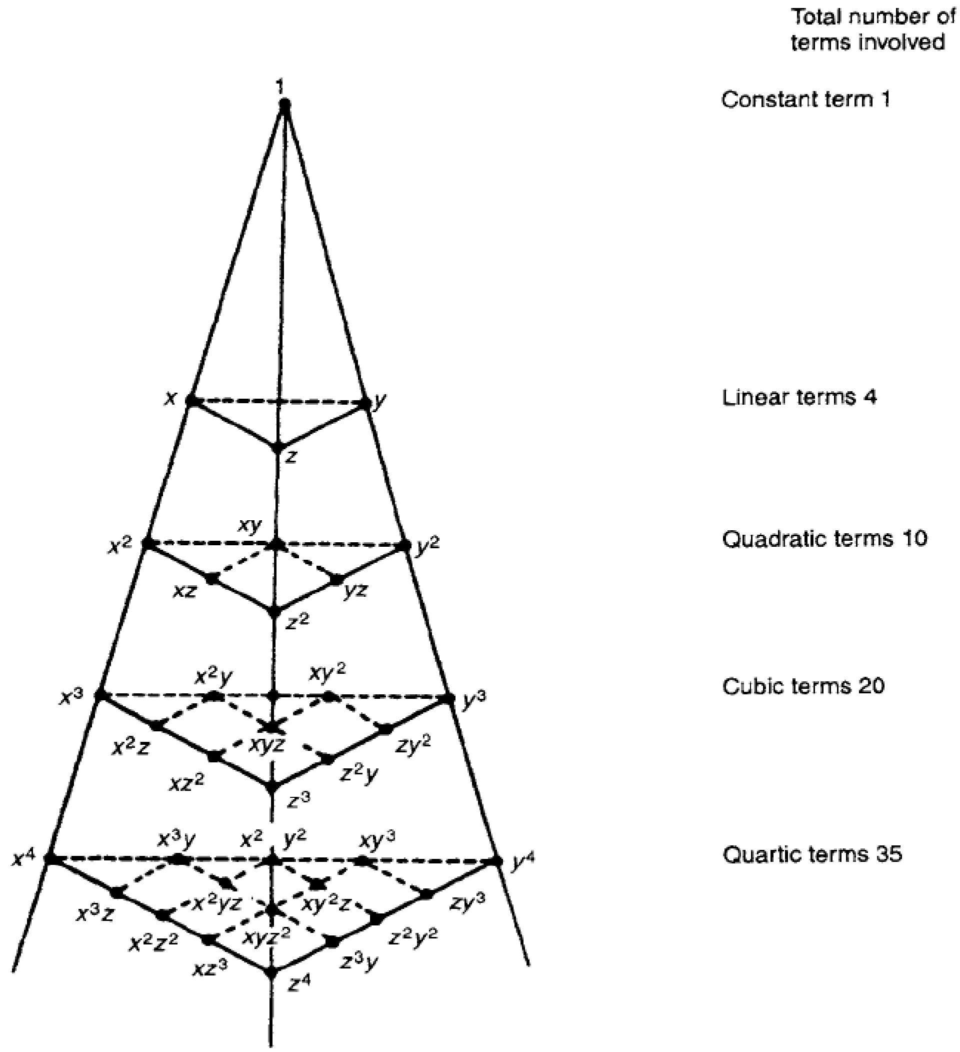
- The first consideration, namely, the convergence requirements is to be satisfied by the interpolation polynomial, is given in the next section.

- According to the second consideration, i.e., the field variable representation within an element, and hence the polynomial, should not change with a change in the local coordinate system (when a linear transformation is made from one Cartesian coordinate system to another). This property is called **geometric isotropy or geometric invariance or spatial isotropy.** In order to achieve geometric isotropy, the polynomial should contain terms that do not violate symmetry in figures below, which are known as **Pascal triangle** in the case of two dimensions and **Pascal tetrahedron** in the case of three dimensions.



(a) In two dimensions (Pascal triangle)

Thus, in the case of a two-dimensional simplex element (triangle) the interpolation polynomial should include terms containing both x and y but not only one of them, in addition to the constant term. In the case of a two-dimensional complex element (triangle), if we neglect the term x^3 (or x^2y) for any reason, we should not include y^3 (or xy^2) also in order to maintain geometric isotropy of the model. Similarly, in the case of a three dimensional simplex element (tetrahedron), the approximating polynomial should contain terms involving x, y, and z in addition to the constant term.



(b) In three dimensions (Pascal tetrahedron)

- The final consideration in selecting the order of the interpolation polynomial is to make the total number of terms involved in the polynomial equal to the number of nodal degrees of freedom of the element.

(a) CONVERGENCE REQUIREMENTS

Since the finite element method is a numerical technique, we obtain a sequence of approximate solutions as the element size is reduced successively. This sequence will converge to the exact solution if the interpolation polynomial satisfies the following convergence requirements.

(i.) The **field variable must be continuous within the elements.** This requirement is easily satisfied by choosing continuous functions as interpolation models. Since polynomials are inherently continuous, the polynomial type of interpolation models satisfies this requirement.

(ii.) **The interpolation function should allow for rigid body displacement and for a state of constant strain with in the element**

The uniform or constant value of the field variable is the most elementary type of variation. Thus, the interpolation polynomial must be able to give a constant value of the field variable within the element when the nodal values are numerically identical.

In the case of solid mechanics and structural problems, this requirement states that the assumed displacement model must permit the rigid body (zero strain) and the constant strain states of the element.

(iii.) The field variable $\phi(x)$ and its partial derivatives up to one order less than the highest order derivative appearing in the functional $I(\phi)$ must be continuous at element boundaries or interfaces.

In the case of general solid and structural mechanics problems, this requirement implies that the element must deform without causing openings, overlaps, or discontinuities between adjacent elements. In the case of beam plate, and shell elements, the first derivative of the displacement (slope) across inter element boundaries also must be continuous.

The elements whose interpolation polynomials satisfy the requirements (i) and (iii) are called **"compatible" or "'conforming" elements** and those satisfying condition (ii) are called **"complete" elements.**

If the interpolation polynomial satisfies all three requirements, the approximate solution converges to the correct solution when we refine the mesh and use an increasing number of smaller elements.

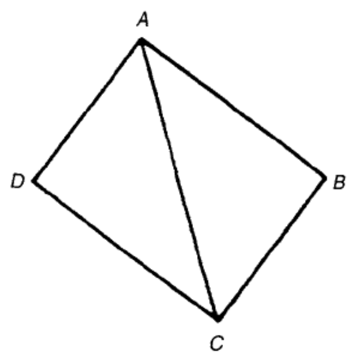
In order to prove the convergence mathematically, the refinement has to be made in a regular fashion so as to satisfy the following conditions:

(i.) All previous (coarse) meshes must be contained in the refined meshes.

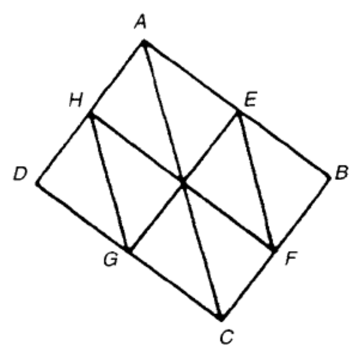
(ii.) The elements must be made smaller in such a way that every point of the solution region can always be within an element.

(iii.) The form of the interpolation polynomial must remain unchanged during the process of mesh refinement.

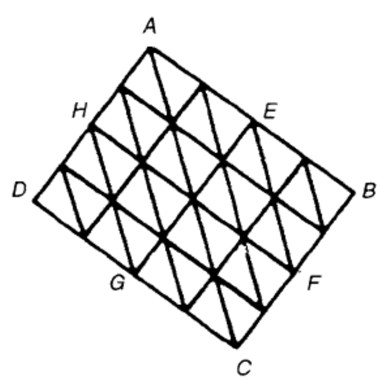
Conditions (i) and (ii) are illustrated in the figure below, in which a two-dimensional region (in the form of a parallelogram) is discretized with an increasing number of triangular elements.



(a) Idealization with 2 elements



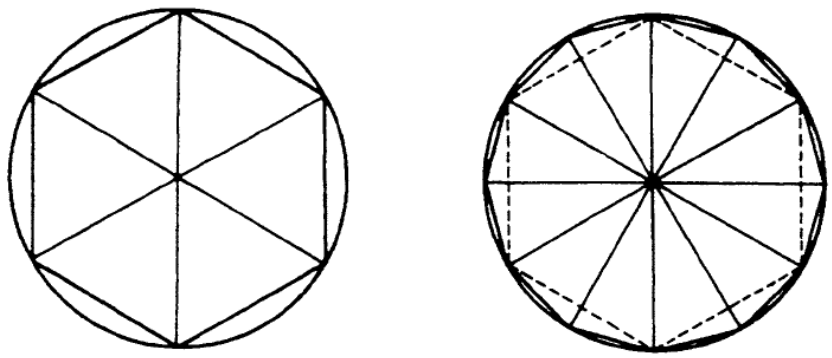
b) Idealization with 8 elements



(c) Idealization with 32 elements

Figure: All Previous Meshes Contained in Refined Meshes.

For a curved boundary, it can be seen that conditions (i) and (ii) are not satisfied if we use elements with straight boundaries.



(a) Idealization with 6 elements (b) Idealization with 12 elements

Figure: Previous Mesh Is Not Contained in the Refined Mesh.

Convergence

When modeling a problem using a finite element program, how do we know that the solution that we get is correct? Just because we get a solution, this does not mean that the solution is correct

The word convergence is used because the output from the finite element program is converging on a single correct solution. In order to check the convergence of the solution, at least two solutions to the same problem are required. The solution from the finite element program is checked with a solution of increased accuracy. If the more accurate solution is dramatically different from the original solution, then the solution is not converged. However, if the solution does not change much (less than a few percent difference) then the solution is considered converged. It is very important to check whether the solution has converged. Convergence is tested differently depending on which solution method is used. The two available methods are the **p-method and the h-method.**

Convergence Using H-Method:

Simple shape functions and many small elements are used in h-method problems. In order to increase the accuracy of the solution, more elements must be added. This means creating a finer mesh.

As an initial run, a course mesh is used to model the problem. A solution is obtained. To check this solution, a finer mesh is created. The mesh must always be changed if a more accurate solution is desired. The problem is run again to obtain a second solution. If there is a large difference between the two solutions, then the mesh must be made even finer and then solve it again. This process is repeated until the solution is not changing much from run to run.

When using an h-method finite element program, the user must run two or more solutions to ensure that the solution has converged. The user runs the solution with one mesh and then changes the mesh and reruns the solution.

The density of the mesh only needs to be increased in the areas of the part where stresses are very high or the stresses change quickly over a small distance. In areas of the part where stress variations are not very high, few elements are required to accurately model the problem. So, when checking convergence, it is only really necessary to create a finer mesh in areas of stress concentration.

Convergence Using P-Method:

Large elements and complex shape functions are used in p-method problems. In order to increase the accuracy of the solution, the complexity of the shape function must be increased. The mesh does not need to be changed when using the p-method.

Increasing the polynomial order increases the complexity of the shape function. As an initial run, the solution might be solved using a first order polynomial shape function. A solution is obtained. To check the solution the problem will be solved again using a more complicated shape function. For the second run, the solution may be solved using a second order polynomial shape function. A second solution is obtained. The output from the two runs is compared. If there is a large difference between the two solutions, then the solution should be run using a third order polynomial shape function. This process is repeated until the solution is not changing much from run to run.