CHAPTER 3

REAL EIGENVALUE ANALYSIS

The usual first step in performing a dynamic analysis is determining the natural frequencies and mode shapes of the structure with damping neglected. These results characterize the basic dynamic behavior of the structure and are an indication of how the structure will respond to dynamic loading.

Natural Frequencies

The natural frequencies of a structure are the frequencies at which the structure naturally tends to vibrate if it is subjected to a disturbance. For example, the strings of a piano are each tuned to vibrate at a specific frequency. Some alternate terms for the natural frequency are characteristic frequency, fundamental frequency, resonant frequency, resonance frequency, and normal frequency.

Mode Shapes

The deformed shape of the structure at a specific natural frequency of vibration is termed its normal mode of vibration. Some other terms used to describe the normal mode are mode shape, characteristic shape, and fundamental shape. Each mode shape is associated with a specific natural frequency.

Natural frequencies and mode shapes are functions of the structural properties and boundary conditions. A cantilever beam has a set of natural frequencies and associated mode shapes (Figure 3-1). If the structural properties change, the natural frequencies change, but the mode shapes may not necessarily change. For example, if the elastic modulus of the cantilever beam is changed, the natural frequencies change but the mode shapes remain the same. If the boundary conditions change, then the natural frequencies and mode shapes both change. For example, if the cantilever beam is changed so that it is pinned at both ends, the natural frequencies and mode shapes change (see Figure 3-2).

3.1 Reasons to Compute Normal Modes

There are many reasons to compute the natural frequencies and mode shapes of a structure. One reason is to assess the dynamic interaction between a component and its supporting structure. For example, if a rotating machine, such as an air conditioner fan, is to be installed on the roof of a building, it is necessary to determine if the operating frequency of the rotating fan is close to one of the natural frequencies of the building. If the frequencies are close, the operation of the fan may lead to structural damage or failure.

Decisions regarding subsequent dynamic analyses (i.e., transient response, frequency response, response spectrum analysis, etc.) can be based on the results of a natural frequency analysis. The important modes can be evaluated and used to select the appropriate time or frequency step for integrating the equations of motion. Similarly, the results of the eigenvalue analysis—the natural frequencies and mode shapes—can be used in modal frequency and modal transient response analyses (see Chapters 5 and 6).

The results of the dynamic analyses are sometimes compared to the physical test results. A normal modes analysis can be used to guide the experiment. In the pretest planning stages, a normal modes analysis can be used to indicate the best location for the accelerometers. After the test, a normal modes analysis can be used as a means to correlate the test results to the analysis results.

Design changes can also be evaluated by using natural frequencies and normal modes. Does a particular design modification cause an increase in dynamic response? Normal modes analysis can often provide an indication.

Eigenvalue analysis is the basis for many types of dynamic response analyses.

In summary, there are many reasons to compute the natural frequencies and mode shapes of a structure. All of these reasons are based on the fact that real eigenvalue analysis is the basis for many types of dynamic response analyses. Therefore, an overall understanding of normal modes analysis as well as knowledge of the natural frequencies and mode shapes for your particular structure is important for all types of dynamic analysis.

3.2 Overview of Normal Modes Analysis

The solution of the equation of motion for natural frequencies and normal modes requires a special reduced form of the equation of motion. If there is no damping and no applied loading, the equation of motion in matrix form reduces to

$$[M]\{\ddot{u}\} + [K]\{u\} = 0 \tag{3-1}$$

where $[M]$ = mass matrix

$[K]$ = stiffness matrix

This is the equation of motion for undamped free vibration. To solve Eq. (3-1) assume a harmonic solution of the form

$$\{u\} = \{\phi\}\sin\omega t \tag{3-2}$$

where $\{\phi\}$ = the eigenvector or mode shape

ω = is the circular natural frequency

Aside from this harmonic form being the key to the numerical solution of the problem, this form also has a physical importance. The harmonic form of the solution means that all the degrees of freedom of the vibrating structure move in a synchronous manner. The structural configuration does not change its basic shape during motion; only its amplitude changes.

In harmonic motion all DOFs move in a synchronous manner.

If differentiation of the assumed harmonic solution is performed and substituted into the equation of motion, the following is obtained:

$$-\omega^2[M]\{\phi\}\sin\omega t + [K]\{\phi\}\sin\omega t = 0 \tag{3-3}$$

which after simplifying becomes

$$\left([K] - \omega^2[M]\right)\{\phi\} = 0 \tag{3-4}$$

This equation is called the eigenequation, which is a set of homogeneous algebraic equations for the components of the eigenvector and forms the basis for the eigenvalue problem. An eigenvalue problem is a specific equation form that has many applications in linear matrix algebra. The basic form of an eigenvalue problem is

$$[A - \lambda I]x = 0 \tag{3-5}$$

where A = square matrix

λ = eigenvalues

I = identity matrix

x = eigenvector

In structural analysis, the representations of stiffness and mass in the eigenequation result in the physical representations of natural frequencies and mode shapes. Therefore, the eigenequation is written in terms of K, ω, and M as shown in Eq. (3-4) with $\omega^2 = \lambda$.

There are two possible solution forms for Eq. (3-4):

1. If $\det\left([K] - \omega^2[M]\right) \neq 0$, the only possible solution is

$$\{\phi\} = 0 \tag{3-6}$$

This is the trivial solution, which does not provide any valuable information from a physical point of view, since it represents the case of no motion. ("det" denotes the determinant of a matrix.)

2. If $\det\left([K] - \omega^2[M]\right) = 0$, then a non-trivial solution $\left(\{\phi\} \neq 0\right)$ is obtained for

$$\left([K] - \omega^2[M]\right)\{\phi\} = 0 \tag{3-7}$$

From a structural engineering point of view, the general mathematical eigenvalue problem reduces to one of solving the equation of the form

$$\det\left([K] - \omega^2[M]\right) = 0 \tag{3-8}$$

or

$$\det\left([K] - \lambda[M]\right) = 0 \tag{3-9}$$

where $\lambda = \omega^2$

The determinant is zero only at a set of discrete eigenvalues λ_i or ω_i^2. There is an eigenvector $\{\phi_i\}$ which satisfies Eq. (3-7) and corresponds to each eigenvalue. Therefore, Eq. (3-7) can be rewritten as

$$\left[K - \omega_i^2 M\right]\{\phi_i\} = 0 \qquad i = 1,\ 2,\ 3... \qquad (3\text{-}10)$$

Each eigenvalue and eigenvector define a free vibration mode of the structure. The i-th eigenvalue λ_i is related to the i-th natural frequency as follows:

$$f_i = \frac{\omega_i}{2\pi} \qquad (3\text{-}11)$$

where f_i = i-th natural frequency

$\omega_i = \sqrt{\lambda_i}$

The number of eigenvalues and eigenvectors is equal to the number of degrees of freedom that have mass or the number of dynamic degrees of freedom.

There are a number of characteristics of natural frequencies and mode shapes that make them useful in various dynamic analyses. First, when a linear elastic structure is vibrating in free or forced vibration, its deflected shape at any given time is a linear combination of all of its normal modes

$$\{u\} = \sum_i \{\phi_i\}\ \xi_i \qquad (3\text{-}12)$$

where $\{u\}$ = vector of physical displacements

$\{\phi_i\}$ = i-th mode shape

ξ_i = i-th modal displacement

Second, if $[K]$ and $[M]$ are symmetric and real (as is the case for all the common structural finite elements), the following mathematical properties hold:

Generalized Mass

$$\{\phi_i\}^T[M]\{\phi_j\} = 0 \qquad \text{if } i \neq j \qquad (3\text{-}13)$$

$$\{\phi_j\}^T[M]\{\phi_j\} = m_j = \text{j-th generalized mass} \qquad (3\text{-}14)$$

and