Steel Structures (SE 505)

M.Sc. Structural Engineering

Lec.#2

Plastic Analysis and Design of Structures

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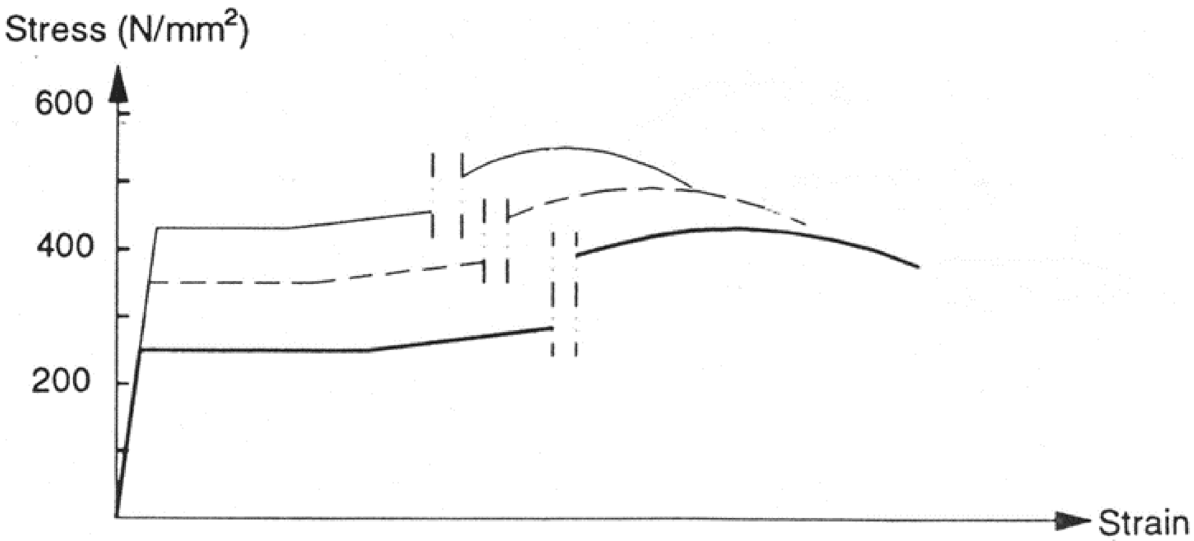
Steel Structures

Plastic Analysis of Plane Trusses

Typical stress-strain curves for structural steels with yield stresses σ_y of 250 MPa, 350 MPa and 430 MPa:



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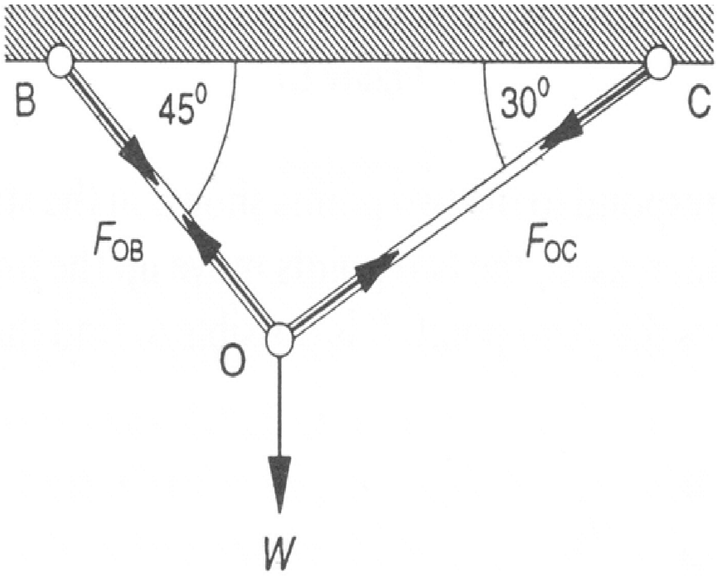
Plastic Analysis of Plane Trusses (cont'd)



$$F_{OB}\cos 45^0 = F_{OC}\cos 30^0$$

$$F_{OB} = 1.22\ F_{OC}$$

$$F_{OB}\sin 45^0 + F_{OC}\sin 30^0 = W$$

$$F_{OB} = 0.897\ W$$

$$F_{OC} = 0.732\ W$$

<u>Statically Determinate</u>

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Plastic Analysis of Plane Trusses (cont'd)

$F_{OB} = 0.897\ W$

$F_{OC} = 0.732\ W$

Let the cross-section area of member OB be equal to A and that of member OC equal to $2A$:

Stress in OB = $0.897\ \frac{W}{A}$

Stress in OC = $0.366\ \frac{W}{A}$

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$$\text{Stress in OB} = 0.897\,\frac{W}{A}$$

$$\text{Stress in OC} = 0.366\,\frac{W}{A}$$

As the load *W* increases, the stress in member OB reaches the yield point first:

$$\text{Stress in OB} = \sigma_y = 0.897\,\frac{W}{A}$$

During yielding in the plateau range (plastic flow), member OB cannot take on more loading as the stress remains constant at σ_y.

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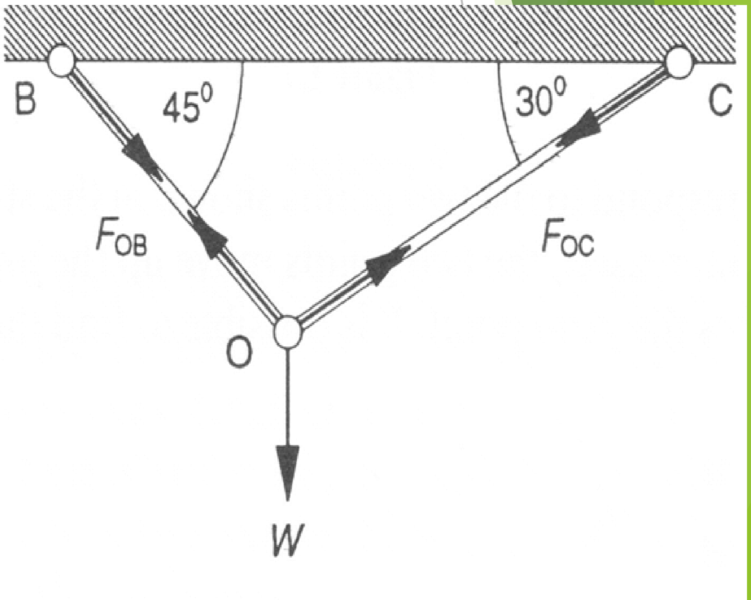
Plastic Analysis of Plane Trusses (cont'd)

As member OB reaches the yield plateau, i.e. its axial force F_{OB} remains constant at $\sigma_y A$ with increasing W, it is impossible to satisfy the equilibrium equations



$$F_{OB}\cos 45^0 = F_{OC}\cos 30^0$$

$$F_{OB}\sin 45^0 + F_{OC}\sin 30^0 = W$$

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Plastic Analysis of Plane Trusses (cont'd)

Plastic Analysis of Plane Trusses (cont'd)

The collapse load W_c of the statically determinate truss is therefore reached when member OB yields:

Stress in OB = $\sigma_y = 0.897 \dfrac{W_c}{A}$

$$W_c = 1.115 A \sigma_y$$

- In a statically determinate truss, collapse occurs when the most highly stressed member yields.
- There is no load redistribution to maintain equilibrium.

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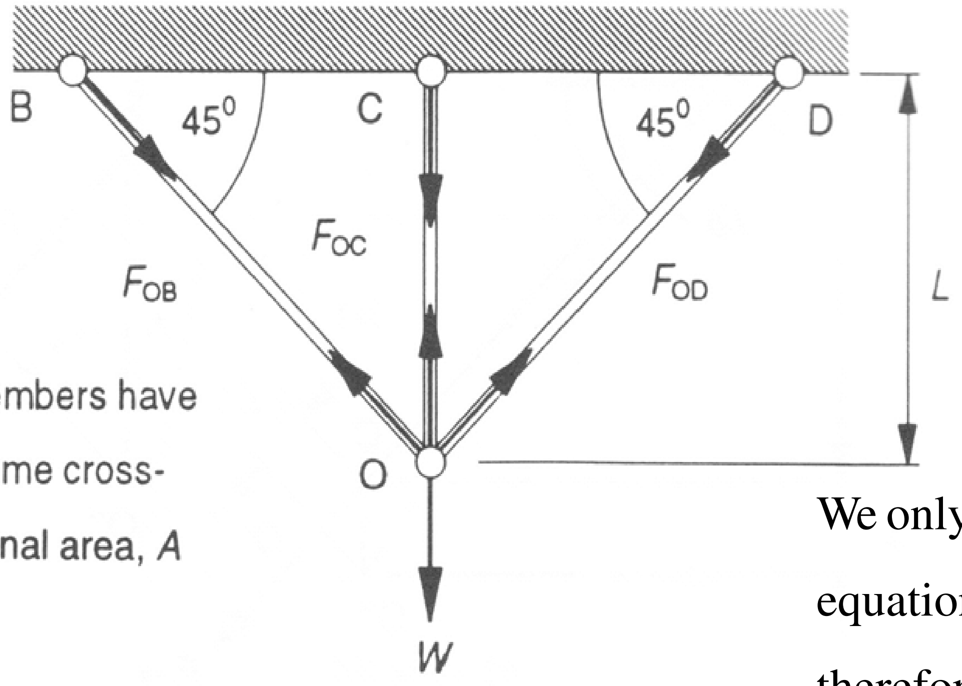


All members have the same cross-sectional area, A

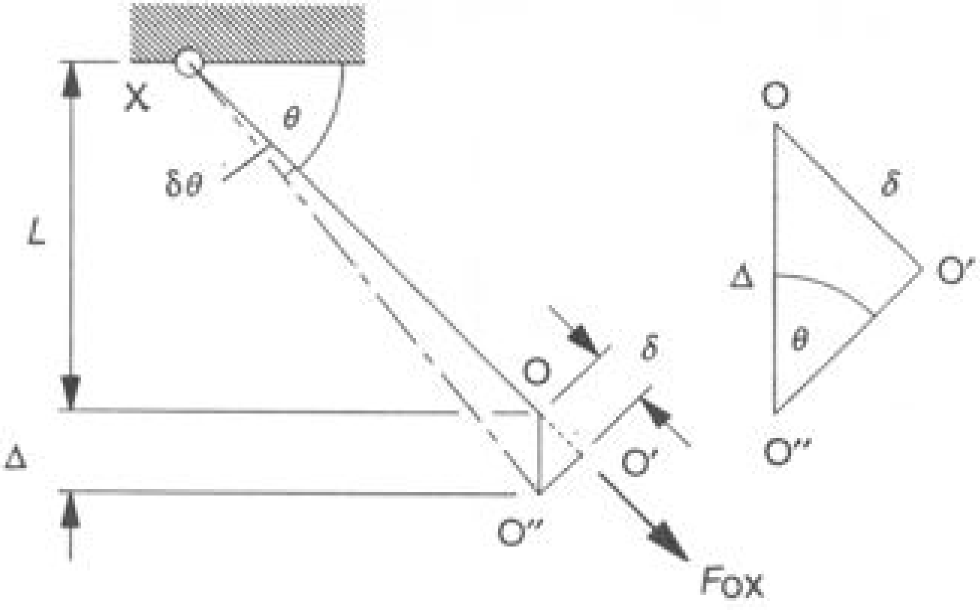
Statically indeterminate

$$F_{OB} \sin 45° + F_{OC} + F_{OD} \sin 45° = W$$

$$F_{OB} = F_{OD}$$

We only have two equilibrium equations with three unknown forces, therefore we need to supplement the equilibrium equations with a kinematic compatibility equation in order to determine the member forces.

Plastic Analysis of Plane Trusses (cont'd)



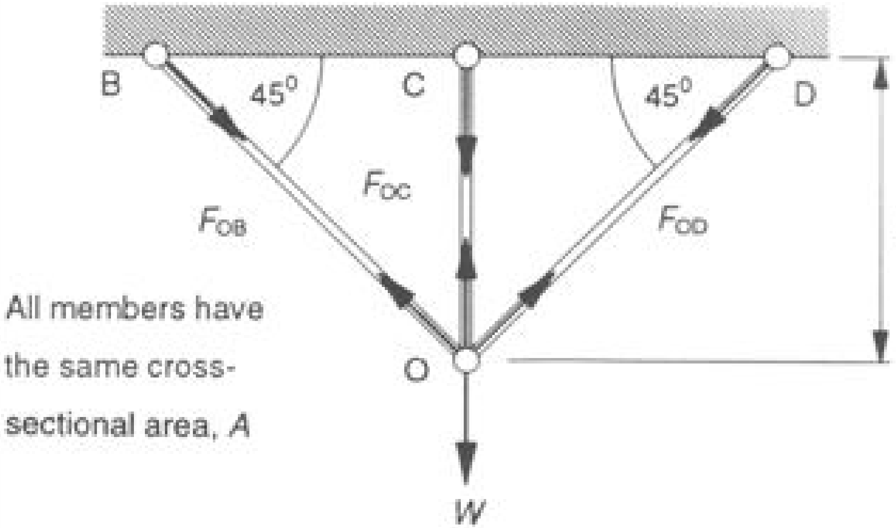
$$\delta = \Delta \sin\theta$$

$$\frac{F_{OX}}{A} \div \frac{\delta}{OX} = E$$

$$F_{OX} = \frac{AE}{L}\Delta \sin^2\theta$$

$$OX = L/\sin\theta$$

Plastic Analysis of Plane Trusses (cont'd)



$$\delta = \Delta \sin \theta$$

$$\delta_{OB} = \delta_{OD} = \Delta \sin 45^\circ = \frac{\Delta}{\sqrt{2}}$$

$$\delta_{OC} = \Delta \sin 90^\circ = \Delta$$

$$F_{OX} = \frac{AE}{L} \Delta \sin^2 \theta$$

$$F_{OB} = F_{OD} = \frac{AE}{L} \Delta \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{AE\Delta}{2L}$$

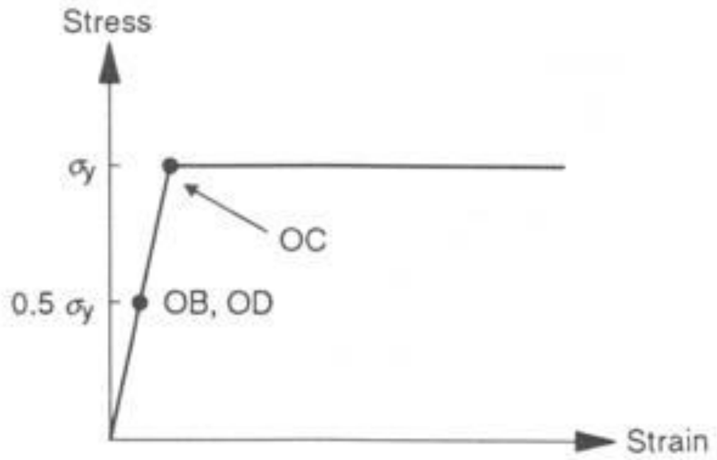
$$F_{OC} = \frac{AE\Delta}{L}$$

$$2F_{OB} = F_{OC}$$

Plastic Analysis of Plane Trusses (cont'd)

$$F_{OB} = F_{OD}$$

$$2F_{OB} = F_{OC}$$



$$F_{OC} = 0.585W$$

$$F_{OB} = F_{OD} = 0.293W$$

$$\frac{0.585W_1}{A} = \sigma_y$$

$$W_1 = 1.709A\sigma_y$$

Plastic Analysis of Plane Trusses (cont'd)

After the first yield point due to W_1, the force in member OC remains the same as the applied load W increases:

$$F_{OC} = A\sigma_y$$

For equilibrium:

$$F_{OB} \sin 45^\circ + F_{OC} + F_{OD} \sin 45^\circ = W$$

$$F_{OB} = F_{OD}$$

$$F_{OB} = 0.707(W - A\sigma_y)$$

$$F_{OB} = 0.707(W - A\sigma_y) \qquad F_{OB} = F_{OD}$$

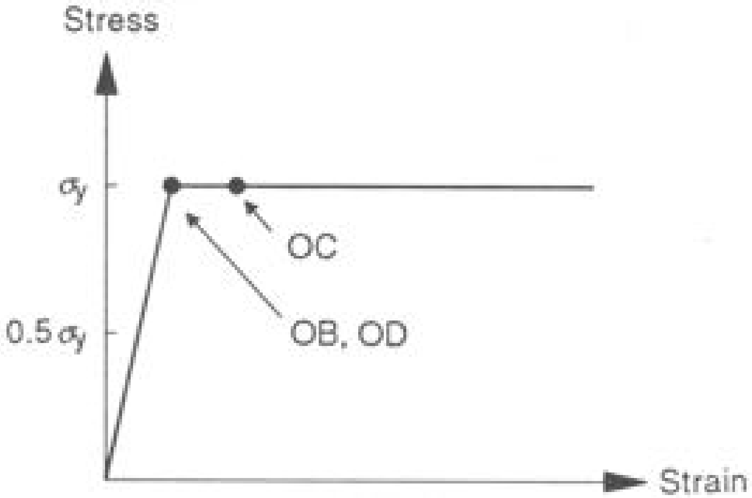
When members OB and OD yield: $F_{OB} = A\sigma_y$

$$0.707(W_2 - A\sigma_y) = A\sigma_y$$

$$W_2 = 2.414A\sigma_y$$

Member OC yields at:

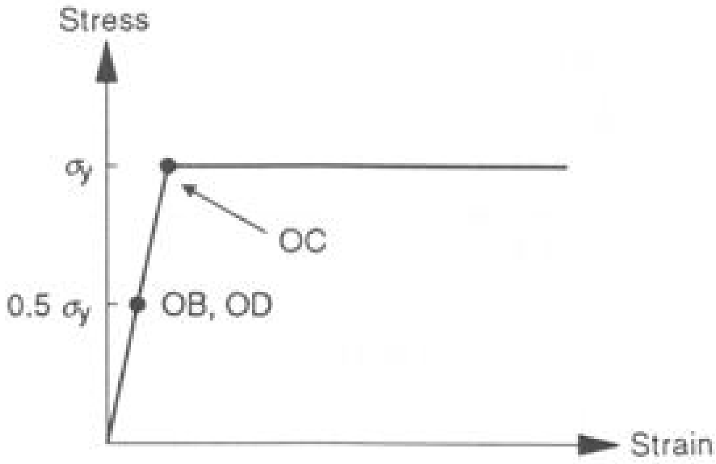
$$W_1 = 1.709A\sigma_y$$



- In a statically indeterminate truss, the degree of indeterminacy (redundancy) is reduced by one each time a member yields.
- As a member yields, additional loading results in redistribution of internal forces.

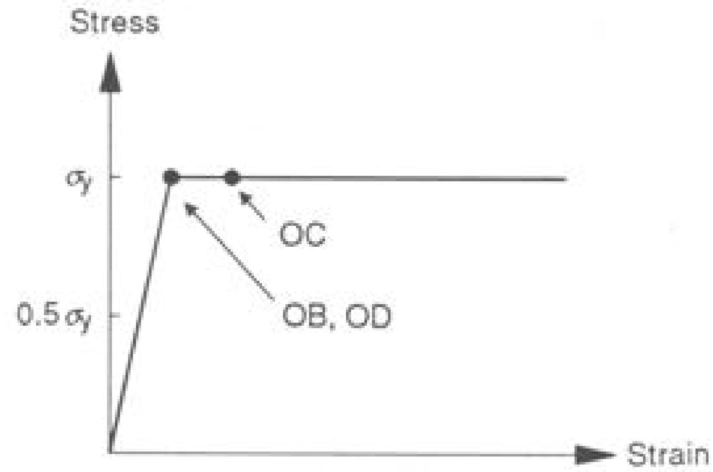
At fully elastic state:

$$2F_{OB} = F_{OC}$$



At plastic collapse:

$$F_{OC} = A\sigma_y \quad F_{OB} = A\sigma_y$$



Steel Structures

Margin of Safety in Plastic Design

ASD: $P_w \text{ (Allowable working stress)} = 0.66P_y$

$$F = M_p / M_y = P_u / P_y$$

$$F = 1.12$$ Shape function for W section

$$P_w = 0.66 \times \frac{P_u}{1.12} = 0.59P_u$$

$$F.O.S = \frac{P_u}{0.59P_u} \cong 1.69$$

(1.67 in new specification)

Steel Structures

Margin of Safety in Plastic Design (contd…)

LRFD/PD:

$$\text{Live Load} = \frac{1.6}{0.9} = 1.78 \qquad \text{Dead Load} = \frac{1.2}{0.9} = 1.33$$

For 3 live load to 1 dead load ratio: Average FOS = 1.67

In concrete design, the overall FOS has reduced with time. As ACI code does not support ASD, direct comparison can not be made.

F.O.S is not less in LRFD/PD, it is quite sufficient.

Load factors and resistance factor are same in LRFD and plastic design

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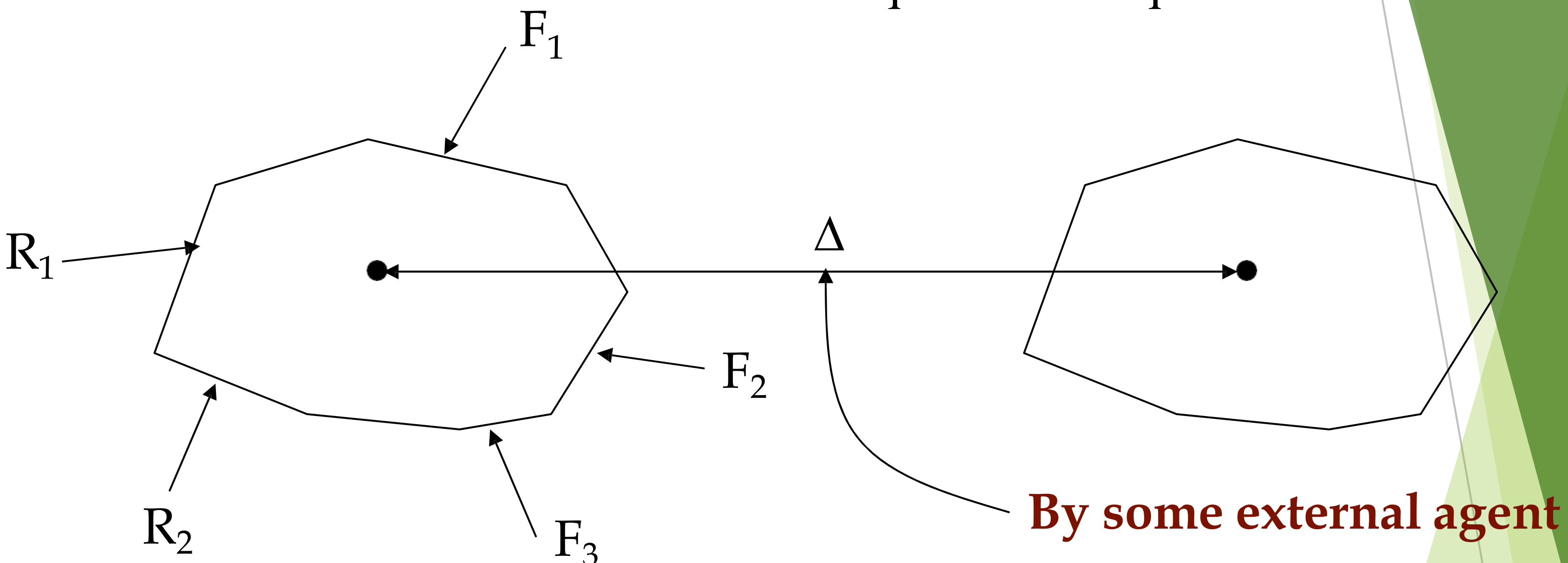
Principle of Virtual Work

For a deformable body, the total external work is equal to the total internal work, for every system of virtual forces and stresses that satisfy the equations of equilibrium.



Work done by loads + Work done by reactions = 0

Work done by loads = Work done by reactions

Principle of virtual work states that in equilibrium the virtual work of forces applied to a system is zero.

Steel Structures

Lower Bound Theorem

For a given structural system, the lower bound method gives an ultimate load that is either **actual or lower than the actual.**

- The equilibrium conditions are satisfied at all the points of the structure.

Upper Bound Theorem

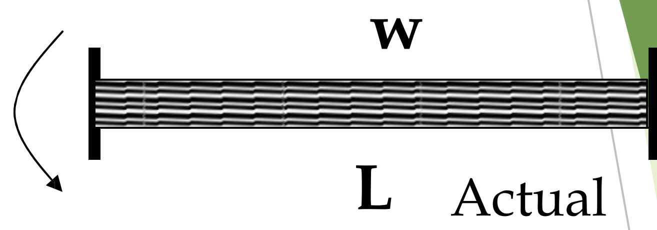
For a given structural system, the upper bound method gives an ultimate load that is either **actual or higher than the actual.**

- The equilibrium conditions are satisfied at the selected points and not at every point.

Steel Structures

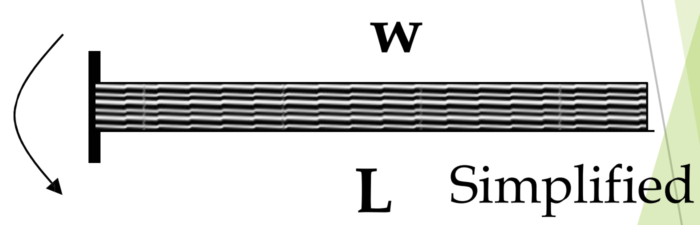
Upper /Lower Bound Theorem

1 $w_c = \dfrac{12M_u}{L^2}$ $\quad M_u = \dfrac{w_c L^2}{12}$

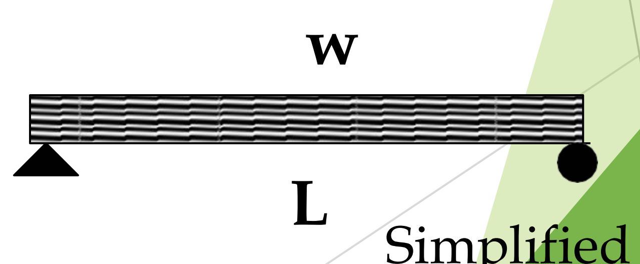


$\dfrac{w_c L^2}{12}$

2 $w_c = \dfrac{2M_u}{L^2}$ $\quad M_u = \dfrac{w_c L^2}{2}$



3 $w_c = \dfrac{8M_u}{L^2}$



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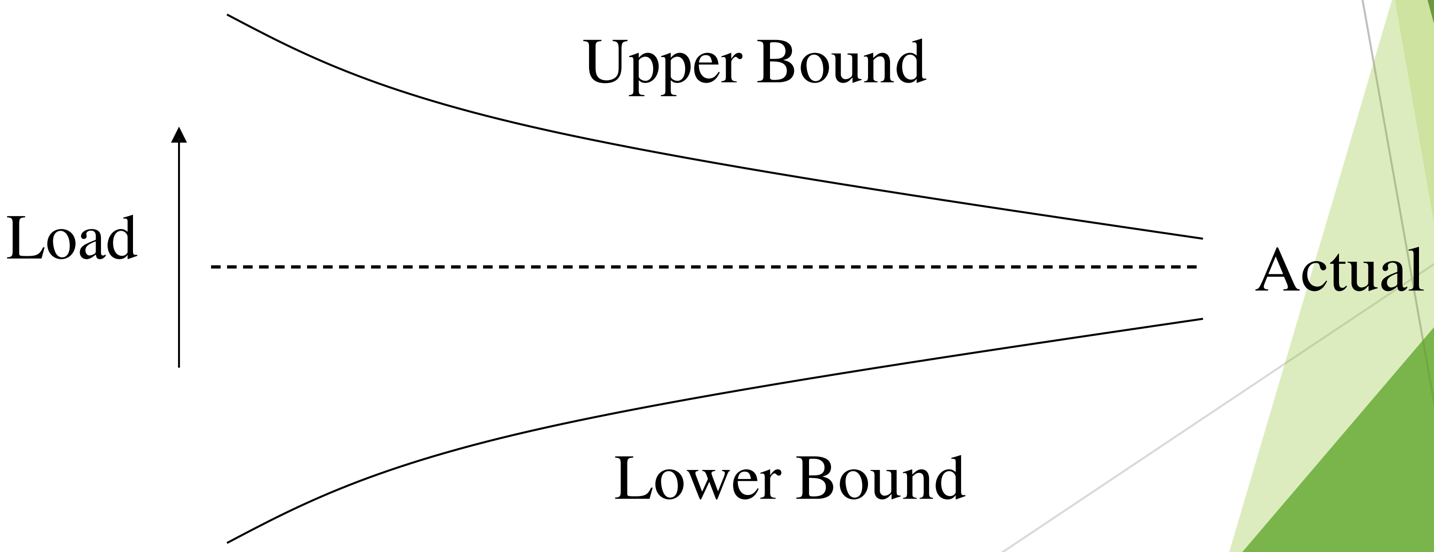
Upper /Lower Bound Theorem

1 ⟹ Actual Solution

2 & 3 ⟹ Safer than **'1'** , Lower bound



Steel Structures

Statical or Equilibrium Method of Analysis

The objective is to find out an equilibrium moment diagram in which $M \leq M_p$ such that a mechanism is formed.

PROCEDURE

1. Calculate degree of indeterminacy of the structure and decide the redundant removing which the structure is changed into stable and determinate structure (called primary structure).
2. Draw bending moment diagram for applied load for the determinate structure.
3. Draw bending moment diagram for the structure loaded by the redundants (H).

Steel Structures

4. Plastic hinges may assumed to be formed at the points where the redundant moment is maximum and applied load moment is zero or minimum. For the actual collapse mechanism, even a hinge is not formed at this location, the method will automatically correct itself.

5. Sketch composite moment diagram in such a way that a mechanism is formed. Sketch the mechanism.

6. Compute the value of ultimate load by solving equilibrium equation for each mechanism possible, as an upper bound.

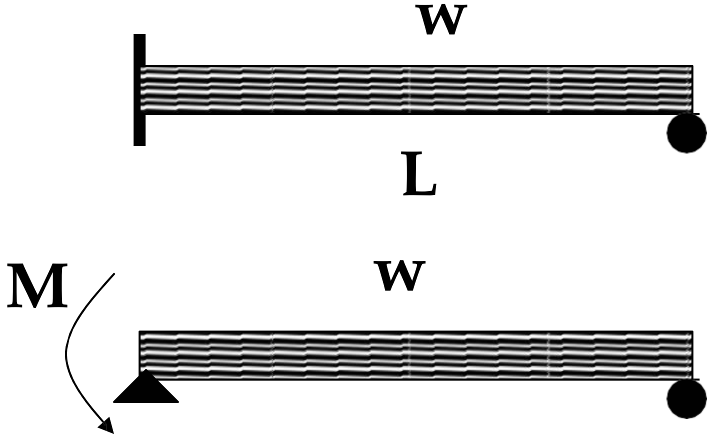
7. The collapse mechanism not involving any applied load can not be critical.

8. Collapse load is taken as the minimum of all the collapse loads calculated above.

9. Check to see that $M \leq M_p$ throughout the structure.

Steel Structures

Example: Solve the given propped cantilever by equilibrium method.



Solution:

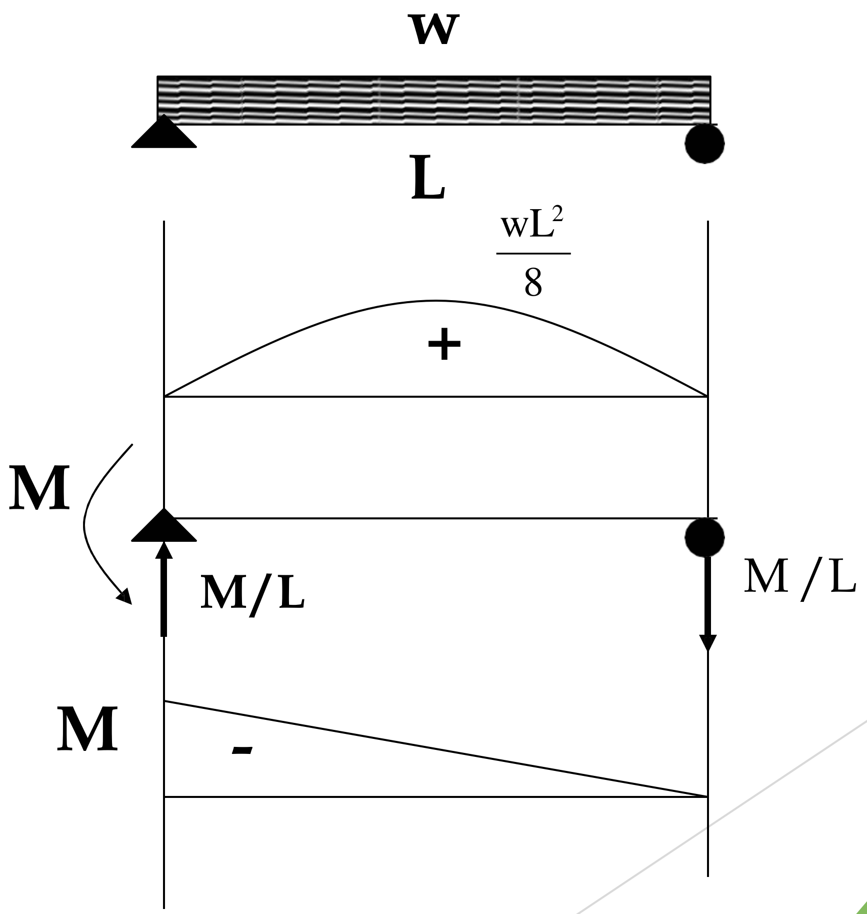
Maximum number of hinges = I + 1

Let end moment 'M' be the redundant

Steel Structures

Statical or Equilibrium Method of Analysis

Solution: (contd…)

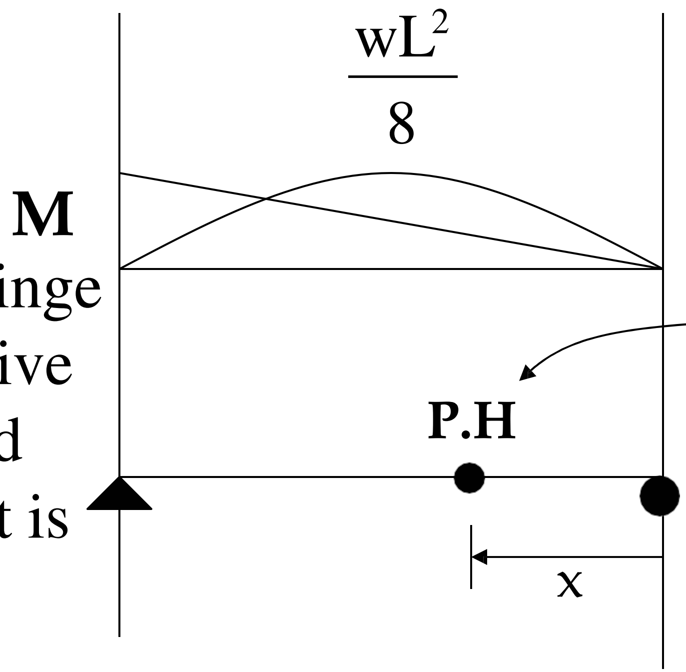


Steel Structures

Statical or Equilibrium Method of Analysis

Solution: (contd…)

Positive plastic hinge will form where +ive moment is max and redundant moment is min.



Negative plastic hinge will form where +ive moment is min and redundant moment is max.

Formed where difference between two moments is maximum.

Full Plastic moment capacity = M_P

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For the first hinge $M = M_P$ —— 1

For the second hinge $M_x - \dfrac{M_P}{L} x = M_P$ —— 2

$$\left(\frac{wL}{2}x - \frac{wx^2}{2}\right) = M_P\left(1 + \frac{x}{L}\right)$$

$$\left(\frac{wL}{2} - \frac{wx}{2}\right) = M_P\left(\frac{1}{x} + \frac{1}{L}\right) \text{ or } \left(\frac{wL}{2} - \frac{wx}{2}\right) = \frac{M_P}{L}\left(\frac{L+x}{x}\right) \quad \text{—— 3}$$

This hinge will form at the location where shear force is zero, that is:

$$\left(\frac{wL}{2} - wx - \frac{M_p}{L}\right) = 0 \qquad \frac{M_p}{L} = \frac{wL}{2} - wx \quad \text{—— 4}$$

Steel Structures

Statical or Equilibrium Method of Analysis

Solution: (contd…)

OR $$M^+ = M_x - \frac{M_P}{L}x$$

$$M^+ = \frac{wL}{2}x - \frac{wx^2}{2} - \frac{M_P}{L}x$$

$$\frac{\partial M^+}{\partial x} = 0 \Rightarrow \quad \frac{wL}{2} - wx - \frac{M_P}{L} = 0$$

$$\frac{M_P}{L} = \frac{wL}{2} - wx \qquad (4)$$

Steel Structures

- Statical or Equilibrium Method of Analysis
- Solution: (contd…)
- From Eqn. 3 and 4

$$\frac{wL}{2} - \frac{wx}{2} = \left(\frac{wL}{2} - wx\right)\left(\frac{L+x}{x}\right)$$

$$x^2 + 2xL - L^2 = 0$$

$$x = 0.414L$$

Put in Eq: 4

$$\frac{M_P}{L} = \frac{wL}{2} - w \times 0.414L$$

$$M_P = 0.086wL^2$$

Steel Structures

Statical or Equilibrium Method of Analysis

Solution: (contd…)

$$w = \frac{M_P}{0.086L^2}$$

$$w_c = \frac{11.63M_P}{L^2}$$

Relation between M_P & M_s

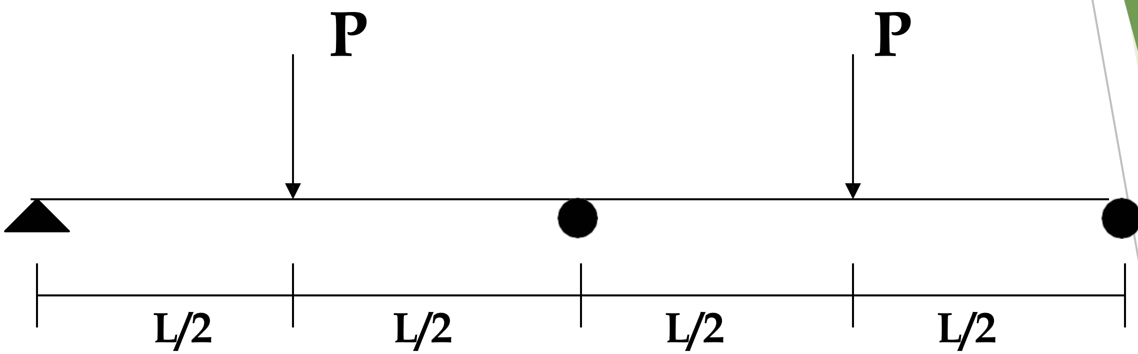
$$M_s = \frac{wL^2}{8} \Rightarrow wL^2 = 8M_s$$

So $$M_P = 8M_s(0.086) \Rightarrow M_P = 0.688M_s$$

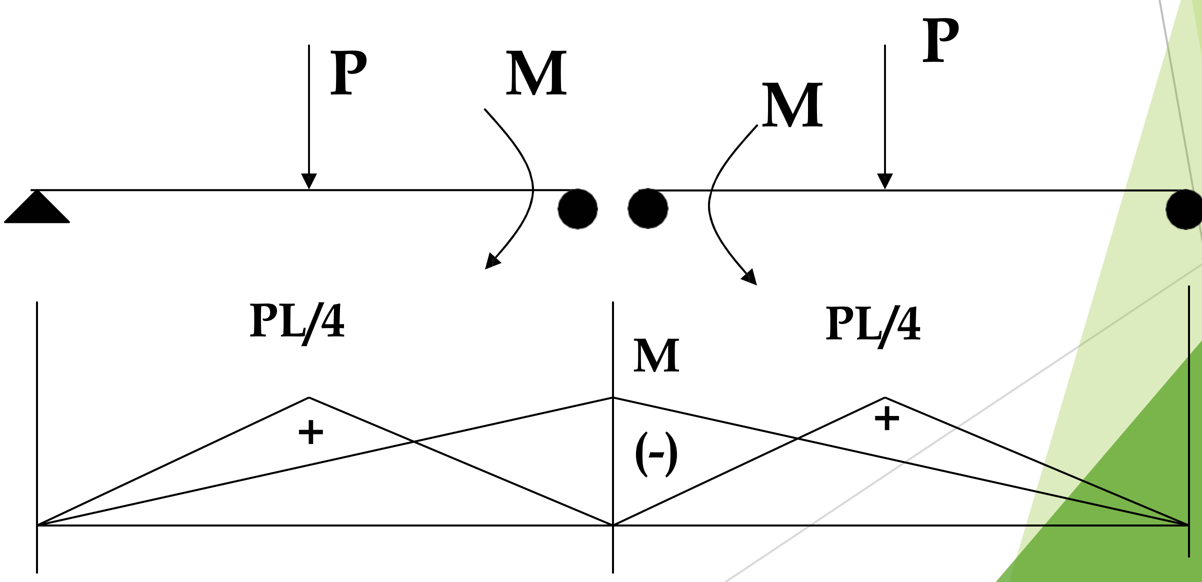
Steel Structures

Statical or Equilibrium Method of Analysis

Example:



Solution:



Steel Structures

Statical or Equilibrium Method of Analysis

Solution: (contd…)

First Plastic Hinge will form at the interior support because redundant moment is maximum here and applied moment is zero.

$$M = M_P$$

For hinge under the point load

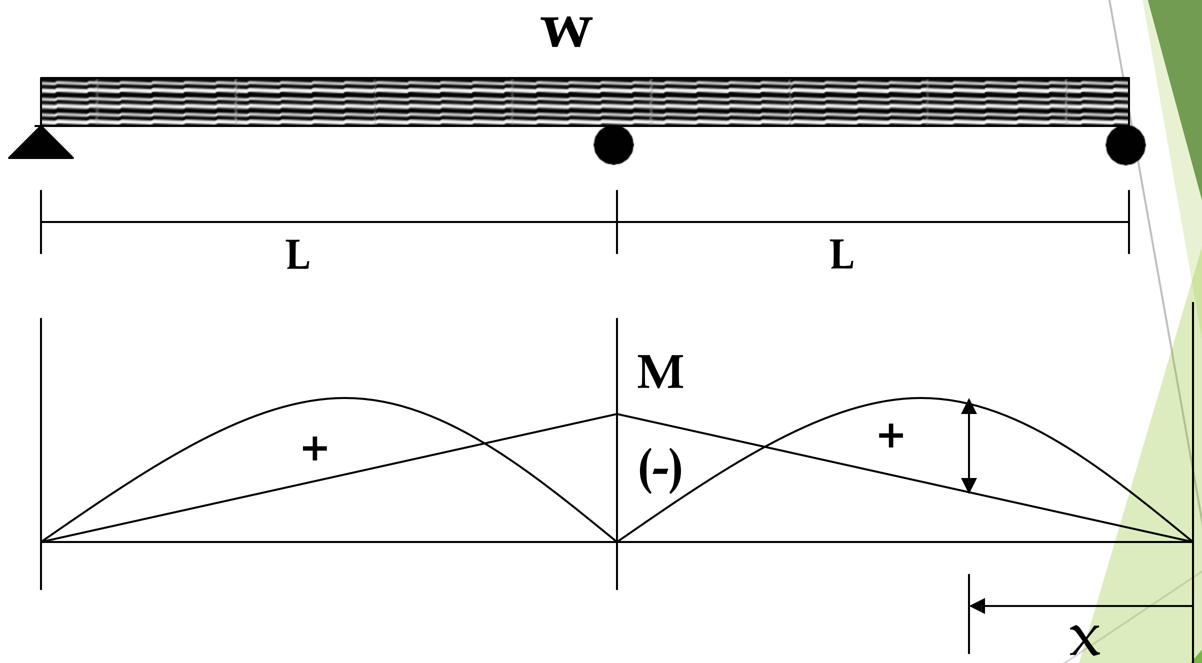
$$\frac{PL}{4} - \frac{M_P}{2} = M_P$$

$$\frac{PL}{4} = \frac{3}{2} M_P \Rightarrow \quad \boxed{P_c = \frac{6M_P}{L}}$$

Steel Structures

Statical or Equilibrium Method of Analysis

Example:



Equation is same as for propped cantilever. Result is also same.

$$M_P = 0.086wL^2$$

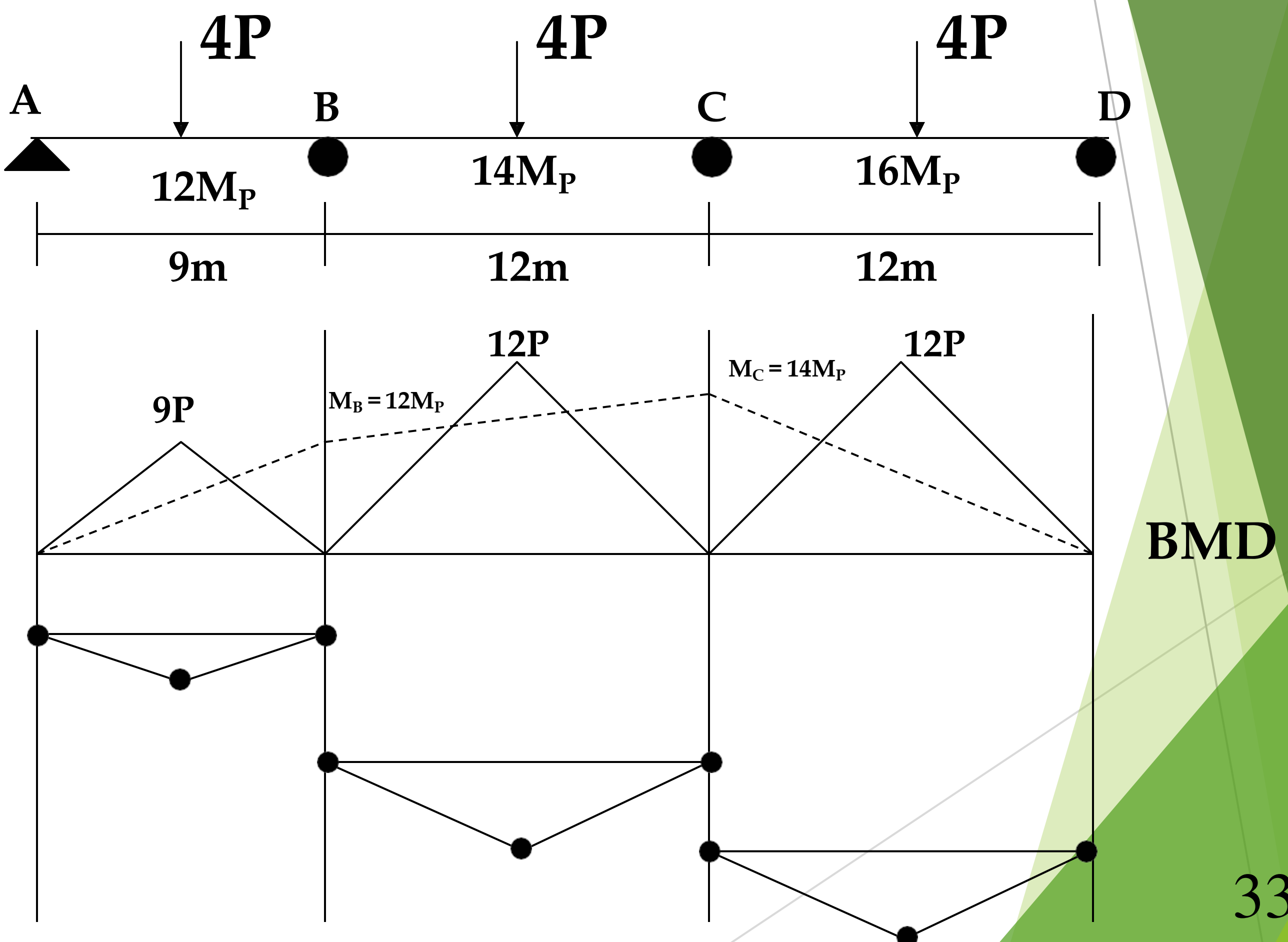
$$w_c = \frac{11.63M_P}{L}$$

Steel Structures

Statical or Equilibrium Method of Analysis

Example:

Find Collapse Load.



Number of possible mechanism = I+1

Mech-1

Mech-2

Mech-3

Steel Structures

Statical or Equilibrium Method of Analysis

Solution

Let we consider M_B and M_C as redundant.

Point B

$\mathbf{M_B = 12M_P}$, if two members of different strength are meeting plastic hinge will be formed in weaker section.

Point C

$$\mathbf{M_C = 14M_P}$$

Steel Structures

- Statical or Equilibrium Method of Analysis
- Solution: (contd…)
- Mechanism -1 **(span AB)**

$$9P_C - \frac{12M_P}{2} = 12M_P$$

$$9P_C = 18M_P$$

$$\boxed{P_C = 2M_P}$$

Mechanism -2 (span BC)

$$12P_C - \frac{12M_P + 14M_P}{2} = 14M_P$$

$$\boxed{P_C = 2.25M_P}$$

Steel Structures

- Statical or Equilibrium Method of Analysis
- Solution: (contd…)
- Mechanism -3 (span CD)

$$9P_C - \frac{14M_P}{2} = 16M_P$$

$$12P_C = 23M_P$$

$$\boxed{P_C = 1.917M_P}$$

The final answer is smallest out of these three.

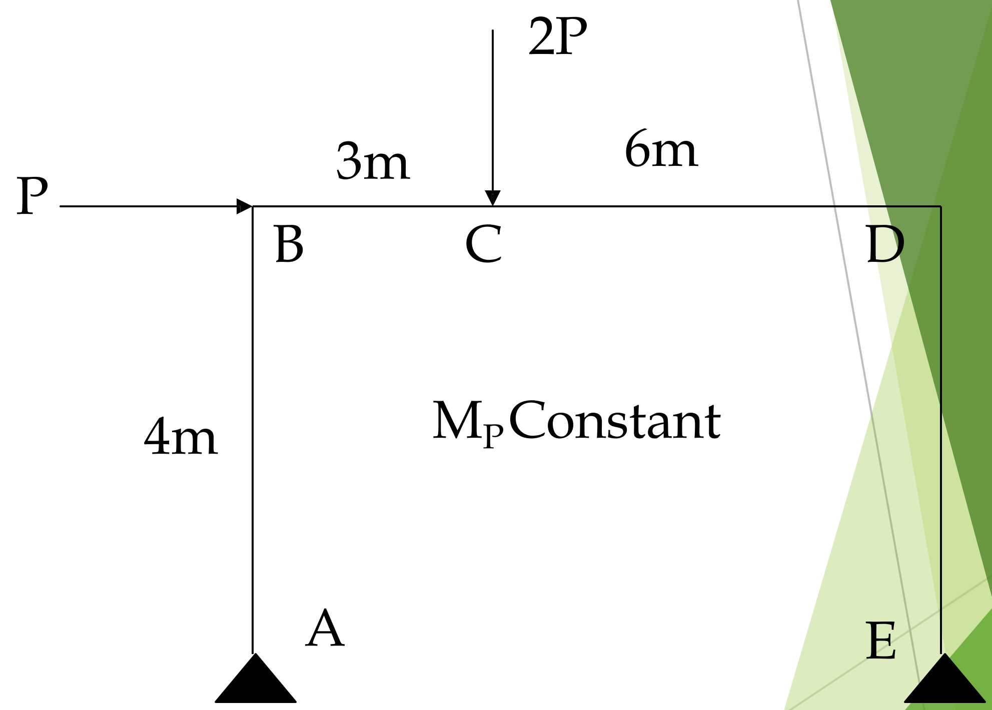
$$\boxed{P_C = 1.917M_P}$$

Steel Structures

Statical or Equilibrium Method of Analysis

Example:

Determine the collapse load.



Solution:

I = 1

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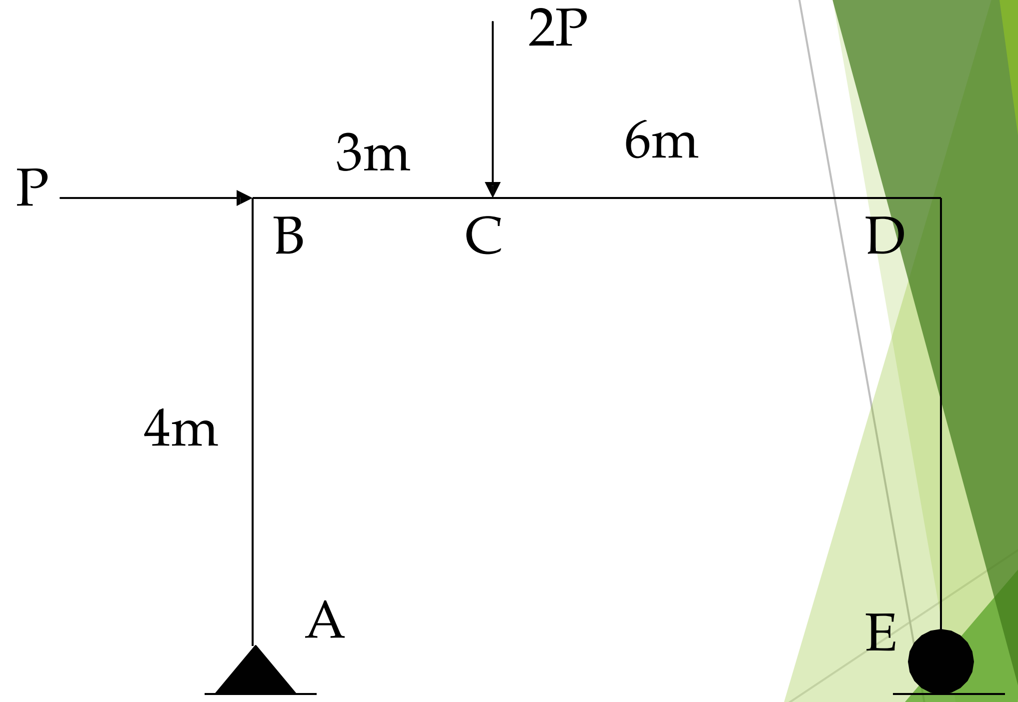
Statical or Equilibrium Method of Analysis

Solution:

Considering the horizontal thrust as redundant

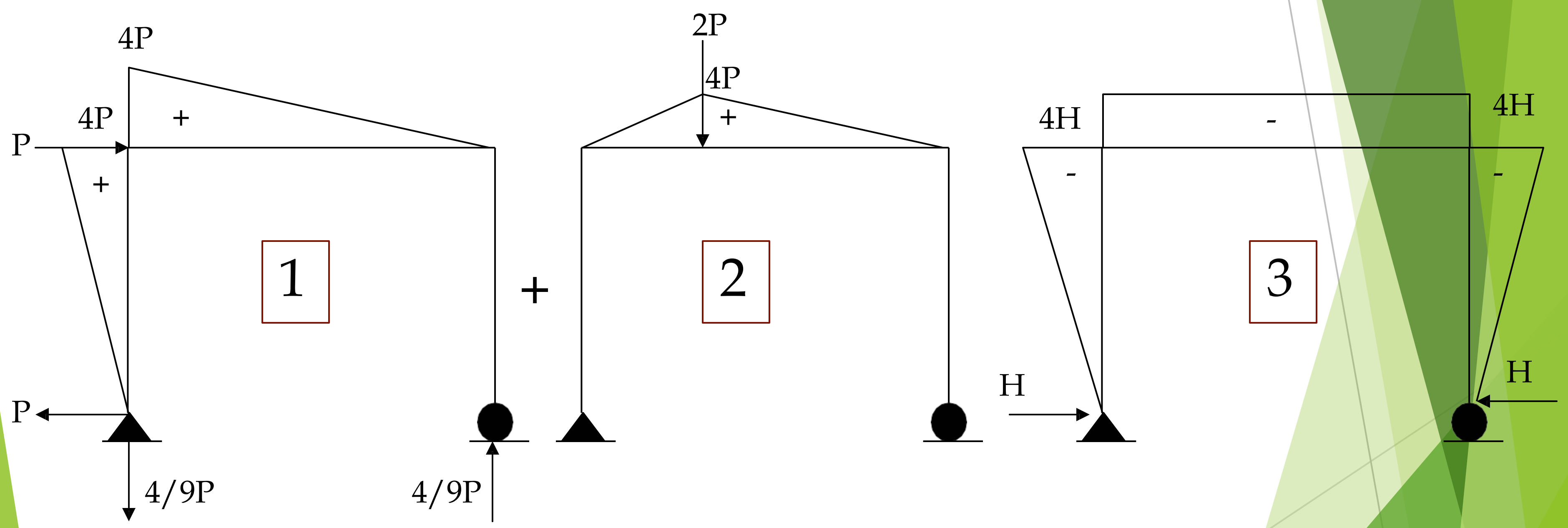
Primary Structure



Steel Structures

Statical or Equilibrium Method of Analysis

Solution: (contd…)



Steel Structures

Statical or Equilibrium Method of Analysis

Solution: (contd…)

Mechanism -1

Point D has maximum negative moment with no positive moment so first negative hinge will form there.

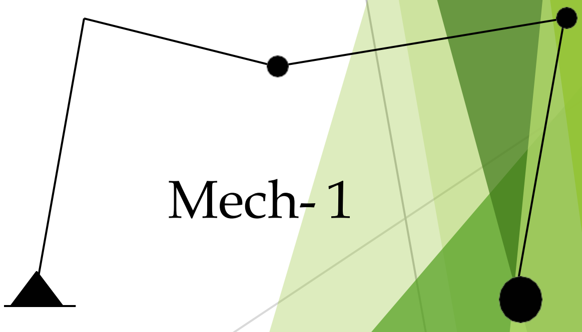
$$M_D = 4H = M_P$$

Positive hinge under load point C.

$$4P \times \frac{6}{9} + 4P - M_P = M_P$$

$$\left(\frac{8+12}{3}\right) P_C = 2M_P$$

$$P_C = 0.3M_P$$



Steel Structures

- Statical or Equilibrium Method of Analysis
- Solution: (contd…)
- Mechanism-2 (Positive Hinge at B)

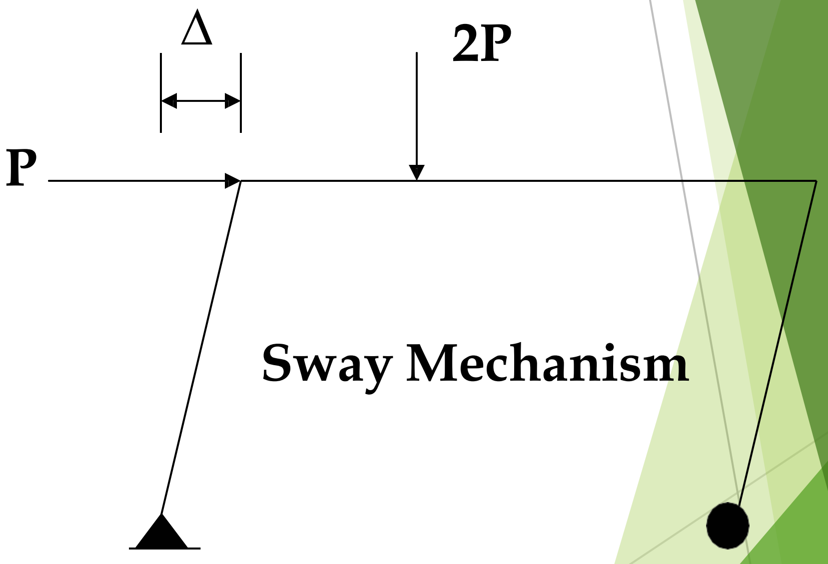
$$4P_C + 0 - 4H(M_P) = M_P$$

$$4P_C = 2M_P$$

$$P_C = 0.5M_P$$

Final Answer is the smallest

$$P_C = 0.3M_P$$

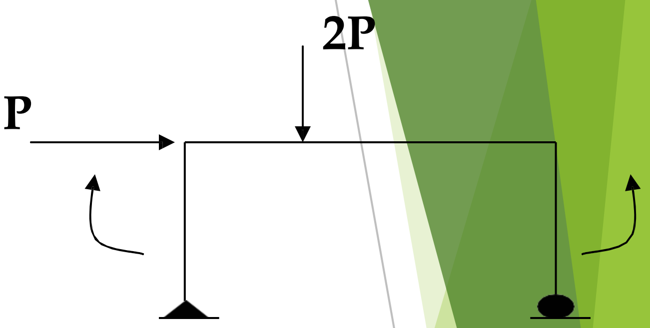


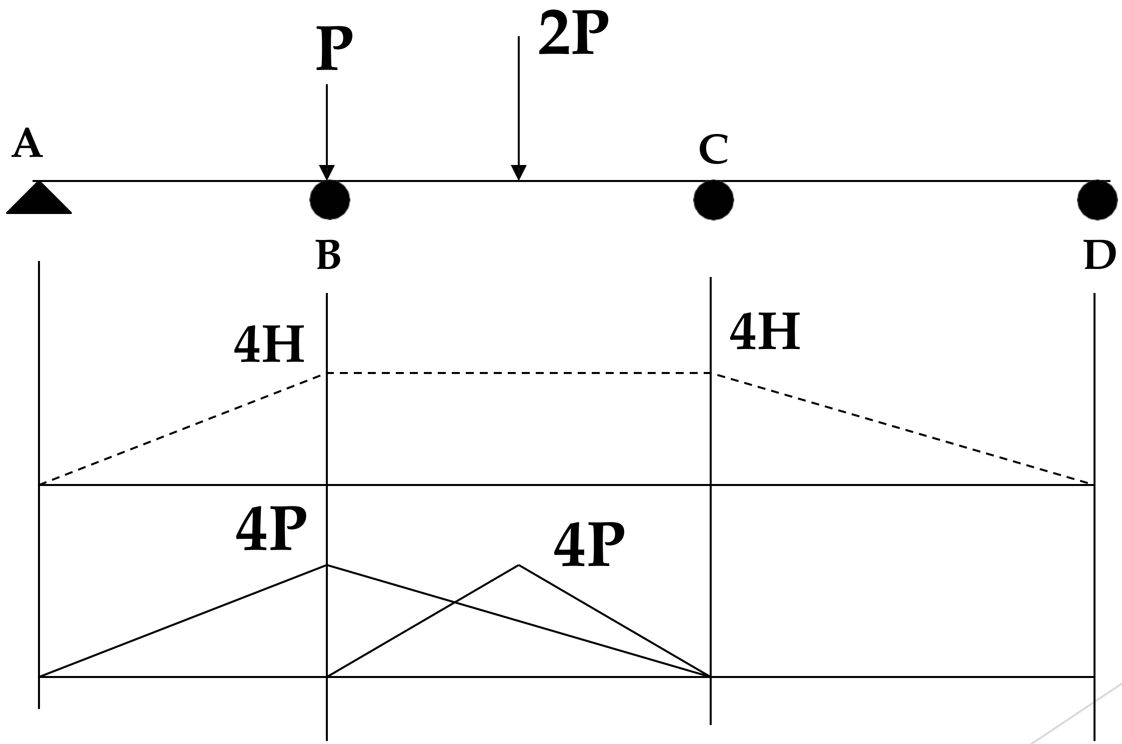
Steel Structures

Statical or Equilibrium Method of Analysis

Solution: (contd…)

Other Method

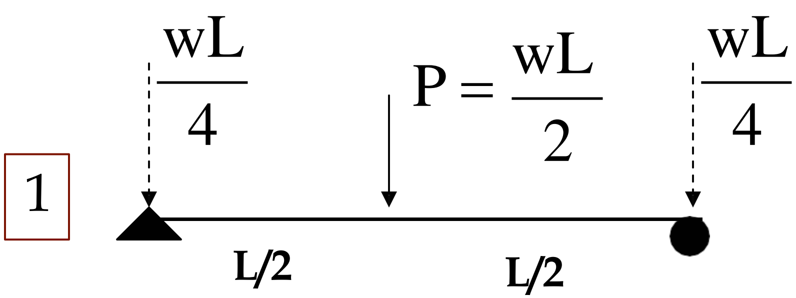


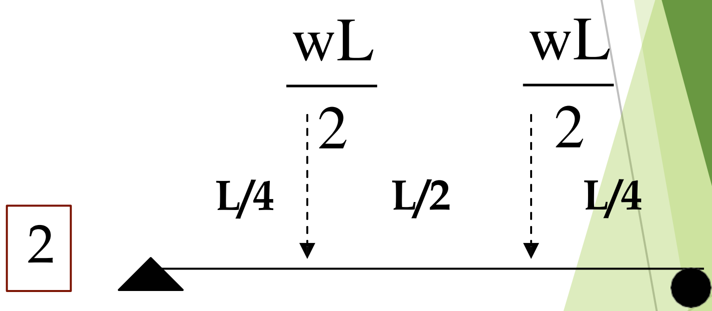


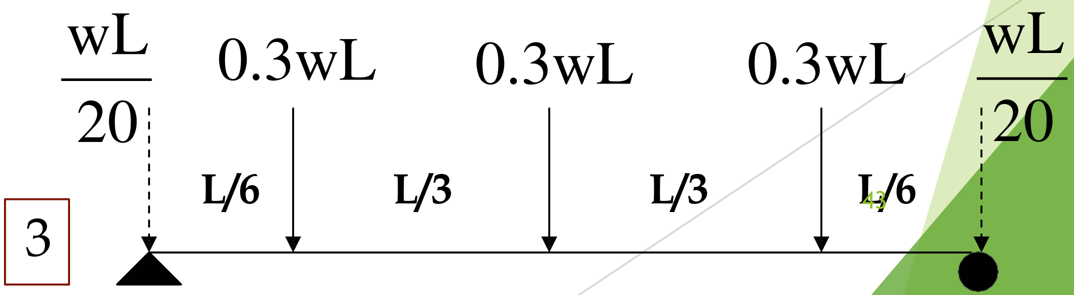
Steel Structures

Equivalent UDL Loading

1. Keep the same simply supported moment.
   Simply supported moment due to point load = $\frac{wL^2}{8}$
2. Distance between two equivalent point loads must be equal to double of distance of edge load from the near support.
3. Total load must be equal to wL



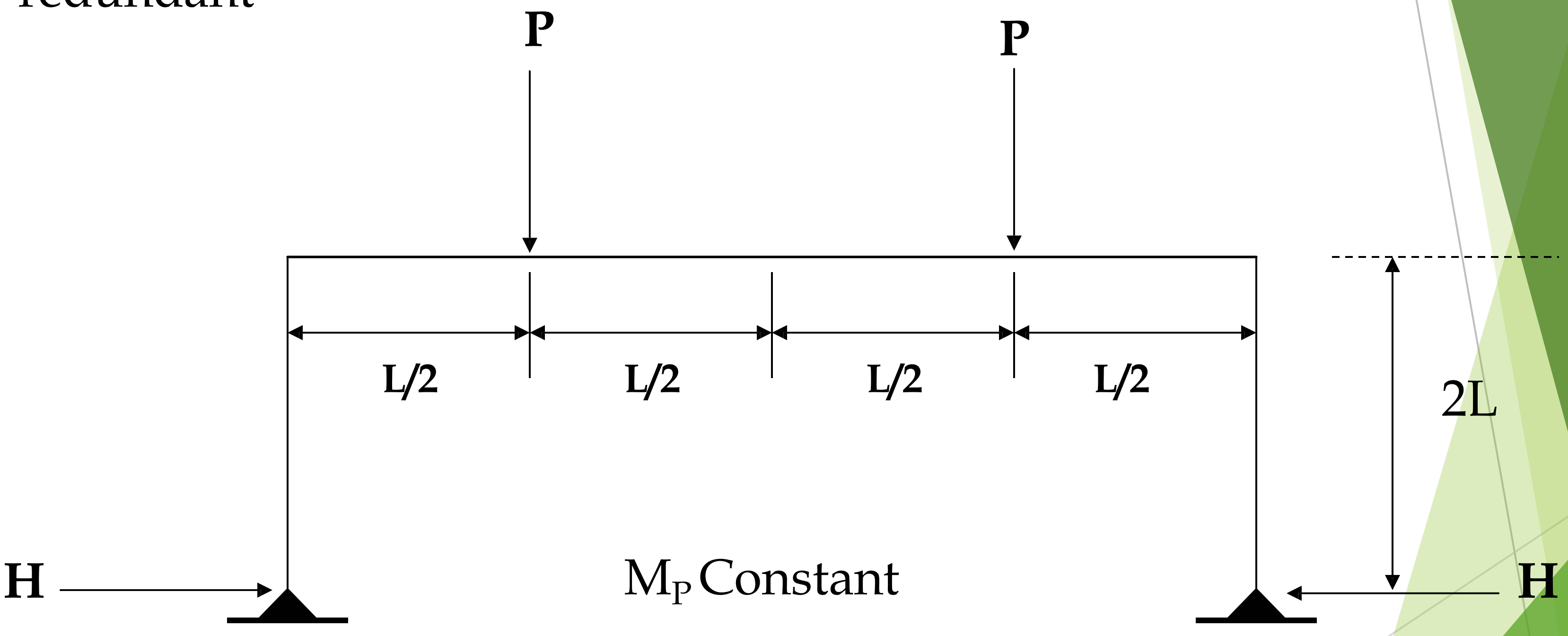




Steel Structures

Example:

Find the collapse load. Horizontal thrust H can be considered as the redundant



I = 1

Concluded