Steel Structures

M.Sc. Structural Engineering

SE-505

Plastic Analysis and Design of Structures

Lec. #3

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Steel Structures

Mechanism Method of Analysis

The objective is to find out a mechanism (independent or composite) such that $\boldsymbol{M \leq M_p}$ throughout the structural element (or at least at the selected points.

1. Determine the total degree of indeterminacy of the structure (I).
2. Determine the location of all possible hinges and thus the number of possible hinges (*H*). Hinges may form at concentrated point loads, connections, and points of zero shear in case of members subjected to UDL.
3. The number of independent mechanisms (*M*) is calculated as under:

$$M = H - I$$

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4. Independent mechanisms include beam mechanism, sway mechanism, gable mechanism and joint mechanism.

5. Sketch each mechanism one by one and give convenient virtual deformation (displacement or rotation) to any point, assuming all the hinges of this mechanism to be just formed.

6. Calculate the external and internal works done. External work done is equal to the load multiplied with its corresponding deformation. Internal work will be done only at the plastic hinges and calculated as the plastic moment multiplied with the rotation.

External WD (W_E) = Σ load x virtual displacement

Internal WD (W_I) = Σ Plastic moment at hinge x virtual hinge rotation

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6. Using the principle of virtual work, evaluate the collapse load for the mechanism under consideration.

$$W_E = W_I$$

7. Solve all the independent mechanisms and then form composite mechanisms by combining two or more independent and other composite mechanisms.

   The composite mechanisms must be formed in such a way that plastic hinges are eliminated reducing the internal work done and hence the collapse load. Lesser Internal work-done gives lesser critical load.

8. Find the smallest value of collapse load.

9. Check, $M \leq M_p$ at all sections, if possible.

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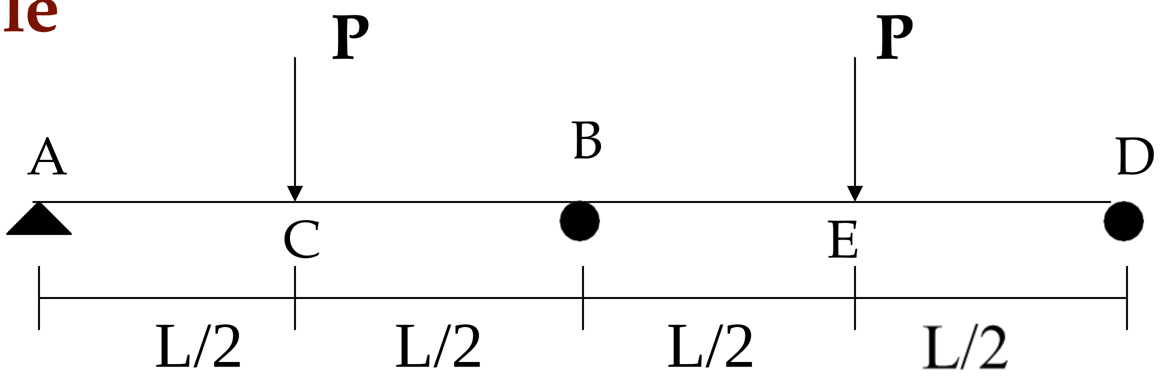
Mechanism Method of Analysis

Example



Determine the collapse Load.

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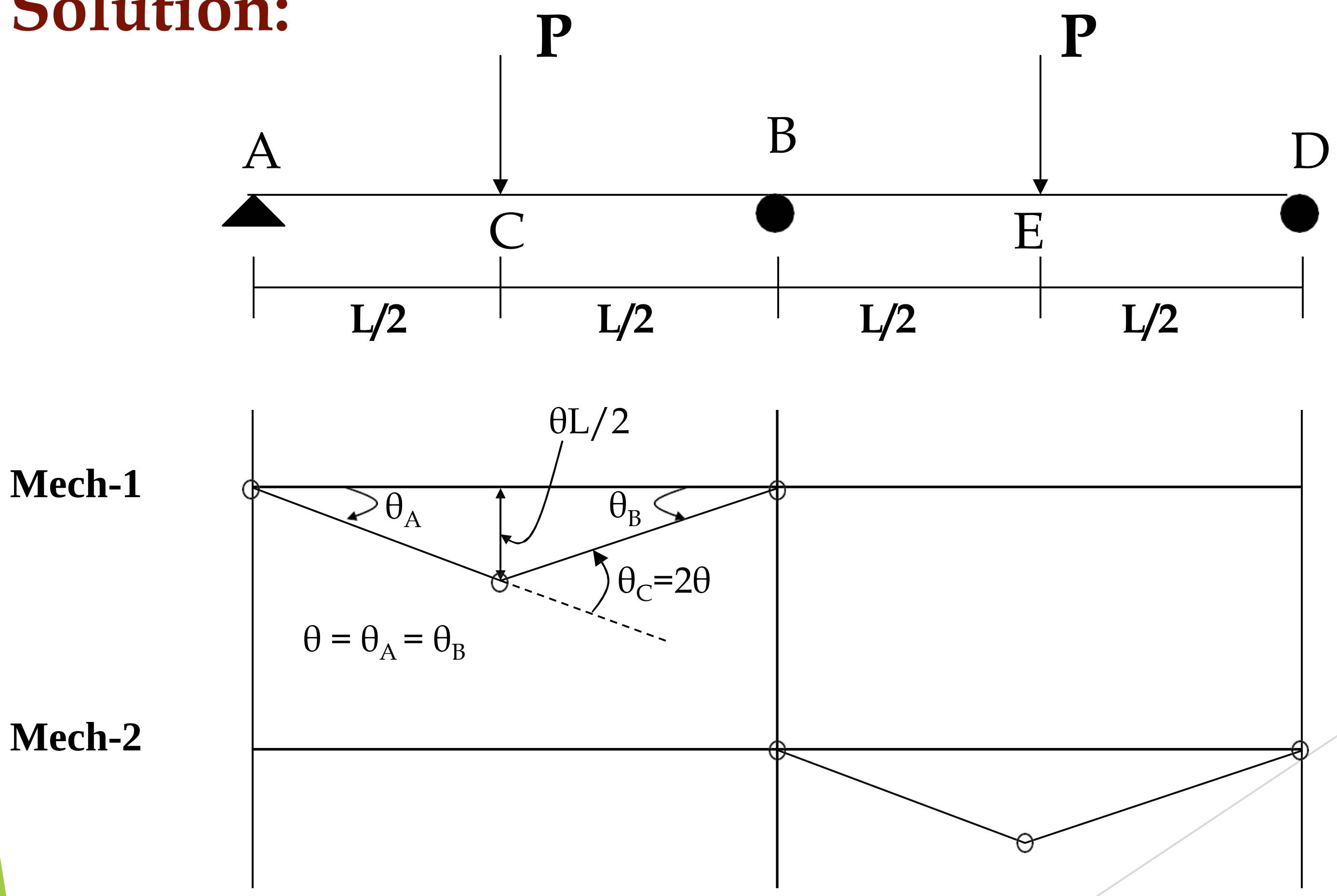
Mechanism Method of Analysis

Solution:



θ_A is the virtual displacement given by us other displacements are as as a result of θ_A

I = 1

No. of Possible Hinges, H = 3

No. of Independent Mechanisms, M = H – I = 2

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Mechanism Method of Analysis

Solution: (contd…)

$$W_E = P\theta \frac{L}{2}$$

$$W_I = 0 \times \theta + 2\theta \times M_P + \theta \times M_P$$

$$W_I = 3\theta \times M_P$$

Virtual Work Principle $W_I = W_E$

$$3M_P\theta = P\theta \frac{L}{2}$$

$$\boxed{P_C = \frac{6M_P}{L}}$$

Same for other mechanism - 2

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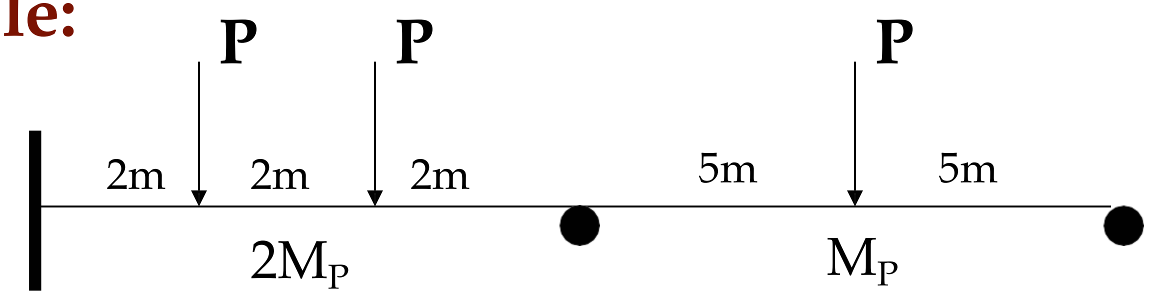
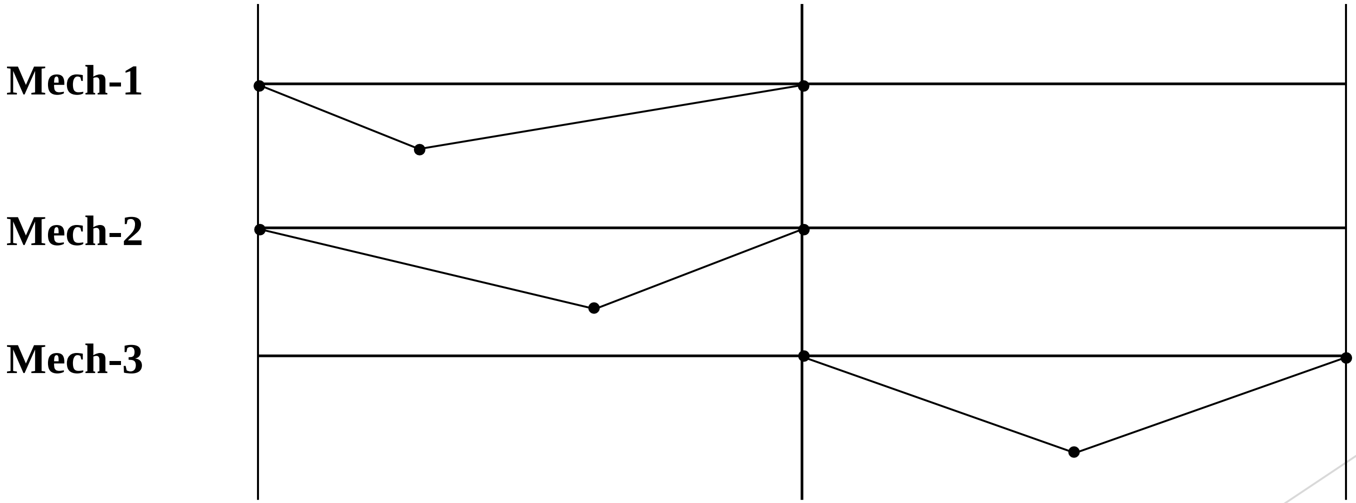
Mechanism Method of Analysis

Example:





I = 2

No. of Possible Hinges, H = 5

No. of Independent Mechanisms, M = H – I = 3

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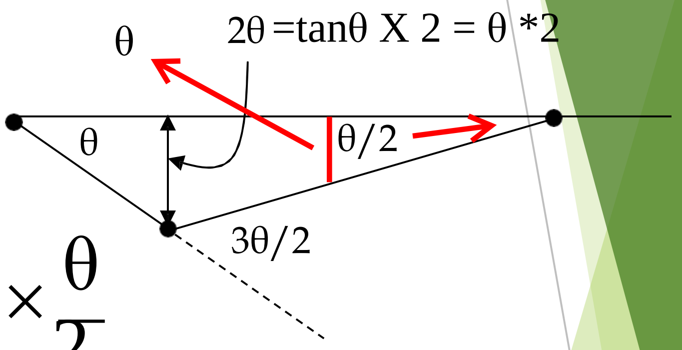
Mechanism Method of Analysis

Solution: (contd…)

Beam Mechanism - 1



$$W_E = P\times 2\theta + P\times\theta = 3P\theta$$

$$W_I = 2M_P\times\theta + 2M_P\times\frac{3\theta}{2} + M_P\times\frac{\theta}{2}$$

$$W_I = 5.5M_P\theta$$

$$W_I = W_E$$

$$5.5M_P\theta = 3P\theta$$

$$\boxed{P = 1.83M_P}$$

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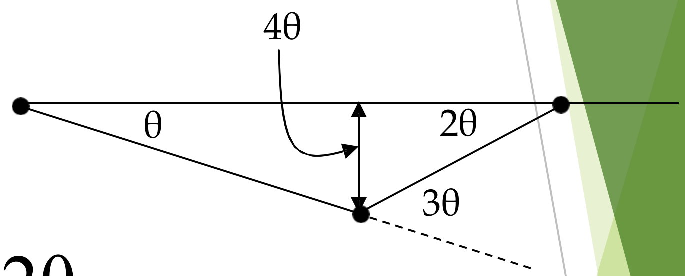
Mechanism Method of Analysis

Solution: **(contd…)**

Beam Mechanism - 2



$$W_E = P \times 2\theta + P \times 4\theta = 6P\theta$$

$$W_I = 2M_P \times \theta + 2M_P \times 3\theta + M_P \times 2\theta$$

$$W_I = 10M_P\theta$$

$$W_I = W_E$$

$$10M_P\theta = 6P\theta$$

$$\boxed{P = 1.67M_P}$$

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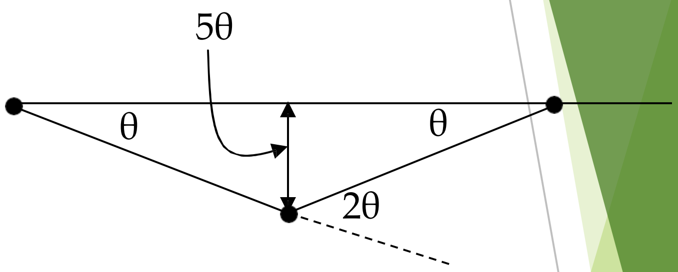
Mechanism Method of Analysis

Solution: (contd…)

Beam Mechanism - 3



$$W_E = P \times 5\theta = 5P\theta$$

$$W_I = M_P \times \theta + M_P \times 2\theta + 0$$

$$W_I = 3M_P\theta$$

$$W_I = W_E$$

$$3M_P\theta = 5P\theta$$

$$\boxed{P = 0.6M_P}$$

Final answer is the smallest

$$\boxed{P_C = 0.6M_P}$$

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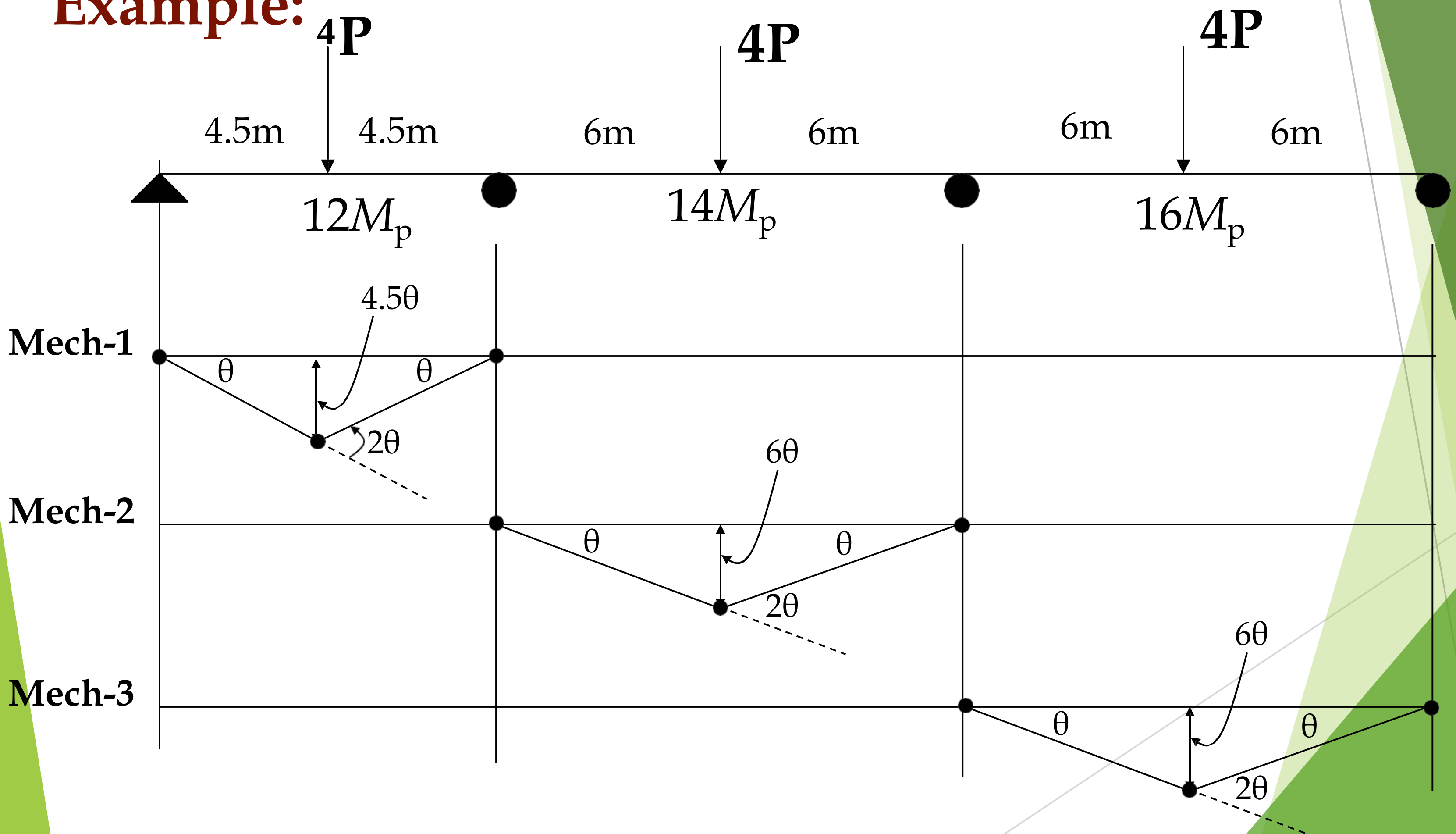
Mechanism Method of Analysis

Example:



I = 2

H = 5

M = 3

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Mechanism Method of Analysis

Solution: **(contd…)**

Beam Mechanism - 1

$$W_E = 4P \times 4.5\theta = 18P\theta$$

$$W_I = 0 + 12M_P \times 2\theta + 12M_P \times \theta$$

$$W_I = 36M_P\theta$$

$$W_I = W_E$$

$$36M_P\theta = 18P\theta$$

$$\boxed{P = 2M_P}$$

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Mechanism Method of Analysis

Solution: (contd…)

Beam Mechanism - 2

$$W_E = 4P \times 6\theta = 24P\theta$$

$$W_I = 12M_P\theta + 14M_P \times 2\theta + 14M_P\theta$$

$$W_I = 54M_P\theta$$

$$W_I = W_E$$

$$54M_P\theta = 24P\theta$$

$$\boxed{P = 2.25M_P}$$

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Mechanism Method of Analysis

Solution: **(contd…)**

Beam Mechanism - 3

$$W_E = 24P\theta$$

$$W_I = 14M_P\theta + 16M_P \times 2\theta + 0$$

$$W_I = 46M_P\theta$$

$$W_I = W_E$$

$$46M_P\theta = 24P\theta$$

$$\boxed{P = 1.92M_P}$$

$$\boxed{P_C = 1.92M_P}$$

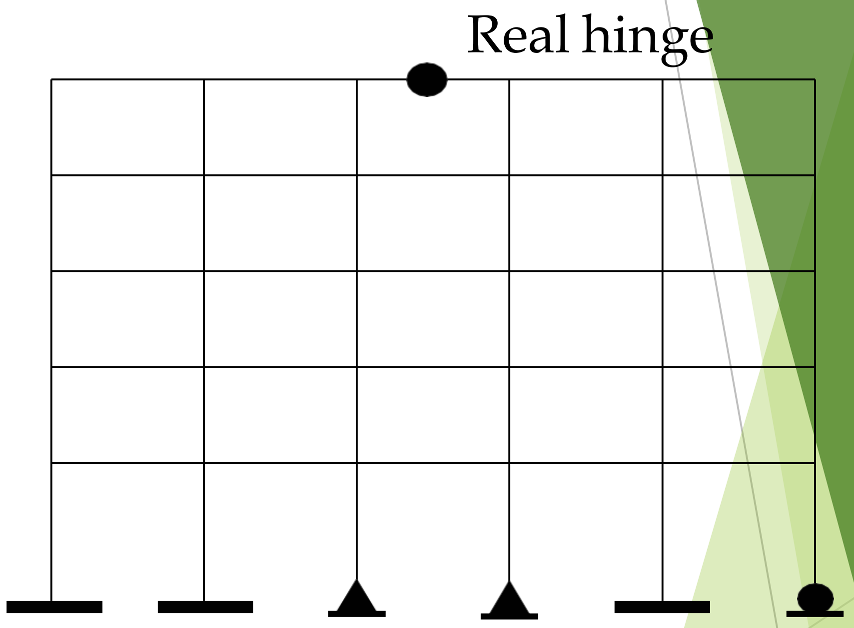
Steel Structures

Total indeterminacy (I) for Rigid Frames

I for rigid frame = no. of closed loops x 3 – Releases at the bottom from fixed end – any real hinge in the super structure

$$I = (5 \times 5) \times 3 - 4 - 1$$

$$I = 70$$



Releases at the bottom from fixed end means value of **1 for hinge and 2 for roller** support. Here 2 hinges and 1 roller so total value is **4.**

Steel Structures

Elimination of Hinges in Mechanism Method

Combined mechanisms are obtained in such a way that number of plastic hinges are reduced.

There are two different ways in which hinges can be eliminated:

1. If all members of the joint rotate through same angle in different direction in mechanisms to be added. **Direction of members in 1 mechanism is reverse of other.**

2. When a member rotates by equal amount but in opposite direction in the mechanisms to be combined.

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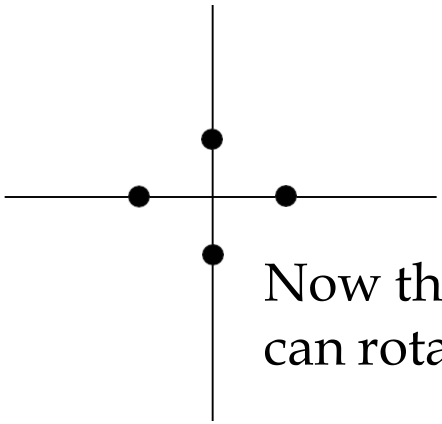
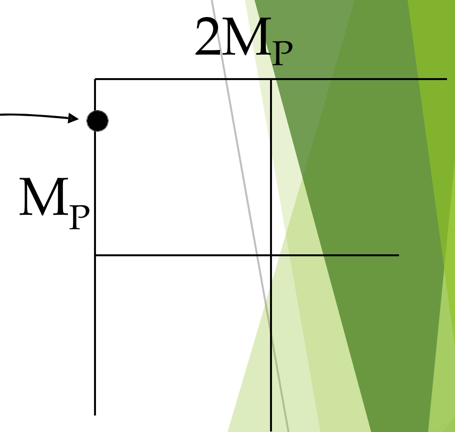
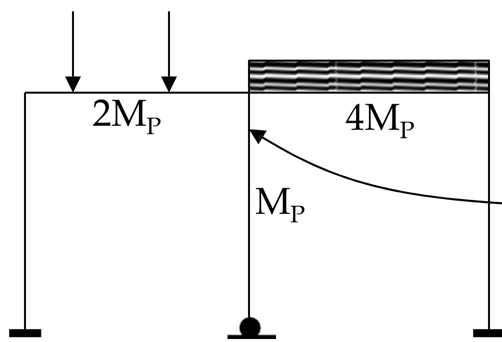
Joint Mechanism: It can only form if more than 2 members are meeting at a joint. Joint mechanism shows the rotation of a joint w.r.t the structure.



Now this joint can rotate

Hinge will be in column as its M_P is smaller





Hinge may not form in this column although it is the weakest, as it can't take moment

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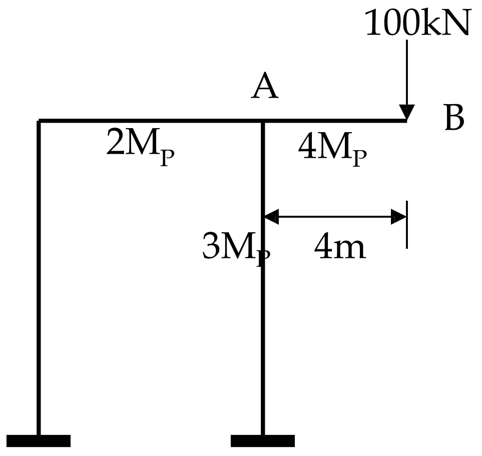
Joint Mechanism **(contd…)**

Location of hinge depends on loading and end conditions of other members.

Let, $M_P = 100$ kN-m

$M_A = 100 \times 4 = 400$ kN-m $= 4M_P$

Hinge will form in member AB although its strength is greater than other members.



Joint mechanism determines location of hinges at joint of more than two members.

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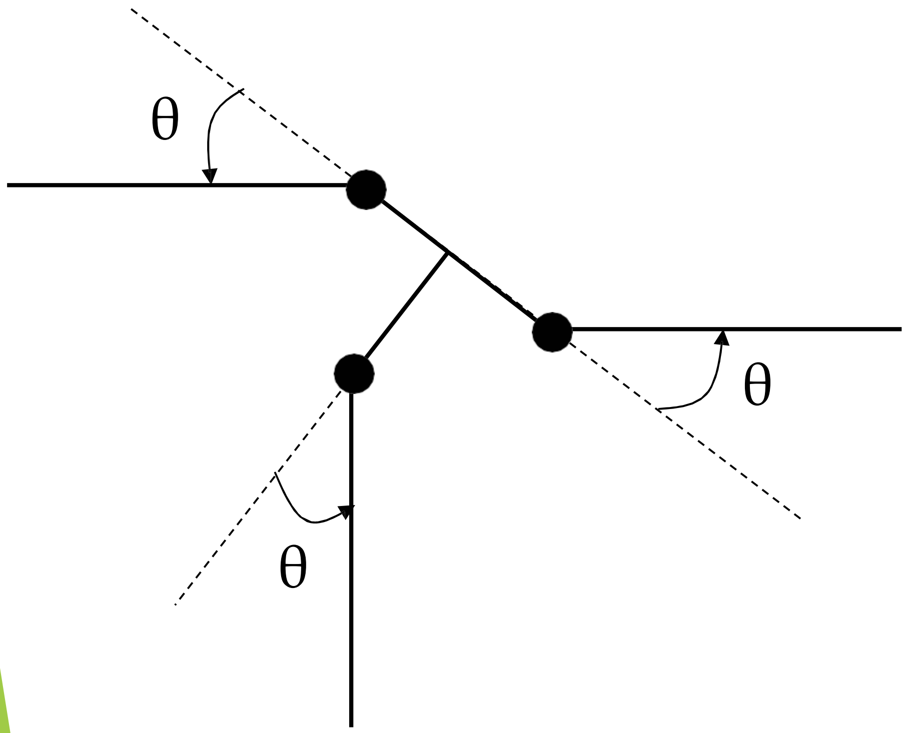
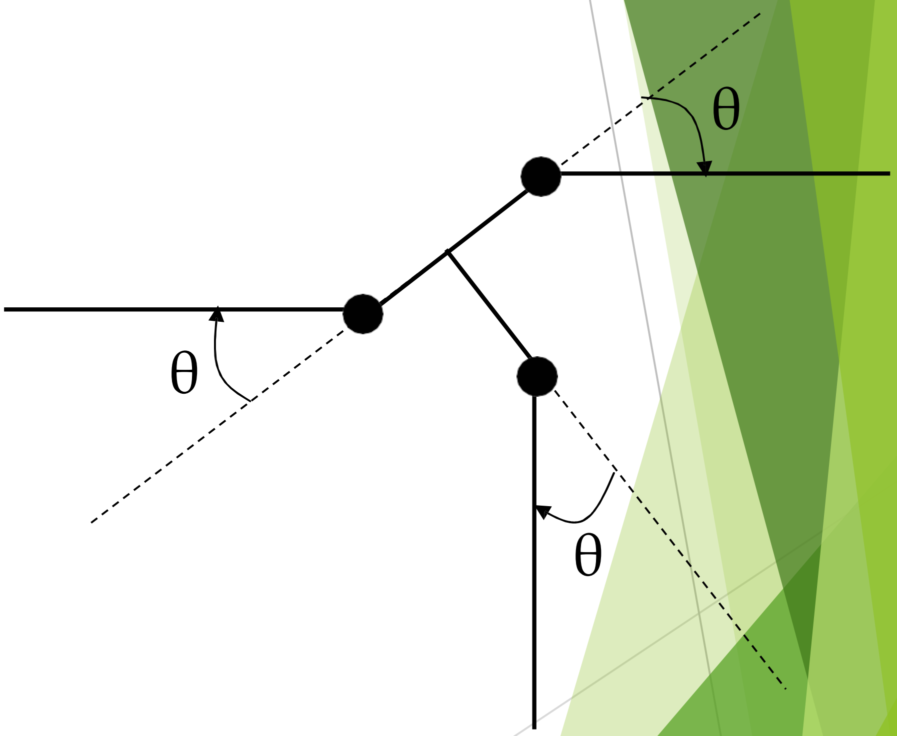
Joint Mechanism (contd…)



CCW rotation of joint



CW rotation of joint

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Mechanism Method of Analysis

Example: Calculate the collapse load

Solution:

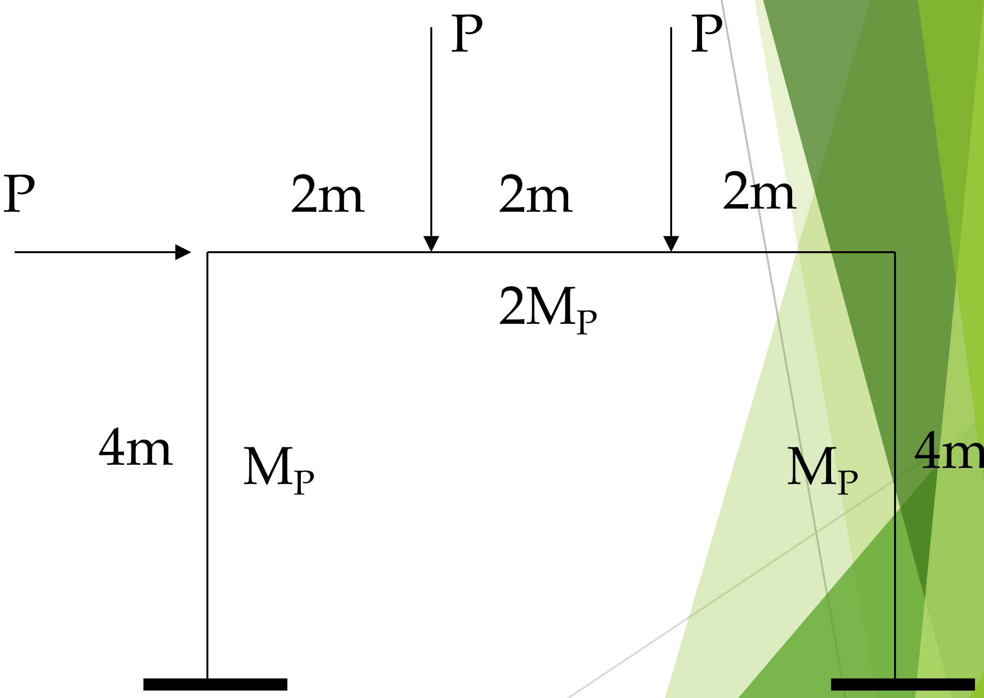
$$I = 3$$

$$H = 6$$

$$M = 6 - 3 = 3$$

2 Beam mechanism and
1 sway mechanism

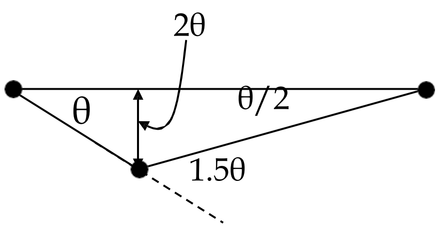
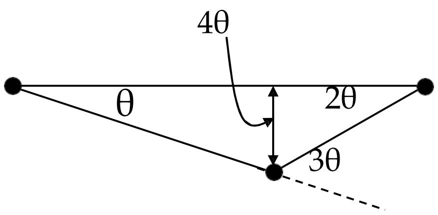
Joint mechanism is not required as no joint is having more than 2 members.



Steel Structures

Mechanism Method of Analysis

Solution: **(contd…)**

Mech #	Type of Mech	Shape of Mech	W_E	W_I	P_C
1	Beam		$P\times 2\theta+$ $P\times\theta$ $=3P\theta$	$M_P\times\theta+$ $2M_P\times 1.5\theta$ $+0.5M_P\theta$ $=4.5M_P\theta$	$1.5M_P$
2	Beam		$P\times 4\theta+$ $P\times 2\theta$ $=6P\theta$	$M_P\times\theta+$ $2M_P\times 3\theta$ $+M_P\times 2\theta$ $=9M_P\theta$	$1.5M_P$

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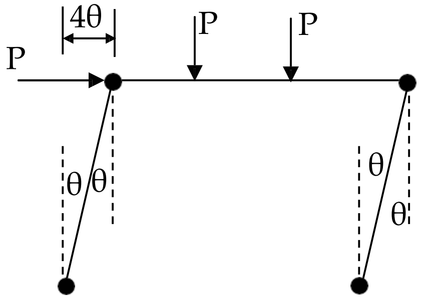
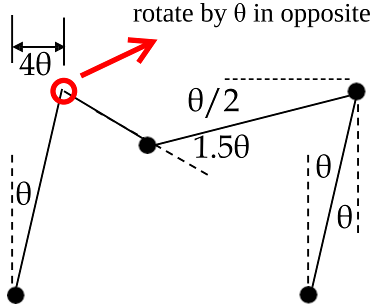
Mechanism Method of Analysis

Solution: **(contd…)**

Sum of individual cases

Mech #	Type of Mech	Shape of Mech	W_E	W_I	P_C
3	Sway		$P \times 4\theta$	$4M_P \times \theta$ **4 times MP x θ**	$1.0M_P$
4	Mech 1 + Mech 3	This joint is eliminated because vertical member rotate by θ in opposite direction in mech 1 & 3. 	$3P\theta +$ $4P\theta =$ $7P\theta$ Sum of individual cases	$4.5M_P\theta +$ $4M_P \times \theta$ - $M_P\theta - M_P\theta$ $= 6.5M_P\theta$	$0.93M_P$

Sum of individual cases minus WI of eliminated

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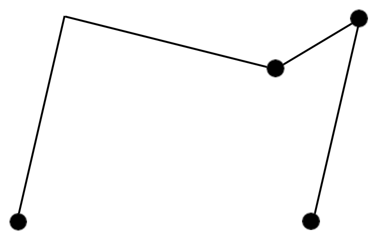
Mechanism Method of Analysis

Solution: **(contd…)**

Mech #	**Type of Mech**	**Shape of Mech**	W_E	W_I	P_C
5	Mech 2 + Mech 3		$6P\theta + 4P\theta = 10P\theta$	$13M_P\theta - M_P\theta - M_P\theta = 11M_P\theta$	$1.10M_P$

Right hinge at the joint can't be eliminated because column and beam are rotating in opposite directions.

$$\boxed{P_C = 0.93M_P}$$

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Mechanism Method of Analysis

Example: Calculate the collapse load

Solution:

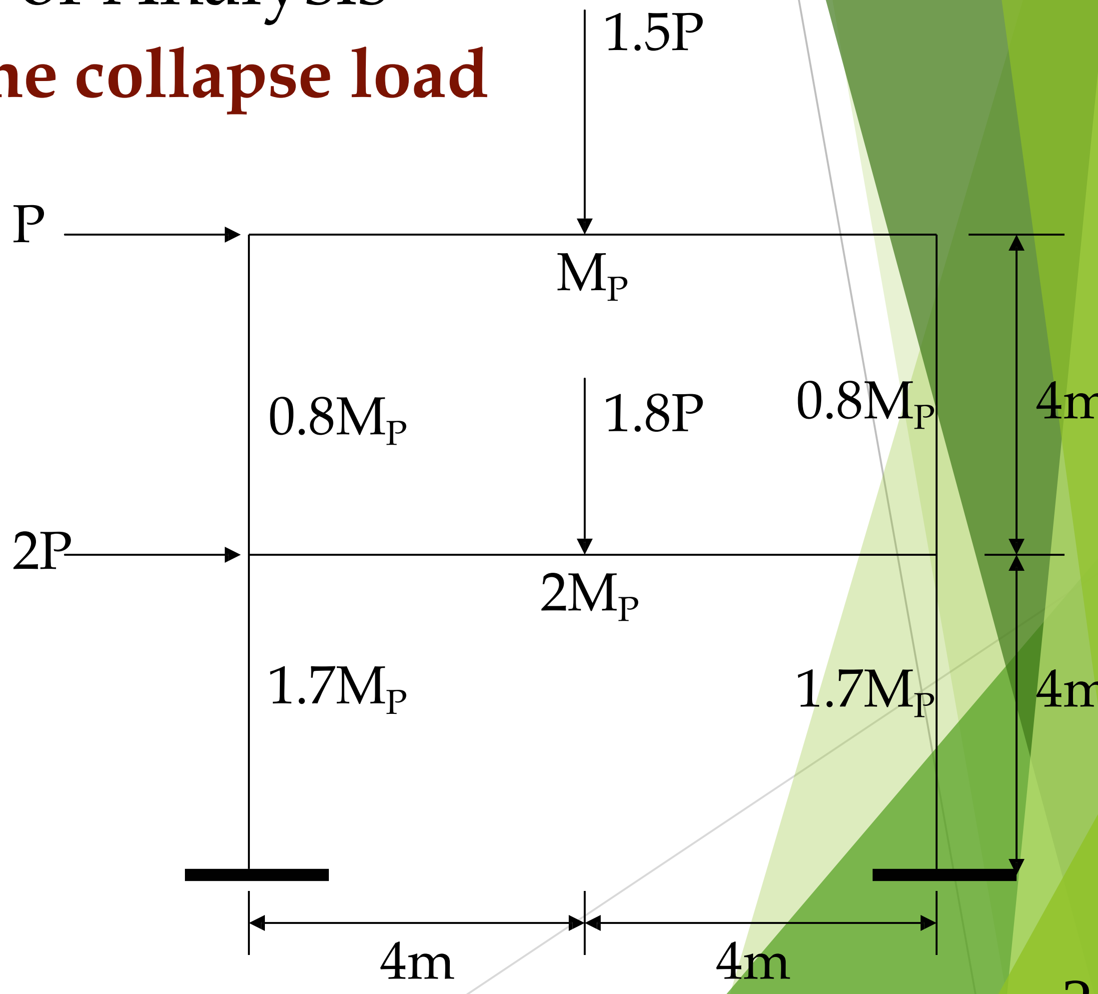
$I = 3m+r-3j=3*8+6-3*8 = 6$

$H = 12$

$M = 12 - 6 = 6$

2 Beam mechanism and
2 sway mechanism
2 Joint mechanism

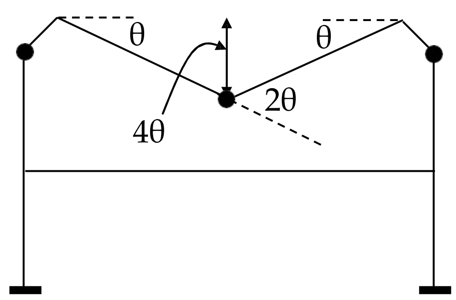
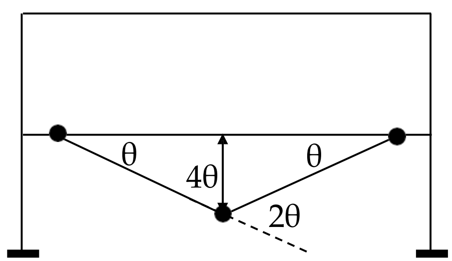
Consider 1 hinge at the junction of 2-members and equal to no. of members at other joints.



Steel Structures

Mechanism Method of Analysis

Solution: **(contd…)**

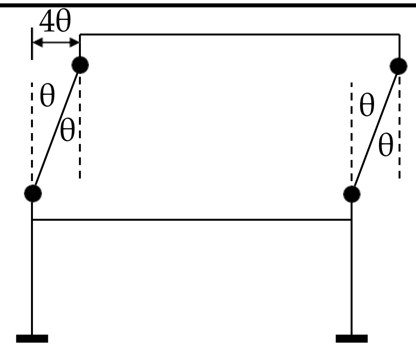
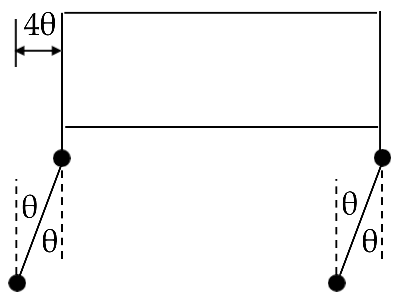
Mech #	Type of Mech	Shape of Mech	W_E	W_I	P_C
1	Beam		$1.5P\times 4\theta$ $= 6P\theta$	$0.8M_P\theta +$ $M_P\times 2\theta$ $+0.8M_P\theta$ $=3.6M_P\theta$	$0.6M_P$
2	Beam		$1.8P\times 4\theta$ $=7.2P\theta$	$2M_P\times\theta +$ $2M_P\times 2\theta$ $+2M_P\times\theta$ $=8M_P\theta$	$1.11M_P$

$2M_P$ is used because rotation is only in beam.

Steel Structures

Mechanism Method of Analysis

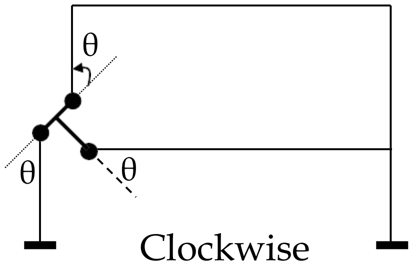
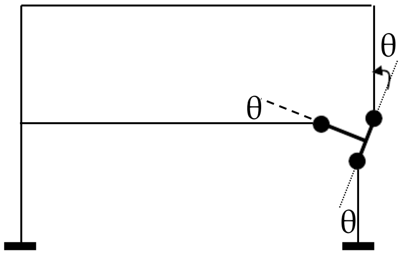
Solution: **(contd…)**

Mech #	Type of Mech	Shape of Mech	W_E	W_I	P_C
3	Sway of top story		$P \times 4\theta$	$0.8M_P \times 4\theta$	$0.8M_P$
4	Sway of bottom story		$P \times 4\theta +$ $2P \times 4\theta$ $= 12P\theta$	$1.7M_P \times 4\theta$ $= 6.8M_P\theta$	$0.567M_P$

Steel Structures

Mechanism Method of Analysis

Solution: (contd…)

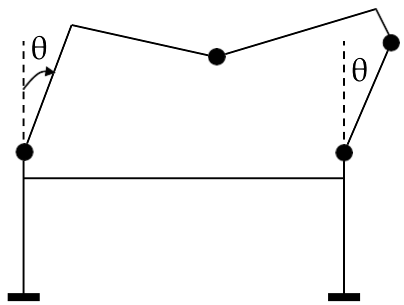
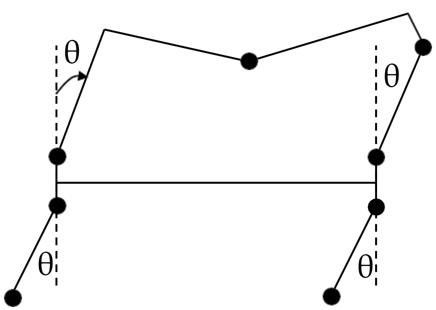
Mech #	Type of Mech	Shape of Mech	W_E	W_I	P_C
5	Left Joint Mech		0	$0.8M_P\theta +$ $2M_P\theta +$ $1.7M_P\theta$ $= 4.5M_P\theta$	∞
6	Right Joint Mech		0	$4.5M_P\theta$	∞

These joint mechanisms are fictitious & don’t occur independently, will occur in some combination

Steel Structures

Mechanism Method of Analysis

Solution: **(contd…)**

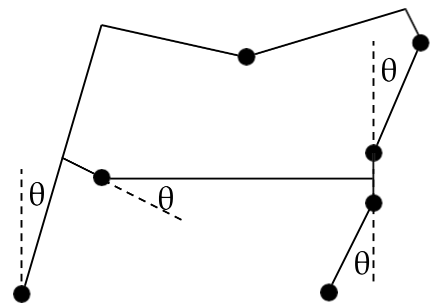
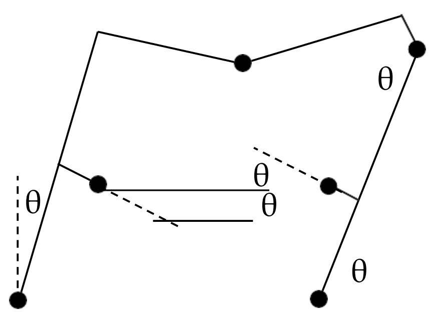
Mech #	Type of Mech	Shape of Mech	W_E	W_I	P_C
7	1 + 3		$6P\theta + 4P\theta$ $= 10P\theta$	$3.6M_P\theta +$ $3.2M_P\theta -$ $0.8M_P\theta -$ $0.8M_P\theta$ $= 5.2M_P\theta$	$0.52M_P$
8	7+4		$10P\theta + 12P\theta$ $= 22P\theta$	$5.2M_P\theta +$ $6.8M_P\theta =$ $12M_P\theta$	$0.545M_P$

We need not to combine mech – 1 & 4 as no hinge will be eliminated.

Steel Structures

Mechanism Method of Analysis

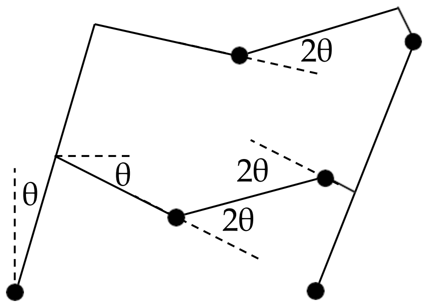
Solution: **(contd…)**

Mech #	Type of Mech	Shape of Mech	W_E	W_I	P_C
9	8 + 5		$22P\theta + 0$ $= 22P\theta$	$12M_P\theta +$ $4.5M_P\theta -$ $2\times0.8M_P\theta$ - $2\times1.7M_P\theta$ $=11.5M_P\theta$	$0.523M_P$
10	9+6		$22P\theta$	$11.5M_P\theta +$ $4.5M_P\theta -$ $2\times0.8M_P\theta$ - $2\times1.7M_P\theta$ $=11M_P\theta$	$0.50M_P$

Steel Structures

Mechanism Method of Analysis

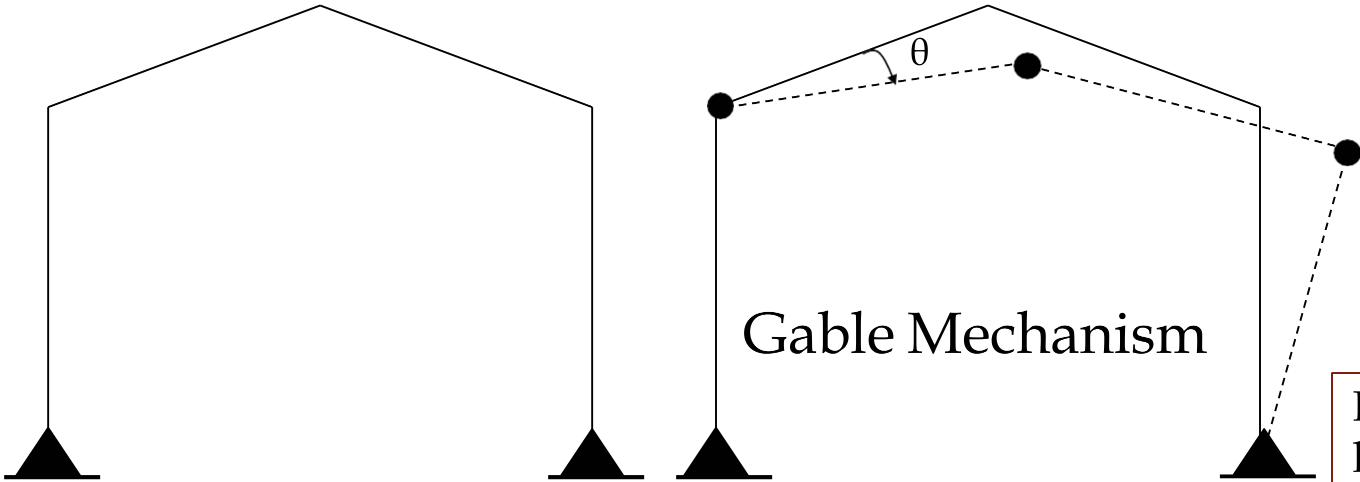
Solution: **(contd…)**

Mech #	Type of Mech	Shape of Mech	W_E	W_I	P_C
11	10 + 2		$7.2P\theta + 22P\theta = 29.2P\theta$	$8M_P\theta + 11M_P\theta - 2\times 2M_P\theta = 15M_P\theta$	$0.514M_P$

$$\boxed{P_C = 0.50M_P}$$

Steel Structures

Gable Mechanism



- One hinge forms at crown and two at the beam joints.
- Somewhat similar to sway mechanism.
- wind ward side column does not move.
- The hinge formed at the crown sinks the frame.
- Gable mechanism is separate/independent mechanism.

If an angle θ is known we cannot determine other angles by some simple method. For this we need to understand Instantaneous Center of Rotation

Influence of Axial Force On Plastic Moment Capacity

- In columns axial force is present simultaneous with moments.
- In beams of frames, smaller axial force in addition to moment may be present.
- Stresses are consumed by axial force hence plastic moment capacity reduces.

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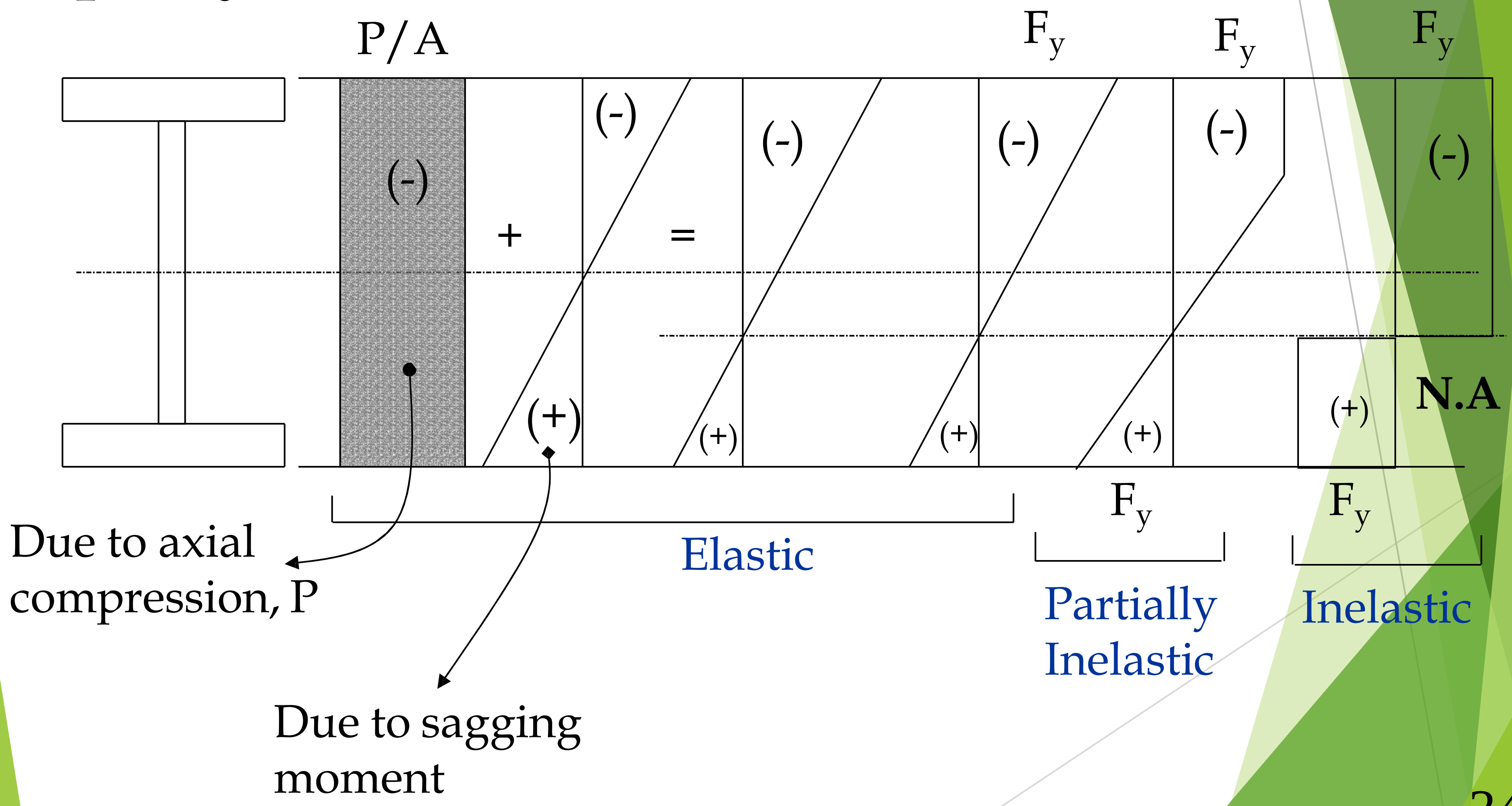
Influence of Axial Force On Plastic Moment Capacity



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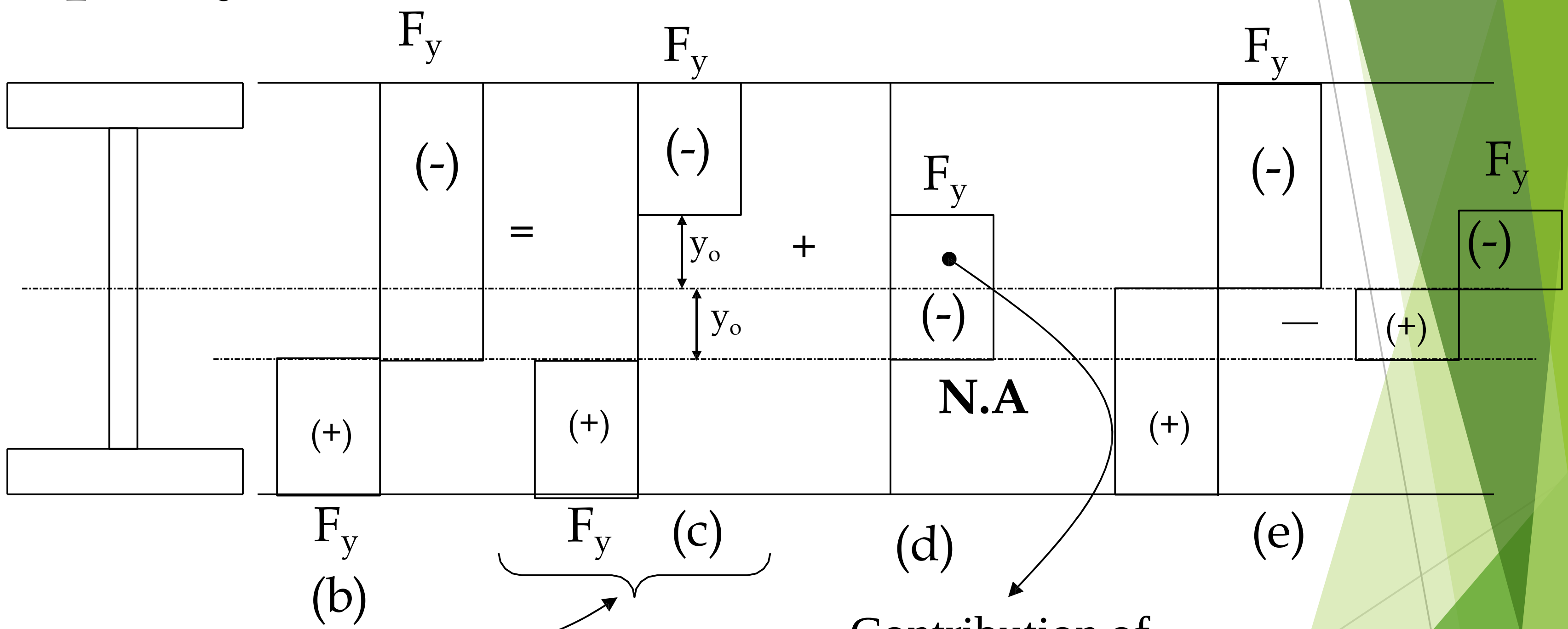
Influence of Axial Force On Plastic Moment Capacity



Contribution of moment. Upper area is equal to lower area

Contribution of Axial force

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Influence of Axial Force On Plastic Moment Capacity (contd…)

Case-I N.A Is Within The Web

Applicable for smaller values of axial load, P.

From diagram "d"

$$P = t_w \times 2y_o \times F_y$$

$$\boxed{y_o = \frac{P}{2t_w F_y}}$$

If $y_o \leq \frac{d - 2t_f}{2}$ Then N.A. is with in Web

Influence of Axial Force On Plastic Moment Capacity

Case-I N.A Is Within The Web (contd…)

Dia: (e)

Reduced moment capacity, M_{PC} = F_Y x Area of diagram on top or bottom x Distance between centers of comp. & tension area.

$M_{PC} = F_Y$ (Z for total section – Z for central portion of $2y_o$ height)

$$M_{PC} = F_y \times \left[\left(Z_x - \frac{t_w (2y_o)^2}{4} \right) \right]$$

$$= F_y \times \left(Z_x - t_w y_o^{\ 2} \right)$$

$$= M_P - F_y t_w y_o^{\ 2}$$

M_P is full plastic moment capacity in the absence of Axial force

Steel Structures

- Influence of Axial Force On Plastic Moment Capacity
- Case-I N.A Is Within The Web (contd…)

$$M_{PC} = M_P - F_y t_w \frac{P^2}{4t_w{}^2 F_y{}^2}$$

$$= M_P - \frac{P^2}{4t_w F_y}$$

$$\frac{M_{PC}}{M_P} = 1 - \frac{P^2}{4t_w F_y (z \times F_y)}$$

$$= 1 - \frac{P^2}{4t_w F_y (z \times F_y)} \times \frac{A^2}{A^2}$$

Influence of Axial Force On Plastic Moment Capacity

Case-I N.A Is Within The Web (contd…)

(1) ———— $$\boxed{\frac{M_{PC}}{M_P} = 1 - \frac{A^2}{4t_w z} \times \left(\frac{P}{P_y}\right)^2}$$ $\therefore P_y = A \times F_y$

P_y = Max. axial compressive load in the absence of bending moment & buckling

If P = 0, P/P_y = 0 then M_{PC}/M_P = 1, **$M_{PC} = M_P$**

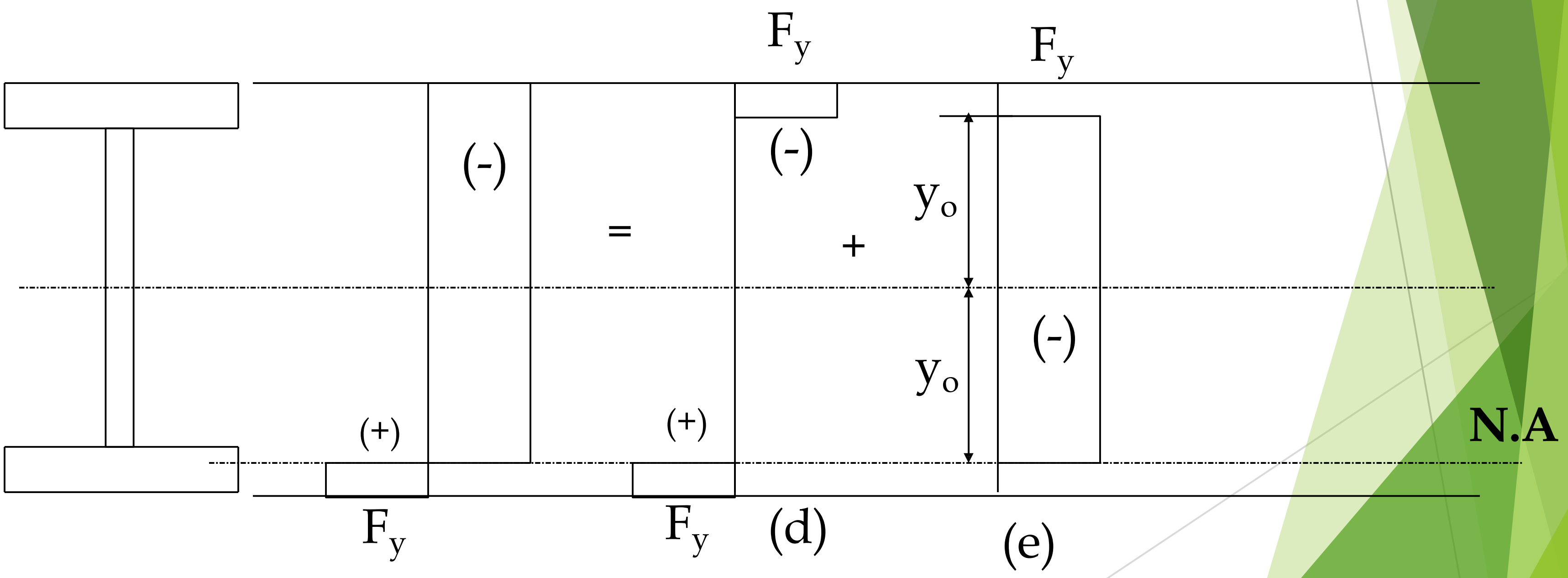
Eq: 1 is applicable when $y_o \leq \frac{d - 2t_f}{2}$

OR $P \leq F_y.t_w.2y_o \;= F_y.t_w(d-2t_f)$

Influence of Axial Force On Plastic Moment Capacity (contd…)

Case-II N.A Is Within The Flange

Applicable when P is high and M is less.



Steel Structures

Influence of Axial Force On Plastic Moment Capacity

Case-II N.A Is Within The Flange (contd…)

Lever arm in the diagram $= 2y_o + \dfrac{d-2y_o}{2} = y_o + \dfrac{d}{2}$

Let "A" be the total area of section

$$P = F_y\left[A - b_f\left(d - 2y_o\right)\right]$$

$$= F_y\left[A - b_f d - 2y_o b_f\right]$$

$$y_o = \frac{1}{2b_f}\left(\frac{P}{F_y} + b_f d - A\right)$$

$$y_o = \frac{P}{2b_f F_y} - \frac{A}{2b_f} + \frac{d}{2}$$

$$y_o = \frac{d}{2} - \frac{A}{2b_f}\left(1 - \frac{P}{P_y}\right)$$

If y_o = d/2, $1-P/P_y=0$
so $P/P_y = 1$

Steel Structures

Influence of Axial Force On Plastic Moment Capacity

Case-II N.A Is Within The Flange (contd…)

$$M_{PC} = F_y\left[\frac{b_f d^2}{4} - \frac{b_f (2y_o)^2}{4}\right]$$

$$= F_y\left[\frac{b_f d^2}{4} - b_f\left(\frac{d}{2} - \frac{A}{2b_f}\left(1 - \frac{P}{P_y}\right)\right)^2\right]$$

$$= F_y\left[-\frac{A^2}{4b_f}\left(1 - \frac{P}{P_y}\right)^2 + \frac{A \times d}{2}\left(1 - \frac{P}{P_y}\right)\right] \times \frac{Z}{Z}$$

Steel Structures

Influence of Axial Force On Plastic Moment Capacity

Case-II N.A Is Within The Flange (contd…)

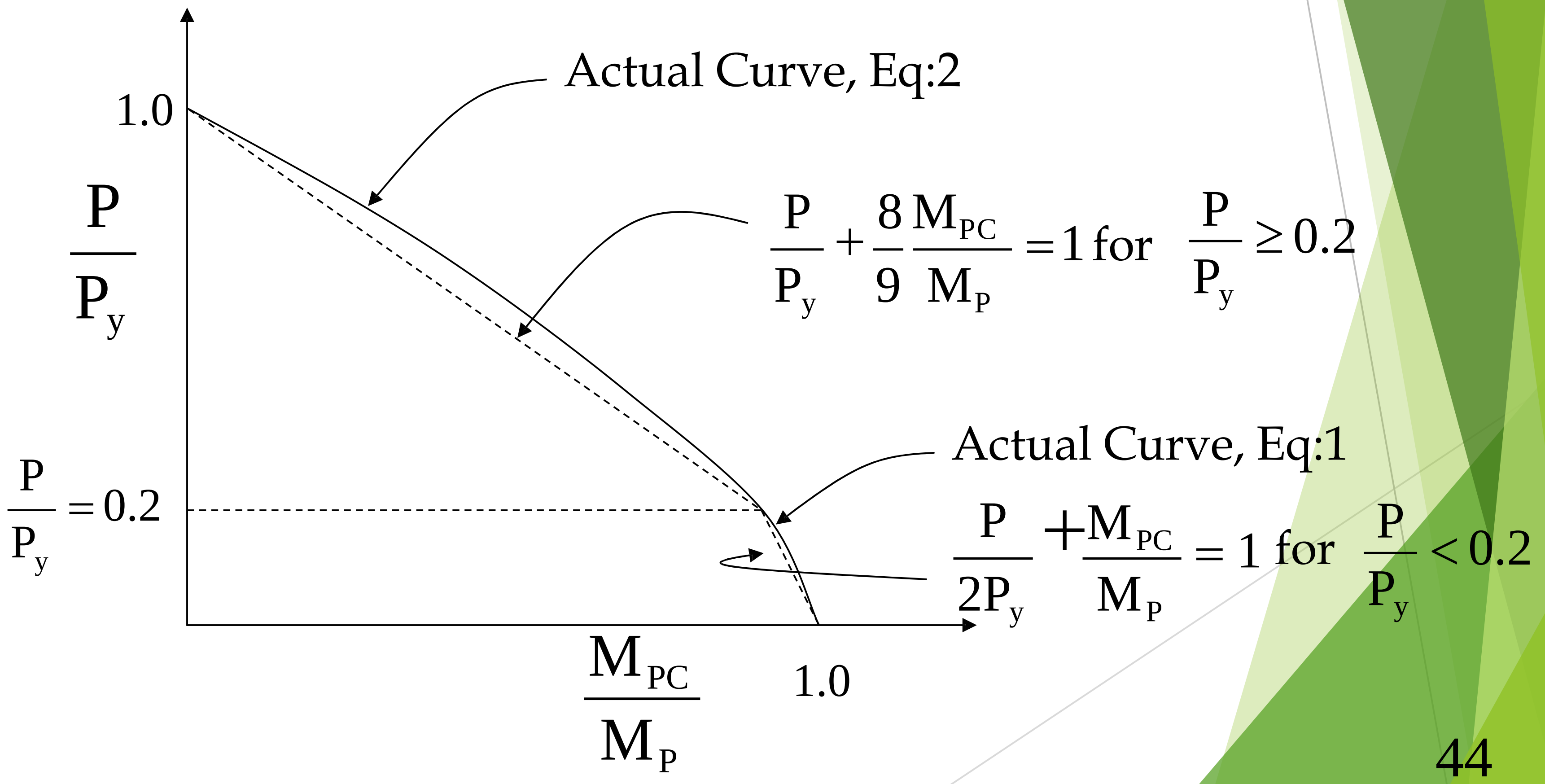
$$\frac{M_{PC}}{M_P} = \frac{A \times d}{2 \times Z}\left(1 - \frac{P}{P_y}\right) + \frac{A^2}{4b_f Z}\left(1 - \frac{P}{P_y}\right)^2$$

This eq is valid for: $y_o > \frac{d}{2} - t_f$ and $P > F_y . t_w (d - 2t_f)$

$$\boxed{\frac{M_{PC}}{M_P} = \frac{A}{2Z}\left[d\left(1 - \frac{P}{P_y}\right) + \frac{A}{2b_f}\left(1 - \frac{P}{P_y}\right)^2\right]} \qquad (2)$$

Steel Structures

Influence of Axial Force On Plastic Moment Capacity



Steel Structures

Influence of Axial Force On Plastic Moment Capacity

Actual curve is difficult to use so specifications allow to approximate the original equations by approximate equations.

For $P/P_y = 0.2$ draw Hz. Line. Where this line cuts the original curve join that point with 1.0 value point on both axis. Then equation can be made for the straight lines.

AISC Interaction Equations

$$\frac{P_r}{P_c} + \frac{8}{9}\left(\frac{M_{rx}}{M_{cx}} + \frac{M_{ry}}{M_{cy}}\right) \leq 1.0 \qquad \text{For} \quad \frac{P_r}{P_c} \geq 0.2$$

Steel Structures

Influence of Axial Force On Plastic Moment Capacity

$$\frac{P_r}{2P_c} + \frac{M_{rx}}{M_{cx}} + \frac{M_{uy}}{M_{cy}} \leq 1.0 \qquad \text{For} \quad \frac{P_r}{P_c} < 0.2$$

P_r = required factored axial compressive strength (LRFD)
= required allowable axial compressive strength (ASD)

P_c = design axial compressive strength (LRFD) = $\phi_c P_n$
= allowable axial compressive strength (ASD) = P_n / Ω

M_r = required factored flexural strength (LRFD)
= required allowable flexural strength (ASD)

Steel Structures

M_c = design flexural strength (LRFD) = $\phi_b M_n$
= allowable flexural strength (ASD) = M_n / Ω

ϕ_c = resistance factor for compression = 0.9
ϕ_b = resistance factor for flexure = 0.9

Ω_c = safety factor for compression = 1.67
Ω_b = safety factor for flexure = 1.67

M_t = Required nominal flexural strength for no axial load

$= M_{PC} / \phi_b$

Suppose $$\frac{P_u}{\phi_c P_n} > 0.2$$

$$\frac{P_u}{\phi_c P_n} + \frac{8}{9} \frac{M_{PC}}{\phi_b M_P} = 1.0$$

Steel Structures

Influence of Axial Force On Plastic Moment Capacity

After simplification

$$M_P = M_t\left(\frac{P_u}{\phi_c P_n} + 0.889\right)$$

$$Z_x = Z_t\left(\frac{P_u}{\phi_c P_n} + 0.889\right)$$

For $\dfrac{P_u}{\phi_c P_n} \le 0.2$

$$Z_x = Z_t\left(0.5\frac{P_u}{\phi_c P_n} + 1.00\right)$$

Approximate interaction equations to reduce the trials

Steel Structures

Design Procedure For Column (beam column)

1. Select section without considering the axial force. $Z_t = \frac{(M_P)_{req}}{0.9F_y}$
2. Calculate $\phi_c \mathbf{F_{cr}}$ & $\phi_c \mathbf{P_n}$ for the trial section. Also calculate $\mathbf{P_u} / \phi_c \mathbf{P_n}$ and decide which interaction equation is applicable.
3. Find out increased value of Z. Because due to axial force there is reduction in plastic moment capacity so we need to calculate the reduction.
4. Select section for this plastic section modulus.
5. Apply interaction equations. If L.H.S < 1.01, section is O.K. If too much less than 1.0 , the section is uneconomical.

Steel Structures

Example:

Design a beam column having required plastic moment of 300 kN-m and an axial force (factored) of 500 kN. A-36 steel is to be used. F_y = 250 MPa. Effective length, KL = 2.0m

Solution:

$$Z_t = \frac{(M_P)_{req}}{0.9F_y} = \frac{300 \times 10^6}{0.9 \times 250}$$

$$= 1333 \times 10^3 \text{mm}^3$$

From beam selection table

W 530 x 66

$L_P = 1.60\text{m} < L_u = 2.0$ m

[Not better as a column]

W 360 x 79

$L_P = 2.44\text{m} > L_u = 2.0$ m

In this way bracing is not required. In beams we can provide bracing but in case of column it may be difficult .

Trial Section W 360 x 79

A = 10,100 mm² r_x = 150mm, r_y = 48.8mm

KL_u/r_{min} = 2000/48.8 = 41 < 200 O.K.

Steel Structures

Solution:

$$\phi_c F_{cr} = 205.82\text{MPa}$$

$$\phi_c P_n = \frac{205.82 \times 10{,}100}{1000} \cong 2079\text{kN}$$

$$\frac{P_u}{\phi_c P_n} = \frac{500}{2079} = 0.24 > 0.2$$

New value of "Z"

$$Z = Z_t \left(\frac{P_u}{\phi_c P_n} + 0.889 \right)$$

In this equation we can use the original value of Z.

Solution:

$$Z = 1333 \times 10^3 (0.24 + 0.889)$$

$$= 1506 \times 10^3 \text{mm}^3$$

Now select the section according to this new value of Z

W 360 x 91, A = 11,500mm^2,
Z = 1671 x 10^3 mm^3, $\phi_b M_P$ = 376 kN-m
r_x = 152 mm, r_y = 62.2 mm, L_p = 3.11 m

$L_P > L_u$. Check local stability yourself.

$$\frac{KL_u}{r_{min}} = 33 \quad \Rightarrow \phi_c F_{cr} = 212.38 \text{MPa}$$

Steel Structures

Solution:

$$\phi_c P_n = \frac{212.38 \times 11500}{1000} = 2442 kN$$

$$\frac{P_u}{\phi_c P_n} = \frac{500}{2442} = 0.205 > 0.2$$

$$\frac{P_u}{\phi_c P_n} + \frac{8}{9}\frac{M_{ux}}{\phi_b M_{nx}} = 0.914 < 1.01$$

Final Selection: W 360 x 91

Concluded