Steel Structures

M.Sc. Structural Engineering

SE-505

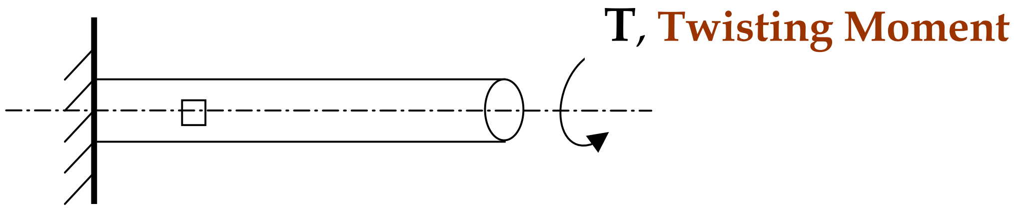
Lecture # 3

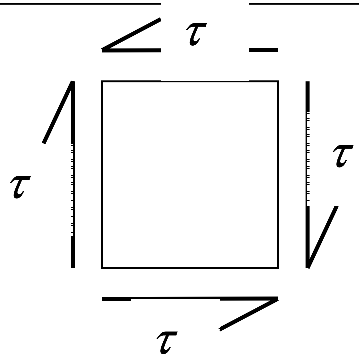
Design for Torsion

Steel Structures

Torque

- Moment about longitudinal axis
- Corresponding deformation produced is twist or torsion.





Torque can be resisted in two different ways

1. Pure Torsion (St. Venant Torsion)
2. Warping Torsion

Steel Structures

Pure Torsion

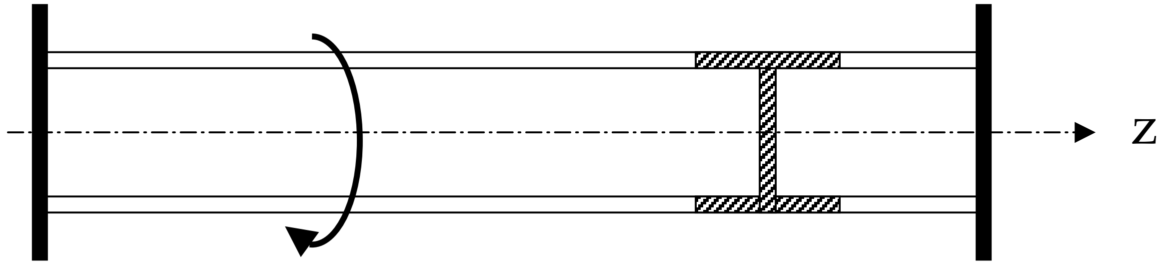
In this case the various cross-sections along the length of the member rotate relative to each other causing twist of the member.

Any particular cross section twists as a whole

Typical example is the torque applied on a circular rod.

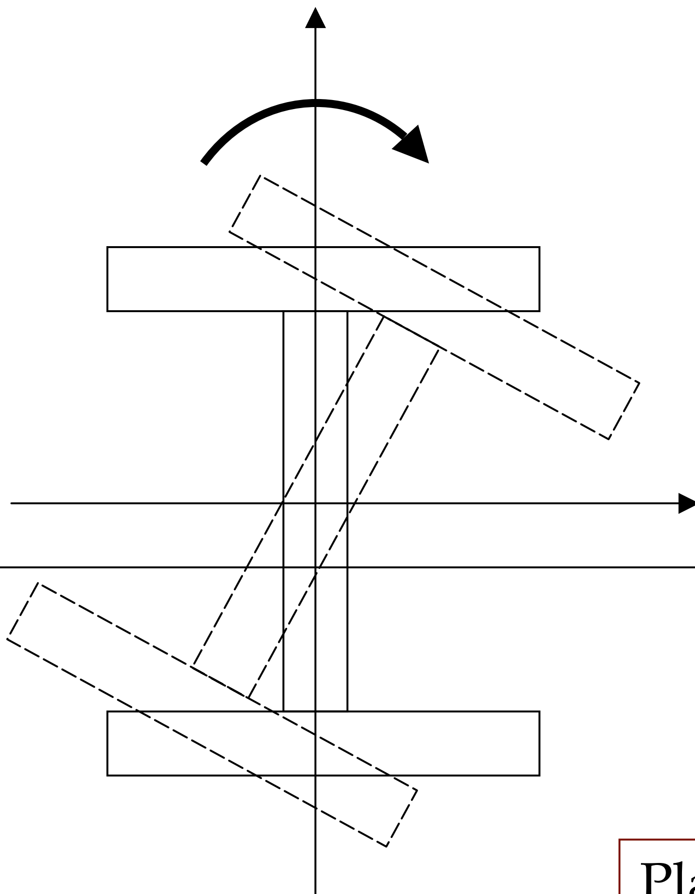
Warping Torsion

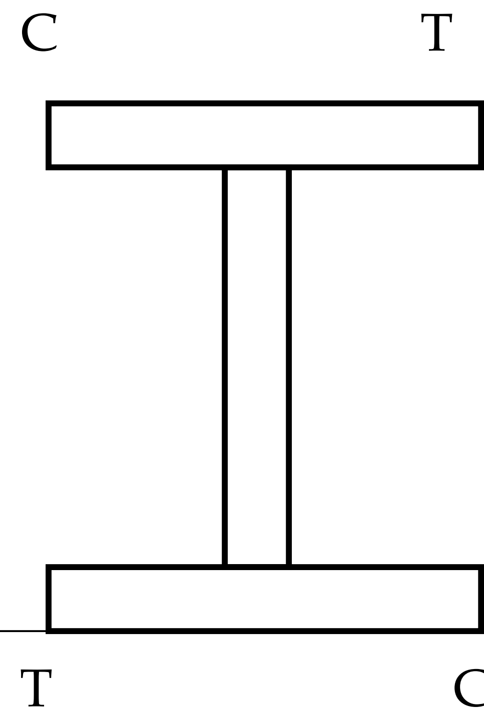
The whole cross-sections do not rotate as a whole



Steel Structures

Warping Torsion (contd…)





Under the Action of Torque

Plane section do not remain plane in warping torsion

Steel Structures

Torsion Formula for Circular Section

(Pure Torsion)

1. Plane section remains plane.
2. Radial lines remain straight.
3. Moment is applied along longitudinal axis.
4. Material remains elastic.

Steel Structures

Torsion Formula for Circular Section (contd…)

θ = total rotation of any section w.r.t. the reference point.

ϕ = change of angle per unit length

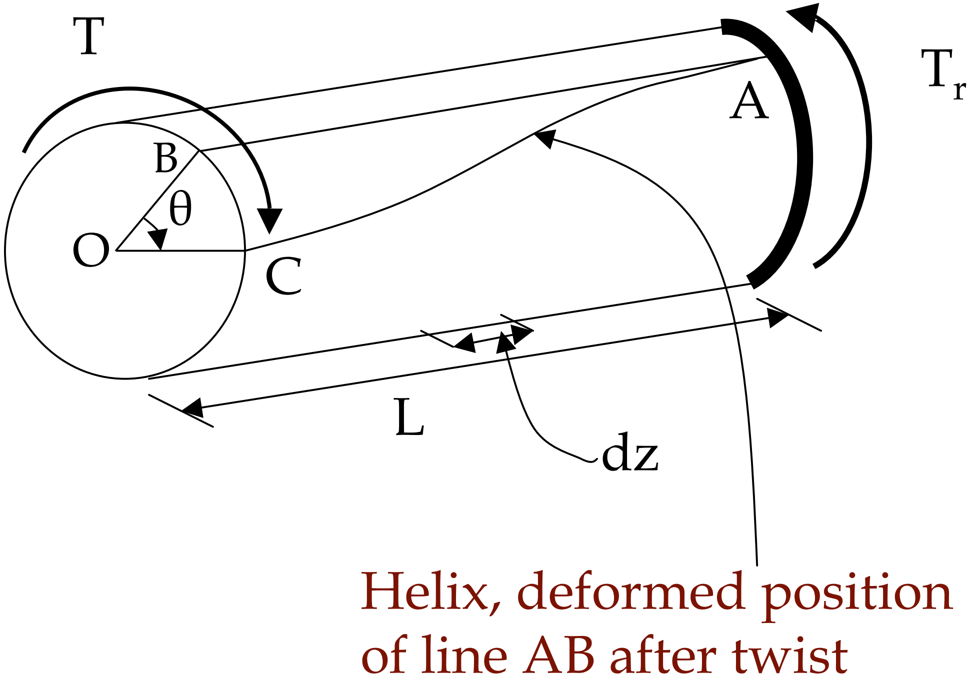
$\phi = \theta/L$ for linear increase

$\phi = d\theta/dz$ in general

ρ = radial distance up to any point where stresses are to be calculated.

τ = shear stress at any point

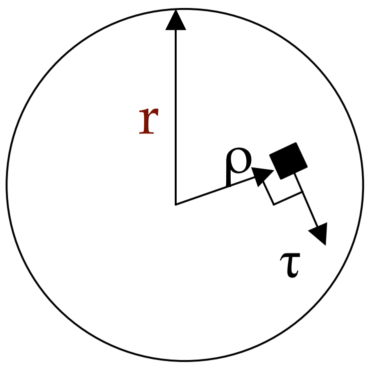
γ = shear strain at any point



Steel Structures

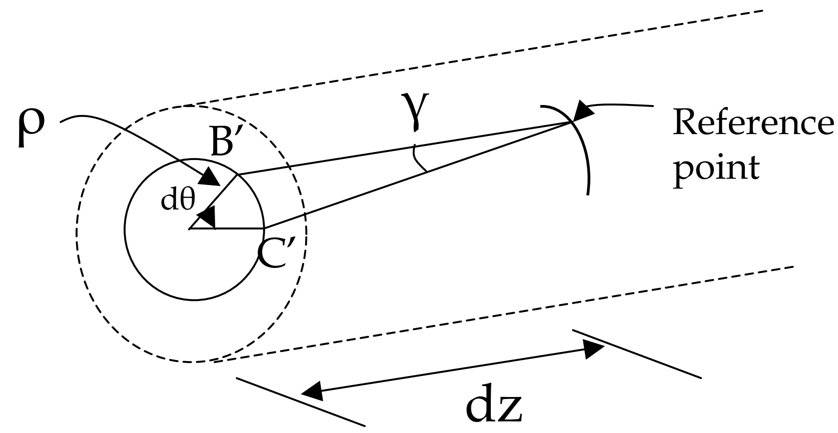
Torsion Formula for Circular Section (contd…)

Shear stress due to pure torsion is always perpendicular to the redial distance at that point.



$$\gamma = \frac{B'C'}{dz} \Rightarrow \gamma = \frac{\rho d\theta}{dz}$$

$$\gamma = \rho \frac{d\theta}{dz} \Rightarrow \gamma = \rho\varphi$$



Steel Structures

Torsion Formula for Circular Section (contd…)

$$dT = \rho \times \tau dA$$

$$= \rho \times (\gamma G) dA \qquad \because \tau = \gamma G$$

$$= \rho \times (\rho \phi) \times G dA$$

$$= \rho^2 \times \phi \times G dA$$

$$T = T_r = \int_A \rho^2 \times \phi \times G dA$$

$$T = \phi \times G \int_A \rho^2 dA$$

$$T = \phi \times GJ$$

GJ = torsional rigidity

$$J = I_x + I_y$$ For circular section

Now

$$\tau = \gamma G$$

$$\tau = \rho \left(\frac{T}{GJ} \right) G$$

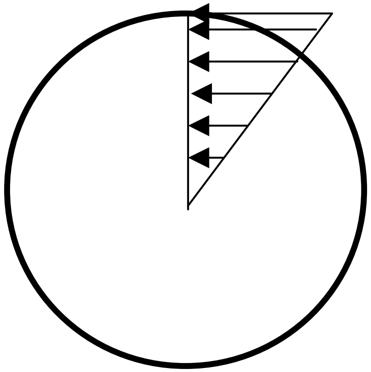
$$\tau = \frac{T \times \rho}{J}$$

Steel Structures

Torsion Formula for Circular Section (contd…)

$$\tau \propto \rho$$

$$\tau_{\max} = \frac{Tr}{J}$$



Shear stress
due to torsion

Steel Structures

Pure Torsion For Non-Circular Section

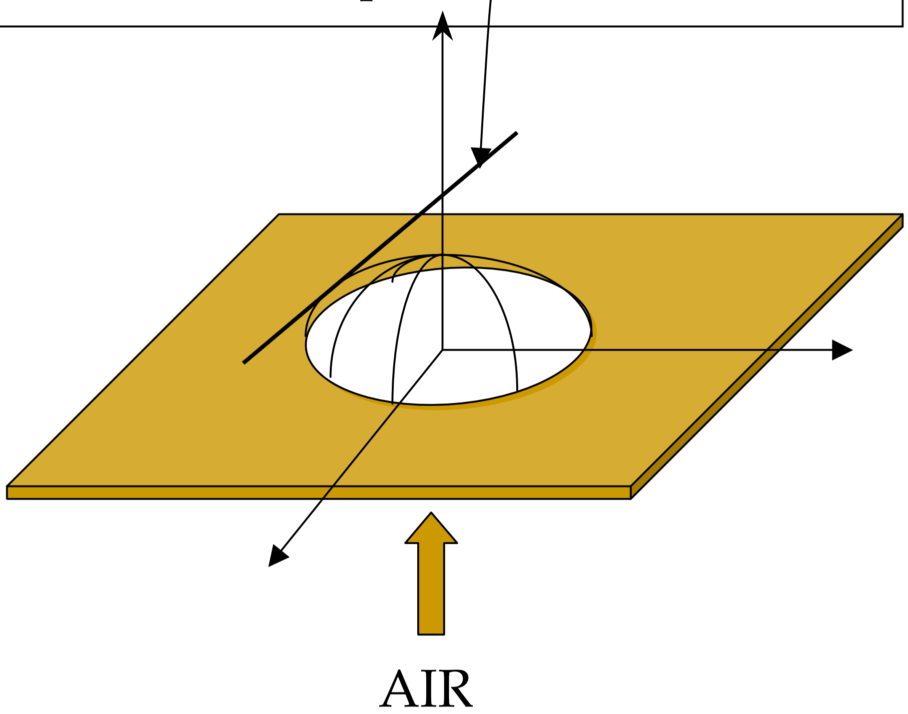
(Experimental Method)

Soap Film Analogy

The volume between the bubble and the original plane (by the analogy of governing differential equation) is proportional to the total torque resistance (applied).

Steeper the slope of tangent at any point greater will be the shear stress.

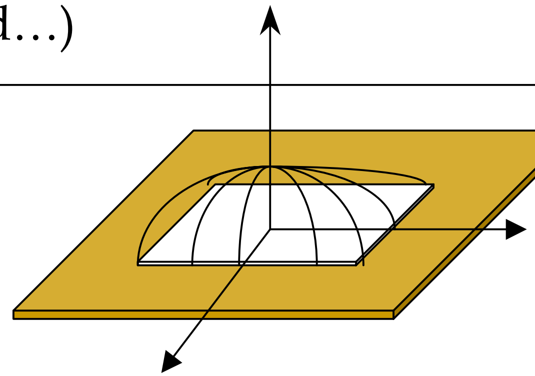
SFA is more useful for noncircular and irregular section for which formulas are difficult to derive.

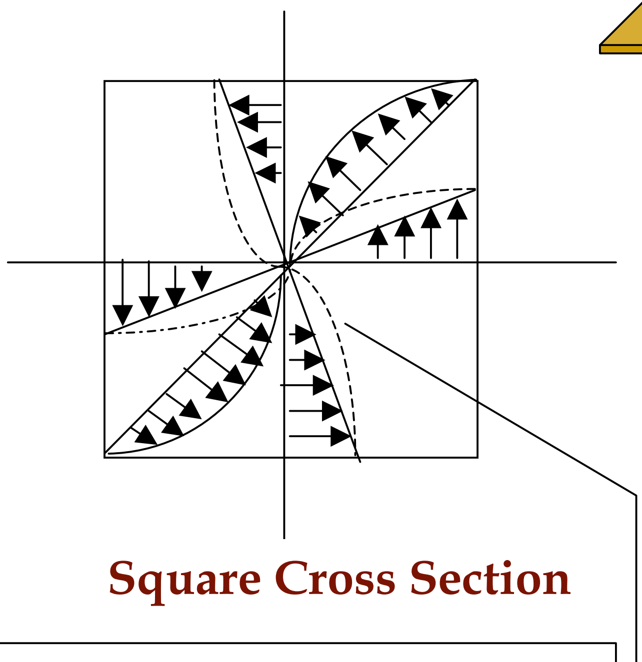


Steel Structures

Pure Torsion For Non-Circular Section

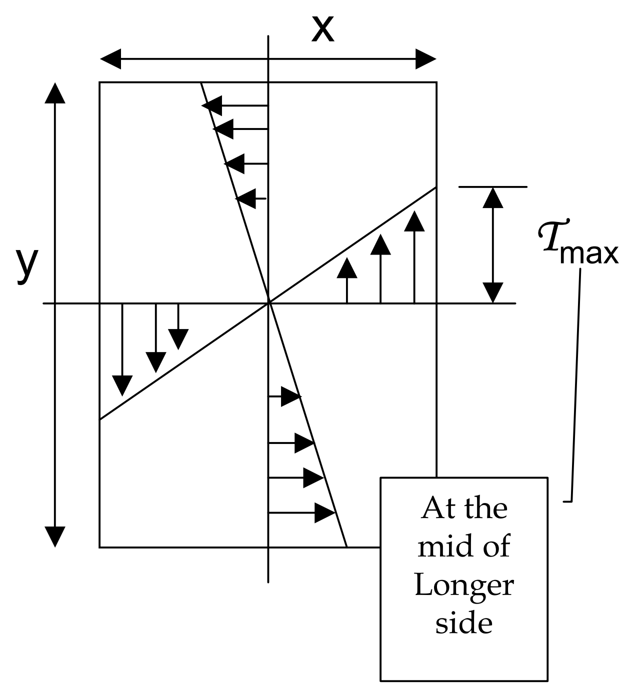
Soap Film Analogy (contd…)





Square Cross Section

When material behaves in-elastically

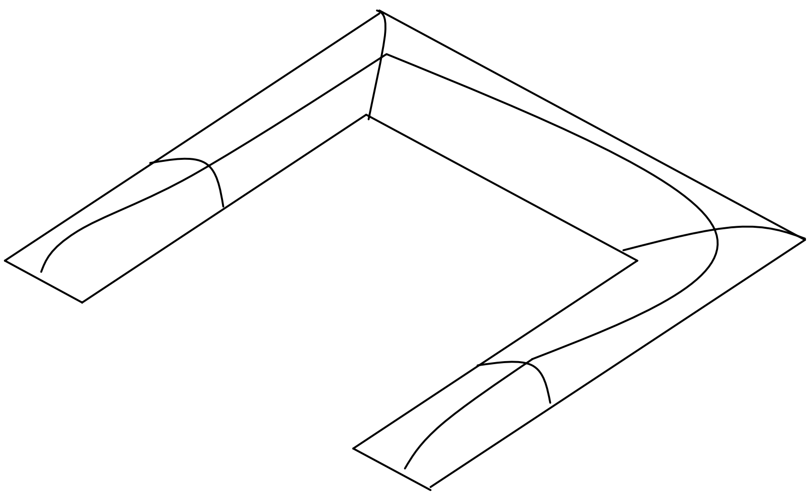


Rectangular Cross Section

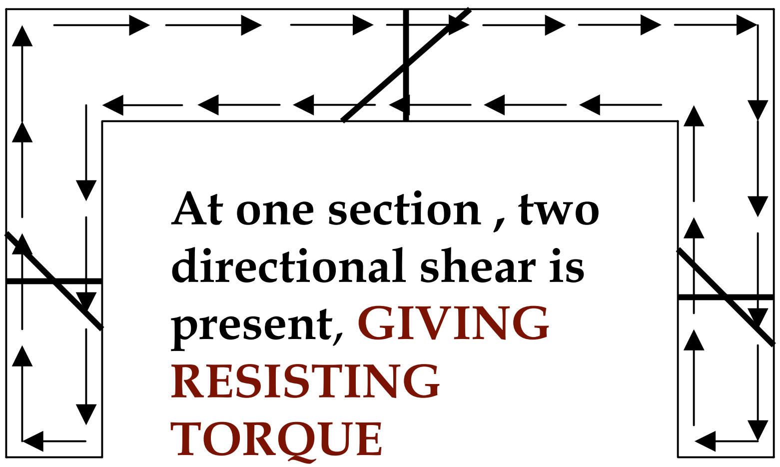
Steel Structures

Pure Torsion For Non-Circular Section

Soap Film Analogy (contd…)



Soap Film



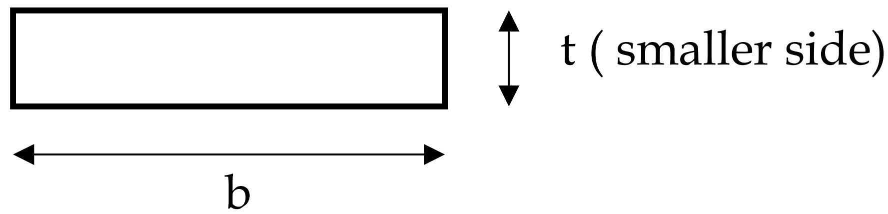
Steel Structures

Pure Torsion For Non-Circular Section

By Timoshenko

$$\tau_{max} = \frac{Tt}{\alpha b t^3} = \frac{Tt}{C}$$

Valid for Rectangular Section only



- C, Torsion constant $= \frac{bt^3}{3}$
- α depends on b/t ratio.

α = 1/3 for practical section with large b/t ratio.

b/t	1.0	1.5	2.0	3.0	5.0	∞
α	.208	.219	.246	.267	.290	1/3

For section consisting of more than one rectangular

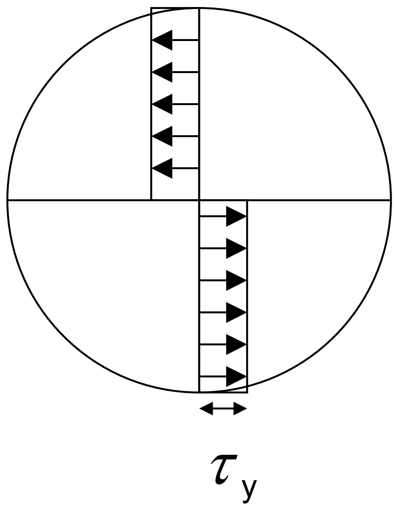
$$C = \Sigma \frac{bt^3}{3}$$

Steel Structures

Plastic Torsion

Whole the section will yield in torsion, $\tau = \tau_y$

Plastic analysis assumes **uniform shear intensity** all around the surface and all around the cross section.

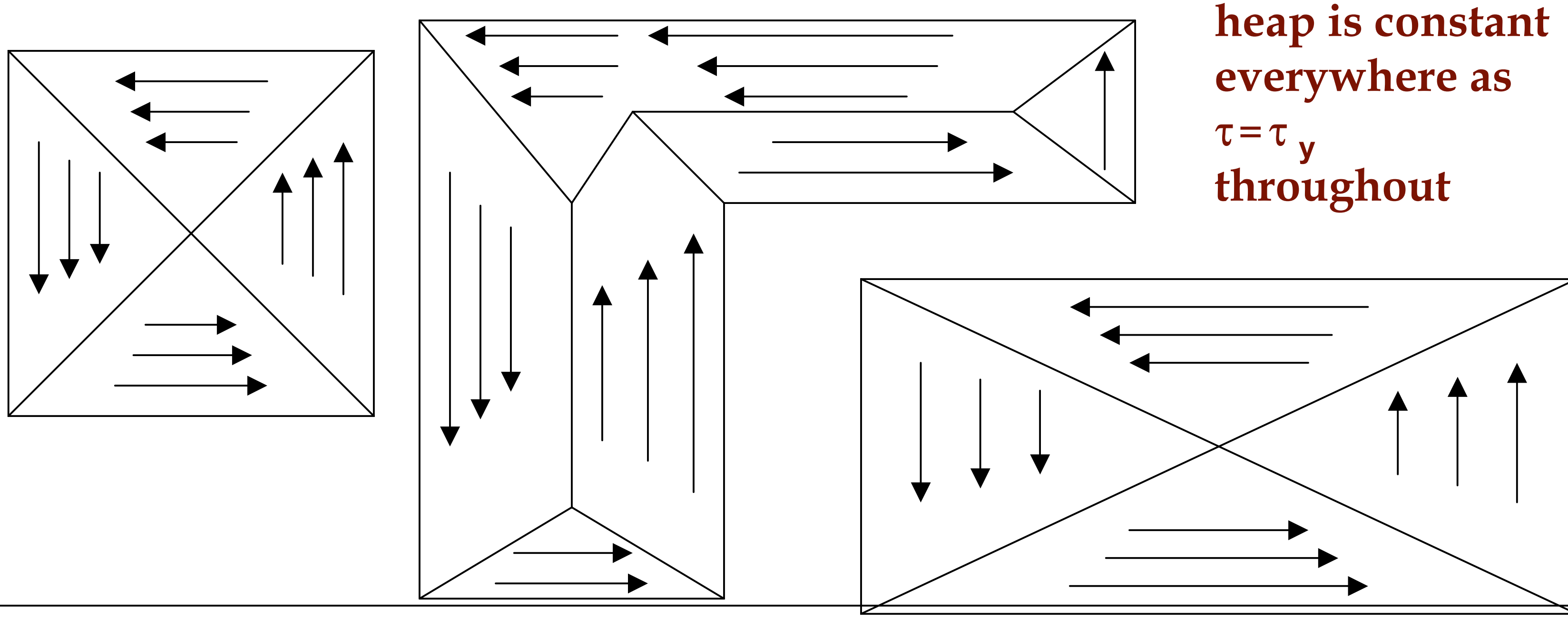


Plastic torsions can be envisioned in terms of SAND HEAP ANALOGY

Steel Structures

Sand Heap Analogy

Put sand on a plate having a shape same as that of cross section (Circular, Rectangular, Irregular)



Slope of sand heap is constant everywhere as $\tau = \tau_y$ throughout

Steel Structures

Sand Heap Analogy (contd…)

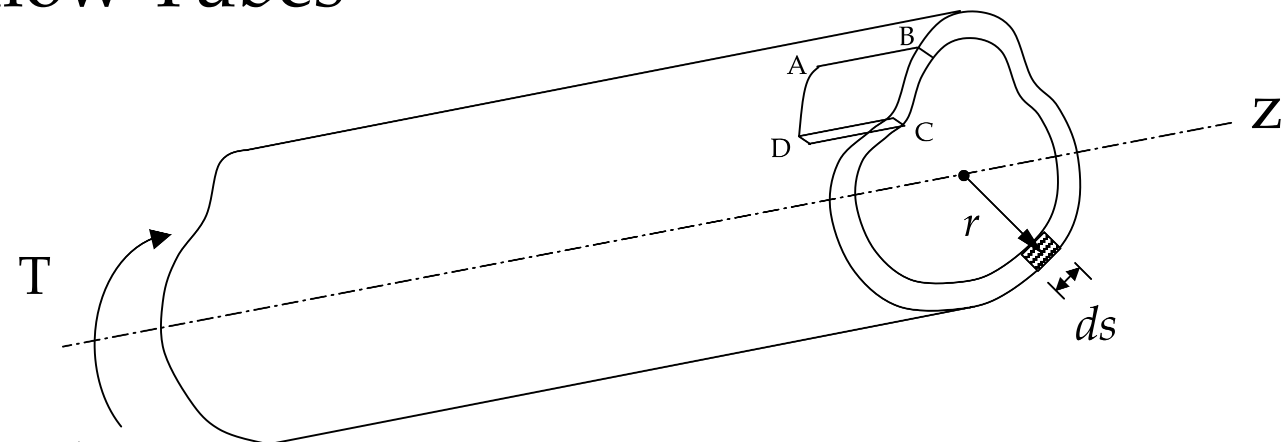
Volume under the sand heap is proportional to the torque.

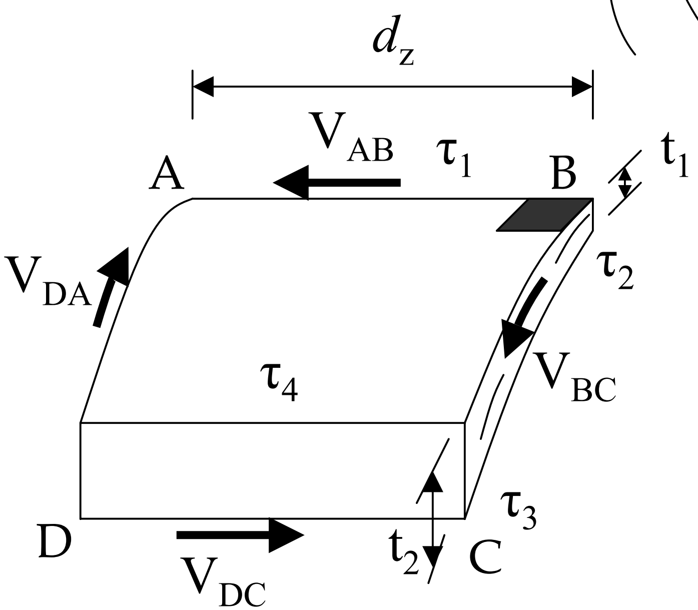
$$\tau_{p_{\max}} = \frac{Tt}{\alpha_p b t^3}$$

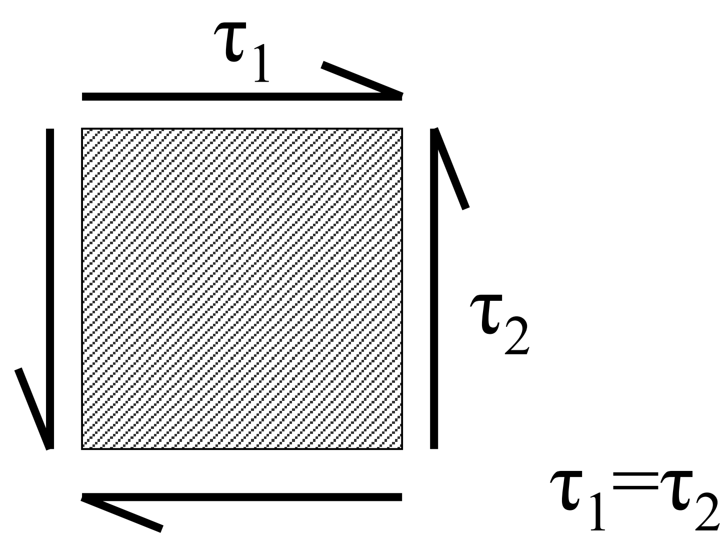
α_p = 0.33 for b/t = 1.0
= 0.5 for b/t = ∞

Steel Structures

Torsion in Hollow Tubes







Steel Structures

Torsion in Hollow Tubes

V = Resultant shear force at a face

τ_1 remains constant throughout the length

$$V_{AB} = \tau_1 \times t_1 \times dz$$

$$V_{CD} = \tau_4 \times t_2 \times dz$$

$$\sum F_z = 0 \quad \Rightarrow \quad V_{AB} = V_{CD}$$

To maintain equilibrium

$$\tau_1 \times t_1 = \tau_4 \times t_2$$

For equilibrium of infinitesimal element at corner B, $\tau_1 = \tau_2$

Similarly, at corner C, $\tau_3 = \tau_4$

Steel Structures

Torsion in Hollow Tubes (contd…)

$$\tau_1 \times t_1 = \tau_4 \times t_2 \quad \Rightarrow$$

$$\tau_2 \times t_1 = \tau_3 \times t_2$$

Shear stress is more in the portion where thickness is less but $\tau \times t$ remains constant

The product $\tau \times t$ is referred to as the shear flow, q having units of N/mm. The shear flow remains constant around the perimeter of the tube.

This term comes from an analogy to water flowing in a loop of pipes having different diameters, where the total discharge remains the same.

Steel Structures

Torsion in Hollow Tubes (contd…)

$$\tau \times t = q$$ (Shear flow)

$$q_B = q_C$$

In general shear flow is same throughout the cross section.

Torsional shear force acting on *ds* length of wall = $q \times ds$

Resisting moment of this force = $r \times q \times ds$

Integrating this differential resisting torque around the perimeter gives the total resisting torque.

$$T = \int_P r \times q \times ds$$

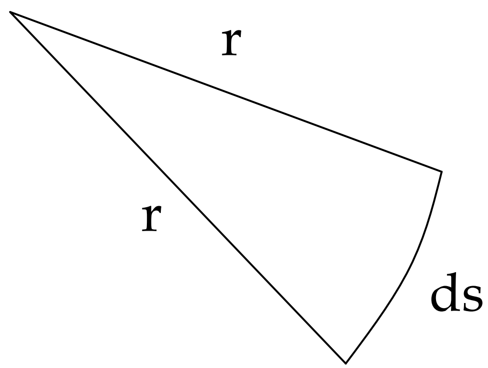
Steel Structures

Torsion in Hollow Tubes (contd…)

$$T = q\int_P r \times ds$$

$$T = 2q\int_P \frac{r \times ds}{2}$$

$$T = 2q \times A_o$$



A_o = Area enclosed by shear flow path

$$T = 2\tau \times t \times A_o$$

$$\boxed{\tau = \frac{T}{2A_o t}}$$

For hollow closed tube

Steel Structures

Shear Center

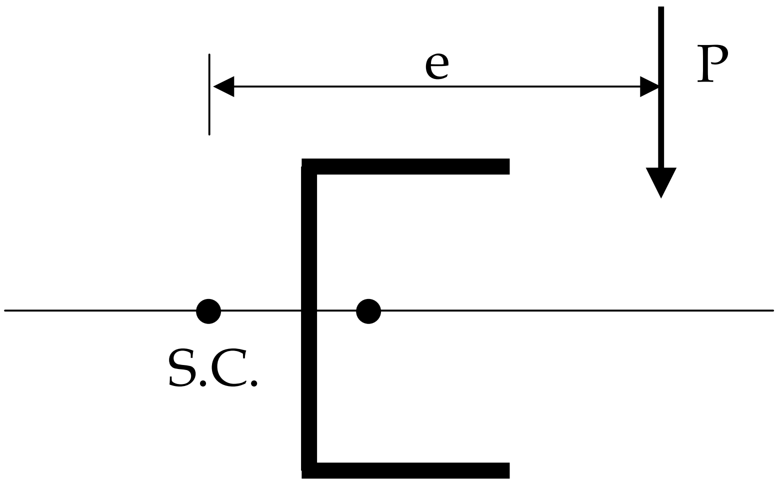
"Shear center is defined as the point in the cross-sectional plane of a beam through which the transverse loads must pass so that the beam bends without twisting."

In other words, loads applied through the shear center will cause no torsional stresses to develop.

$$\int_0^n (\tau t) r \, ds = 0$$

$$T = P \times e$$

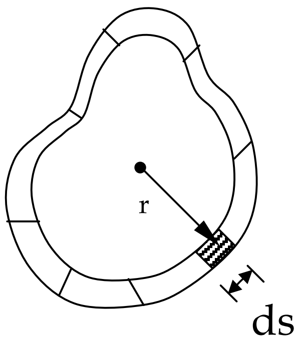
"e" is from Shear Center



Steel Structures

Shear Center (contd…)

$$T = \int_0^n (\tau \times t) \times r \times ds = 0$$



Closed Thin Walled Section

Magnitude of Shear Flow for Transverse Loads Through Shear Center

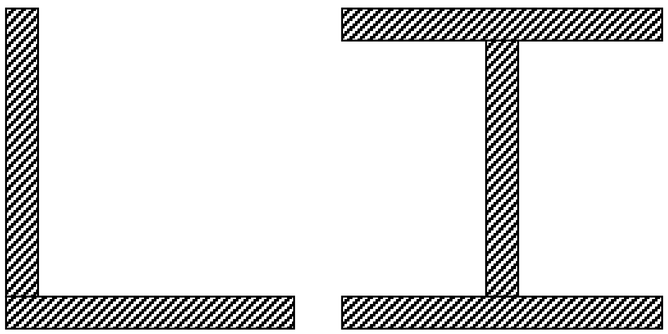
$$q = \frac{VQ}{I} \quad (1)$$

Valid for sections having $I_{xy} = 0$

"I" is about the axis of bending

$$q = \frac{V_y}{I_x I_y - I_{xy}^2}\left(I_y \int_0^s ytds - I_{xy} \int_0^s xtds \right)$$

If we put $I_{xy} = 0$, we will get (1)



Open Thin Walled Section

Steel Structures

Shear Flow In Thin Walled Open Sections Due to Applied Shear Force

Rules For Plotting Shear Flow Diagram

1. The shear flow in the part of element parallel to the applied shear is always in a direction opposite to this applied shear.

2. Shear flow due to direct shear occurs in one direction through thin walls of open sections.

3. At junction of elements, incoming shear flow is equal to outgoing shear flow.

Steel Structures

Rules For Plotting Shear Flow Diagram (contd…)

4. The value of shear flow is zero at free tips of the element and more shear flow is generated as more area is added.
5. Shear flow is assumed to be generated on one side of the neutral axis and consumed/absorbed on the other side.
6. Shear flow generated is proportional to the first moment of the area added.

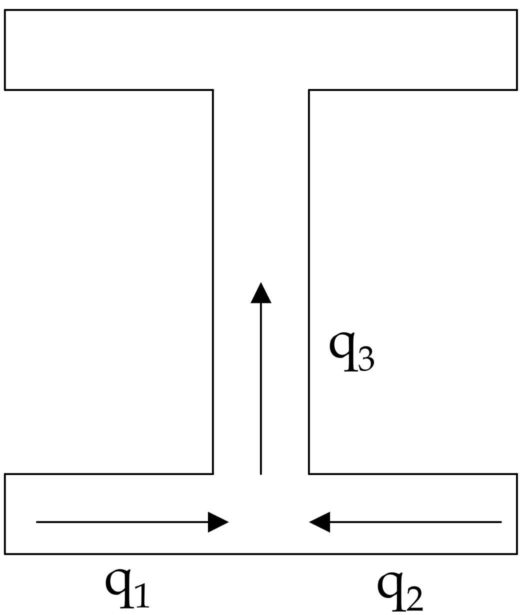
Steel Structures

7. Shear flow increases linearly for the elements perpendicular to the load and parabolically for the elements parallel to the load.

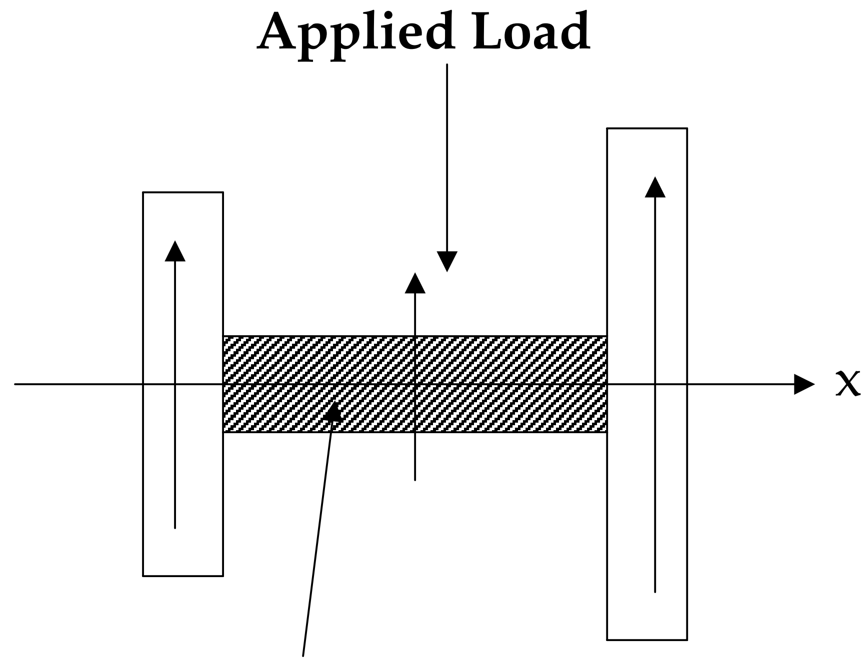
8. Shear flow is considered zero for elements which have insignificant contribution in corresponding "I" value.

Steel Structures

Rules For Plotting Shear Flow Diagram (contd…)



$$q_3 = q_1 + q_2$$

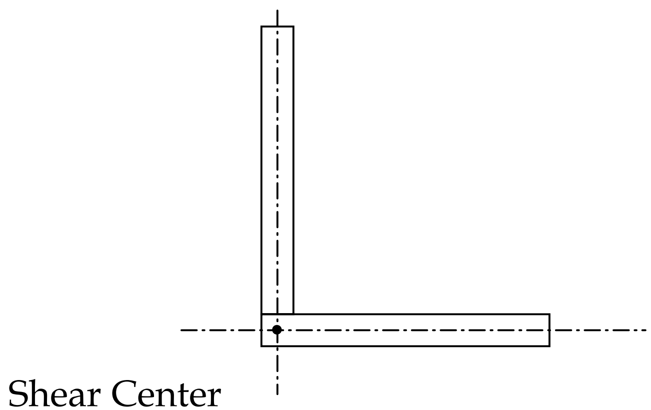


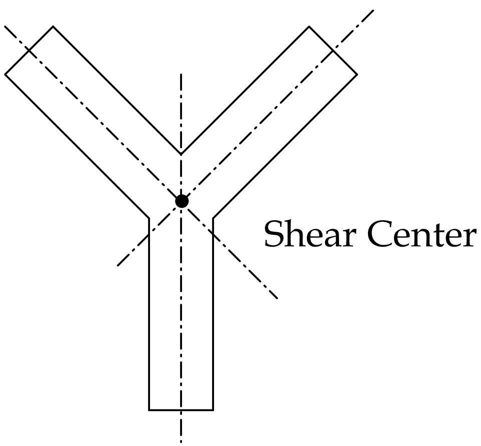
I_x is very small so this portion can be neglected

Steel Structures

General Rules For Locating Shear Center

1. Shear center always lie on axis of symmetry.
2. If two axes of symmetry exist for a section, S.C. will be at the intersection of these two axis.
3. If the centerlines of all the elements of a section intersect at a single point this is the shear center.
4. Shear center of “Z” section is at the centroid.





Steel Structures

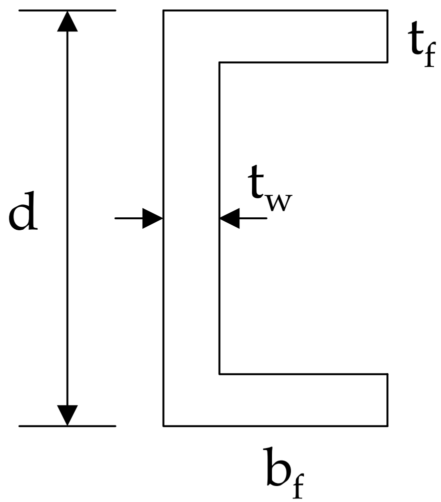
Procedure to Locate Shear Center

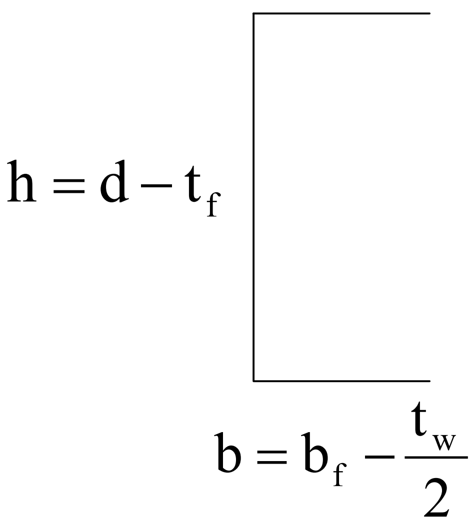
1. To find horizontal location (e_x) apply vertical load (V) at e_x from reference point.
2. Plot shear flow diagram due to applied load.
3. Find the internal shear force in each element.
4. Apply $\sum M = 0$ at convenient location and find e_x
5. Similarly apply horizontal load at a vertical distance "e_y" from reference point (say centroid) and repeat the above procedure to calculate "e_y"
6. The distances "e_x" and "e_y" locate the shear center.

Steel Structures

Example:

Locate the Shear Center for the given channel section.



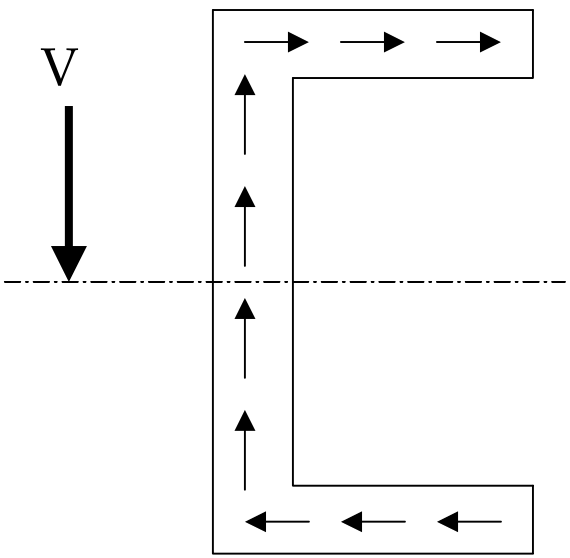


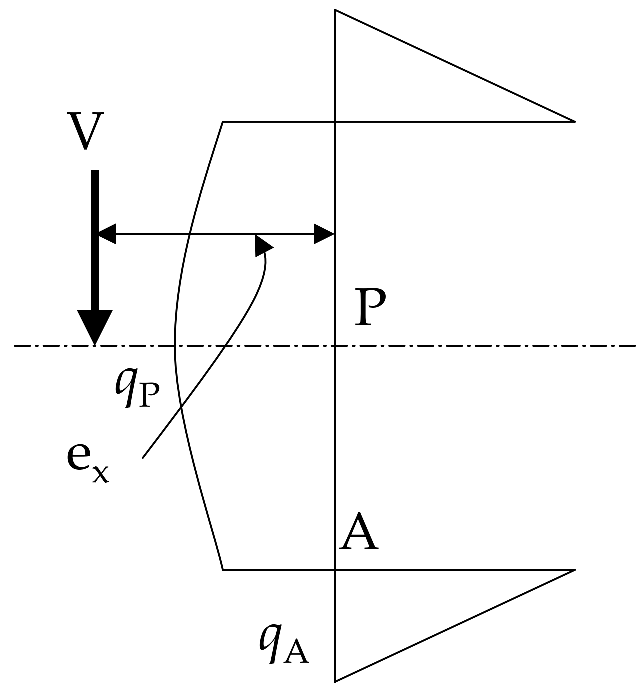
Centerline Representation

Steel Structures

Solution

By symmetry about z-axis, the shear center must lie at half the depth. Only horizontal location is to be found.





$$q = \frac{VQ}{I}$$

Steel Structures

Point A

$$Q = \left(b \times t_f\right) \times \frac{h}{2}$$

$$q_A = \frac{V}{I_x}\left(b \times t_f\right) \times \frac{h}{2}$$

Point P

$$q_P = q_A + \frac{V}{I_x} \times \left(\frac{h}{2} \times t_w\right) \times \frac{h}{4}$$

$$q_P = \frac{V}{I_x} \times \left(bt_f \times \frac{h}{2} + \frac{h^2}{8} \times t_w\right)$$

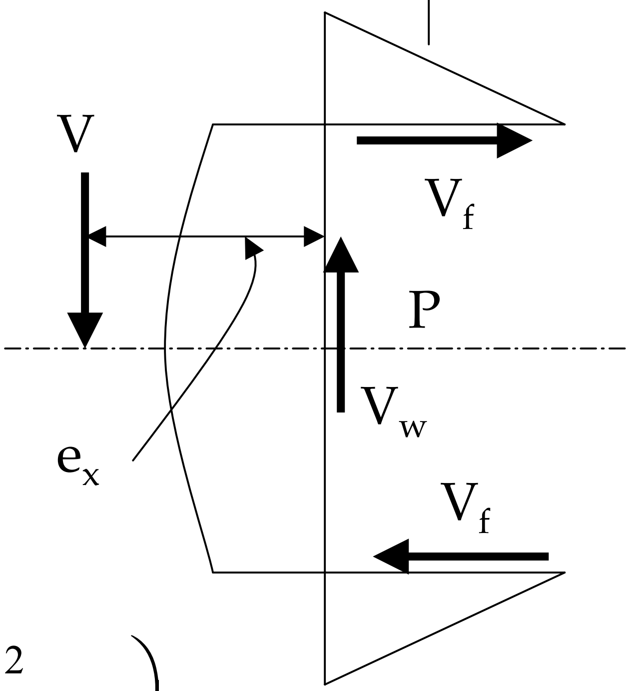
Steel Structures

Solution

Shear force in flange

$$V_f = \frac{1}{2} \times \frac{V}{I_x} \times \frac{b t_f h}{2} \times b$$

$$V_f = \frac{V}{I_x} \times \frac{b^2 t_f h}{4}$$



Shear force in web

$$V_w = \frac{V}{I_x} \times \frac{b t_f h}{2} \times h + \frac{2}{3}\left(\frac{V}{I_x} \times \frac{t_w h^2}{8} \times h\right)$$

$$V_w = \frac{V}{I_x}\left(\frac{b t_f h^2}{2} + \frac{t_w h^3}{12}\right)$$

Steel Structures

Solution

$$\Sigma M_P = 0$$

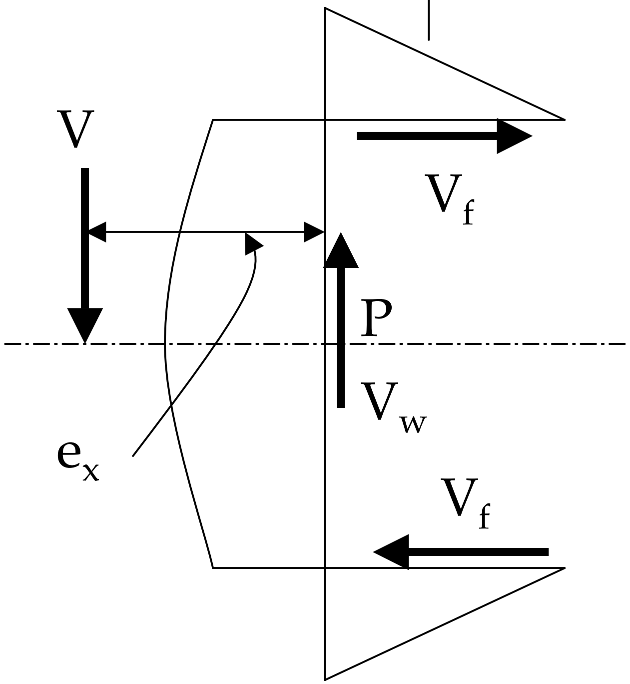
$$V \times e_x - V_f \times \frac{h}{2} - V_f \times \frac{h}{2} = 0$$

$$V \times e_x = V_f \times h$$

$$e_x = \frac{h}{V}\left(\frac{V}{I_x} \times \frac{b^2 t_f h}{4}\right)$$

$$e_x = \frac{b^2 t_f h^2}{4I_x}$$

Positive means on the assumed left side.

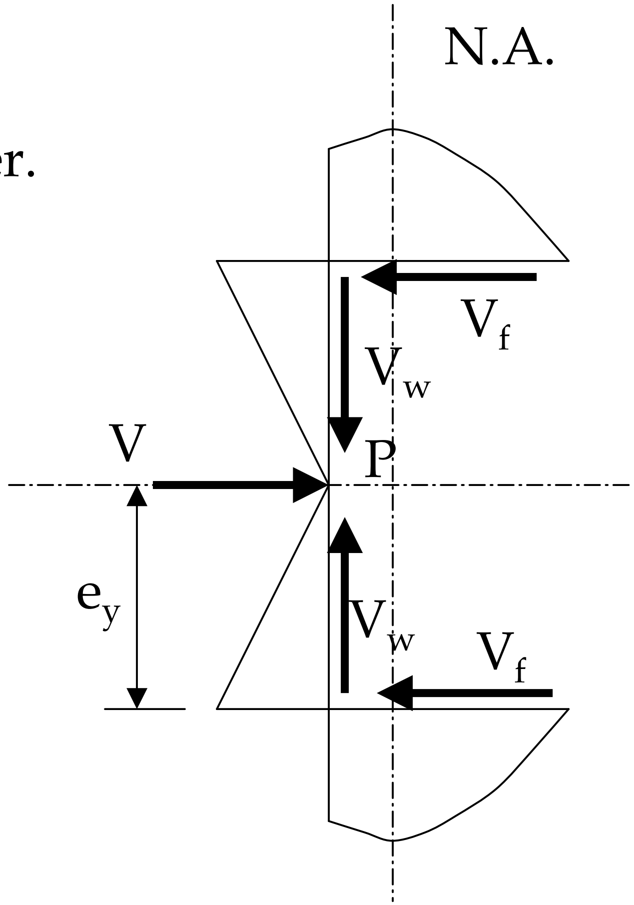


Steel Structures

Solution

For vertical location of shear center.

$$e_y = \frac{h}{2}$$



Applied Torque = Load x Perpendicular distance from S.C.

Steel Structures

Differential Equation for Torsion of I-Shaped Sections

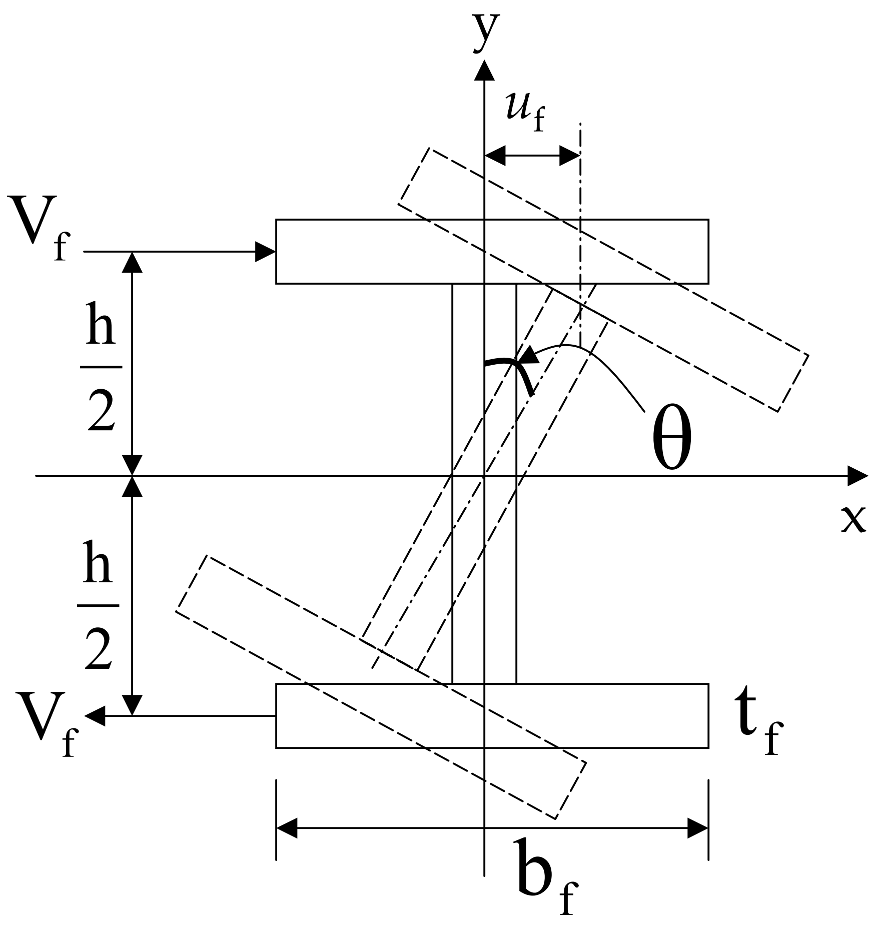
u_f = lateral deflection of one of the flanges

θ = twist angle at the selected section

V_f = Shear force in flange due to torsion. (internal force developed)

θ is smaller and is in radians, so

$$u_f \cong \theta \times \frac{h}{2} \qquad (1)$$



Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

The lateral curvature relationship of one flange alone is:

$$\frac{d^2 u_f}{dz^2} = -\frac{M_f}{EI_f} \qquad \text{————} \quad (2)$$

M_f = Lateral Bending moment on one flange

I_f = Moment of inertia of one flange about y-axis of beam

$$I_f = \frac{t_f b_f^{\,3}}{12}$$

Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

$$V = \frac{dM}{dz} \Rightarrow V_f = \frac{dM_f}{dz} \qquad (3)$$

Differentiating (1)

$$\frac{d^3 u_f}{dz^3} = \frac{-V_f}{EI_f} \qquad (4)$$

$$V_f = -EI_f \frac{d^3 u_f}{dz^3}$$

$$V_f = -EI_f \frac{(h/2) d^3\theta}{dz^3} \Longrightarrow V_f = -EI_f \frac{h}{2} \frac{d^3\theta}{dz^3} \qquad (5)$$

Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

Torsion resistance due to warping

$$M_w = V_f \times h$$

$$M_w = -EI_f \frac{h}{2} \times \frac{d^3\theta}{dz^3} \times h$$

$$= -EI_f \frac{h^2}{2} \times \frac{d^3\theta}{dz^3}$$

$$M_w = -EC_w \times \frac{d^3\theta}{dz^3} \qquad (6)$$

Warping Constant

$$C_w = I_f \frac{h^2}{2}$$

$$C_w = \frac{t_f b_f^3}{12} \times \frac{h^2}{2}$$

$$C_w = \frac{I_y}{2} \times \frac{h^2}{2}$$

$$C_w = \frac{I_y h^2}{4}$$

Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

Torsion resistance due to Pure torsion

$$M_s = GJ \times \frac{d\theta}{dz}$$

For Circular Section

OR

$$M_s = GC \times \frac{d\theta}{dz}$$

For Non-Circular Section

Total Torque Applied $M_z = M_s + M_w$

$$M_z = GC \times \frac{d\theta}{dz} - EC_w \frac{d^3\theta}{dz^3} \qquad (8)$$

Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

Dividing by "$-EC_w$"

$$\frac{d^3\theta}{dz^3} - \frac{GJ}{EC_w} \times \frac{d\theta}{dz} = -\frac{M_z}{EC_w} \qquad (9)$$

$$\frac{d^3\theta}{dz^3} - \lambda^2 \frac{d\theta}{dz} = -\frac{M_z}{EC_w} \qquad (10)$$

Non homogeneous differential equation

where

$$\lambda^2 = \frac{GC}{EC_w} \Rightarrow \lambda = \sqrt{\frac{GC}{EC_w}} \qquad (11)$$

λ^2 = Ratio of pure torsion rigidity to warping torsion rigidity

Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

Total Solution

$$\theta = \theta_h + \theta_P$$

θ = Total Solution
θ_h = Homogeneous Solution
θ_P = Particular Solution

Homogeneous Equation

$$\frac{d^3\theta}{dz^3} - \lambda^2 \frac{d\theta}{dz} = 0$$

Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

Trial Function

$$\theta_h = Ae^{mz}$$

"A", "m" are constants. "z" is independent variable

$$\frac{d^3\theta}{dz^3} = Am^3 e^{mz}$$

$$\Rightarrow \quad Am^3 e^{mz} - \lambda^2 \times Ame^{mz} = 0$$

$$Ae^{mz}\left(m^3 - \lambda^2 m\right) = 0$$

For non-trivial solution $\quad A \neq 0$

Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

$$m^3 - \lambda^2 m = 0$$

$$m\left(m^2 - \lambda^2\right) = 0$$

Possible Solutions: $m = 0$, $m = +\lambda$, $m = -\lambda$

Sum of all solutions is total homogeneous solution

$$\theta_h = A_1 e^{\lambda z} + A_2 e^{-\lambda z} + A_3 e^{o}$$

$$= A_1 e^{\lambda z} + A_2 e^{-\lambda z} + A_3$$

We know

$\sinh(x) + \cosh(x) = e^x$ and $\sinh(x) - \cosh(x) = e^{-x}$

Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

$$\theta_h = A_1\left[\sinh(\lambda z) + \cosh(\lambda z)\right] + A_2\left[\cosh(\lambda z) - \sinh(\lambda z)\right] + A_3$$

$$\theta_h = \sinh(\lambda z)(A_1 - A_2) + \cosh(\lambda z)(A_1 + A_2) + A_3$$

$$\theta_h = A\sinh(\lambda z) + B\cosh(\lambda z) + C \qquad (13)$$

Homogeneous solution

Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

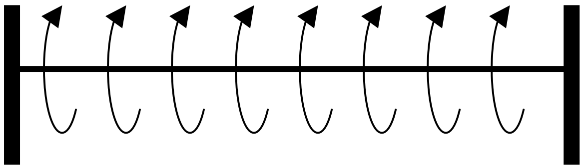
Particular solution

Consider M_z to be constant or linearly varying along the length

$M_z = f(z)$ [Constant or function of first degree]. θ_p may assumed to be a polynomial of degree up to 2, as twist due to pure torque is first integral of moment.

Let

$$\theta_P = f_1(z) \quad (14)$$



Uniform torque

e.g. $f_1(z) = Dz^2 + Ez + F$

Polynomial of second order. One order higher that applied torque.

Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

Try this particular integral in (10)

$$\frac{d^3 f_1(z)}{dz^3} - \lambda^2 \frac{df_1(z)}{dz} = -\frac{1}{EC_w} f(z)$$

As $M_z = f(z)$

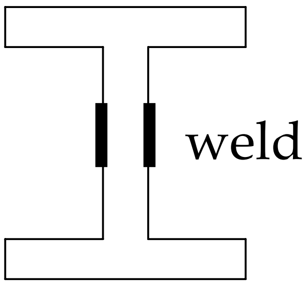
Polynomial of Ist order

$$\lambda^2 \frac{df_1(z)}{dz} = \frac{1}{EC_w} f(z) \qquad (15)$$

Boundary conditions

1- Torsionally Simply Supported

$$\theta = 0 \qquad \frac{d^2\theta}{dz^2} = 0 \qquad \frac{d\theta}{dz} \neq 0$$



Flanges can bend laterally

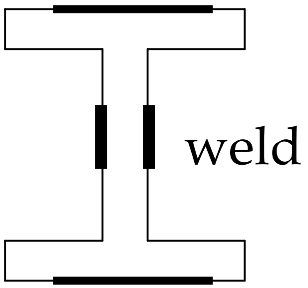
Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

This is equivalent to deflection and moment made equal to zero for simple support for bending. Change of twist $d\theta$ / dz may have any value at the end.
Flange may displace at the end but web is held at its position.

2- <u>Torsionally Fixed End</u>

$$\theta = 0 \quad \frac{d^2\theta}{dz^2} \neq 0 \quad \frac{d\theta}{dz} = 0$$



The constant of integration will be evaluated for individual cases.

Both Flanges and Web are connected

Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

After getting the value of constants and full solution for θ, the stresses may be evaluated as follows:

<u>Pure Torsional Shear Stress</u>

$$v_s = \frac{Tr}{C}$$

where

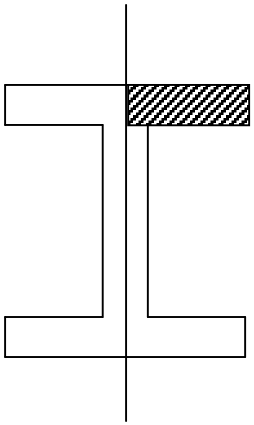
$$T = GC\frac{d\theta}{dz}$$

$$\boxed{v_s = Gt\frac{d\theta}{dz}} \qquad (16)$$

Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

Warping Shear Stress



No stress in the web

$$v_w = \frac{V_f Q_f}{I_f t_f}$$

From (5)

$$(v_w)_{\text{max. mag.}} = \frac{\left(EI_f \frac{h}{2} \times \frac{d^3\theta}{dz^3}\right) \times \left(t_f \times \frac{b_f}{2} \times \frac{b_f}{4}\right)}{I_f t_f}$$

$$(v_w)_{\text{max. mag.}} = E \frac{b_f^{\,2} h}{16} \frac{d^3\theta}{dz^3} \qquad (16)$$

Steel Structures

Differential Equation for Torsion of I-Shaped Section (contd…)

Normal Warping Stress

(in the flanges)

$$f_{bw} = \frac{M_f x}{I_f}$$

$$(M_f)_{mag} = EI_f \frac{d^2 u_f}{dz^2} \Rightarrow (M_f)_{mag} = EI_f \frac{h}{2} \frac{\mathrm{d}^2\theta}{\mathrm{d}z^2}$$

$$(M_f)_{mag} = E \frac{C_w}{h} \frac{\mathrm{d}^2\theta}{\mathrm{d}z^2}$$

$$(f_{bw})_{\max} = \frac{EI_f \frac{h}{2} \frac{d^2\theta}{dz^2} \times \frac{b_f}{2}}{I_f} \Rightarrow \boxed{(f_{bw})_{\max} = E \frac{hb_f}{4} \frac{d^2\theta}{dz^2}}$$

$(f_{bw})_{\max}$ is at the tips of flange

Steel Structures

DESIGN AND ALLOWABLE TORSION STRENGTHS

The design and allowable torsion strengths are below:

Design torsional strength in LRFD = $\phi_t T_n$

Allowable torsional strength in ASD = T_n / Ω_t

Resistance factor for torsion in LRFD = $\phi_t = 0.9$

Safety factor for torsion in ASD = $\Omega_t = 1.67$

Steel Structures

The nominal torsional strength (T_n) according to the limit states of torsional yielding and torsional buckling is:

$$T_n = F_n C$$

The following nomenclature may be used in the further discussion:

C = torsion constant

= $2(B - t)(H - t) - 4.5(4 - \pi)t^3$ for rectangular HSS

Steel Structures

C = $\dfrac{\pi(D-t)^2 t}{2}$ for round HSS

B = overall width of rectangular HSS

H = overall height of HSS

h = clear distance between the flanges less the inside corner radius on each side

D = outside diameter of round HSS

L = length of the member

Steel Structures

F_n For Round HSS

$$F_n = F_{cr} = \text{larger of } \frac{1.23E}{\sqrt{\frac{L}{D}\left(\frac{D}{t}\right)^{5/4}}} \text{ and } \frac{0.60E}{\left(\frac{D}{t}\right)^{3/2}}$$

but the value should not exceed $0.6F_y$

F_n For Rectangular HSS

i) For $\frac{h}{t} \leq 2.45\sqrt{\frac{E}{F_y}}$ $\quad F_n = F_{cr} = 0.6F_y$

Steel Structures

ii) For $2.45\sqrt{\frac{E}{F_y}} < \frac{h}{t} \leq 3.07\sqrt{\frac{E}{F_y}}$

$$F_n = F_{cr} = 0.6F_y\left(2.45\sqrt{\frac{E}{F_y}}\right)\frac{h}{t}$$

iii) For $3.07\sqrt{\frac{E}{F_y}} < \frac{h}{t} \leq 260$

$$F_n = F_{cr} = \frac{0.458\pi^2 E}{(h/t)^2}$$

Steel Structures

F_n For Other Sections

a) For the limit state of yielding under normal stress:

$$F_n = F_y$$

b) For the limit state of shear yielding under shear stress:

$$F_n = 0.6F_y$$

c) For the limit state of buckling

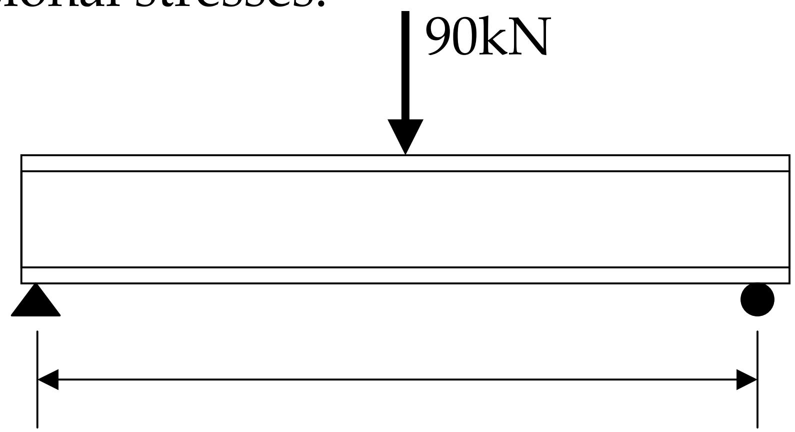
$$F_n = F_{cr}$$

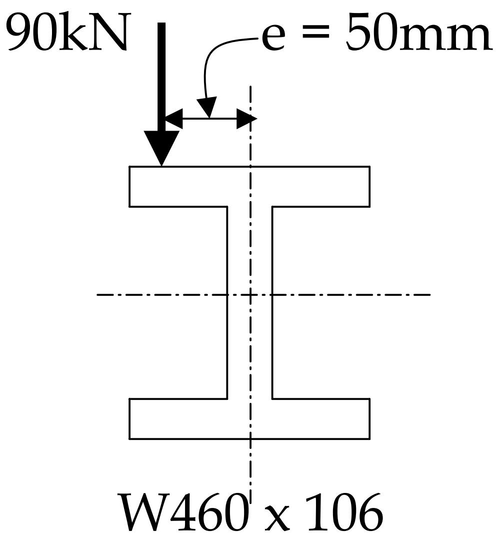
where F_{cr} for buckling is to be determined by detailed analysis.

Steel Structures

Example:

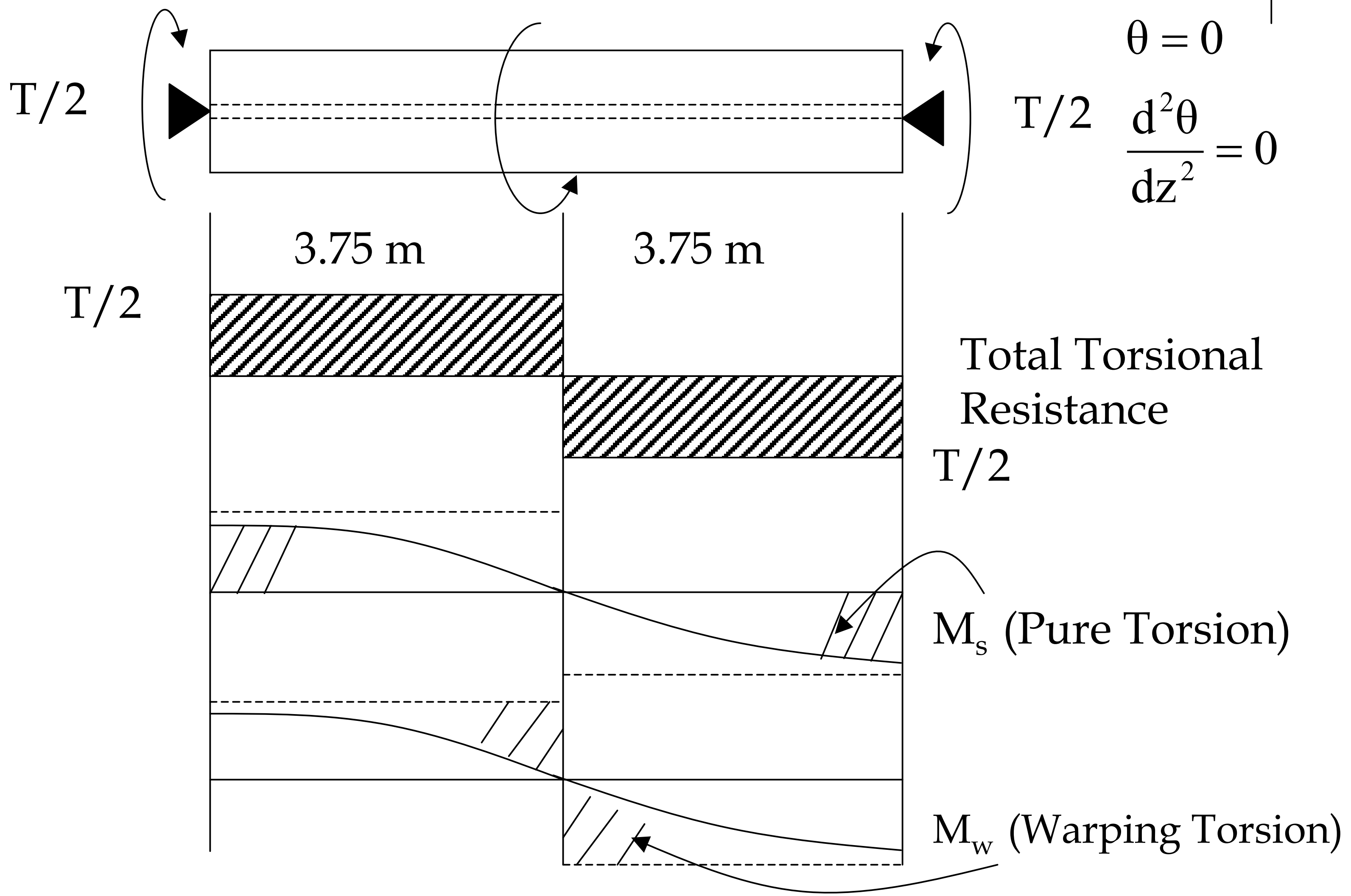
A W460 x 106 simply supported beam of span 7.5 m is subjected to a concentrated load of 90kN at mid-span at an eccentricity of 50mm from the plane of the web. The ends of the member are simply supported with respect to torsional restrain. Develop the expression for the angle θ and compute combined bending and torsional stresses.





Steel Structures

Solution:



Steel Structures

Solution: (contd…)

$$f(z) = T/2$$

$$T = P \times e = 90 \times 50$$

$$= 4500 kN - mm$$

$$\theta_P = C_1 + C_2 z = f_1(z)$$

- One order ahead

$$\frac{d^3 f_1(z)}{dz^3} - \lambda^2 \frac{df_1(z)}{dz} = -\frac{1}{EC_w} f(z)$$

where $\lambda^2 = \frac{GJ}{EC_w}$

$$0 - \lambda^2 (C_2) = -\frac{1}{EC_w} \frac{T}{2}$$

$$C_2 = \frac{T}{2} \frac{1}{EC_w} \times \frac{EC_w}{GJ} = \frac{T}{2GC}$$

So, the particular solution is:

$$\theta_P = C_1 + \frac{T}{2GC} \times z$$

Steel Structures

Solution: (contd…)

The total solution is

$$\theta = A\sinh(\lambda z) + B\cosh(\lambda z) + C + C_1 + \frac{T}{2GC}\times z$$

$$\boxed{\theta = A\sinh(\lambda z) + B\cosh(\lambda z) + \frac{T}{2GC}\times z + C}$$

Boundary Conditions

$$z = 0,\quad \theta = 0 \text{ ——— (I)}$$

$$z = 0,\quad \frac{d^2\theta}{dz^2} = 0 \text{ ——— (II)}$$

$$z = \frac{L}{2},\quad \frac{d\theta}{dz} = 0 \text{ ——— (III)}$$

Steel Structures

Solution: (contd…)

To apply the boundary condition first we have to take I[st] and 2[nd] derivatives

$$\frac{d\theta}{dz} = A\lambda \cosh(\lambda z) + B\lambda \times \sinh(\lambda z) + \frac{T}{2GC}$$

$$\frac{d^2\theta}{dz^2} = A\lambda^2 \sinh(\lambda z) + B\lambda^2 \times \cosh(\lambda z)$$

$$\frac{d^3\theta}{dz^3} = A\lambda^3 \cosh(\lambda z) + B\lambda^3 \times \sinh(\lambda z)$$

$$(\mathrm{I}) \Rightarrow 0 = 0 + \mathrm{B} + 0 + \mathrm{C} \quad \Longrightarrow \quad \mathrm{B} + \mathrm{C} = 0$$

$$(\mathrm{II}) \Rightarrow 0 = 0 + \mathrm{B}\lambda^2 \times 1 \quad \Longrightarrow \quad \boxed{\mathrm{B} = 0}$$

$$\Longrightarrow \quad \boxed{\mathrm{C} = 0}$$

Steel Structures

Solution: (contd…)

$$(\text{III}) \Rightarrow \quad 0 = A\lambda \cosh\left(\lambda \frac{L}{2}\right) + 0 + \frac{T}{2GC}$$

$$A = -\frac{T}{2GC\lambda}\left(\frac{1}{\cosh\dfrac{\lambda L}{2}}\right)$$

$$\theta = \frac{-T}{2GC\lambda}\left(\frac{1}{\cosh(\lambda L/2)}\right) \times Sinh(\lambda z) + \frac{T}{2GC} z$$

Steel Structures

Solution: (contd…)

$$\theta = \frac{T}{2GC\lambda}\left(\lambda z - \frac{\sinh(\lambda z)}{\cosh(\lambda L/2)}\right)$$

$$\frac{d\theta}{dz} = \frac{T}{2GC\lambda}\left(\lambda - \frac{\lambda\cosh(\lambda z)}{\cosh(\lambda L/2)}\right)$$

$$\frac{d\theta}{dz} = \frac{T}{2GC}\left(1 - \frac{\cosh(\lambda z)}{\cosh(\lambda L/2)}\right)$$

$$\frac{d^2\theta}{dz^2} = \frac{T\lambda}{2GC}\left(\frac{-\sinh(\lambda z)}{\cosh(\lambda L/2)}\right) \qquad \frac{d^3\theta}{dz^3} = \frac{T\lambda^2}{2GC}\left(-\frac{\cosh(\lambda z)}{\cosh(\lambda L/2)}\right)$$

Steel Structures

Solution: (contd…)

W 460 x 106

S_x = 2080 x 10^3 mm^3
I_x = 48, 700 x 10^4 mm^4
C = J = 145 x 10^4 mm^4
C_w = 12,62,119 x 10^6 mm^6
1/λ = 1501 mm
L = 7500 mm

h = d – t_f = 448.4 mm
t_f = 20.6 mm
t_w = 12.6 mm
b_f = 194 mm
d = 469 mm

Steel Structures

Solution: (contd…)

z	λz	Sinh(λz)	Cosh(λz)
0	0	0	1.000
0.1L	0.5	0.521	1.128
0.2L	0.999	1.174	1.542
0.3L	1.499	2.127	2.350
0.4L	1.999	3.623	3.759
0.5L	2.498	6.038	6.120

$$G = \frac{E}{2(1+\nu)} = \frac{2{,}00{,}000}{2(1+0.3)} = 76{,}923\text{MPa}$$

$$GC = 76{,}923 \times 145 \times 10^4 = 1115 \times 10^8\ N - mm^2$$

Steel Structures

Solution: (contd…)

Pure Torsional Shear Stress

$$v_s = Gt\frac{d\theta}{dz}$$

$$v_s = \frac{Tt}{2J}\left(1 - \frac{\cosh(\lambda z)}{\cosh(\lambda L/2)}\right)$$

$$v_s = \frac{4500 \times 1000t}{2 \times 145 \times 10^4}\left(1 - \frac{\cosh(\lambda z)}{\cosh(2.49)}\right)$$

$$v_s = 1.552t\left(1 - \frac{\cosh(\lambda z)}{6.120}\right)$$

Steel Structures

Solution: (contd…)

Maximum pure torsional shear stress is at the ends

$$\left(v_s\right)_{\substack{\max.\\ z=0,L}} = 1.55t\left(1-\frac{\cosh\left(\lambda\times 0\right)}{6.120}\right)$$

$$\left(v_s\right)_{\substack{\max.\\ z=0,L}} = 1.297t$$

$$\left(v_s\right)_{\substack{\max.\\ \text{for flange}}} = 1.297t_f = 1.297\times 20.6 = 26.72MPa$$

$$\left(v_s\right)_{\substack{\max.\\ \text{for web}}} = 1.297t_w = 1.297\times 12.6 = 16.34MPa$$

Steel Structures

Solution: (contd…)

Warping Shear Stress

In flanges

$$\left(v_w\right)_{\max} = \frac{E b_f^{\,2} h}{16} \frac{d^3\theta}{dz^3}$$

$$= \frac{E b_f^{\,2} h}{16} \times \frac{T\lambda^2}{2GC}\left(-\frac{\cosh(\lambda z)}{\cosh(\lambda L/2)}\right)$$

$$= \frac{T}{2C_w} \frac{b_f^{\,2} h}{16}\left(-\frac{\cosh(\lambda z)}{6.12}\right) \qquad \lambda^2 = \frac{GC}{EC_w}$$

$$= -0.307\cosh(\lambda z)$$

Steel Structures

Solution: (contd…)

Along the length maximum value will occur at z=L/2

$$\left(v_w\right)_{\max} \text{at midspan}_{z=L/2} = -0.307\cosh\left(\lambda \times \frac{L}{2}\right)$$

$$= -1.88 MPa$$

$$\left(v_w\right) \text{at ends}_{z=0} = -0.31 MPa$$

Steel Structures

Solution: (contd…)

Normal Warping Stress

$$\left(f_{bw}\right)_{\max} = \frac{Eb_f h}{4}\frac{d^2\theta}{dz^2}$$

$$\left(f_{bw}\right)_{\max} = \frac{Eb_f h}{4} \times \frac{T\lambda}{2GC}\left(-\frac{\sinh(\lambda z)}{6.12}\right)$$

$$\left(f_{bw}\right)_{\max} = -9.56\sinh(\lambda z)$$

As flanges are simply supported at ends, the maximum stress will be at mid-span

$$\left(f_{bw}\right)_{\max} = -9.56\sinh\left(\lambda \times \frac{L}{2}\right) = -57.69\,MPa$$

Steel Structures

Solution: (contd…)

Maximum Normal Stress due to Ordinary Flexure

$$f_b = \frac{M}{S_x} = \frac{PL/4}{S_x}$$

$$= \frac{90,000 \times 7500/4}{2080 \times 10^3}$$

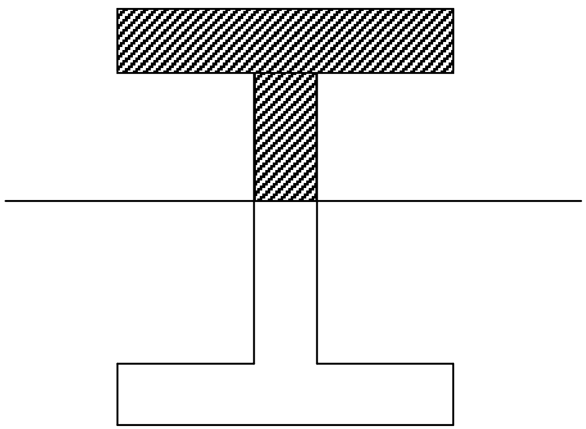
$$\boxed{f_b = 81.13 MPa}$$

Steel Structures

Solution: (contd…)

<u>Shear Stress due to Ordinary Bending</u>

$$v = \frac{VQ}{Ib}$$



At the N.A.:

$$\nu = \frac{45000\left(194 \times 20.6 \times \frac{448.4}{2} + 12.6 \times \frac{427.8}{2} \times \frac{427.8}{4}\right)}{48{,}700 \times 10^4 \times 12.6}$$

$$v = 8.68 MPa$$

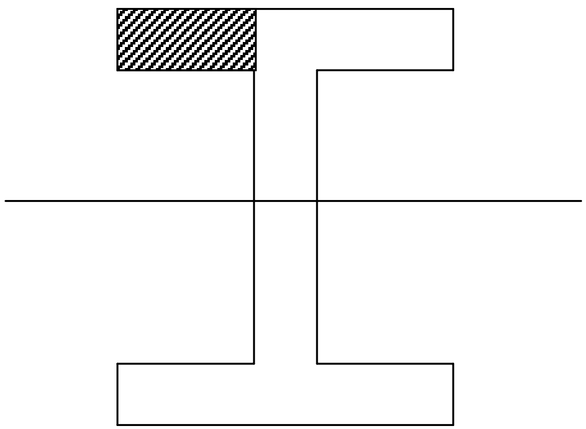
Steel Structures

Solution: (contd…)

At face of Web:

With in flange at edge of web

$$v = \frac{VQ}{Ib}$$



$$v = \frac{45000\left(\frac{194-12.6}{2}\times 20.6\times\frac{448.4}{2}\right)}{48{,}700\times 10^{4}\times 12.6} = 1.88MPa$$

$$v = 1.88MPa$$

Steel Structures

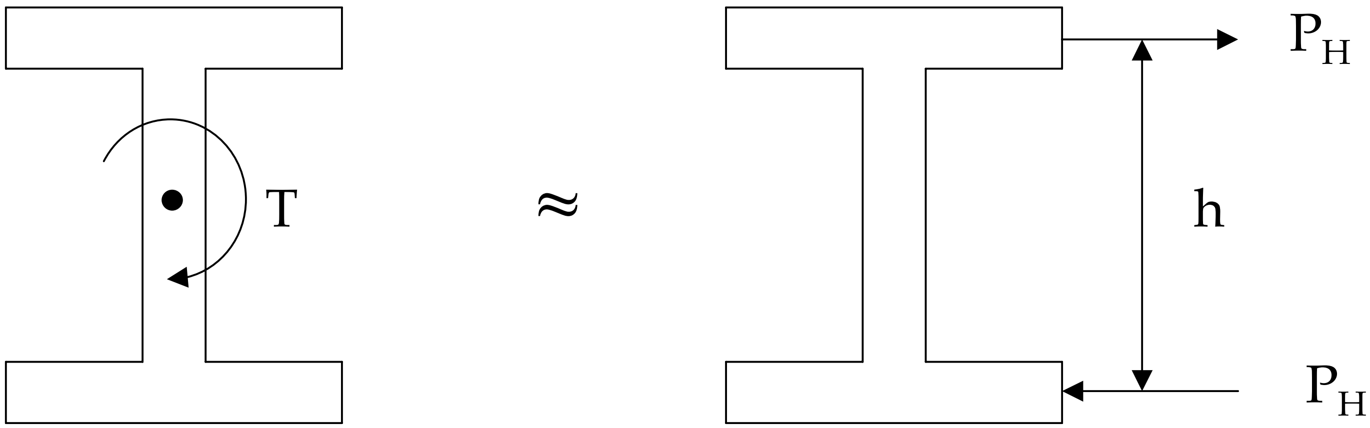
Summary of Stresses

Type of Stress	Support	Mid Span
Normal Stress • Vertical Bending, f_b • Torsional Bending, f_{bw}	 0 0	 81.13 57.69 **Sum = 138.82 MPa < 0.9 x 250 = 225 MPa O.K.**
Shear Stress in Web • Pure Torsion, ν_s • Vertical Bending, ν	 16.34 8.68 Sum = 25.02 MPa **< 0.9x 0.6x 250 = 135 OK**	 0 8.68 Sum = 8.68 MPa **< 0.9x 0.6x 250 = 135 OK**
Shear Stress in Flange • Pure Torsion, ν_s • Warping Torsion, ν_w • Vertical Bending, ν	 26.72 0.31 1.88 Sum = 28.91 MPa **< 135 MPa OK**	 0 1.88 1.88 **Sum = 3.76 MPa < 135 MPa, O.k.**

Results: Beam is safe in flexure, torsion and shear at all the sections

Steel Structures

Analogy Between Warping Torsion and Lateral Bending



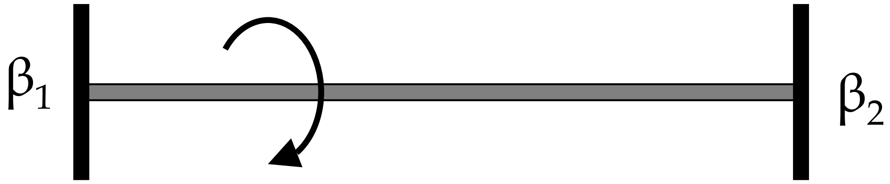
$$P_H x\ h = T$$

$$P_H = T/h$$

Steel Structures

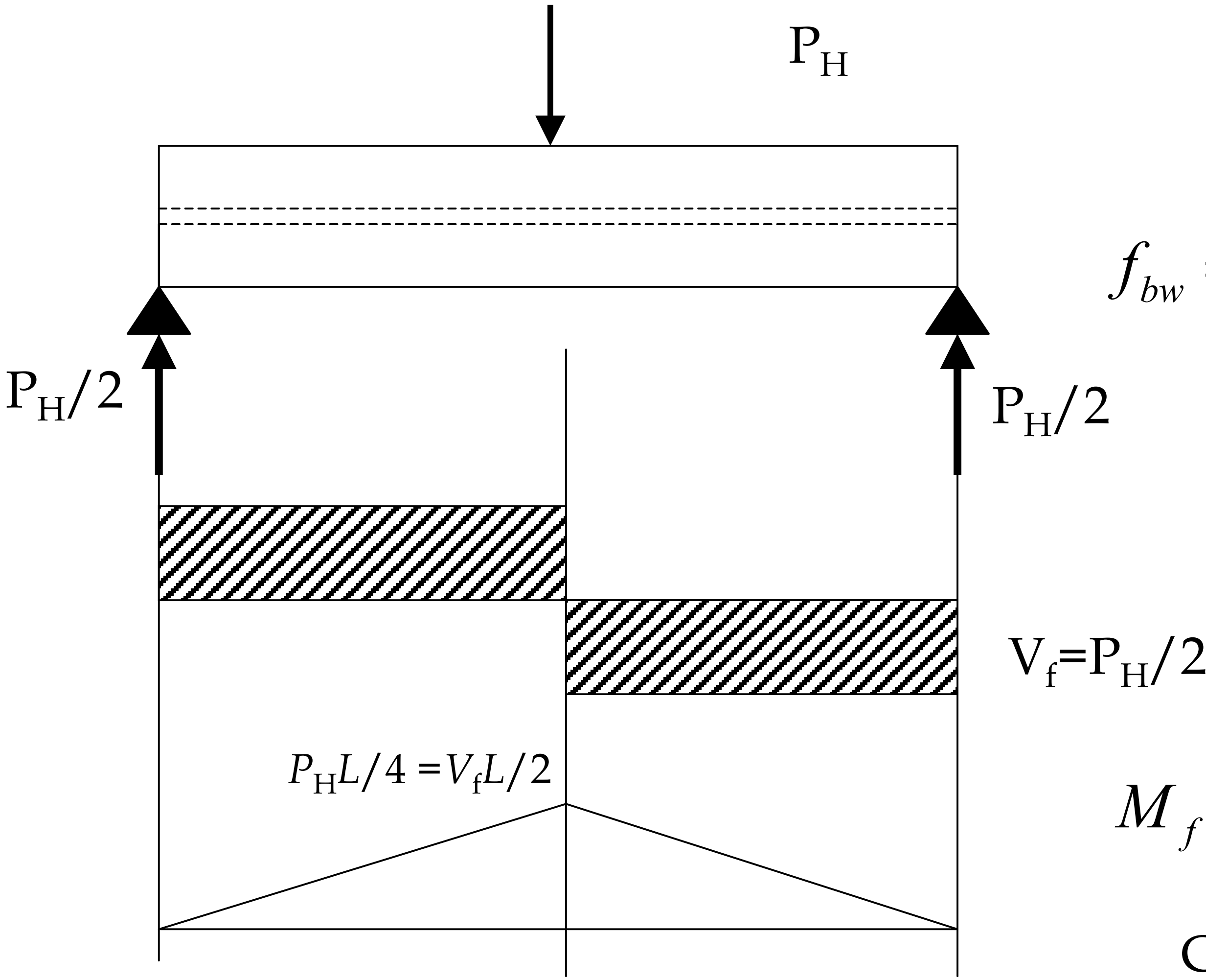
Analogy For Torsion (contd…)

- Because the differential equation solution is time consuming, and really suited only for analysis, design of a beam to include torsion is most conveniently done by making the analogy between torsion and ordinary bending
- It is assumed that all the torque is resisted by warping torsion which is not the actual situation (solution will be approximate).
- β factor is used to reach near to actual solution.
- β factors are problem specific values, depending on end conditions.
- Tables have been proposed for β factor to cover different situations.
- β factor tables are available on Page # 476 & 477, (Salmon & Johnson)



Steel Structures

Analogy For Torsion (contd…)



$$v_s = \frac{T \times t}{C}$$

$$f_{bw} = \frac{M_f}{S_y/2} = \frac{2M_f}{S_y}$$

$$v_w = \frac{V_f Q_f}{I_f t_f}$$

$$M_f = \beta \times V_f \times \frac{L}{2}$$

Correction factor

Steel Structures

Example:

Select a W section for a beam to carry 9kN/m dead load including the self weight, and a live load of 24 kN/m. The load is applied at an eccentricity of 175mm from center of web. The simply supported span is 8.0 m. Assume that ends of beam are simply supported for torsion.

Solution:

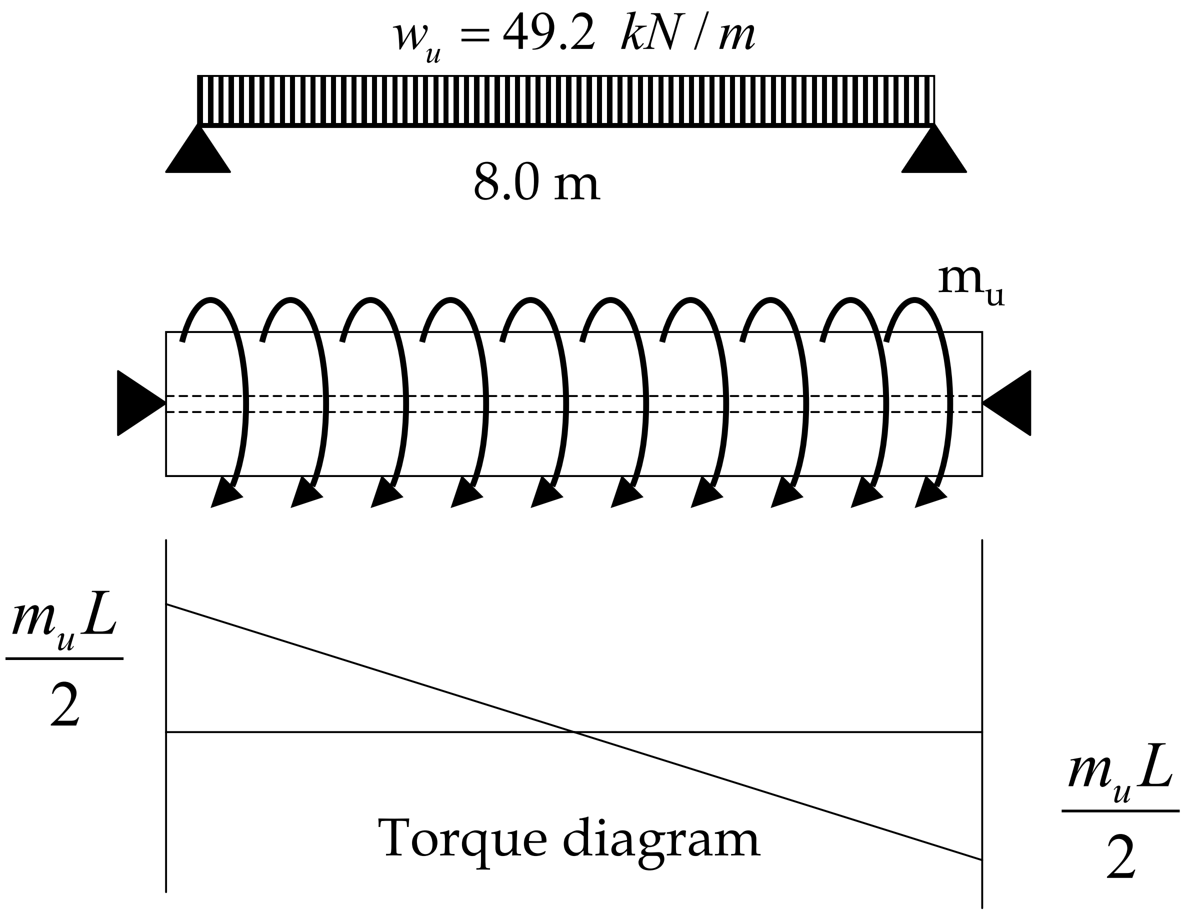
$$w_u = 1.2D + 1.6L$$

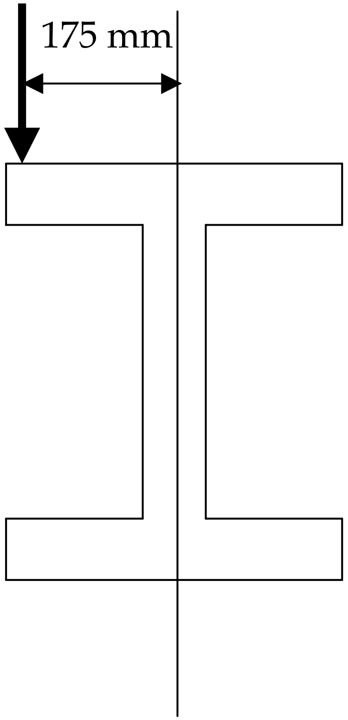
$$= 1.2 \times 9 + 1.6 \times 24$$

$$= 49.2 \quad kN/m$$

Steel Structures

Solution: (contd…)





Steel Structures

Solution: (contd…)

$$M_{ux} = \frac{49.2 \times 8.0^2}{8} = 393.60\ \ kN - m$$

$$m_u = 49.2 \times \frac{175}{1000} = 8.61\ \ kN - m / m$$

$$d_{\min} = \frac{L}{22} = \frac{8000}{22} = 364mm$$

Let $h \cong 364mm$

Assume $\lambda L = 3.0$ Initial assumption

Steel Structures

Solution: (contd…)

$$M_f = \beta \frac{w_H L^2}{8}$$

$$M_f = \beta \frac{m_u}{h} \times \frac{L^2}{8} \qquad w_H = \frac{m_u}{h}$$

z = 0.5L, a = 0.5

From table 8.6.8, P # 477

$$\lambda L = 3 \Rightarrow \beta = 0.51$$

$$M_f = 0.51 \times \frac{8.61}{0.364} \times \frac{8.0^2}{8} = 96.51 \text{ kN} - \text{m}$$

Steel Structures

Solution: (contd…)

$$\left(S_x\right)_{req} = \frac{M_{ux}}{\phi_b F_y} + \frac{2M_f\left(S_x / S_y\right)}{\phi_b F_y}$$

Approximate value

$$\left(S_x\right)_{req} = \frac{393.60\times10^6}{0.9\times250} + \frac{2\times96.51\times10^6\left(2.75\right)}{0.9\times250}$$

$$\left(S_x\right)_{req} = 4109\times10^3\,mm^3$$

Steel Structures

Solution: (contd…)

Where high torsional strength is required, W360 sections are preferable because these usually give less stresses due to torsional warping.

Check conditions of compact section

Trial Section

W 360 x 237

b_f = 395 mm	$b_f/2t_f$ = 6.5	I_y = 31100 x 10^4 mm^4
t_f = 30.2 mm	h/t_w = 13.7	t_w = 18.9 mm
S_y = 1580 x 10^3 mm^3	d = 380 mm	C = J = 824 x 10^4 mm^4
S_x = 4160 x 10^3 mm^3	I_x = 79100 x 10^4 mm^4	$1/\lambda$ = 1735 mm

Steel Structures

Solution: (contd…)

Assuming that $L_b \le L_p$, no problem of LTB

$$\lambda L = \frac{1}{1735} \times 8000 = 4.61$$

$$\beta = 0.27 + \frac{0.37 - 0.27}{1} \times (5.0 - 4.61) = 0.309$$

$$h = d - t_f = 380 - 30.2 = 349.8mm$$

λL	β
4	0.37
5	0.27

$$M_f = \beta \frac{m_u}{h} \times \frac{L^2}{8}$$

$$= 0.309 \times 8.61 \times \frac{8.0^2}{8} \times \frac{1}{0.3498} = 60.85kN - m$$

Steel Structures

Solution: (contd…)

Normal Bending Stress At Mid-span

$$f_{un} = \frac{M_{ux}}{S_x} + \frac{2M_f}{S_y}$$

$$= \frac{393.60 \times 10^6}{4160 \times 10^3} + \frac{2 \times 60.85 \times 10^6}{1580 \times 10^3}$$

$$= 171.64 MPa$$

$$< \phi_b F_y = 225 MPa$$ **O.K.**

Steel Structures

Solution: (contd…)

<u>Shear Stress</u>

- Warping torsion…………Critical at center
- Vertical bending…………Critical at ends
- Pure torsion……………...Critical at ends.

Warping Shear Stress At Mid-Span:

$$v_w = \frac{V_f Q_f}{I_f t_f}$$

$$V_f = \beta \frac{m_u}{h} = 0.309 \times \frac{8.61 \times 1000}{0.3498} = 7606 \text{ N}$$

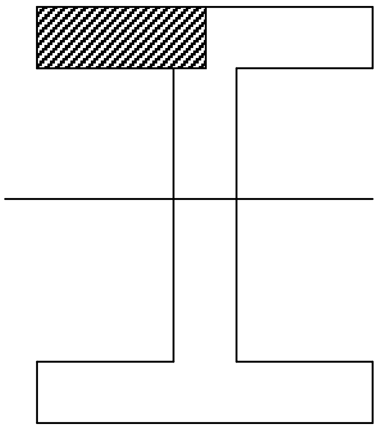
Steel Structures

Solution: (contd…)

$$Q_f = \frac{b_f}{2} \times t_f \times \frac{b_f}{4} = 589 \times 10^3\, mm^3$$

$$I_f = \frac{I_y}{2} = 15550 \times 10^4\, mm^4$$

$$v_w = \frac{7606 \times 589 \times 10^3}{15550 \times 10^4 \times 18.9} = 1.524\,MPa \quad < 135\text{MPa}$$



O.K.

Total shear stress = 1.524 MPa, because there is no applied shear at the center and there is no simple torsion ***O.K.***

Steel Structures

Solution: (contd…)

Web Shear Stress (end section):

$$v = \frac{(wL/2) \times Q}{I_x \times t_w}$$

At N.A.

$$Q = \left(b_f \times t_f \times \frac{h}{2} \right) + t_w \left(\frac{h - t_f}{2} \right) \times \left(\frac{h - t_f}{4} \right) = 2328 \times 10^3 \, mm^3$$

$$v = \frac{(49.2 \times 8.0/2) \times 1000 \times 2328 \times 10^3}{79100 \times 10^4 \times 18.9}$$

$$= 30.65 MPa$$

Steel Structures

Solution: (contd…)

Pure Torsion (end section):

$$v_s = \frac{T \times t_w}{C}$$

$$T = \frac{m_u L}{2} = 34.44 \text{ kN} - \text{mm}$$

$$v_s = \frac{34.44 \times 10^6 \times 18.9}{824 \times 10^4} = 79.00 MPa$$

Including small warping contribution in the same formula

Total Shear stress at end section $= 30.65 + 79.00 = 109.65 MPa$

$< 135 \text{MPa}$ O.K.

Steel Structures

Flange Shear Stress (end section):

$$v = \frac{(wL/2) \times Q}{I_x \times t_w}$$

At Junction of Web and Flange

$$Q = \frac{b_f}{2} \times t_f \times \frac{b_f}{4} = 1043 \times 10^3 mm^3$$

$$v = \frac{196.8 \times 1000 \times 1043 \times 10^3}{79100 \times 10^4 \times 30.2}$$

$$= 8.59 MPa$$

Steel Structures

Pure Torsion (end section):

$$v_s = \frac{T \times t_w}{C}$$

$$T = \frac{m_u L}{2} = 34.44 \ \text{kN} - \text{mm}$$

$$v_s = \frac{34.44 \times 10^6 \times 30.2}{824 \times 10^4} = 113.62 MPa$$

Including small warping contribution in the same formula

Total shear stress at end section $= v + v_s + v_s$

$$= 8.59 + 113.62 + 0 = 122.21 MPa$$

$$< 135 \text{MPa}$$ O.K.

Steel Structures

Table. Values of λ and C		
Designation	$1/\lambda$	$C = J$ ($\times 10^4$ mm^4)
W360 × 216	1869	633
× 237	1735	824
× 262	1600	1100
× 287	1483	1450
× 314	1389	1860
× 347	1288	2480
× 382	1196	3290
× 421	1118	4330
× 463	1046	5660
× 509	980	7410
× 551	932	9240
× 592	892	11400
× 634	853	13800

Steel Structures

Concluded