Steel Structures

M.Sc. Structural Engineering

SE-505

Minimum Weight Design

The simple plastic design procedure starts by assigning ratios to the plastic moments of resistance of the various members, approximately targeting economy of the structure.

By this procedure, the member economy at the most may be satisfied.

However, the capacity of individual members may also affect the design of other members and overall economy of the structure becomes a complex phenomenon.

The equilibrium method of analysis involves plotting of simply supported moment diagram and a reactant line.

The position of this reactant line may be adjusted by selecting M_p – values at some critical sections and a number of design solutions may thus be obtained.

Factors other than strength may also be introduced to decide the best design.

Examples of such factors are limiting deflections, minimum total weight, availability of sections, convenience of fabrication and minimum total cost.

The most useful criterion is that of minimum total weight and will be discussed here.

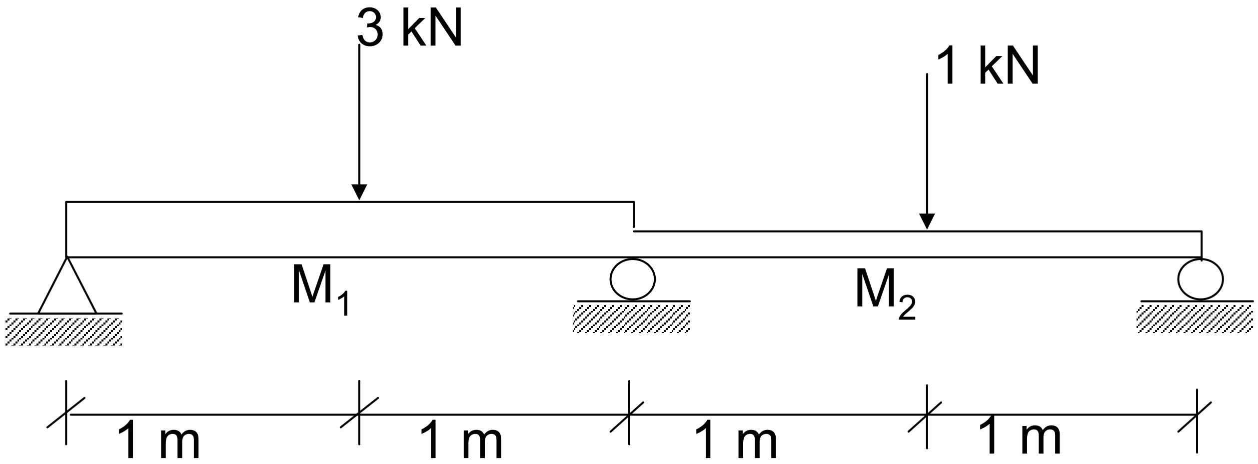
It is necessary to adopt some relationship between weight per unit length "g" and plastic moment "M".

In general, $g \propto M^n$ or $g = k M^n$
(*k* is just a scale factor)

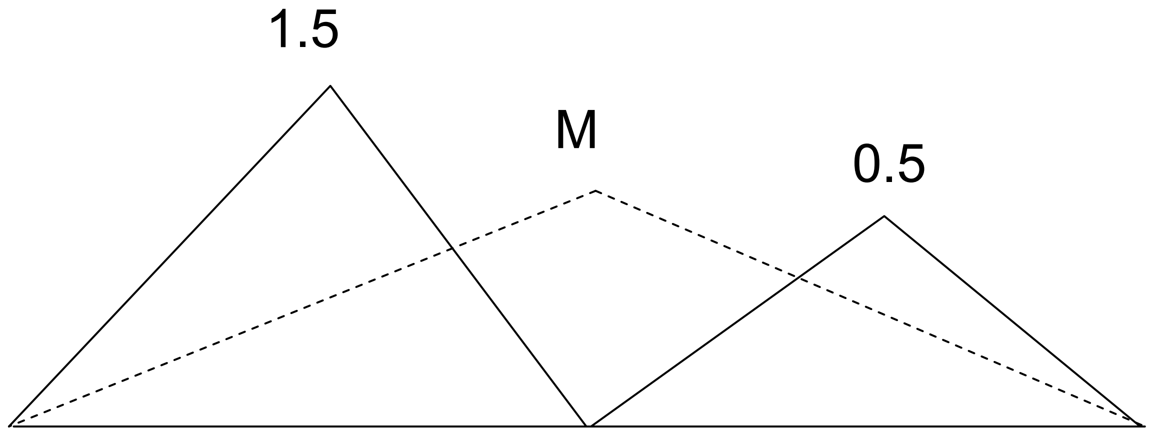
For W and S sections, $n \approx 0.6$

For a structure consisting of a number of prismatic members of length L_i with full plastic moment M_i, the total weight G can be expressed as:

$$G = k \sum L_i M_i^n$$



3 kN
1 kN
M1
M2
1 m
1 m
1 m
1 m



1.5
M
0.5

Consider the example of a two bay continuous beam with equal spans as shown in the figure.

The total weight of the continuous beam, to some scale (k may be considered equal to 1), can be expressed as follows:

$G = M_1^n + M_2^n$ (I)

Considering reactant line with

$M_2 = M \quad : \quad M_1 = 3 / 2 – 1 / 2 M$ (II)

Combining equations I and II:

$G = (M_1 = 3 / 2 – 1 / 2 M)^n + M^n$ (III)

The minimum weight problem is resolved into finding the value of M to give the least value of G from Eq. III.

1. Minimum Value of M_2

As already considered M is the moment at the central support and is equal to smaller moment capacity out of the two beams, M_2.

For required strength of the right panel:

$$M_2 + 0.5\,M_2 = 0.5$$

$$\Rightarrow \quad (M_2)_{min} = 1 / 3$$

2. Maximum Value of M_2

The maximum value at the most can be equal to the moment capacity of the larger section, $(M_2)_{max} = M_1$

3. Minimum Value of M_1

The value will be the minimum when the maximum value of M_2 and hence M is used.

$$0.5\, M_2 + M_1 = 1.5$$

$$0.5\, M_1 + M_1 = 1.5$$

$$\Rightarrow \quad (M_1)_{min} = 1$$

4. Maximum Value of M_1

The value will be the maximum when the minimum value of M_2 (= 1 / 3)is used.

$$0.5\,M_2 + M_1 = 1.5$$

$$0.5 \times 1/3 + M_1 = 1.5$$

$$\Rightarrow \quad (M_1)_{max} = 4/3$$

To minimize the weight, we get:

$$\frac{dG}{dM} = n\left[M^{n-1} - \frac{1}{2}\left(\frac{3}{2} - \frac{1}{2}M\right)^{n-1}\right] = 0$$

$$M^{n-1} = \frac{1}{2}\left(\frac{3}{2} - \frac{1}{2}M\right)^{n-1}$$

$$M = \left(\frac{1}{2}\right)^{\frac{1}{n-1}}\left(\frac{3}{2} - \frac{1}{2}M\right) = \left(\frac{1}{2}\right)^{\frac{n}{n-1}}(3 - M)$$

$$2^{\frac{n}{n-1}}M + M = 3$$

$$\left[1 + (2)^{\frac{n}{n-1}}\right]M = 3$$

$$M_{min} = \frac{3}{1 + (2)^{\frac{n}{n-1}}}$$

For n = 0.6, M_{min} = 0.784

However, $M_{min} = M_2$ is known to be 1 / 3, giving M_1 equal to 4 / 3.

The answer is not correct and, for the value of M less than this limit, *dG* / *dM* is positive showing a continuous decrease in the weight.

In this example, differentiation has not given the final solution, but merely has indicated that M_2 should be reduced and M_1 increased correspondingly.

With unequal spans and different loads, M_2 might have to be increased and a uniform beam design might result.

In either case, a general trend only is indicated, and the least-weight design occurs when other considerations enter into the problem.

It should be noted that other considerations (such as equilibrium condition, $M_2 \geq 1/3$) are in no way dependent on the form of the function assumed to give the total weight o the structure.

Other simpler functions for the weight may also lead to same minimum design, Provided that they are not too different from the true expression.

Further, the sections for a particular data do not vary over full range of sections available.

For example, function of linear form may be taken:

$$g = a + k M \qquad \text{V}$$

$$G = \sum L_i (a + k M_i)$$

$$= a \sum L_i + k \sum L_i M_i$$

The term $a \sum L_i$ is constant for a structure and cannot be varied. Further, the constant "k" is a scale factor on the weight W. We may consider:

$$G = (M_1 + M_2)\, 2L \qquad \text{VI}$$

or $\quad G = M_1 + M_2$

as 2L is a constant multiplier.

From Eq. II,

$$G = (1.5 - 0.5M) + M$$

$$= 1.5 + 0.5\,M_2 \qquad \text{since } M = M_2$$

Again $dG\,/\,dM_2$ does not yield a solution. However, it is quite clear hat smallest possible value of M_2 will give the minimum weight. The same solution as before is obtained.

METHODS OF INEQUALITIES

The same example of two span beam will be used. The fact that the bending moment at each of the three critical sections must be not greater than the full plastic value can be expressed as:

Midspsan: $M_1 \geq 1.5 - 0.5M$ (left span)
and $M_2 \geq 0.5 - 0.5M$ (right span) VII

Central Support: $M_1 \geq M$ and $M_2 \geq M$ VIII

Inequalities VIII are required as it is not known in the general that whether M_1 is greater or M_2 is greater. For loading to give opposite sign moment, we have

$-M_1 \geq 1.5 - 0.5M$; $-M_2 \geq 0.5 - 0.5M$; $-M_1 \geq M$
and $-M_2 \geq M$ IX

In the present case, signs are known and inequalities IX are not required. Inequalities VII (last two) and VIII (first two) may be simplified as:

i) $-M + M_1 \geq 0$

ii) $-M + M_2 \geq 0$

iii) $M + 2M_1 - 3 \geq 0$

iv) $M + 2M_2 - 1 \geq 0$ X

Adding (i) with (iii) and (iv) and also (ii) with (iii) and (iv) to eliminate M, we get:

i) $3M_1 + 3 \geq 0$

ii) $M_1 + 2M_2 - 1 \geq 0$

iii) $2M_1 + M_2 - 3 \geq 0$

iv) $3M_2 - 1 \geq 0$ XI

Considering $M_1 \geq 1$, the 2nd inequality gives $2M_2 \geq$ –ve value, which is always satisfied and is redundant.

Putting $M_1 = G - M_2$, we get:

$$G - M_2 - 1 \geq 0$$

$$2G - M_2 - 3 \geq 0$$

$$M_2 - 1/3 \geq 0 \qquad \text{XII}$$

Adding 3rd inequality to 1st and 2nd:

$$G - 4/3 \geq 0 \quad \text{or} \quad G \geq 4/3$$

$$2G - 10/3 \geq 0 \quad \text{or} \quad G \geq 5/3 \qquad \text{XIII}$$

$$G \geq 5/3 \qquad \text{XIV}$$

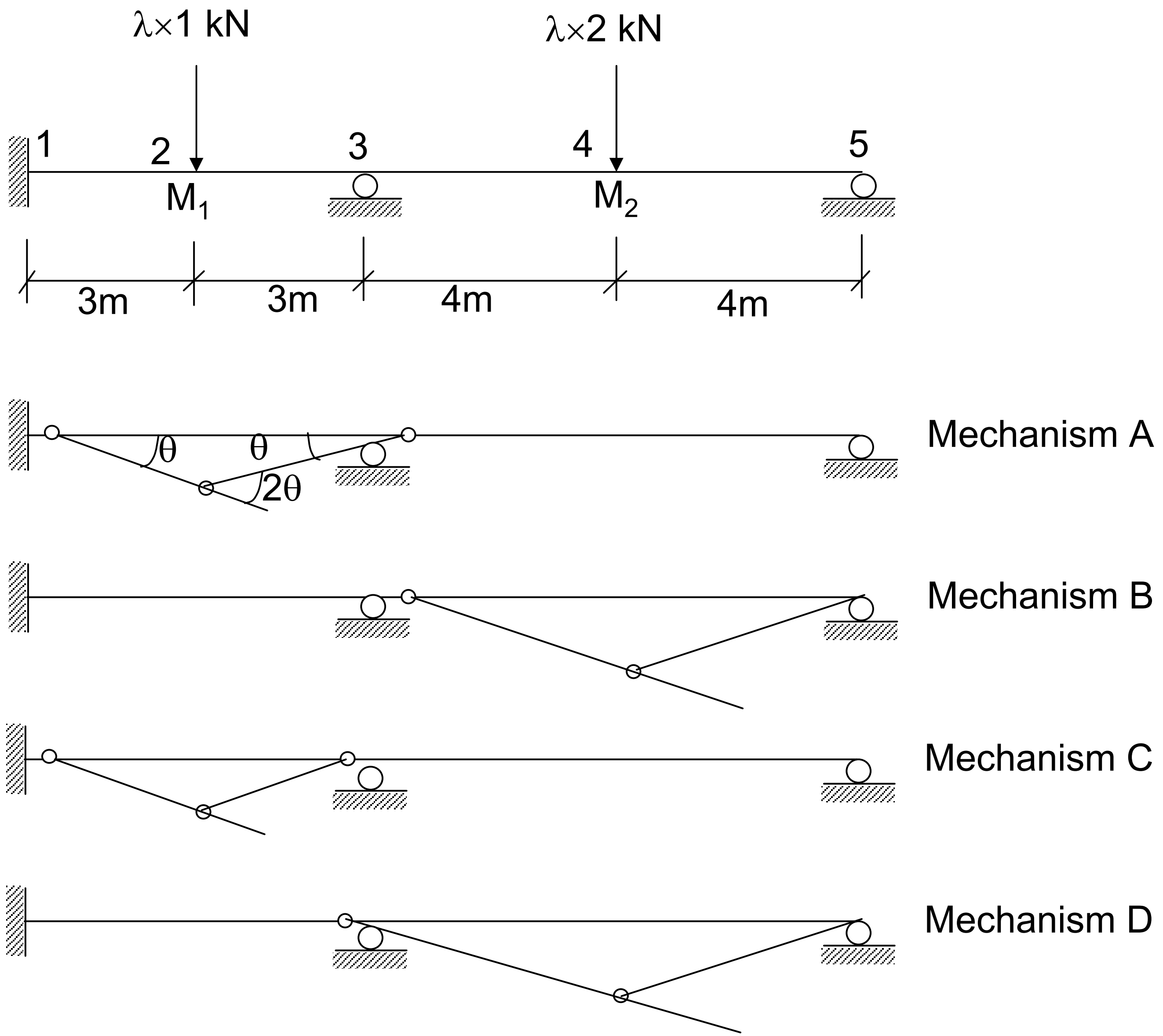
Putting G = 5/3 in 2nd of inequalities XII:

$$M_2 \le 2 \times 5/3 - 3 \text{ or } \quad M_2 \le 1/3$$

The 3rd inequality gives $M_2 \ge 1/3$.

Hence the only possible value is $M_2 = 1/3$, implying that $M_1 = 4/3$.

Example



λ×1 kN
λ×2 kN
1
2
3
4
5
M1
M2
3m
3m
4m
4m
θ
θ
2θ
Mechanism A
Mechanism B
Mechanism C
Mechanism D

Consider the design of the two-span beam with the plastic moments of the left-side and right-side spans being M_1 and M_2, respectively.

It is required to determine the values of M_1 and M_2 to give minimum total weight, G, for the given loads.

The factor λ is a load factor that must be one for minimum weight.

$$G = k\,(6\,M_1 + 8\,M_2)$$

There are four possible mechanisms A, B, C and D.

Mechanisms A and B are postulated on the assumption that $M_1 > M_2$ and mechanisms C and D on the assumption that $M_1 < M_2$.

It is necessary to postulate both sets of mechanisms since it is not known at the start which span should have the larger plastic moment.

Note that every mechanism in this case can be the final mechanism provided that the M_p values are selected accordingly.

The load factors for these mechanisms are as under:

$$\lambda_A = \frac{3M_1 + M_2}{3} \qquad \lambda_B = \frac{3M_2}{8}$$

$$\lambda_C = \frac{4M_1}{3} \qquad \lambda_D = \frac{M_1 + 2M_2}{8}$$

For each λ = 1, the equations may be plotted on a $M_1 - M_2$ space.

$$\lambda_A = 1 = \frac{3M_1 + M_2}{3} \qquad 3 = 3M_1 + M_2$$

For $M_1 = 0$, $M_2 = 3$

For $M_2 = 0$, $M_1 = 1$

$$\lambda_B = 1 = \frac{3M_2}{8} \qquad M_2 = 2.7$$

$$\lambda_C = 1 = \frac{4M_1}{3} \qquad M_1 = 0.75$$

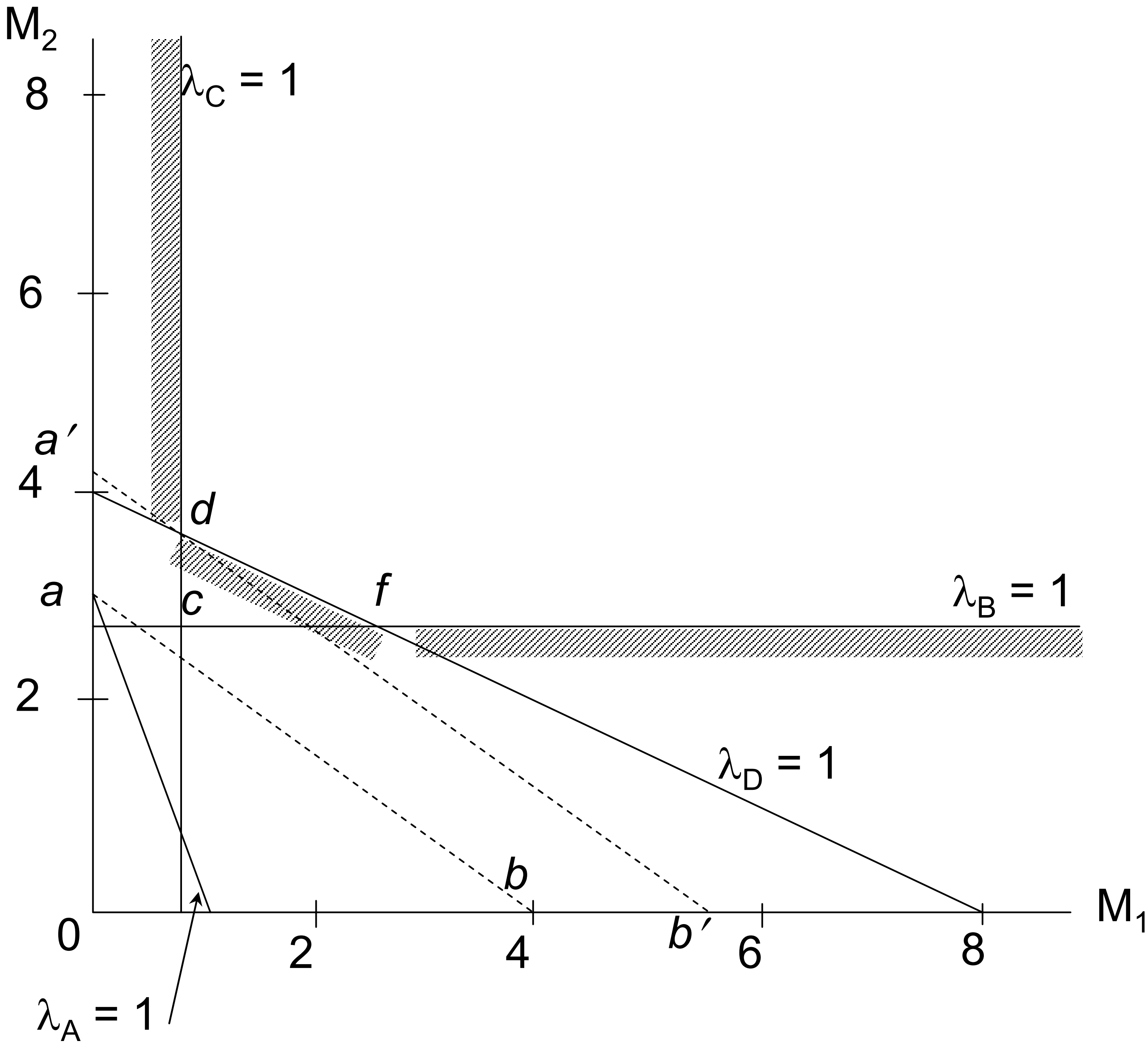
$$\lambda_D = 1 = \frac{M_1 + 2M_2}{8} \qquad 8 = M_1 + 2M_2$$

For $M_1 = 0$, $M_2 = 4$

For $M_2 = 0$, $M_1 = 8$

Combinations of M_1 and M_2 to the left and below the mechanism lines show relative values of M_1 and M_2 that are safe (λ at least equal to 1.0).

The area bounded by these mechanism lines forms a permissible region in which any design in which any safe design must lie.


M2
8
λC = 1
6
a′
4
d
a
c
f
λB = 1
2
λD = 1
b
0
2
4
b′
6
8
M1
λA = 1

Mechanism A will not actually occur as when left span is stronger, the right span may fail because it has more loads.

The weight function (as shown by dashed line in the figure) may also be plotted by a straight line anywhere but maintaining the correct $M_1 : M_2$ ratio. Any set of parallel lines may then be drawn.

$$G / k = 6 M_1 + 8 M_2$$

When $M_1 = 0$, $M_2 = G / 8k$

When $M_2 = 0$, $M_1 = G / 6k$

M_1 intercept : M_2 intercept = G / 6k : G / 8k

= 4 (horizontal) : 4 (vertical)

A parallel line that just touches the permissible region represents the minimum weight design.

The point of intersection in this example is the point "d".

For this point, $M_1 = 0.75$

and from mechanism D, $8 = M_1 + 2 M_2$

$M_2 = 29/8$

Hence $G = k(6 M1 + 8 M2) = 33.5 k$

Any other point, such as “f”, gives greater G.

$M_2 = 8 / 3$

and from mechanism D, $8 = M_1 + 2 M_2$

$M_2 = 8 / 3$

Hence $G = k (6 M_1 + 8 M_2) = 37.3 k$

In the above example, slope of the weight line *a'b'* which touches the boundary of the permissible region at "d" is intermediate between the slopes of the mechanism lines C and D which intersect at "d".

Weight line has the form:

$$G = k\,(6M_1 + 8M_2) \qquad \text{I}$$

Mechanism C: $3 = 4M_1$ II

Mechanism D: $8 = M_1 + 2M_2$ III

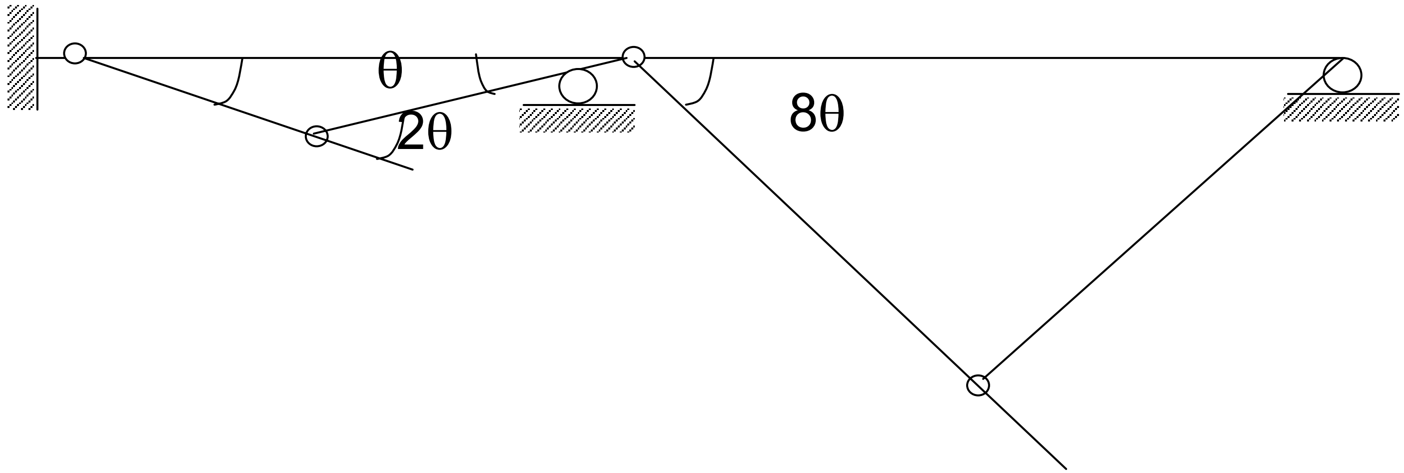
The statement that *a'b'* has a slope intermediate between those of Eqs. II and III mean algebraically that it must be possible to combine these equations with positive multiplier μ with Eq. III.

Adding $\mu \times$ Eq. III and Eq. II, following is obtained:

$$3 + 8\mu = (4 + \mu) M_1 + 2 \mu M_2 \qquad \text{IV}$$

The resulting equation is identical in slope with Eq. I if $\mu = 8$, making Eq. IV the following:

$$69 = 12 M_1 + 16 M_2$$



The addition of mechanism C to eight times mechanism D leads to the above mechanism. In this mechanism, the total hinge rotation $\sum\theta$ associated with any plastic moment of resistance M is proportional to the total lengths of members with that plastic moment.

Span 1: $\sum\theta = 4 + 8 = 12$ $\sum L = 6$

Span 2: $\sum\theta = 16$ $\sum L = 8$

Uniqueness Of Minimum Weight

For "j" members of a structure,

$$G = k \sum_j M_j L_j \qquad \text{I}$$

Suppose it is possible to postulate a mechanism involving total plastic hinge rotations ϕ_j associated with plastic moments M_j, together with corresponding displacements Δ_i associated with the loads W_i, such that in the work equation

$$\sum_j W_j \phi_j = \sum_i M_i \Delta_i$$

The condition $\phi_j = \alpha_j L_j$ is satisfied where "α" is a constant. Any plastic rotation ϕ_j is composed of individual rotations ϕ_{jk} at points h_{jk}, so that

$$\phi_j = \sum_k \phi_{jk}$$

Then, providing the bending moments throughout the structure satisfy the equilibrium and yield conditions for plastic collapse, the structure is a minimum weight structure for the given loads.

A design thus gives the minimum weight if it satisfies the following four conditions:

(i) Equilibrium condition.

(ii) Yield condition ($M \leq M_p$).

(iii) Mechanism condition (collapse mechanism is produced).

(iv) Plastic hinge condition ($\phi_j = \alpha L_j$).

Upper Bound On Minimum Weight

Any design for which a set of bending moments satisfying conditions (a) and (b) is available gives an upper bound on the minimum weight.

In terms of the graphical representation, any design satisfying the equilibrium and yield conditions must lie within the permissible region, and cannot lie nearer to the origin than the tangent weight line.

Any design for which a set of moments satisfying conditions (a), (b) and (c) is available will just collapse under the specified loads, and therefore also give an upper bound on the minimum weight.

In terms of the graphical representation, such a design lies on the boundary of the permissible region.

Lower Bound On Minimum Weight

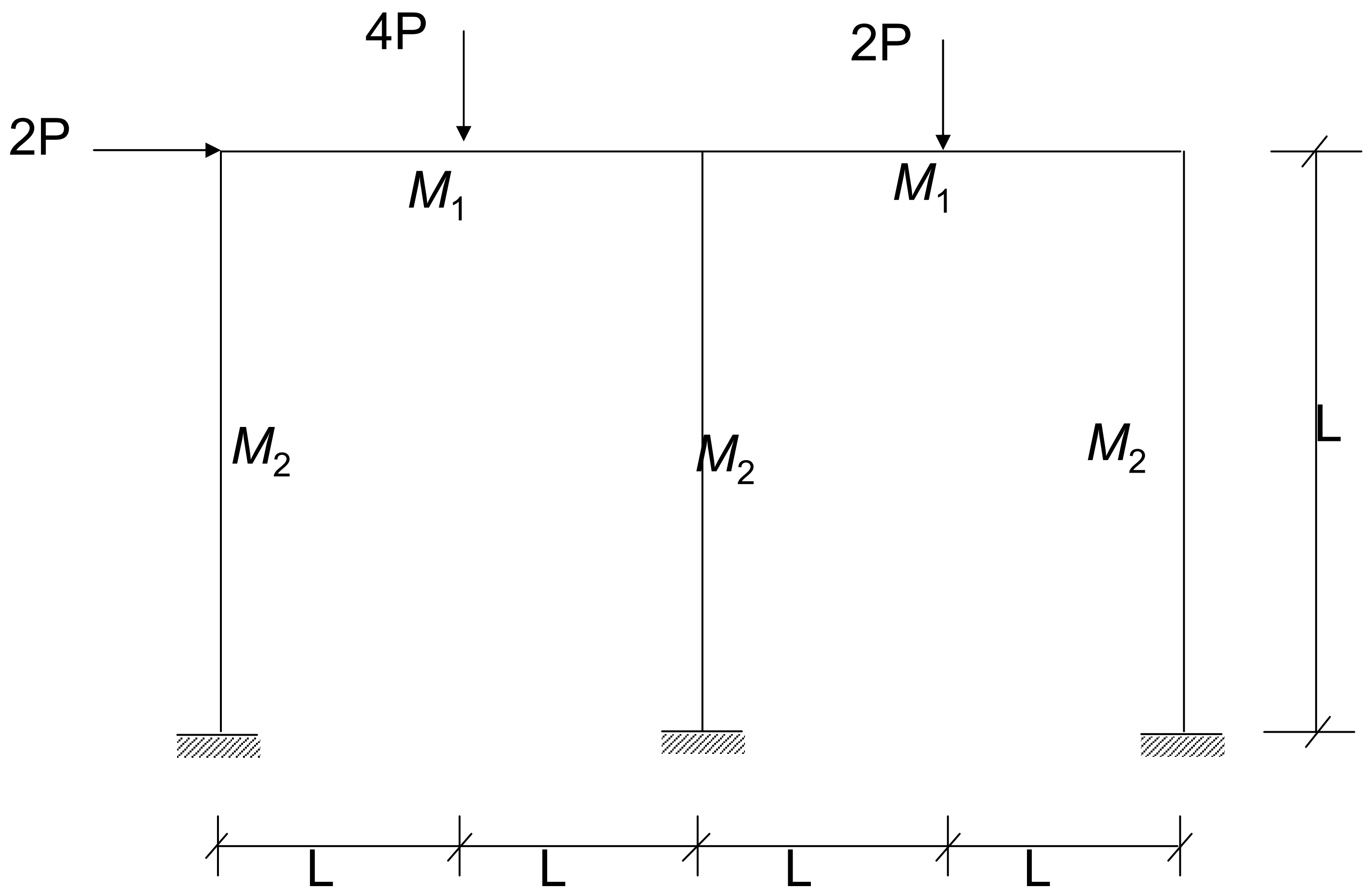
Any design satisfying conditions (c) and (d) provides a lower bound on the minimum weight.

For example, a weight line may be drawn through “e” of the previous example, representing a combination of mechanisms B and C, and the corresponding weight is a lower bound on the minimum weight.

In the absence of the first two conditions, the design will be unsafe.

Example

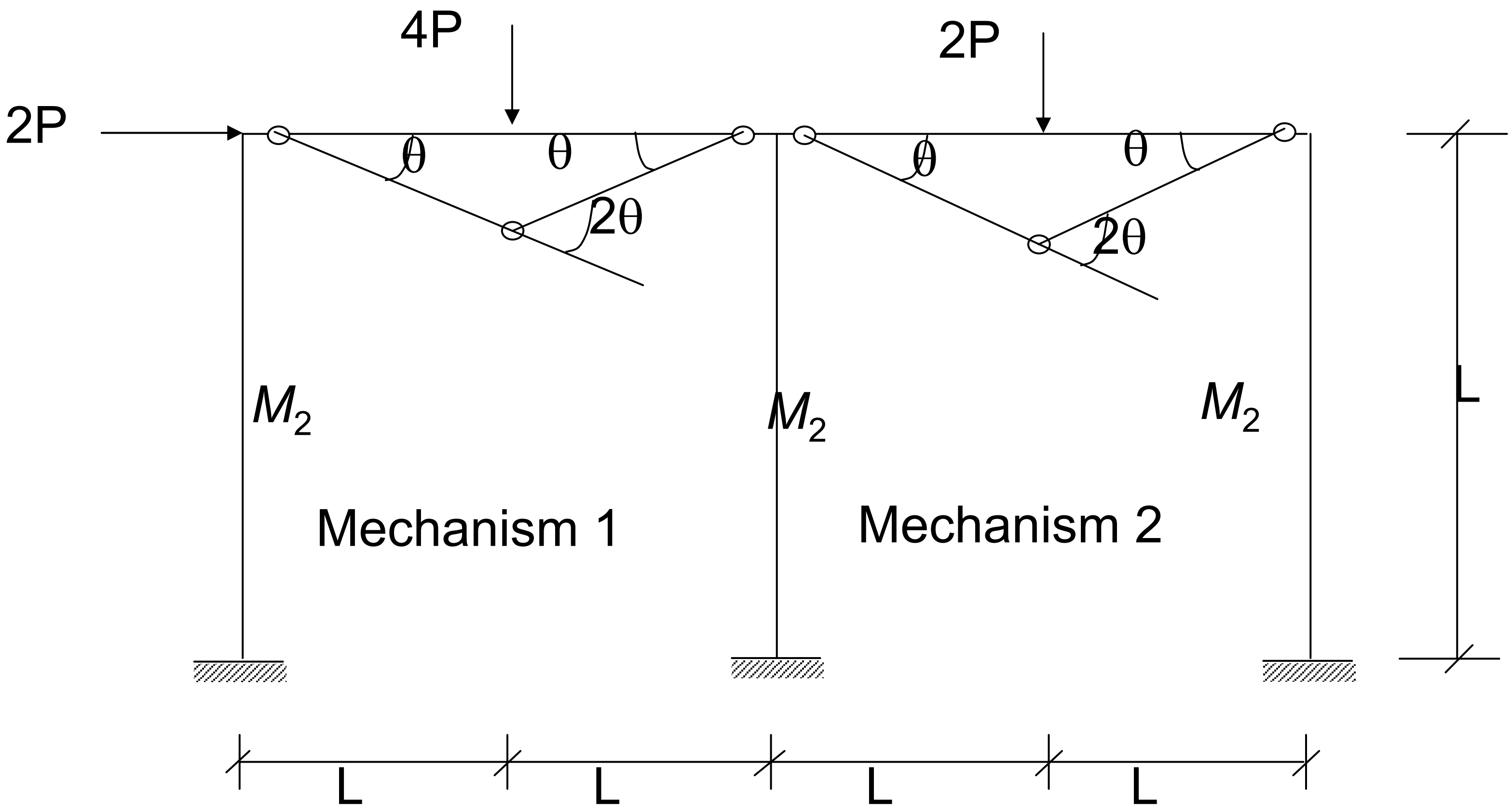
Design the following frame for minimum weight.



Solution

$$G = k\,(4L \times M1 + 3L \times M2)$$

$$= 4\,M1 + 3\,M2 \qquad \text{(omitting the constants)}$$



Mechanism 1

$$4\,P \times \theta\,L = 4\,M_1\,\theta$$

$$M_1 = PL$$

If PL is considered equal to 1.0, then $M_1 = 1$

Eq. II

Mechanism 2

$$2\,P \times \theta\,L = 4\,M_1\,\theta$$

$$M_1 = 0.5\,PL$$

If PL is considered equal to 1.0, then $M_1 = 0.5$

Eq. III

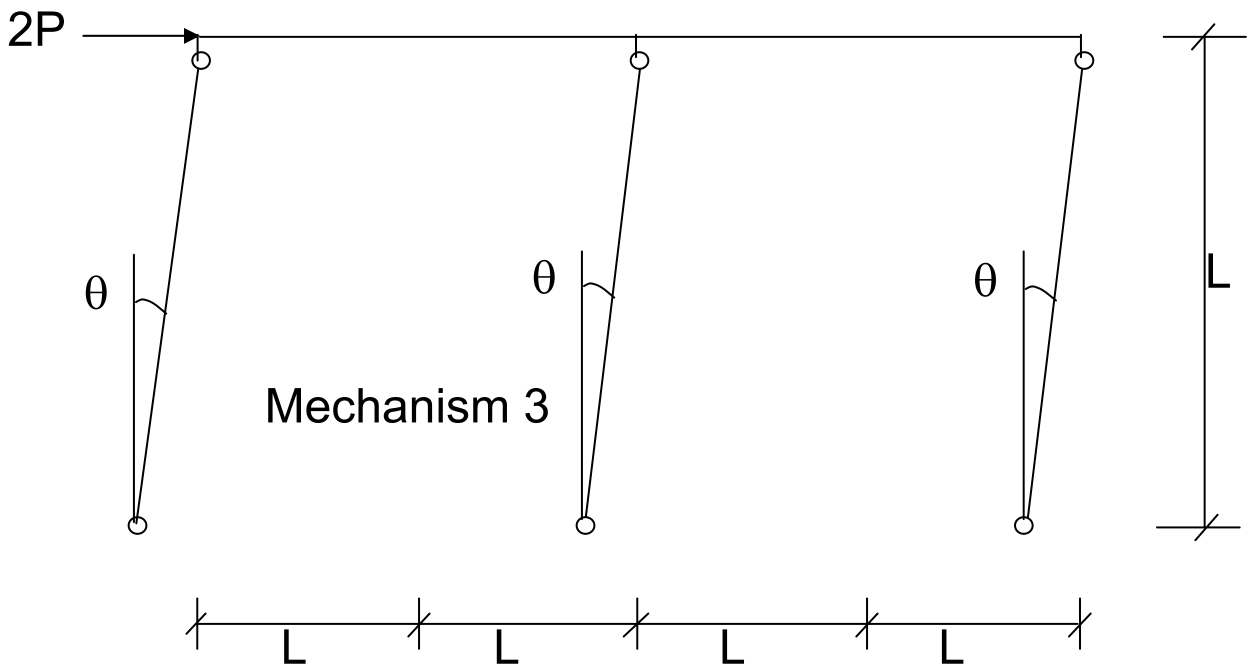
Mechanism 3

$$2P \times \theta L = 6 M_2 \theta$$

$$M_2 = 0.33 PL$$

If PL is considered equal to 1.0, then

$$M_1 = 0.33 \qquad \text{Eq. IV}$$



Mechanism 4

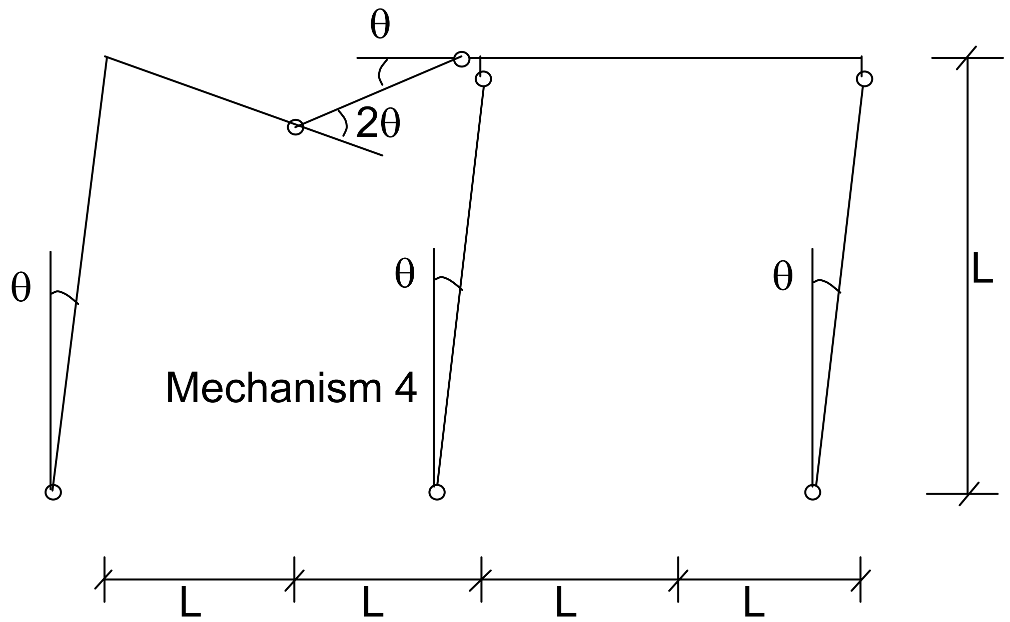
$$2\,P \times \theta\,L + 4\,P \times \theta\,L = 3\,M_1\,\theta + 5\,M_2\,\theta$$

If PL is considered equal to 1.0, then

$3M_1 + 5M_2 = 6$ Eq. V

If $M_2 = 0$, $M_1 = 2$

and if $M_1 = 0$, $M_2 = 6/5 = 1.2$



Mechanism 5

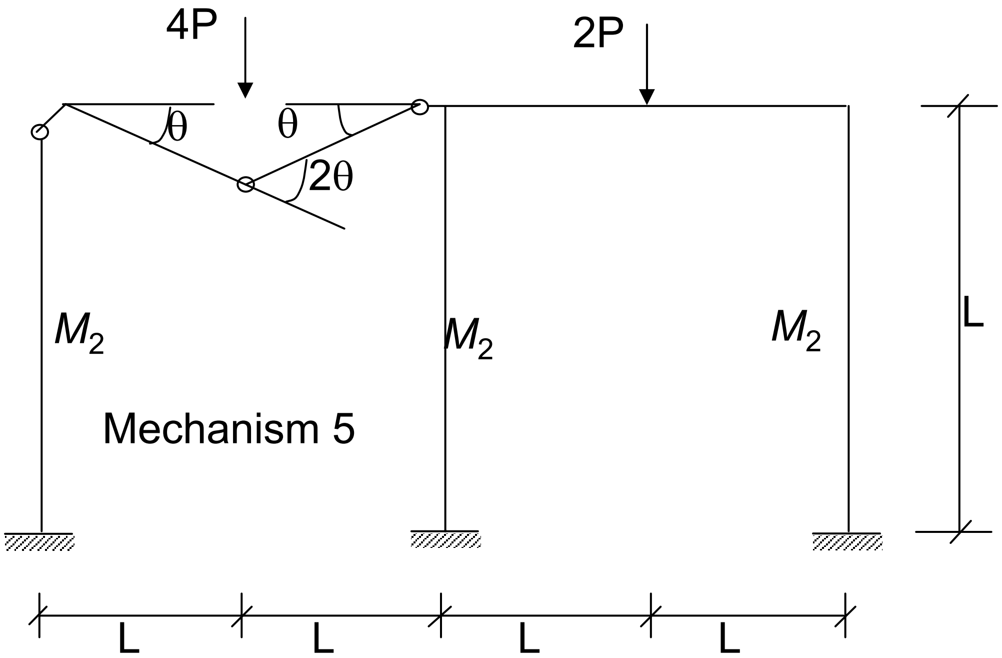
$$4P \times \theta L = 3M_1\theta + M_2\theta$$

If PL is considered equal to 1.0, then

$$3M_1 + M_2 = 4 \qquad \text{Eq. VI}$$

If $M_2 = 0$, $M_1 = 1.33$

and if $M_1 = 0$, $M_2 = 4$



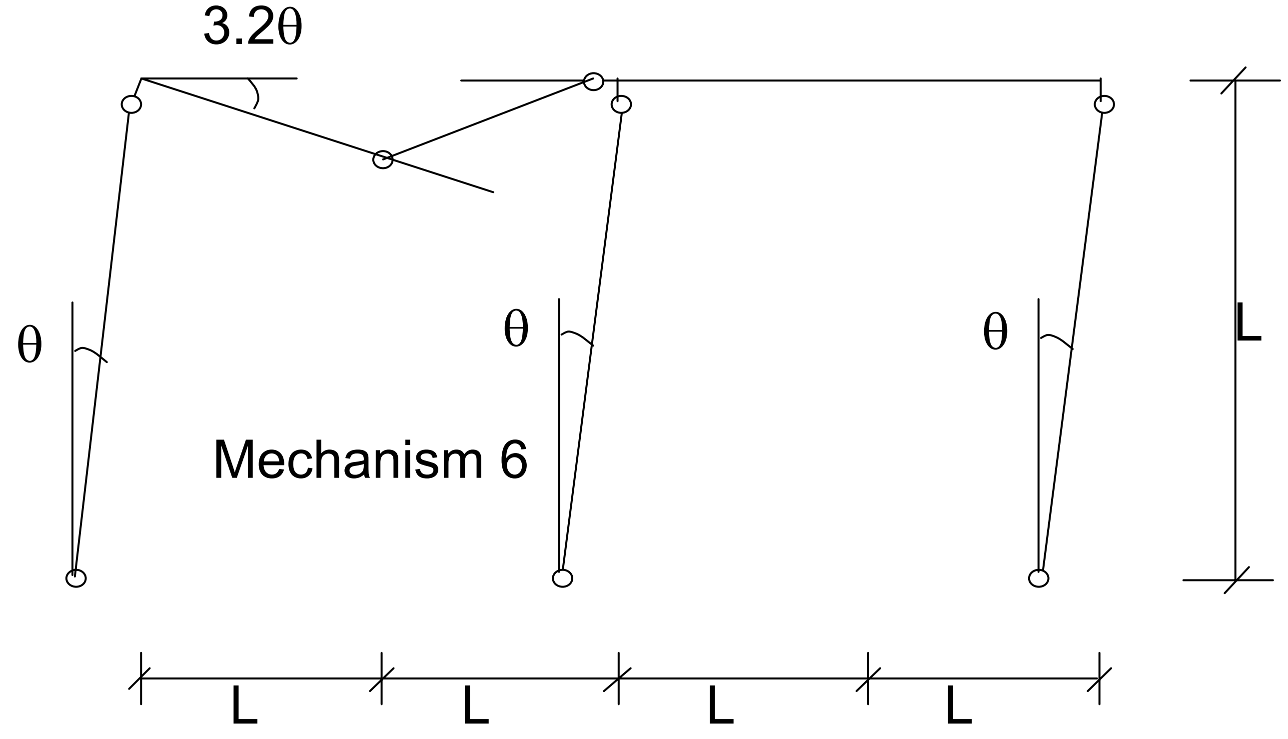
Mechanism 6

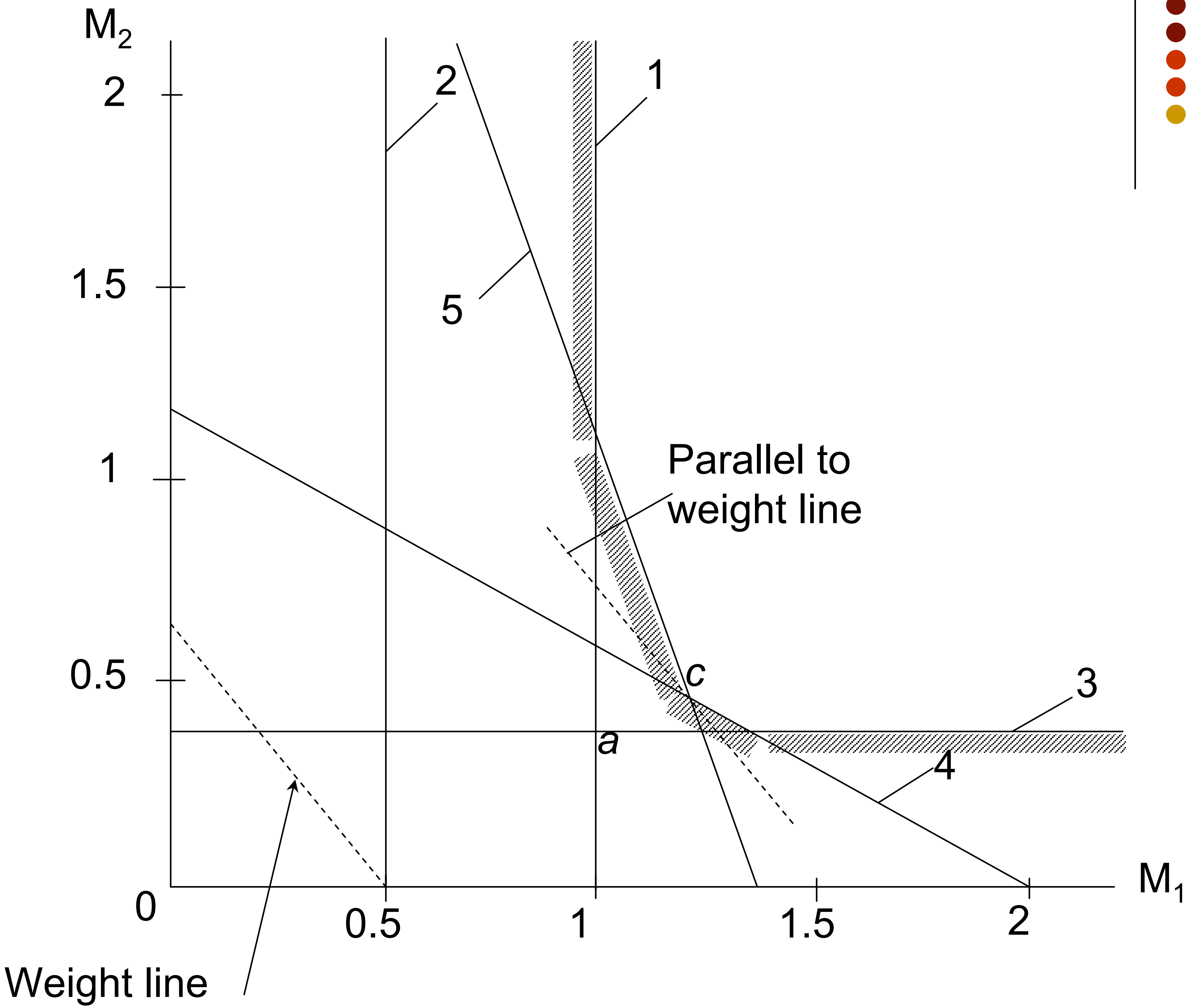
$= \text{Mechanism 4} + 2.2 \times \text{Mechanism 5}$

$9.6M_1 + 7.2 M_2 = 14.8$ Eq. VI

If $M_2 = 0$, $M_1 = 1.54$

and if $M_1 = 0$, $M_2 = 2.06$





M2
2
1.5
1
0.5
0
0.5
1
1.5
2
M1
1
2
5
Parallel to
weight line
c
a
3
4
Weight line

Weight line has the form:

$$G = 2 M1 + 1.5 M2$$

(By taking 2 as common and eliminating it)

For $G = 1$

If $M1 = 0$, $M2 = 0.67$ and if $M2 = 0$, $M1 = 0.5$

The next step is to maintain M1 and M2 in the same ratio, but to increase their values until a point on the boundary of the permissible region is obtained.

The weight line intersects the permissible region at point c, which is the point of intersection of mechanisms 4 and 5.

Slope of weight line $= \dfrac{dM_2}{dM_1} = -1.33$

Slope of mechanism-4 line $= \dfrac{dM_2}{dM_1} = -0.6$

Slope of mechanism-5 line $= \dfrac{dM_2}{dM_1} = -3$

The slope of the weight line is in-between the slopes of mechanisms 4 and 5. The weight line passing through "c" is obtained by combining Eq. V and μ times Eq. VI. Adding $\mu \times$ Eq. VI and Eq. V, following is obtained:

$3(1+\mu)M_1 + (5+\mu)M_2 = 6 + 4\mu$ Eq. VII

The slope of this equation must be same as that of the weight line.

$$3\,(1 + \mu) + (5 + \mu)\frac{dM_2}{dM_1} = 0$$

or $$\frac{dM_2}{dM_1} = -\frac{3(1+\mu)}{(5+\mu)} = -\ 4/3$$

$$\frac{3(1+\mu)}{(5+\mu)} = 4/3 \quad \text{or} \quad \mu = 2.2 \qquad \text{Eq. VIII}$$

This gives mechanism-6 with the following equation:

$$9.6M_1 + 7.2\,M_2 = 14.8 \qquad \text{Eq. IX}$$

Eq. V – VI gives the intersection point "c":

$$4\,M_2 = 2 \qquad M_2 = 0.5$$

From Eq. V:

$$3\,M_1 + 5 / 2 = 6 \qquad M_1 = 7 / 6$$

$$\therefore \qquad G_{min} = 4\,(7 / 6) + 3\,(1 / 2) = 6.17$$

Concluded