YIELD LINE ANALYSIS OF SLABS

Yield line analysis of slabs is identical to plastic design of frames consisting of skeletal elements.

In frames, ***plastic hinges*** are formed at maximum moment sections.

This means that, at these sections, large inelastic rotations may occur at almost constant resisting moments.

This constant resisting moment is called the ***plastic moment***.

After the formation of initial hinges without loosing the internal stability of indeterminate structures, more loads may be applied due to moment redistribution and utilization of strength of less-stressed sections.

Difference of a fictitious plastic hinge from a real hinge is that, in case of the plastic hinge, free rotations occur at a constant moment level.

Below the plastic moment value, the rotations are, however, locked.

When sufficient number of these hinges is formed adjacent to each other, internal stability of the structure causes infinitely larger deformations causing collapse of the structure.

This condition is referred to as the formation of ***collapse mechanism*** or simply mechanism.

Yield line formation is a similar mechanism that takes place in slabs with the difference that the plastic hinges concentrated at points are replaced by lines of free rotation at constant moments.

When the slab is loaded beyond a certain limit, a fictitious hinge is formed over a certain straight length of the slab having maximum moment; this line is called ***yield line***.

The yield line serves as an axis of rotation for the slab segment and large inelastic rotation may occur at nearly the same moment per unit length measured along a yield line.

YIELD-LINE PATTERN

When sufficient number of yield-lines is formed such that any further load causes very large infinite deformations, exceeding the permissible deflection limit state, the resulting system is called collapse mechanism or simply mechanism or ***yield-line pattern***.

Plastic moment capacity at a yield line is determined by the usual strength or limit state method.

Yield line forms at the section where the flexural reinforcement yields.

The plastic moment capacity at the yield line is assumed equal to the ultimate strength of a section, $\phi_b M_n$, distributed over the length ($\phi_b m_n$ per unit length of the yield line).

The position and orientation of he yield lines in a yield line pattern depends on the boundary conditions, the nature of loading and the geometric dimensions.

The actual analysis of complex systems is either quite lengthy or is not possible.

Hence simplifying assumptions are made to reach at reasonably accurate solutions.

Two general methods may be used for the approximate solutions depending upon the nature of the simplifying assumptions, namely, lower bound and upper bound methods.

For approximate elastic or plastic analysis, certain assumptions are to be made in the procedures.

Either the moments are distributed in the start depending upon experience as in direct design method or the failure pattern (collapse mechanism) is assumed in the start as in the plastic analysis.

Both of these starting assumptions may not give the actual ultimate load and different trials may be required to reach at an answer close to the actual ultimate load.

Similarly detailed analysis of determinate and indeterminate slabs is generally very complicated.

According to the general theory of plasticity, the true ultimate collapse load of a structure lies between two limits, an upper bound and a lower bound of the collapse load.

These limits can be found by well-established methods. A full solution is obtained when both the upper and lower bound solutions converge to a single solution, as shown in Fig. 13.1.

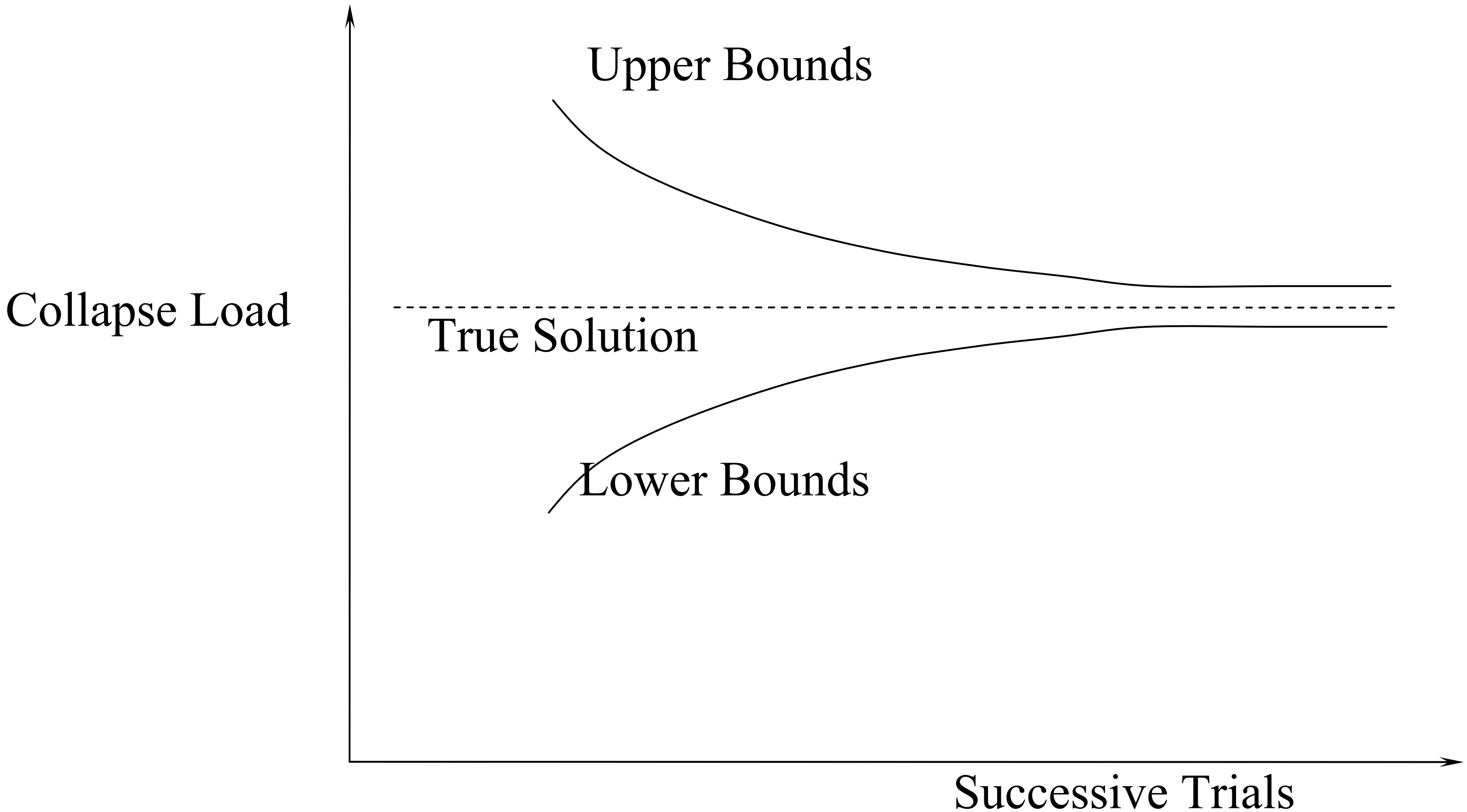


Fig. 13.1. Concept of Upper And Lower Bounds.

Lower Bound Method

For a given slab system the lower bound method gives an ultimate load, which is either correct or low.

That is, the ultimate load is never overestimated and there is no possibility that the ultimate load is below our calculated collapse load.

The distribution of loads or moments is decided at the start depending on the past experience.

The ultimate load is calculated from the equilibrium equations and the postulated distribution of moments.

The lower bound method gives a distribution of moments in the slab system at the ultimate load such that:

1. The equilibrium conditions are satisfied at all points in the slab system.
2. In case of the segment equilibrium method of plastic analysis (which is an upper bound solution), the equilibrium is satisfied only along the yield lines and <u>not within the slab segments</u>.

Hence, this method is not a complete equilibrium solution.

3. The yield strength of the slab sections is not exceeded anywhere in the slab system.

4. The boundary conditions are satisfied.

Upper Bound Method

For a given slab system, the upper bound method gives an ultimate load that is either correct or higher than the actual value.

A reasonable collapse mechanism is assumed in the start depending on experiments and past experience.

The ultimate load is calculated from the equilibrium of the slab segments separated by the yield lines.

The upper bound method gives a collapse mechanism for the slab system at the ultimate load such that:

1. The moments at the plastic hinges are not greater than the ultimate moments of resistance of the sections.

2. The collapse mechanism is compatible with the boundary conditions.

The yield line method of analysis for slabs is an upper bound method, and consequently the failure load calculated for a slab with known flexural resistances may be higher than the true values in case a correct failure pattern is not assumed.

This is certainly a concern, as the designer would naturally prefer to be correct, or at least to be on the safe side.

However, procedures can be incorporated in yield line analysis to help ensure that the calculated capacity is correct.

ADVANTAGES OF YIELD-LINE ANALYSIS

Some of the most significant advantages of the yield-line analysis of slabs are explained here:

1. In Strength Design, the load actions and the corresponding material properties are not fully compatible. Yield-line analysis removes this inconsistency.

2. The slabs designed by yield-line theory become economical as it accounts for the reserve strength characteristics of most concrete structures.

3. Yield-line theory permits, within limits, an arbitrary readjustment of moments found by elastic analysis to arrive at design moments that gives more practical reinforcing arrangements.

4. Yield-line theory gives more general method of analysis and design. It can be used for round and triangular slabs; slabs with large openings, slabs supported on two or three edges only, and slabs carrying concentrated loads.

5. As compared with this, the slabs are lightly reinforced and their rotation capacities are much greater than usually required.

DEMERITS OF YIELD-LINE ANALYSIS

1. The yield line method is an upper bound approach in determining the ultimate flexural strength of slabs; the error in the analysis is on the unsafe side.

2. The yield line approach is basically a tool for review of capacity of a given slab. It can be used for design only in an iterative sense. The capacities of trial designs are calculated by varying amount of reinforcement until a satisfactory arrangement is obtained.

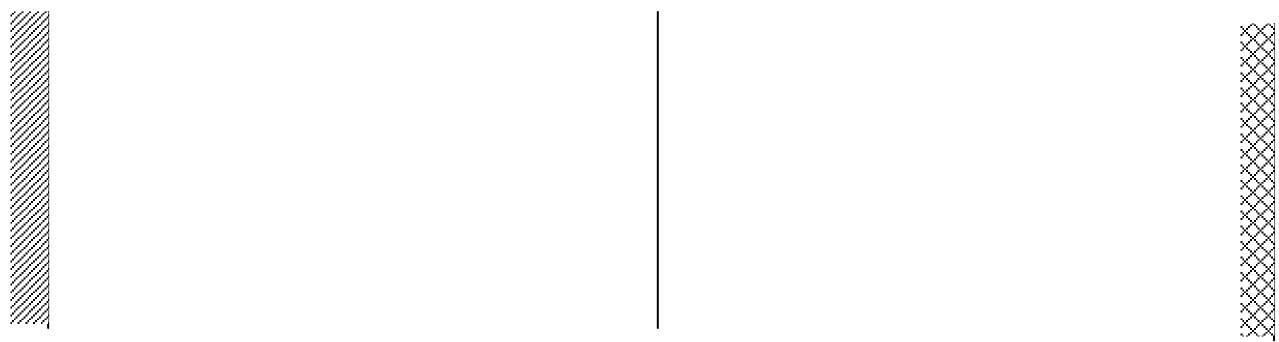
3. Principle of superposition is not valid for any plastic method of analysis. Completely independent analysis is generally required to be performed for different load combinations.

4. For slabs having regular geometries, the calculations are lengthy as compared with traditional elastic methods.

CONVENTIONS TO SHOW BOUNDARY CONDITIONS AND YIELD LINES

The sign convention of Fig. 13.2 will be used throughout for the edge conditions.

A simply supported edge will be shown by line with single hatching, a free edge will be represented by a simple line and a fixed, built-in, or continuous edge will be identified by line with double hatching.



a) Simply Supported Edge

b) Free edge

c)Fixed, Built-in Or Continuous Edge

Fig. 13.2. Conventions to Show Edge Conditions.

Similarly, the notations to show various types of yield lines are shown in Fig. 13.3.

Wavy or bold lines will show positive yield lines.

A positive yield line is formed by sagging curvature of slab and the tension cracks are developed on the lower surface.

Bold dashed lines will show negative yield lines.

A negative yield line is formed by hogging curvature of slab and the tension cracks are developed on the upper surface.

a) Positive Yield Lines

b) Negative Yield Lines

Fig. 13.3. Convention to Show Yield Lines.

RULES TO LOCATE YIELD LINES

<u>Rule No.1</u>

Yield lines are straight lines because they represent the intersection of two planes.

The adjacent plate segments are assumed to rotate as a whole with negligible elastic bending within them.

Hence, each plate segment is a plane.

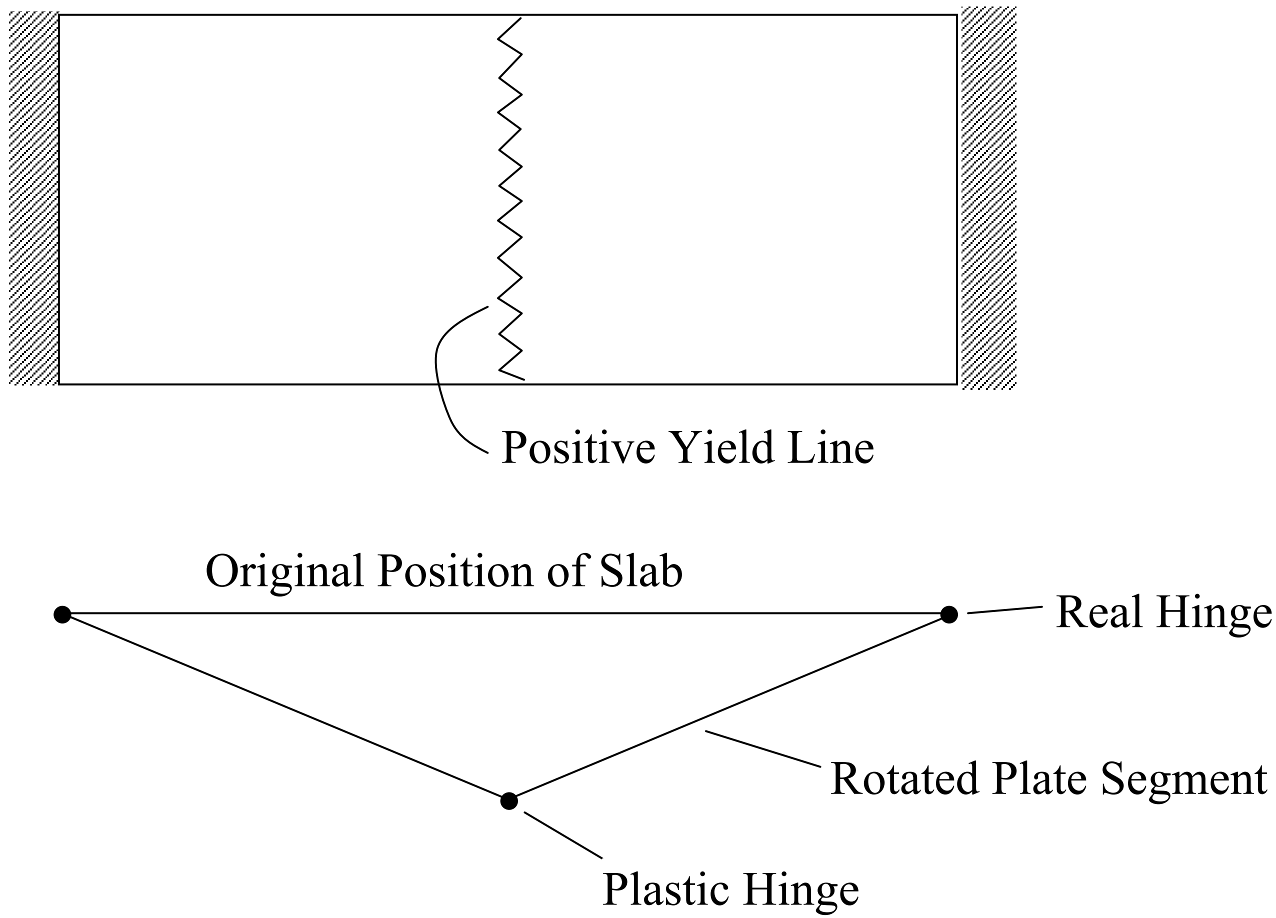


Fig. 13.4. Positive Yield Line in One-Way Simply Supported Slab.

Rule No.2: Yield lines represent axes of rotation.

The slab segments are considered to rotate as rigid bodies in space about these axes of rotation.

Rule No.3: Axes of rotation will be formed at the supported edges of the slab in the following two different ways:

> A negative yield line may form over a fixed edge providing constant resistance to rotation equal to the ultimate flexural capacity of the slab.
>
> An existing axis of rotation is considered at the edge that is simply supported providing zero restraining moment.

Rule No. 4: An axis of rotation will pass over any column support.

Its orientation depends on other considerations.

For example, if columns are present in a row, yield line may pass through the column centerline.

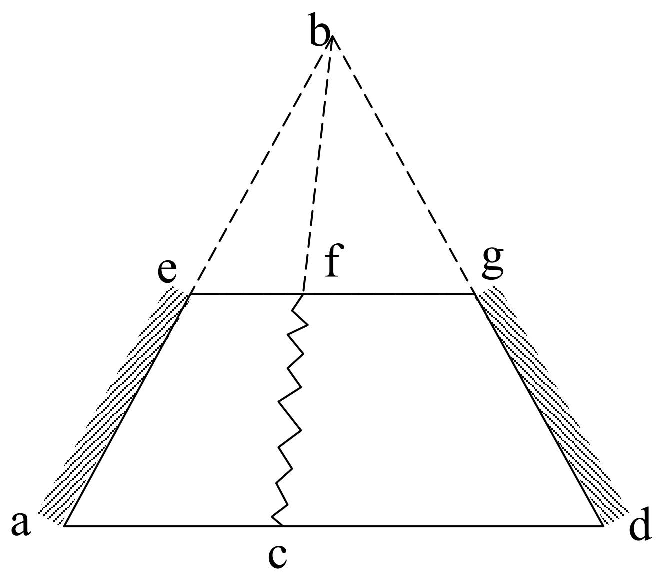
An independent single column may act as a full edge along any direction or it can act as a point load applied from below.

Rule No. 5: Yield lines form under concentrated loads, radiating outward from the point of application of the loads.

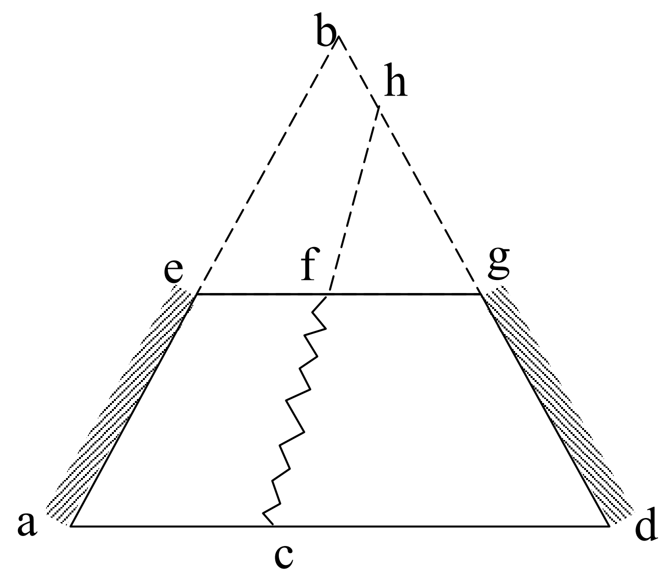
<u>Rule No. 6:</u>

A yield line between two slab segments must pass through the point of intersection of the axes of rotation on the other sides of the adjacent slab segments (Fig. 13.5).

This condition is not required to be satisfied if the three axes of rotation are parallel to each other.



a) Correct Position of Yield Line



b) Incorrect Position of Yield Line

Fig. 13.5. Yield Line Between Two Trapezoidal Slab Segments.

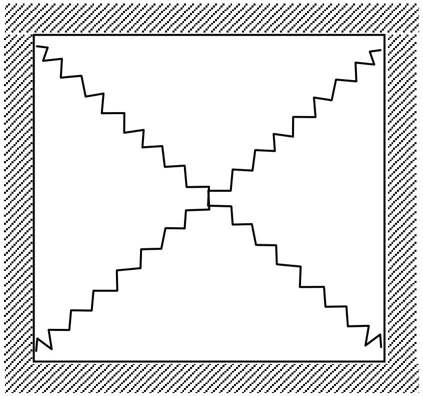
Rule No.7: At a corner, the positive yield line extends towards the point of intersection of the two edges at an angle.

Only exception to this is the formation of the corner levers, which will be discussed later.

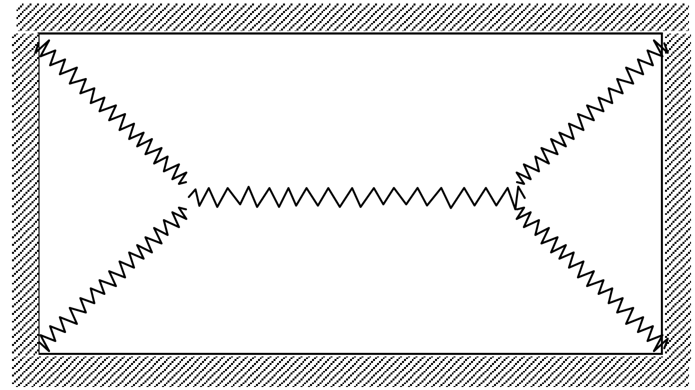
Rule No.8: The positive yield lines are always pushed away from the negative yield lines as compared with the natural axis of rotation.

YIELD LINE PATTERNS

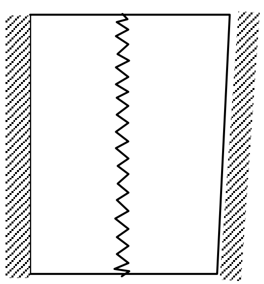
Some typical yield-line patterns developed according to the rules to locate the yield lines are given in Fig. 13.6.



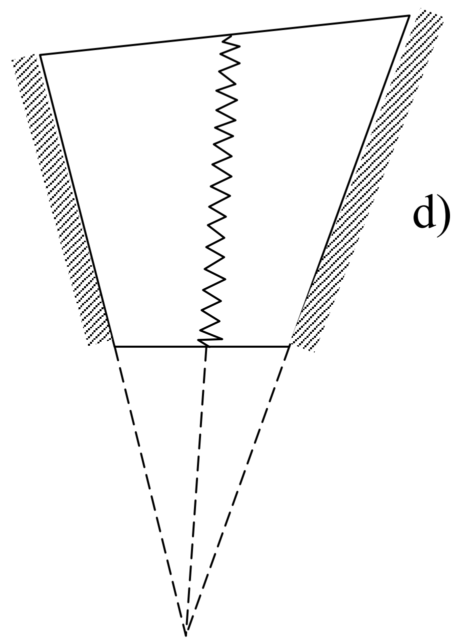
a) Simply Supported Square Slab



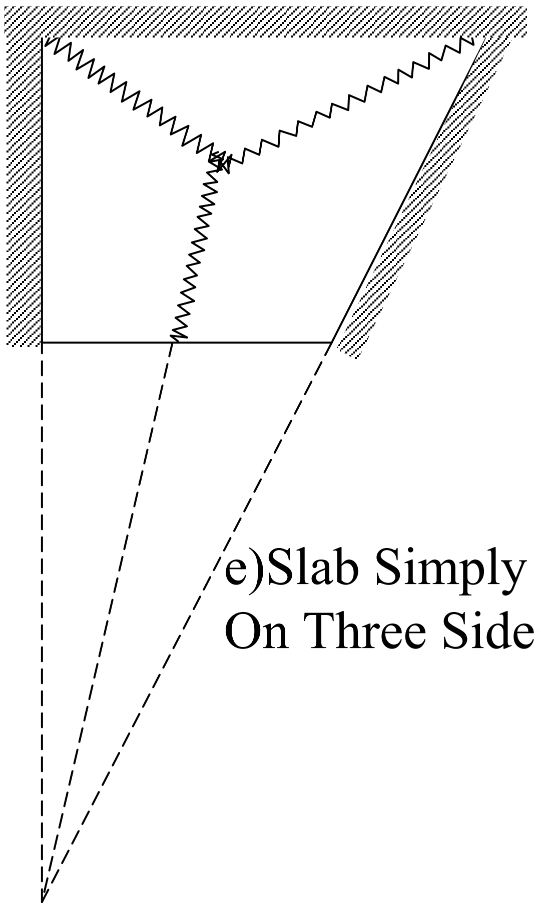
b) Simply Supported Rectangular Slab



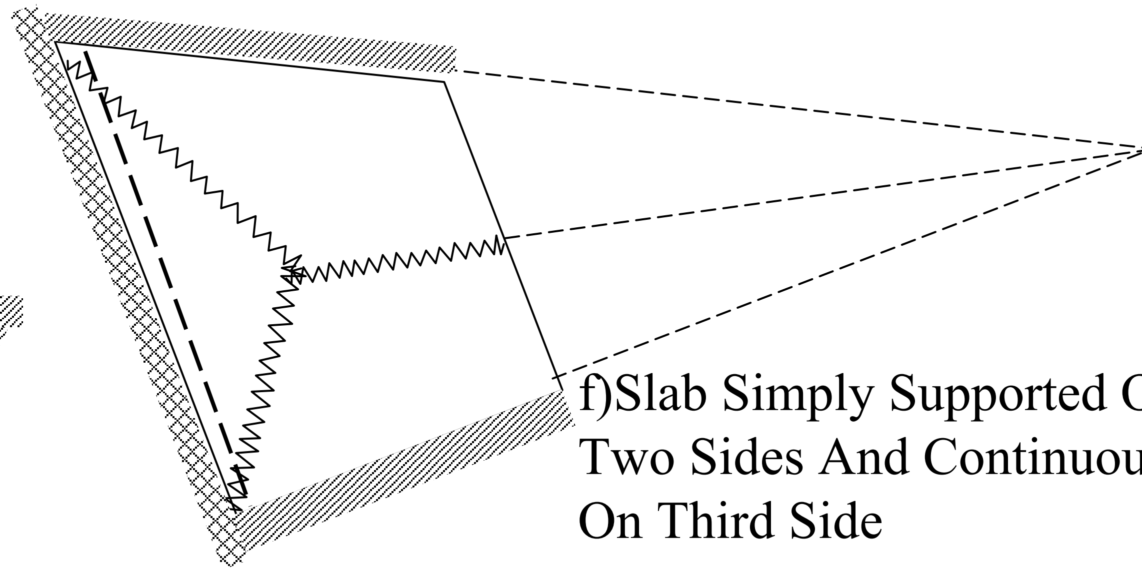
c) One-Way Rectangular Slab



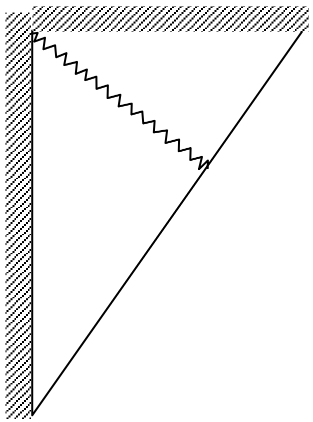
d) One-Way Skew Slab



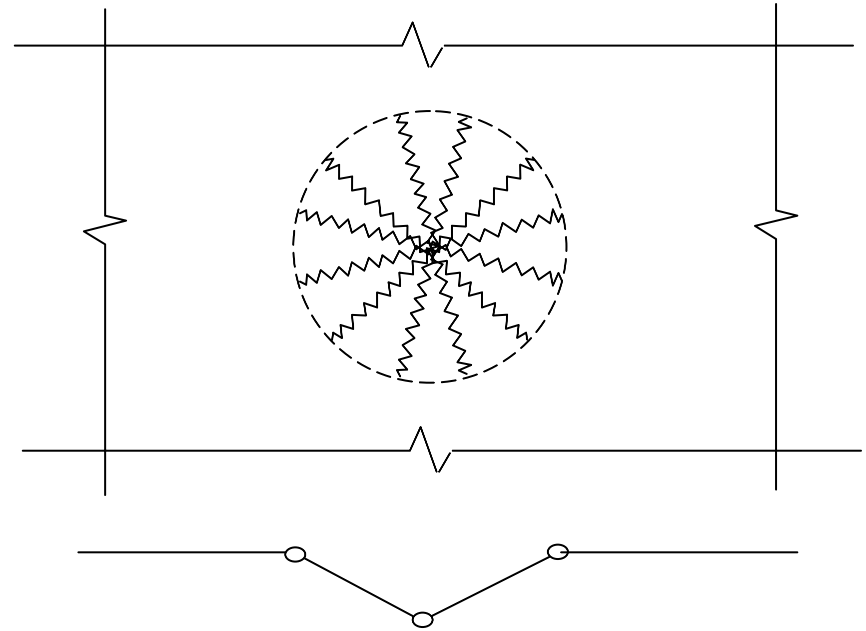
e)Slab Simply Supported
On Three Sides



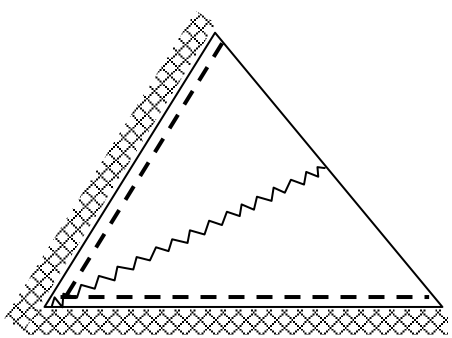
f)Slab Simply Supported On
Two Sides And Continuous
On Third Side



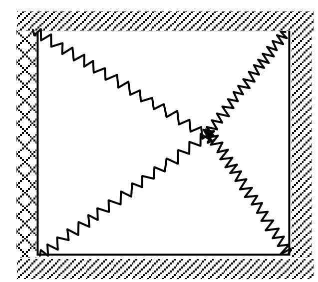
g)Triangular Slab Simply Supported On Two Sides



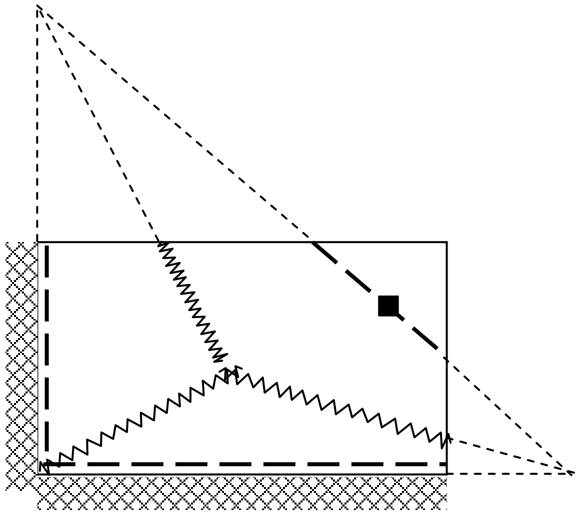
h)Fan Pattern Due To Heavy Concentrated Load



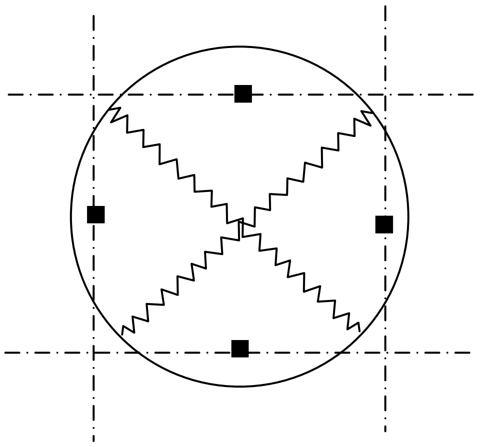
i)Triangular Slab Continuous On Two Sides



j)Square Slab Continuous On One Side And Simply Supported On Other Three Sides

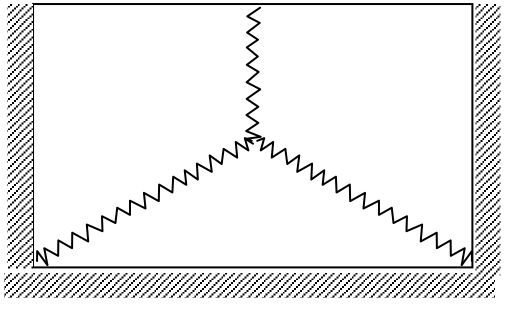


k)Rectangular Slab Continuous On Two Sides And Resting On A Corner Column

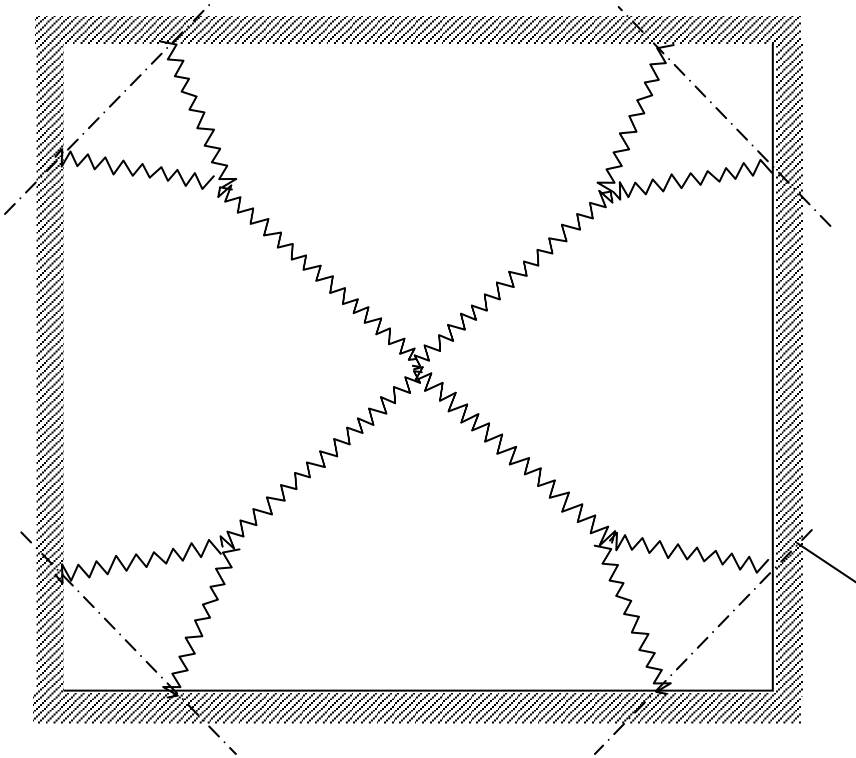


l)Circular Slab Resting On Four Columns

m)Rectangular Slab Simply Supported On Three Sides Having Large Aspect Ratio



n)Rectangular Slab Simply Supported On Three Sides Having Small Aspect Ratio



o)Square Simply Supported Slab
With Corner Levers

Axis of rotation,
becomes negative
yield line if the corner
is held down.

Fig. 13.6. Typical Yield Line Patterns.

FUNDAMENTAL ASSUMPTIONS IN YIELD LINE THEORY

1. Steel reinforcement is fully yielded along the yield lines at failure. The sections are under-reinforced with very small steel ratios allowing large hinge rotations.

2. The resistance per unit width of slab is the nominal flexural strength of the slab; that is, $m_p = m_n$, where m_n is calculated by the usual expression for evaluation of the flexural resistance. For design purposes, m_p is to be taken equal to ϕm_n, with $\phi = 0.90$ for flexure.

3. Bending and twisting moments are uniformly distributed along the yield lines and these give the maximum bending moments perpendicular to the yield lines.

4. Elastic deformations are negligible as compared with plastic deformations.

5. Moment rotation (moment-curvature) curve of critical regions is idealized as elastic-plastic bilinear curve with considerable inelastic rotation to allow full redistribution (Fig. 13.7).

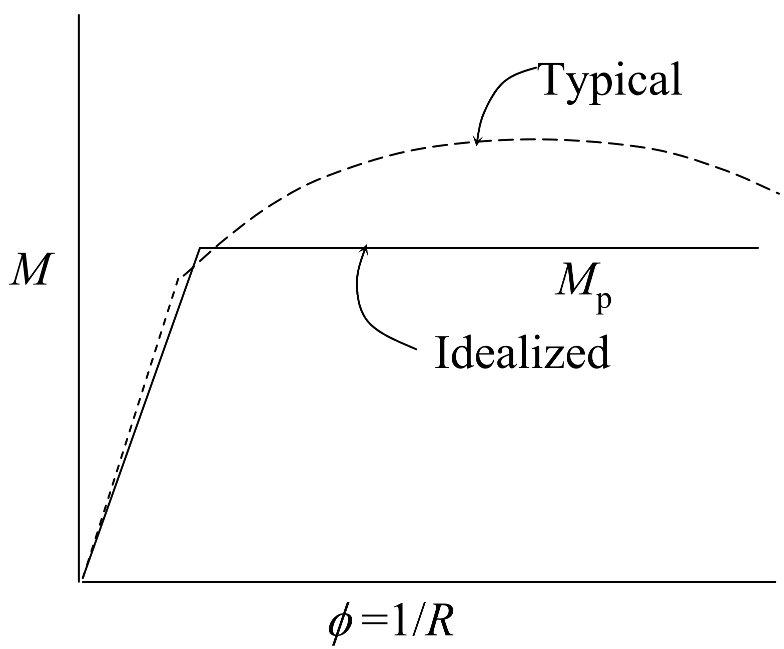


Fig. 13.7. Moment Curvature Relationship.

6. Shear failure, bond failure and compression failure are prevented.

7. The position of yield lines in a yield pattern depends upon the boundary conditions, nature of the load and the geometric dimensions.

CONTRIBUTION OF A STEEL (m_1) ALONG AND ACROSS A SKEWED YIELD LINE

A ***moment vector*** (m_1) shows that the reinforcement is provided in a perpendicular direction to resist a bending moment m_1 given by the right hand rule, as shown in Fig. 13.8.

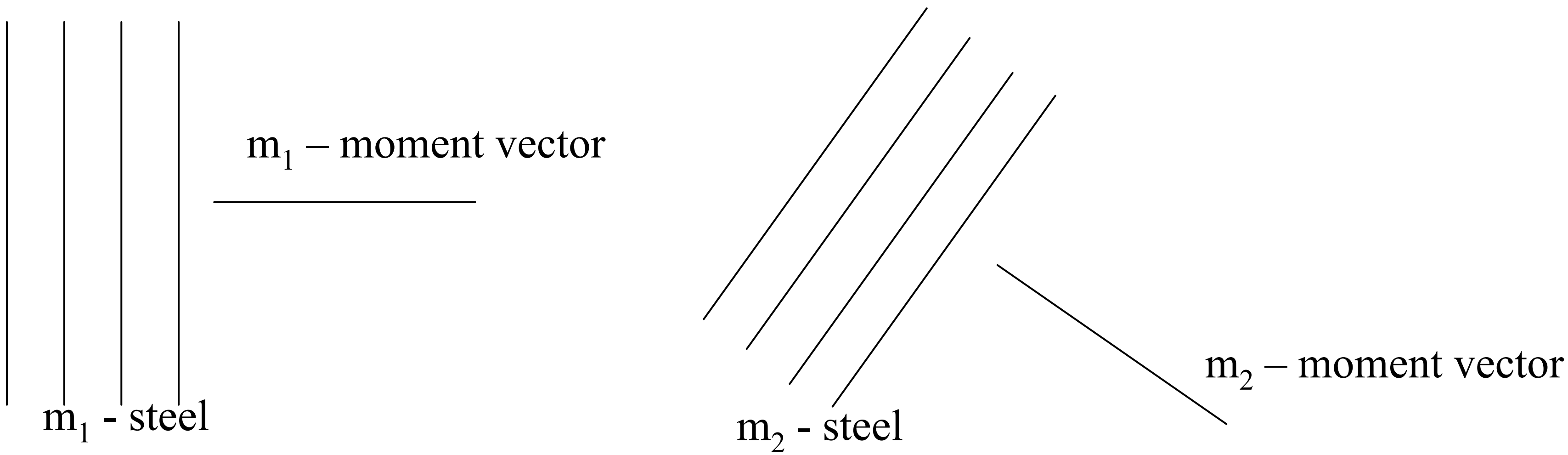


Fig. 13.8. Steel And Its Corresponding resisting Moment Vector.

In Fig. 13.10,

ϕ	=	clockwise angle of the yield line from the moment axis direction,
s'	=	bar spacing along the yield line,
m_n	=	normal yield moment (perpendicular to the yield line, called bending moment),
m_t	=	twisting yield moment (parallel to yield line).

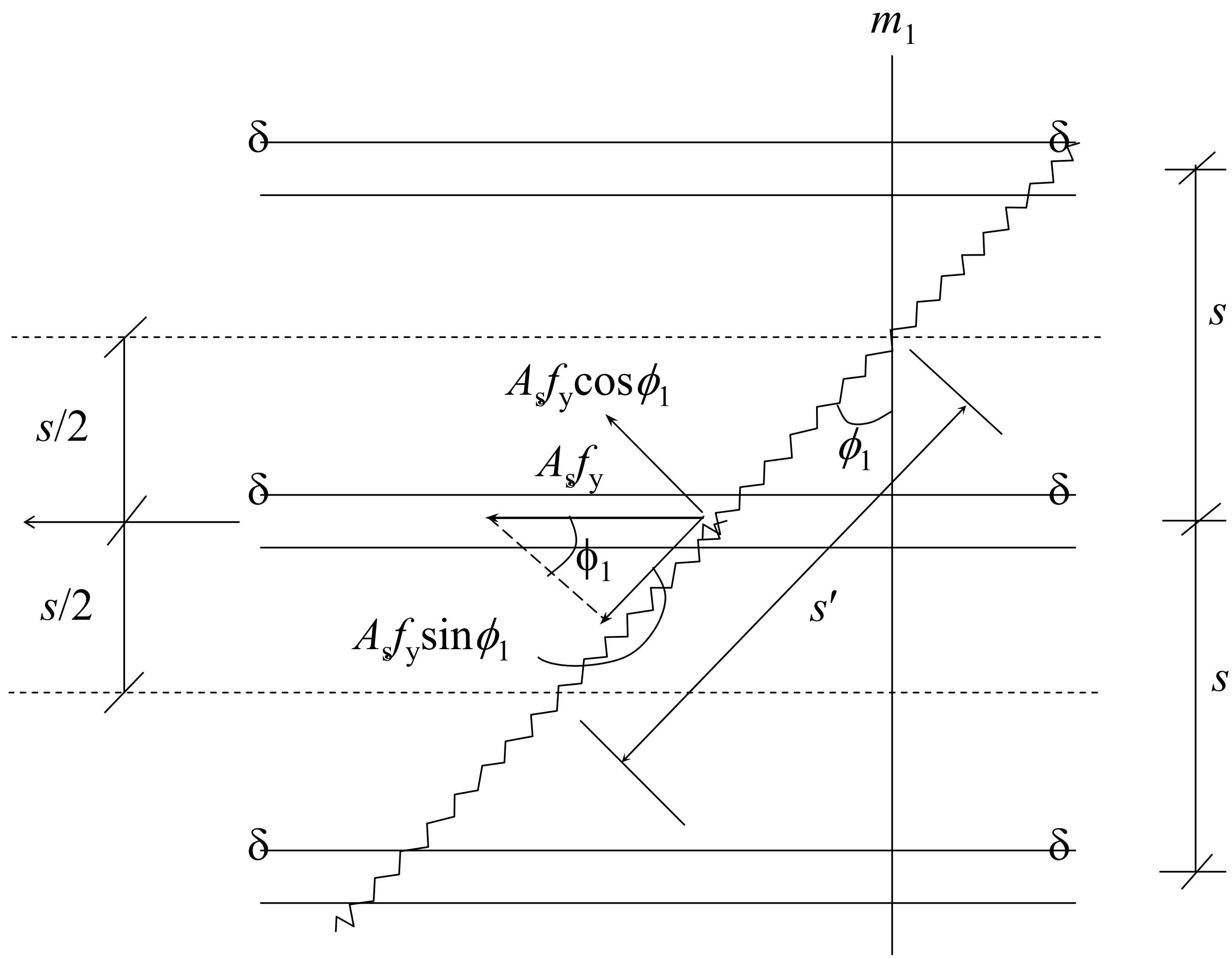


Fig. 13.10. Contribution of Steel Along And Across A Skewed Yield Line.

If the slab yield moment per unit width is denoted by m_1, the total moment over a width equal to s is given by:

$$m_1 \ s = A_s f_y \times z \qquad (1)$$

where z = lever arm between tensile and compressive forces.

The total moment along the yield line for a width equal to s' is:

$$\begin{aligned} m_n \times s' &= A_s f_y \, z \times \cos \phi_1 \\ &= m_1 \, s \cos \phi_1 \qquad (2) \end{aligned}$$

and

$$\begin{aligned} m_t \times s' &= A_s f_y \times z \times \sin \phi_1 \\ &= m_1 \, s \sin \phi_1 \qquad (3) \end{aligned}$$

It can be seen from Fig. 13.10 that $\cos \phi_1 = \dfrac{s}{s'}$

or $s = s' \cos \phi_1$ (4)

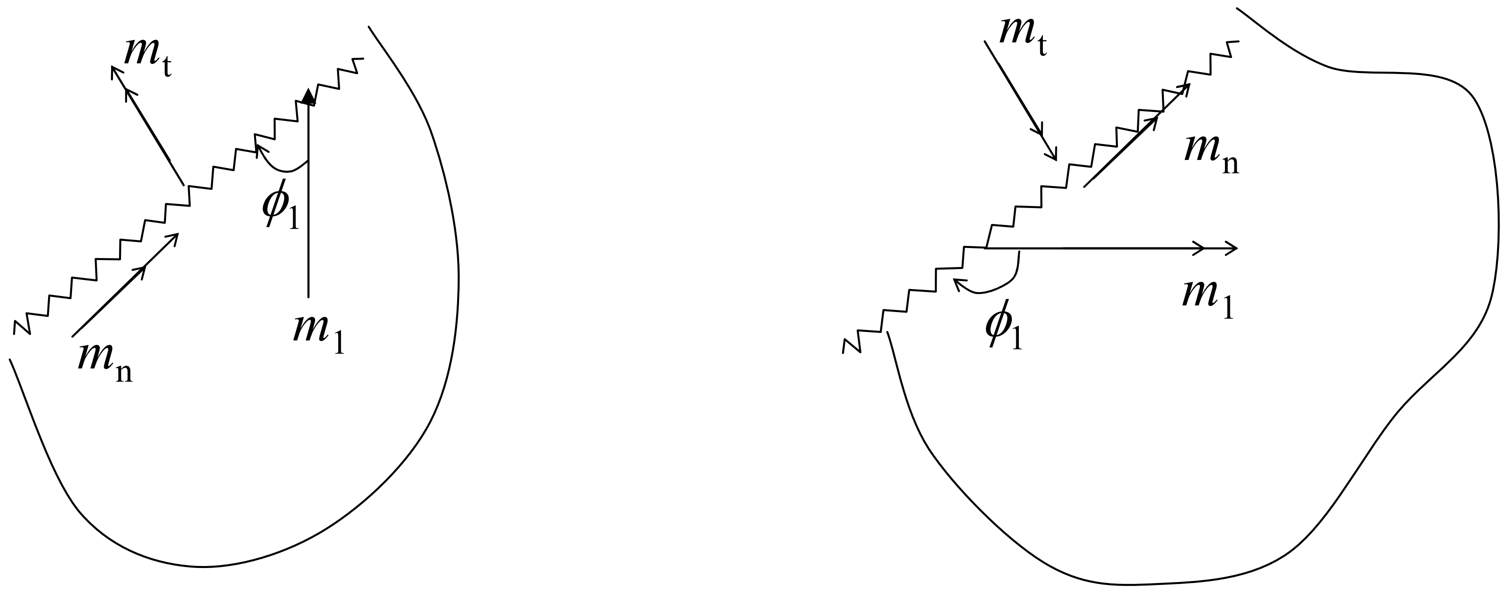
From Eqs. (2) and (4): $m_n \times s' = m_1 s' \cos^2 \phi_1$

or $m_n = m_1 \cos^2 \phi_1$ (5)

From Eqs. (3) and (4): $m_t = m_1 \sin \phi_1 \cos \phi_1$ (6)

The directions of normal moment vector and torsional moment vector are shown in Fig. 13.11, both for acute and obtuse angle of yield line from the resultant moment vector.

The torsional moment m_t is considered positive when its moment vector points away from the section.



a) Yield Line Having Acute Angle

b) Yield Line Having Obtuse Angle

Fig. 13.11. Sense And Direction of Normal And Torsional Moments.

CONTRIBUTION OF STEEL PLACED IN GRID RESOLVED ALONG AND ACROSS A SKEWED YIELD LINE

Consider an orthogonal grid of reinforcement, with angle α between the yield line and the x-axis.

Bars in the x-direction are at a spacing v and have moment resistance m_y per unit length about the y-axis, while bars in the Y-direction are at spacing u and have moment resistance m_x per unit length about the x-axis.

Bars in x-direction produce moment about y-axis and vice versa.

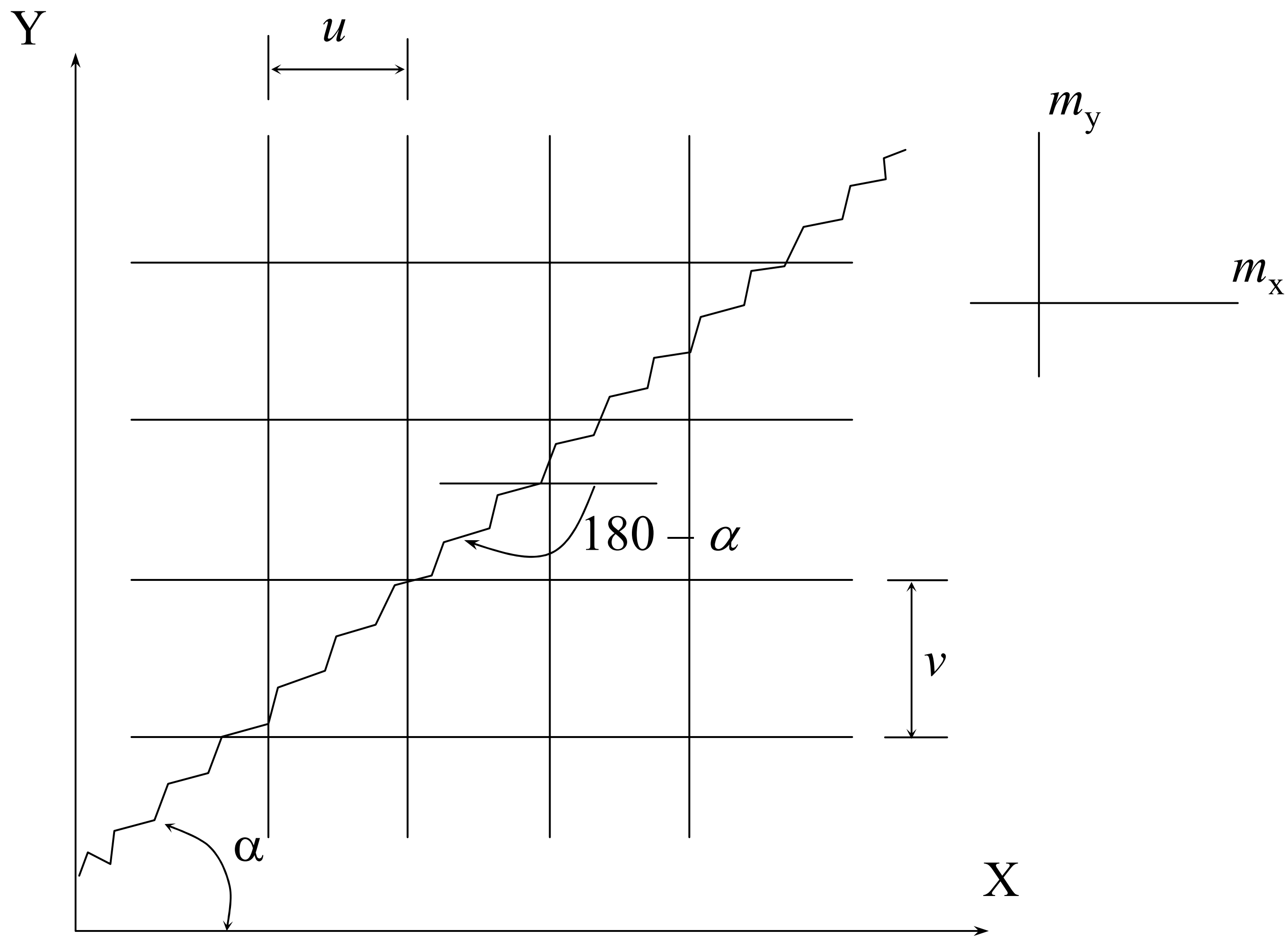


Fig. 13.12. Contribution of Orthogonal Grid of Reinforcement At A Skewed Yield Line.

For the y-direction bars producing m_x,

$\phi_1 \quad = \quad 180 - \alpha$

(clockwise angle between yield line and moment vector)

$$\cos\phi_1 = -\cos\alpha \quad \text{and} \quad \sin\phi_1 = +\sin\alpha$$

and, $m_n \quad = \quad m_x \cos^2\alpha$

$$m_t \quad = \quad -m_x \sin\alpha \cos\alpha$$

For the x-direction bars producing m_y,

$\phi_1 \quad = \quad 90 - \alpha$

$$\cos\phi_1 = \sin\alpha \quad \text{and} \quad \sin\phi_1 = \cos\alpha$$

$$m_{\mathrm{n}} = m_{\mathrm{y}} \cos^2 (90 - \alpha)$$

$$= m_y \, sin^2 \, \alpha$$

and $$m_t = m_y \sin(90 - \alpha) \cos(90 - \alpha)$$

$$= m_y \, sin\alpha \, cos\alpha$$

For both the bars present together,

$$m_n = m_x \, cos^2\alpha + m_y \, sin^2 \, \alpha$$

and $$m_t = -m_x \, sin\alpha \, cos\alpha + m_y \, sin\alpha \, cos\alpha$$

$$= (m_y - m_x) \, sin\alpha \, cos\alpha$$

For the special case when $m_x = m_y = m$, with the same reinforcement provided in each direction,

$$m_n \quad = \quad m(\cos^2 \alpha + \sin^2 \alpha) = \quad m$$

and $\quad m_t \quad = \quad 0$

The slab having the same resistance per unit length ($m_n = m$) in all directions is said to be ***isotropically reinforced slab***.

MAJOR STEPS FOR ANALYSIS

(a) A possible mechanism or yield-line pattern is assumed.

(b) Segment equilibrium or virtual work method is used to find out the ultimate or collapse load for this mechanism.

(c) The geometric dimensions within a mechanism are adjusted to get a minimum collapse load for this mechanism.

(d) The above procedure is repeated for all the possible yield line patterns. The minimum collapse load is considered as the final answer.

SEGMENT EQUILIBRIUM METHOD

1. A suitable yield line pattern is assumed according to previously discussed rules and guidelines producing a collapse mechanism.

2. For the selected collapse mechanism, rigid body movements of slab segments are considered along the yield lines maintaining the deflection compatibility.

3. Each piece or segment is considered as free body and its equilibrium is studied.

4. Unknowns for the solution are the failure load and location and orientation of the yield lines.

5. The number of equilibrium equations required is equal to the number of unknowns. One unknown is always the resisting moment of the slab or the collapse load. Other unknowns are needed to define the locations of yield lines.

6. The resulting equations are solved simultaneously to evaluate all the unknowns.

7. Because the yield moments are principal moments (yielding will start at the maximum moment values), twisting moments are zero along the yield lines, and in most cases the shearing forces are also zero (maximum moment sections are usually associated with zero shear force). Hence, only the unit moment 'm' is generally considered in writing equilibrium equations.

8. All the other possible yield line patterns are investigated turn by turn. The minimum collapse load is the final answer.

Example 13.1: A one way, uniformly loaded and continuous slab panel of 4m span having a positive flexural capacity of 30 kN-m/m and negative flexural capacities of 30 kN-m/m and 40 kN-m/m at the left and the right supports, respectively. Calculate the ultimate load capacity of the slab.

Solution:

The slab system for the given data is shown in Fig. 13.13.

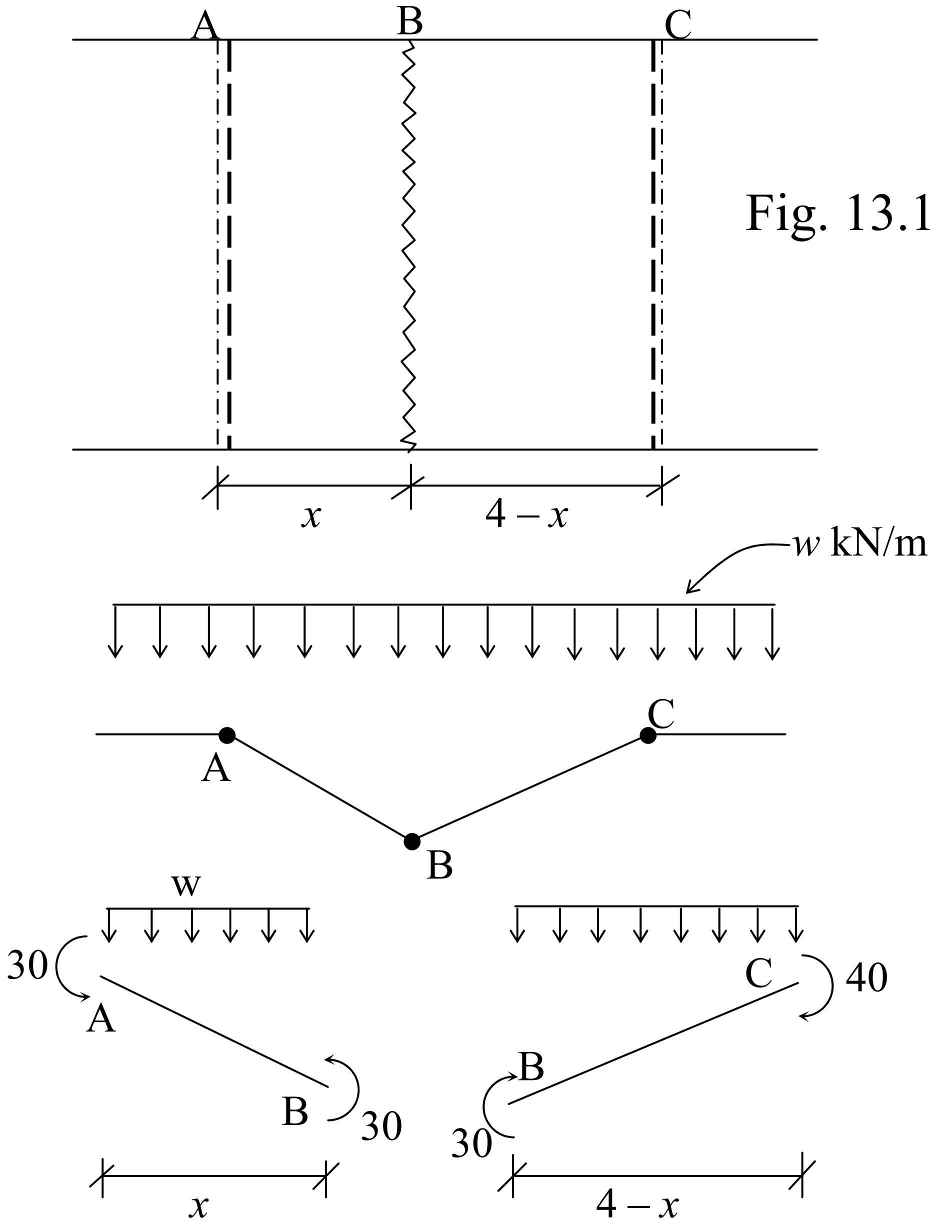


Fig. 13.13. Slab System For Example 13.1.

Equilibrium of Portion AB

$$\Sigma M_A = 0$$

$$\Rightarrow \quad \frac{wx^2}{2} - (30 + 30) = 0$$

$$\frac{wx^2}{2} - (60) = 0 \qquad \text{(I)}$$

Equilibrium of Portion BC

$$\Sigma M_C = 0$$

$$\Rightarrow \quad \frac{w(4-x)^2}{2} - 70 = 0 \qquad \text{(II)}$$

Equations I & II are solved simultaneously for w and x:

From I: $$w = \frac{120}{x^2}$$

Using this value of w in Eq. II, we get:

$$\frac{60}{x^2}(4-x)^2 - 70 = 0$$

$$60(16 + x^2 - 8x) - 70x^2 = 0$$

$$-10x^2 - 480x + 960 = 0$$

$$x^2 + 48x - 96 = 0$$

$$x = \frac{-48 \pm 51.85}{2} = 1.923 \text{ m}$$

(As x cannot be negative)

$$\therefore \quad w = \frac{120}{x^2} = 32.45 \text{ kN/m}^2$$