DYNAMICS OF STRUCTURES

Response to General Dynamic Loading and Transient Response

Introduction

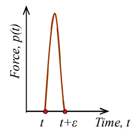
- The analysis of response to general dynamic loading is comparatively more complex
- For linear systems the response to general dynamic loading can be obtained by dividing the force into a series of impulses and the total response is obtained by superposing the response to individual impulse.
- The superposition process involves the evaluation of an integral called the *convolution integral* or *Duhamel's integral*.
- Short duration non-periodic loads are known as Impulsive loads or *shock loads*. Blast load, dynamic loads in automobiles, traveling crane and other mobile machinery may be categorized as shock loads
- The response to these loads is transient in nature and decay rapidly.
- However, from structural engineering point of view the displacement and stresses induced are more important than the duration
- Because of the short duration of response, damping does not have a significant influence and can reasonably be ignored in the analysis

Response to Impulsive Force

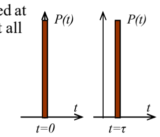
- A large force acting for a very short duration of time is known as impulsive force
- The magnitude of the force may be infinitely large but its time integral (impulse of the force) is finite.

$$I = \int_t^{t+\varepsilon} p(t)\,dt$$

Where ε is very small interval of time during which the impulsive force is acting.



- Mathematically, an impulsive force can be expressed in terms of a *delta function, δ(t)*. The function is centered at t = 0 with an infinitely large value at t = 0 and zero at all other location
- The impulsive force centered at t = 0 and having an impulse equal to *I* is represented by *Iδ(t)*.
- Analogously the impulsive force centered at t = τ of impulse *I* is *Iδ(t- τ)*



Response to Impulsive Force (Cont..)

- The impulse *I* will change the velocity of a system with mass m by:

$$\Delta v = \frac{I}{m}$$

- The response to impulse is a initial velocity problem. For undamped system:

$$u = \frac{I/m}{\omega}\sin \omega t$$

- The response to *I=1.0* is called *unit impulse response*. It is denoted by *h(t)* and is given by:

$$h(t) = \frac{1}{m\omega}\sin \omega t$$

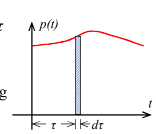
- For an damped system, the response to a unit impulse is:

$$h(t) = \frac{1}{m\omega_d} e^{-\omega\zeta t}\sin \omega_d t$$

Response to General Dynamic Loading

- The response of a linear system to general dynamic loading is obtained by dividing the load into a series of impulses and superposing the response to individual impulse.
- From the figure, the undamped incremental response (*du*) at any time *t* = *τ* where the impulse is, *I* = *p*(*τ*)*dτ* is given by:

$$du = \frac{p(\tau)d\tau}{m\omega}\sin \omega(t-\tau)$$



- The total response at time t is obtained by superposing the impulses from *τ* = 0 to *τ* = *t*, giving:

$$u(t) = \frac{1}{m\omega}\int_0^t p(\tau)\sin \omega(t-\tau)\,d\tau$$

- Similarly for a damped system:

$$u(t) = \frac{1}{m\omega_d}\int_0^t p(\tau)e^{-\omega\zeta(t-\tau)}\sin \omega_d(t-\tau)\,d\tau$$

Response to General Dynamic Loading (Cont..)

- The response to general dynamic loading for both damped and undamped system can me expressed in terms of unit impulse response:

$$u(t) = \int_0^t p(\tau)h(t-\tau)d\tau$$

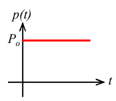
- The above integral is known as *Convolution integral* or *Duhamel's integral*. It provides a general method for the analysis of linear system subjected to any arbitrary loading and form the basis for the development of *Fourier transform method*.
- For simple function of $p(\tau)$ closed form solution can easily be obtained; in other cases, numerical technique must be used.
- For a system with non-zero initial condition, the total response is:

$$u(t) = e^{-\omega\zeta t}\left(u_o \cos\omega_d t + \frac{v_o + \omega\zeta u_o}{\omega_d}\right) + \int_0^t p(\tau)h(t-\tau)d\tau$$

For zero initial condition

Response to a Step Function Load

- A suddenly applied load which remains constant after application. The governing equation of motion is:



$$m\ddot{u} + c\dot{u} + ku = P_o$$

- The particular solution can be obtained by the methods of trials. We assume for our trial solution $u = C$ (constant). Putting in the above equation results in $u = C = P_o/k$.
- The complimentary solution is: $u = e^{-\omega\zeta t}\left(A\cos\omega_d t + B\sin\omega_d t\right)$

  Where A and B are arbitrary constants depend on initial conditions
- The total solution is thus:

$$u = e^{-\omega\zeta t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + \frac{P_o}{k}$$

- For zero initial conditions:

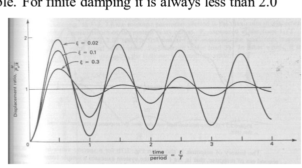
$$A = -\frac{P_o}{k} \qquad B = -\frac{P_o}{k}\frac{\zeta}{\sqrt{1-\zeta^2}}$$

Response to a Step Function Load (Cont..)

- The resulting solution becomes:

$$u = \frac{P_o}{k}\left[1 - e^{-\omega\zeta t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\right]$$

- The maximum value of dynamic load factor, $u/(P_o/k)$ is 2.0 when damping is negligible. For finite damping it is always less than 2.0



Response to a Step Function Load (Cont..)

- The time at which the peak response occurs is obtained by differentiating the above equation and equating to zero:

$$t_p = \frac{n\pi}{\omega_d} - -(n = 0,1,2,.....)$$

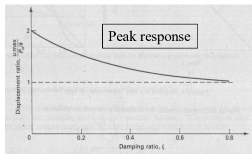
- The first peak occurs when n=1 i.e. $t_p=\pi/\omega_d$. The peak response is given by:

$$u = \frac{P_o}{k}\left(1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}\right)$$

Cos is -ve at Pi

- Thus the peak response is a function of damping only.
- The response to step function can also be obtained using the Duhamel's integral

$$u(t) = \frac{1}{m\omega_d}\int_0^t P_o e^{-\omega\zeta(t-\tau)} \sin\omega_d(t-\tau)d\tau$$



Response to a Ramp Function Load (Cont..)

- A ramp function load is a load that increases linearly with time. Mathematically, it can be expressed as:

$$p(t) = \frac{P_o t}{t_1}$$

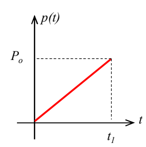
- The Duhamel's integral for ramp function load is:

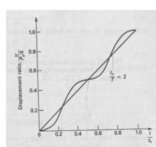
$$u(t) = \frac{1}{m\omega_d}\int_0^t \frac{P_o\tau}{t_1} e^{-\omega\zeta(t-\tau)} \sin\omega_d(t-\tau)d\tau$$

- For undamped system Duhamel's integral simplifies to:

$$u(t) = \frac{1}{m\omega}\int_0^t \frac{P_o t}{t_1}\sin\omega(t-\tau)d\tau = \frac{P_o}{k}\left(\frac{t}{t_1} - \frac{\sin\omega t}{\omega t_1}\right)$$

- The undamped response to ramp function load is shown in figure on right





Response to a Step Function Load with Rise Time

- The response to such a load is obtained from the superposition of the following two ramp functions:
  - A ramp function load applied at t = 0.0 given by:

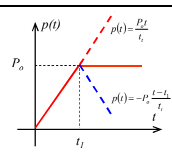
$$u_1(t) = \frac{P_o}{k}\left(\frac{t}{t_1} - \frac{\sin\omega t}{\omega t_1}\right)$$

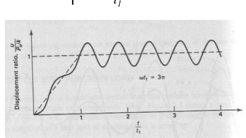
  - An equal but negative ramp function applied at $t = t_1$ given by:

$$u_2(t) = -\frac{P_o}{k}\left(\frac{t - t_1}{t_1} - \frac{\sin\omega(t - t_1)}{\omega t_1}\right)$$

  - The total response is given by;

$$u(t) = \begin{cases} \frac{P_o}{k}\left(\frac{t}{t_1} - \frac{\sin\omega t}{\omega t_1}\right) - - - - - - - - - - - - - (t \le t_1) \\ \frac{P_o}{k}\left(1 - \frac{\sin\omega t}{\omega t_1} + \frac{\sin\omega(t - t_1)}{\omega t_1}\right) - - - - (t > t_1) \end{cases}$$





Response Spectrum

- Response spectrum is a curve drawn for any response quantity of SDOF system against natural period or frequency.

Example 7.1: Suddenly Applied Load that decays Exponentially

- A suddenly applied load with exponential decay is given by: $p(t) = P_o e^{-at}$
- The governing equation of motion for undamped SDOF system is given by:

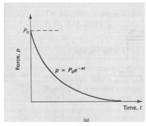
$$m\ddot{u} + ku = P_o e^{-at}$$

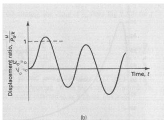
- The solution to this equation for zero initial condition, is obtained by Duhamel's Integral:

$$u(t) = \frac{P_o}{m\omega}\int_0^t e^{-a\tau}\sin\omega(t-\tau)d\tau$$

$$u(t) = \frac{P_o}{k(1 + a^2/\omega^2)}\left(\frac{a}{\omega}\sin\omega t - \cos\omega t + e^{-at}\right)$$

$$u(t) = \frac{P_o}{k(1 + a^2/\omega^2)}\left(\sqrt{1 + a^2/\omega^2}\sin(\omega t + \phi) + e^{-at}\right)$$





- For high value of t, e^{-at} becomes very small and the system vibrates with steady state amplitude of:

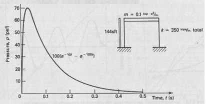
$$u(t) = \frac{P_o}{k\sqrt{1 + a^2/\omega^2}}$$

Example 7.2: Blast Induced Pressure

A blast-induced pressure wave striking a single-story building is represented by $p = 100(e^{-10t} - e^{-100t})$, where p is the pressure in psf. The building face subject to the wind pressure has an area of 144 ft^2. The mass of the building, assumed concentrated at the floor level, is 0.1 kip · s^2/in. and the total stiffness of the columns supporting the floor is 350 kips/in. Find the displacement response of the building and the maximum base shear induced by the pressure wave.

- Natural frequency of the system is given by:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{350}{0.1}} = 59.16 - rad/s$$



- The force applied to the floor level is obtained by multiplying half of the area subjected to wind pressure:

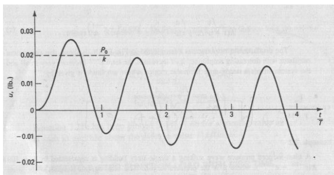
$$P(t) = \frac{A}{2}p = \frac{144}{2}100(e^{-10t} - e^{-100t})/1000 = 7.2(e^{-10t} - e^{-100t})$$

- The applied force has two exponential decay components. Therefore the total response is obtained by summing the response to each component.

Example 7.2: Blast Induced Pressure (Cont..)

$$u(t) = \frac{7.2}{350\left(1+10^2/59.16^2\right)}\left(\frac{10}{59.16}\sin 59.16t - \cos 59.16t + e^{-10t}\right)$$
$$-\frac{7.2}{350\left(1+100^2/59.16^2\right)}\left(\frac{100}{59.16}\sin 59.16t - \cos 59.16t + e^{-100t}\right)$$

$$u(t) = 0.0056\sin 59.16t - 0.0147\cos 59.16t + 0.02e^{-10t} - 0.0053e^{-100t}$$



Response to Shock/Impulsive Loading: Rectangular Pulse

- For $t \le t_1$, the response is obtained from the step function equation:

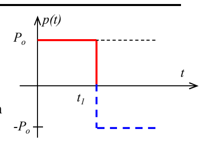
$$u(t) = \frac{P_o}{k}(1-\cos\omega t) - -t \le t_1 \text{.............(1)}$$

- For $t > t_1$, the response is obtained from superposition of above step function applied at t = 0 and equal but negative step function applied at $t = t_1$,

$$u(t) = \frac{P_o}{k}(1-\cos\omega t) - \frac{P_o}{k}\{1-\cos\omega(t-t_1)\}$$

$$u(t) = \frac{P_o}{k}\{\cos\omega(t-t_1) - \cos\omega t\} - -t > t_1 \text{...........(2)}$$

- Or in the 2nd era the response is obtained from the free vibration response due to velocity and displacement at time $t = t_1$.



Meaning P(t) = 0

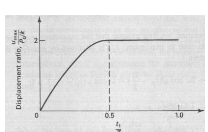
Rectangular Pulse (Cont..)

- The peak response will either be in the first era or the 2nd era of free vibration
- Assuming the peak is in the first era, the time at peak, t_p is obtained by differentiating equation (1) and equating to zero. We get: $t_p = \pi/\omega$
- The solution is valid only when: $t_p = \pi/\omega < t_1$ or $t_1/T > 1/2$
- The peak response is given by: $u_{max} = 2P_o/k$
- If $t_1/T<1/2$, then the peak falls in the free vibration phase. The time at peak in the 2nd era is obtained by differentiating equation (2):

$$\sin\omega(t_p - t_1) - \sin\omega t_p = 0$$
$$\tan\omega t_p = \frac{\sin\omega t_1}{(\cos\omega t_1 - 1)} = -\cot\frac{\omega t_1}{2}$$
$$\omega t_p - \frac{\omega t_1}{2} = \frac{\pi}{2} \Rightarrow t_p = \frac{\pi}{2\omega} + \frac{t_1}{2} = \frac{T}{4} + \frac{t_1}{2}$$

- The peak response is given by:

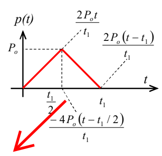
$$u_{max} = 2\frac{P_o}{k}\sin\frac{\pi t_1}{T}$$



Response (shock) spectrum

Triangular Pulse

- The triangular pulse can be represented as the superposition of three ramp function shown in figure
- For $t < t_1/2$ the solution of the first ramp function is the solution of the first era. For $t_1/2 < t < t_1$, summation of the solution of 1st and 2nd ramp functions is the solution of the 2nd era. Similarly for $t > t_1$, summation of the three ramp functions gives the solution for the 3rd era. The solution for the 3rd era can also be obtained from the free vibration using the initial conditions at $t = t_1$.



Its slope is double

2 is used as force becomes P_o in half time

$$u(t)=\begin{cases}\frac{2P_o}{k}\left(\frac{t}{t_1}-\frac{\sin\omega t}{\omega t_1}\right) & t\le\frac{t_1}{2}\\ \frac{2P_o}{k}\left(1-\frac{t}{t_1}-\frac{\sin\omega t}{\omega t_1}+\frac{\sin\omega(t-t_1/2)}{\omega t_1}\right) & \frac{t_1}{2}<t\le t_1\\ \frac{2P_o}{k}\left(-\frac{\sin\omega t}{\omega t_1}+\frac{\sin\omega(t-t_1/2)}{\omega t_1}-\frac{\sin\omega(t-t_1)}{\omega t_1}\right) & t>t_1\end{cases}$$

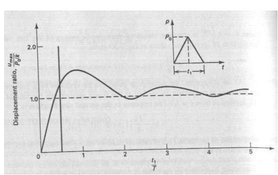
Triangular Pulse (Cont..)

- The peak response may occur in any of three eras, depending upon the ratio t_1/T.
- The peak in each era may be obtained by differentiating and equation to zero the respective equation. For the first era, the time at peak and the peak response (the equations are valid when $t_p < t_1$) are:

$$t_p = \frac{T}{4} + \frac{t_1}{2}$$

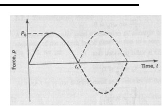
$$u_{max} = \frac{4P_o}{k\omega t_1}\left(1 - \cos\frac{\omega t_1}{2}\right)$$

- Similarly one can get peak response for the 2nd and 3rd eras
- The response spectrum for the triangular pulse is shown in figure



Sinusoidal Pulse

- The sine pulse can be represented as the superposition of two sine waves as shown in figure



- For $t < t_1$ the solution of the sine force is the solution for the first era. For the 2nd era (free vibration era) the solution is obtained as summation of two sine waves:

$$u(t)=\begin{cases}\frac{P_o}{k}\frac{1}{1-\beta^2}(\sin\Omega t-\beta\sin\omega t) & t\le t_1\\ \frac{P_o}{k}\frac{1}{1-\beta^2}\{\sin\Omega t-\beta\sin\omega t+\sin\Omega(t-t_1)-\beta\sin\omega(t-t_1)\} & t>t_1\end{cases}$$

- If the peak response occurs in the 1st era then the time for peak response is:
- The smallest value of t_p other than zero is obtained for n=1 and using the negative sign

$$t_p = \frac{2\pi\omega}{1+\beta} = \frac{2}{1/t_1 + 2/T}$$

Sinusoidal Pulse (Cont..)

- The equation is valid only when:

$$\left(t_p = \frac{2}{1/t_1 + 2/T}\right) \le t_1 \Rightarrow \frac{t_1}{T} \ge \frac{1}{2} \Rightarrow \left(\frac{\pi/\Omega}{2\pi/\omega} = \frac{1}{2\beta}\right) \ge \frac{1}{2} \Rightarrow \beta \le 1$$

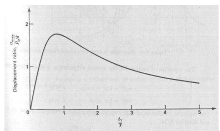
- The peak response for the first era is:

$$u_{\max} = \frac{P_o}{k}\frac{1}{1-\beta^2}\left(\sin\frac{2\pi\beta}{1+\beta} - \beta\sin\frac{2\pi}{1+\beta}\right) ----\beta \le 1$$

- For the 2nd era the peak response is:

$$u_{\max} = \frac{P_o}{k}\frac{2\beta}{1-\beta^2}\cos\frac{\pi}{2\beta} ----\beta > 1$$

- The shock spectrum is shown in the figure

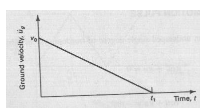


Response to a Ground Motion Pulse

Example 7.3

The base of an undamped single-degree-of-freedom system is subjected to a velocity pulse shown in Fig. E7.3a. Obtain the response of the system and plot the shock spectrum of relative motion.

$$\dot{u}_g = v_o\left(1 - \frac{t}{t_1}\right) \qquad \ddot{u}_g = -\frac{v_o}{t_1}$$



- The governing equation of motion is:

$$m\ddot{u} + ku = \frac{mv_o}{t_1}$$

- The total solution is obtained by summing the Duhamel's Integral and the free vibration response with initial conditions: $u_o = 0 \qquad \dot{u}_o = \dot{u}_t - \dot{u}_{go} = -v_o$

$$u = \begin{cases} -\frac{v_o}{\omega}\sin\omega t + \frac{1}{m\omega}\int_0^t \frac{mv_o}{t_1}\sin\omega(t-\tau)d\tau ---t \le t_1 \\ -\frac{v_o}{\omega}\sin\omega t + \frac{1}{m\omega}\int_0^{t_1} \frac{mv_o}{t_1}\sin\omega(t-\tau)d\tau ---t > t_1 \end{cases}$$

$$u = \begin{cases} \frac{v_o}{\omega}\left\{-\sin\omega t + \frac{1}{\omega t_1}(1-\cos\omega t)\right\} ----------t \le t_1 \\ \frac{v_o}{\omega}\left[-\sin\omega t + \frac{1}{\omega t_1}\{\cos\omega(t-t_1) - \cos\omega t\}\right] ---t > t_1 \end{cases}$$

Response to a Ground Motion Pulse (Cont..)

- Shock Spectrum