DYNAMICS OF STRUCTURES

Single Degree of Freedom System: Forced Harmonic Vibration (Cont..)

Resonance Response: Undamped System

- The harmonic vibration response of undamped system is given by:

$$u(t) = u_0 \cos \omega t + \frac{v_o}{\omega} \sin \omega t + \frac{p_o}{k} \frac{1}{1-\beta^2} (\sin \Omega t - \beta \sin \omega t)$$

- For a simple case of $u_o = v_o = 0$

$$u(t) = \frac{p_o}{k} \frac{1}{1-\beta^2} (\sin \Omega t - \beta \sin \omega t) = \frac{p_o}{k} \left(\frac{\sin \beta \omega t - \beta \sin \omega t}{1-\beta^2} \right)$$

- For β = 1.0, the numerator and the denominator are both zero and the displacement becomes indeterminate. In the limiting case, the problem can be solved by *L'Hospital's rule*

$$\lim_{\beta \to 1} u(t) = \frac{p_o}{k} \lim_{\beta \to 1} \left(\frac{\sin \beta \omega t - \beta \sin \omega t}{1-\beta^2} \right) = \frac{p_o}{k} \lim_{\beta \to 1} \left(\frac{\omega t \cos \beta \omega t - \sin \omega t}{-2\beta} \right)$$

$$\boxed{u(t) = \frac{p_o}{k} \left(\frac{\omega t \cos \omega t - \sin \omega t}{-2} \right) = \frac{p_o}{2k} (\sin \omega t - \omega t \cos \omega t)}$$

Resonance Response: Undamped System (Cont..)

- The response is periodic with period of T = 2π/ω.
- Peak response is obtained by differentiating:

$$u = \frac{p_o}{2k} (\sin \omega t - \omega t \cos \omega t)$$

$$\dot{u} = \frac{p_o}{2k} \{\omega \cos \omega t - (\omega \cos \omega t + \omega t(-\omega \sin \omega t))\}$$

$$\dot{u} = \frac{p_o}{2k} \omega^2 t \sin \omega t$$

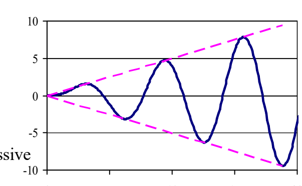
$$\dot{u} = 0 \Rightarrow \omega t = n\pi ----n = 0,1,2,3......$$

- The difference between two successive peaks is given by

$$\boxed{u\left(\frac{2\pi}{\omega} + \frac{n\pi}{\omega}\right) - u\left(\frac{n\pi}{\omega}\right) = \frac{p_0}{2k} 2\pi \cos n\pi = \pm \frac{p_0}{k} \pi}$$

$$\boxed{\frac{uk}{p_o} = \frac{1}{2} (\sin \omega t - \omega t \cos \omega t)}$$

Modification of last equation



Cos and sine is same for both t values. Only t value with Omega causes difference.

Damped Harmonic Vibration

- Damped Harmonic Vibration is governed by:

$$m\ddot{u} + c\dot{u} + ku = p_o \sin \Omega t ----------(1)$$

- The particular solution (steady state response) is of the form:

$$u = G_1 \cos \Omega t + G_2 \sin \Omega t ----------(2)$$

- On putting in eq.1 we get values of G_1 and G_2 the particular solution is thus

$$u = \frac{p_o}{k} \frac{1}{(1-\beta^2)^2 + (2\zeta\beta)^2} \left\{ (1-\beta^2) \sin \Omega t - 2\zeta\beta \cos \Omega t \right\} -----(3)$$

- The complimentary solution (transient response) is:

$$u_{trans} = e^{-\zeta\omega t} (A \cos \omega_d t + B \sin \omega_d t) ----------(4)$$

Damped Harmonic Vibration (Cont..)

- The complete solution is thus:

$$u = \underbrace{e^{-\zeta\omega t} (A \cos \omega_d t + B \sin \omega_d t)}_{\text{Transient}} + \underbrace{G_1 \sin \Omega t + G_2 \cos \Omega t}_{\text{Steady State}} -------(5)$$

where A and B are arbitrary constants determined from the initial conditions:

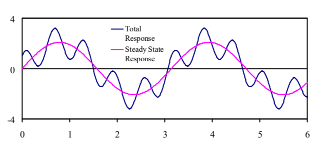
$$A = \frac{p_o}{k} \frac{2\zeta\beta}{(1-\beta^2)^2 + (2\zeta\beta)^2} + u_o \qquad B = \frac{p_o}{k} \frac{\omega}{\omega_d} \frac{2\beta\zeta^2 - \beta(1-\beta^2)}{(1-\beta^2)^2 + (2\zeta\beta)^2} + \frac{v_o + u_o\omega\zeta}{\omega_d} --------(7)$$

- The transient response is given by:

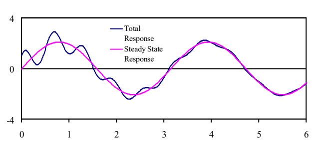
$$u_{trans} = e^{-\zeta\omega t} \left(u_o \cos \omega_d t + \frac{v_o + u_o\omega\zeta}{\omega_d} \sin \omega_d t \right)$$
$$+ \frac{p_o}{k} \frac{e^{-\zeta\omega t}}{(1-\beta^2)^2 + (2\zeta\beta)^2} \left[2\zeta\beta \cos \omega_d t + \frac{\omega}{\omega_d} \left\{ 2\beta\zeta^2 - \beta(1-\beta^2) \right\} \sin \omega_d t \right] ------(8)$$

Damped Harmonic Vibration (Cont..)

- Undamped Harmonic Vibration
  - The transient response does not decay with time



- Damped Harmonic Vibration
  - The transient response decay with time



Damped Harmonic Vibration (Cont..)

- The steady state response (Eq.3) of a damped harmonic vibration can be written in the alternative form as:

$$u = \rho \sin(\Omega t - \phi) -----------(9)$$

- Where ρ is the amplitude of steady state response and *Φ* is the phase angle by which the response lags the exciting force.

$$\rho = \frac{p_o}{k} \frac{1}{\sqrt{(1-\beta^2)^2 + (2\zeta\beta)^2}}, -- \tan \phi = \frac{2\zeta\beta}{1-\beta^2}$$

Damped Harmonic Vibration : Magnification

- Dynamic load factor, i.e. the ratio of dynamic to static displacement, is given by:

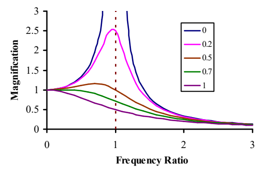
$$D(t) = \frac{1}{\sqrt{\left(1-\beta^2\right)^2 + \left(2\zeta\beta\right)^2}} \sin(\Omega t - \phi)$$

- The amplitude of dynamic load factor called the dynamic magnification factor is:

$$A_D = \frac{1}{\sqrt{\left(1-\beta^2\right)^2 + \left(2\zeta\beta\right)^2}}$$

- The plot between β and A_D shows that A_D is not maximum at β =1.0. The value of β to maximize A_D is obtained as:

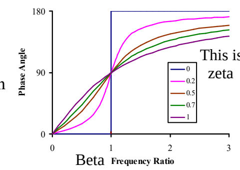
$$\frac{dA_D}{d\beta} = 0 \Rightarrow \beta = \sqrt{1-2\zeta^2} \qquad \left(A_D\right)_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$



Damped Harmonic Vibration : Phase Angle

- At resonance, i.e. β =1.0, all the curves pass through a single point (Φ = 90).
- For β < 1.0 the phase angle is less than 90º.
- For β > 1.0 the phase angle is between 90º and 180º.
- For undamped system the steady state response is in phase (Φ = 0) with the exciting force when β < 1.0. Zeta (0)
- For undamped system the steady state response is in out-of-phase (Φ = 180) with the exciting force when β > 1.0.

$$\tan\phi = \frac{2\zeta\beta}{1-\beta^2}$$

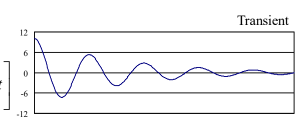


Resonance Response: Damped System

- At resonance (β =1.0)the steady state and transient responses are:

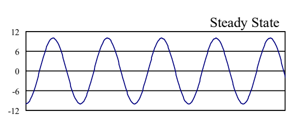
$$u_{steady} = -\frac{p_o}{2\zeta k}\cos\Omega t$$

$$u_{trans} = \frac{p_o}{k}\frac{e^{-\zeta\omega t}}{2\zeta}\left[\cos\omega_d t + \frac{\omega\zeta}{\omega_d}\sin\omega_d t\right]$$



- In the initial phase the transient response will minimize the total response.
- When the transient response diminish, the system vibrates with the exciting force but lagging by 90º.
- For small damping, the total response is:

$$u_{total} = \frac{p_o}{k}\frac{1}{2\zeta}\left(e^{-\zeta\omega t} - 1\right)\cos\omega t$$



Resonance Response: Damped System (Cont..)