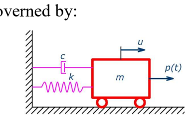
DYNAMICS OF STRUCTURES

Single Degree of Freedom System: Forced Harmonic Vibration

Forced Vibration

- The force vibration response of system is governed by:

  $$m\ddot{u} + c\dot{u} + ku = p(t)$$



- The exciting force may be of:
  - Short duration
  - Long duration
- The response of a system subjected to a short duration exciting force is also of short duration called transient response. Damping in the system causes the vibration to decay and the system returns to rest after a while. This type of response may be critical because of low cycle fatigue.
- The response to long duration force has two components:
  - *Transient component* is due to the initial conditions and present only at the beginning and decays due to damping
  - *Steady state component* lasts as long as the exciting force.

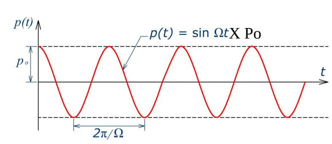
  The failure may be due to fatigue.

Harmonic Forced Vibration

- A harmonic force is given by:

  $$p(t) = p_o \sin \Omega t \_ or \_ p_o \cos \Omega t$$ Rad/sec

  Where p_o is the amplitude of the force and Ω is the frequency of vibrating force.



- Example: Unbalanced rotating Machine
- The study of harmonic force vibration, while useful by itself, also provide an insight into the nature of force vibration of a more general type.

Undamped Harmonic Vibration

- The undamped harmonic vibration of a dynamic system is governed by:

$$m\ddot{u} + ku = p_o \sin \Omega t$$

- The solution of the equation of motion is made with two parts:
  - *Complimentary solution*: obtained by putting p(t) equal to zero and is equivalent to free vibration response of undamped system and is called transient response.

    $$u_c(t) = A\cos\omega t + B\sin\omega t$$

  - *Particular solution*: giving the steady state response of the system. The particular response is of the form:

    $$u_p(t) = G\sin\Omega t$$ → Putting this value of U and U'' in above equation give equation written in step below.

    The value of G is obtained by satisfying the equation of motion:

    $$G\left(k - m\Omega^2\right)\sin\Omega t = p_o \sin\Omega t \Rightarrow G = \frac{p_o}{k - m\Omega^2}$$

  - The complete solution is therefore:

$$u(t) = u_c(t) + u_p(t) = A\cos\omega t + B\sin\omega t + \frac{p_o}{k - m\Omega^2}\sin\Omega t$$

Undamped Harmonic Vibration (Cont..)

- Constant A and B are determined from the initial conditions

$$u(0) = u_o \_ and \_ \dot{u}(0) = v_o$$

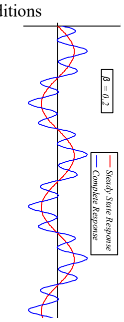
$$u_o = A(1) + B(0) + \frac{p_o}{k - m\Omega^2}(0) = A$$

$$v_o = A\omega(0) + B\omega(1) + \frac{p_o}{k - m\Omega^2}\Omega(1) = B\omega + \frac{p_o\Omega}{k - m\Omega^2}$$

$$u(t) = \underbrace{u_0\cos\omega t + \left(\frac{v_o}{\omega} - \frac{\Omega}{\omega}\frac{p_0}{(k - m\Omega^2)}\right)\sin\omega t}_{\text{Transient}} + \underbrace{\frac{p_o}{k - m\Omega^2}\sin\Omega t}_{\text{Steady State}}$$

$$u(t) = u_0\cos\omega t + \frac{v_o}{\omega}\sin\omega t + \frac{p_o}{k - m\Omega^2}\left(\sin\Omega t - \frac{\Omega}{\omega}\sin\omega t\right)$$

$$u(t) = u_0\cos\omega t + \frac{v_o}{\omega}\sin\omega t + \frac{p_o}{k}\frac{1}{1 - \beta^2}\left(\sin\Omega t - \beta\sin\omega t\right)$$



Where β = Ω/ω is known as the *frequency ratio*.

Of vibrating force — Of structure

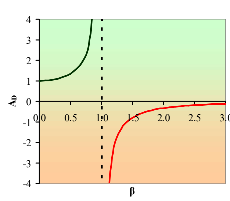
Undamped Harmonic Vibration (Cont..)

- *Dynamic Load Factor, D* is the ratio of maximum dynamic deflection (steady state) to static deflection

$$u_p = \frac{p_o}{k}\frac{1}{1 - \beta^2}\sin\Omega t \qquad u_{static} = \frac{p_o}{k}$$

$$D = \frac{u_p}{u_{static}} = \frac{1}{1 - \beta^2}\sin\Omega t$$

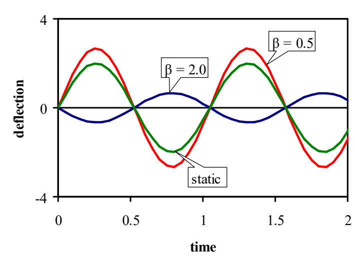
- The amplitude of dynamic load factor A_D called *dynamic magnification factor*, is given by: $A_D = \frac{1}{1 - \beta^2}$



- For small value of β, A_D is approaching 1.0 and for large values A_D is approaching zero
- For β =1.0 the response is infinitely large called *resonance*.

Undamped Harmonic Vibration (Cont..)

- For β <1.0, A_D is +ve and the system is said to be vibrating in phase with the applied force and for β <1.0, A_D is –ve i.e. out-of-phase



Resonance Response: Undamped System

- The harmonic vibration response of undamped system is given by:

$$u(t) = u_0 \cos \omega t + \frac{v_o}{\omega} \sin \omega t + \frac{p_o}{k} \frac{1}{1-\beta^2} \left(\sin \Omega t - \beta \sin \omega t \right)$$

- For a simple case of $u_o = v_o = 0$

$$u(t) = \frac{p_o}{k} \frac{1}{1-\beta^2} \left(\sin \Omega t - \beta \sin \omega t \right) = \frac{p_o}{k} \left(\frac{\sin \beta \omega t - \beta \sin \omega t}{1-\beta^2} \right)$$

- For β = 1.0, the numerator and the denominator are both zero and the displacement becomes indeterminate. In the limiting case, the problem can be solved by L'Hospital's rule

$$\lim_{\beta \to 1} u(t) = \frac{p_o}{k} \lim_{\beta \to 1} \left(\frac{\sin \beta \omega t - \beta \sin \omega t}{1-\beta^2} \right) = \frac{p_o}{k} \lim_{\beta \to 1} \left(\frac{\omega t \cos \beta \omega t - \sin \omega t}{-2\beta} \right)$$

Differentiating w.r.t Beta

$$\lim_{\beta \to 1} u(t) = \frac{p_o}{k} \left(\frac{\omega t \cos \omega t - \sin \omega t}{-2} \right) = \boxed{\frac{p_o}{2k} \left(\sin \omega t - \omega t \cos \omega t \right)}$$

Putting value of Beta

Harmonic response for Beta = 1 & $u_o = v_o = 0$