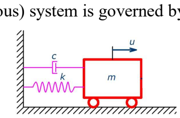
DYNAMICS OF STRUCTURES

Single Degree of Freedom System:
Free Vibration Response

Free Vibration: Viscous Damping

- The free vibration response an damped (viscous) system is governed by:

$$m\ddot{u} + c\dot{u} + ku = 0$$



- A possible solution of the above equation is:

$$u = Ge^{\lambda t}$$

- Substituting the solution into equation of motion:

$$G\lambda^2 me^{\lambda t} + c\lambda e^{\lambda t} + Gke^{\lambda t} = 0$$

$$\lambda^2 m + c\lambda + k = 0$$

$$\Rightarrow \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

$$\lambda = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}$$

Free Vibration: Viscous Damping (Cont..)

- Based on the value of the discriminant, three different cases arise:

$$\lambda = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{4m^2} - \frac{k}{m}}$$

- Critically damped system (discriminant is zero) $\frac{c^2}{4m^2} - \frac{k}{m} = 0 \Rightarrow c = c_{cr} = 2\sqrt{km} = 2m\omega$
- Overdamped system (discriminant is positive) $\frac{c^2}{4m^2} - \frac{k}{m} > 0 \Rightarrow c > (c_{cr} = 2m\omega) \Rightarrow \left(\zeta = \frac{c}{c_{cr}}\right) > 1$
- Underdamped system (discriminant is negative) $\frac{c^2}{4m^2} - \frac{k}{m} < 0 \Rightarrow c < (c_{cr} = 2m\omega) \Rightarrow \left(\zeta = \frac{c}{c_{cr}}\right) < 1$

Critically Damped System

- For critically damped system

$$\frac{c^2}{4m^2}-\frac{k}{m}=0\_\Rightarrow\_c=c_{cr}=2\sqrt{km}=2m\omega$$

The roots of the differential equation are:

$$\lambda_1=\lambda_2=-\frac{c}{2m}=-\frac{c_{cr}}{2m}=-\omega$$

Lambda is equal to omega on for critically damped system

Therefore the solution of equation of motion is:

$$u=(G_1+G_2t)e^{-\omega t}$$

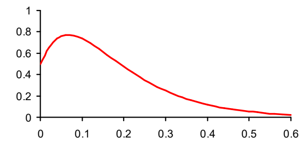
The arbitrary constants are determined from the initial condition. The general solution thus becomes:

$$u=\{u_o+(v_o+\omega u_o)t\}e^{-\omega t}$$

For critically damped

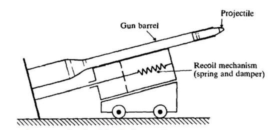
Critically Damped System (Cont..)

- Thus the free vibration of a critically damped system is not oscillatory
- After the initial disturbance the system will come back to its original position without oscillation in maximum time.



- Examples:

  Recoiling Gun and Weighing Scale



An example of critically damped system

Overdamped System

- For overdamped system, damping is greater than critical damping
- The ratio of damping of a system to its critical damping is called damping ratio and is given by:

$$\zeta=\frac{c}{c_{cr}}=\frac{c}{2m\omega}\_\Rightarrow\_c=2m\omega\zeta$$

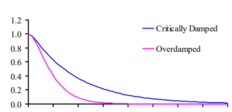
- The roots of characteristic equation may be written as:

$$\lambda=-\omega\zeta\pm\omega\sqrt{\zeta^2-1}=-\omega\zeta\pm\overline{\omega}$$

$$where\_\overline{\omega}=\omega\sqrt{\zeta^2-1}$$

- The general solution of equation of motion is of the form:

$$u=e^{-\omega\zeta t}\left(A\cosh\overline{\omega}t+B\sinh\overline{\omega}t\right)$$



Examples are: Automatic door closer

Underdamped System

- For underdamped system, damping is less than the critical damping, i.e. the damping ratio is less than one
- The roots of characteristic equation may be written as:

$$\lambda=-\omega\zeta\pm i\omega\sqrt{1-\zeta^2}=-\omega\zeta\pm\omega_d$$

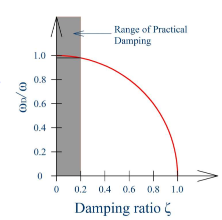
$$where\_\omega_d=\omega\sqrt{1-\zeta^2}$$

ω_d is known as the *damped circular frequency*

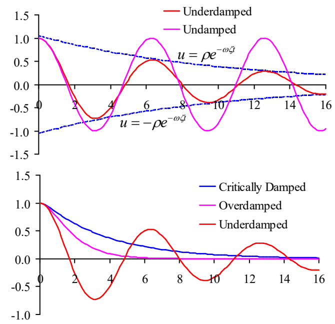
- The general solution of equation of motion is of the form:

$$u=e^{-\omega\zeta t}\left(A\cos\omega_d t+B\sin\omega_d t\right)$$

$$u=e^{-\omega\zeta t}\left(u_o\cos\omega_d t+\frac{v_o+\omega\zeta u_o}{\omega_d}\sin\omega_d t\right)$$



Underdamped System (Cont..)



Phase Plane Diagram

- Under-damped system

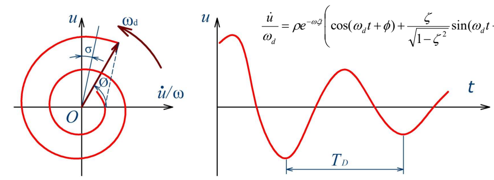
$$u = \rho e^{-\omega\zeta t}\sin(\omega_d t + \phi)$$

where

$$\rho = \sqrt{(u_o)^2 + \left(\frac{v_o + u_o\omega\zeta}{\omega_d}\right)^2}$$

$$\tan\phi = \frac{u_o\omega_d}{v_o + u_o\omega\zeta}$$

$$\frac{\dot{u}}{\omega_d} = \rho e^{-\omega\zeta t}\left(\cos(\omega_d t + \phi) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)\right)$$



$$\frac{e^{-\omega\xi t_1}}{e^{-\omega\xi(t_1 + \frac{2\pi}{\omega_d})}} = \frac{e^{-\omega\xi t_1}}{e^{-\omega\xi t_1} \times e^{-\omega\xi\frac{2\pi}{\omega_d}}} = e^{\omega\xi\frac{2\pi}{\omega_d}}$$

$$\delta = \ln(e^{\omega\xi\frac{2\pi}{\omega_d}}) = \omega\xi\frac{2\pi}{\omega_d} = 2\pi\xi\frac{\omega}{\omega_d} = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

Logarithmic Decrement

- It is the natural log of the ratio of displacement at any time t_1 and $t_1 + 2\pi/\omega_d$
- It represent decay in the magnitude during one cycles
- Logarithmic Decrement is used to calculate the damping ratio of a system

$$u(t_1) = \rho e^{-\omega\zeta t_1}\sin(\omega_d t_1 + \phi)$$

$$u\left(t_1 + \frac{2\pi}{\omega_d}\right) = \rho e^{-\omega\zeta\left(t_1 + \frac{2\pi}{\omega_d}\right)}\sin(\omega_d t_1 + 2\pi + \phi) = \rho e^{-\omega\zeta\left(t_1 + \frac{2\pi}{\omega_d}\right)}\sin(\omega_d t_1 + \phi)$$

$$\delta = \ln\frac{u(t_1)}{u\left(t_1 + \frac{2\pi}{\omega_d}\right)} = \ln\frac{e^{-\omega\zeta t_1}}{e^{-\omega\zeta\left(t_1 + \frac{2\pi}{\omega_d}\right)}} \qquad \frac{e^{-\omega\xi t_1}}{e^{-\omega\xi(t_1 + \frac{2\pi}{\omega_d})}} = \frac{e^{-\omega\xi t_1}}{e^{-\omega\xi t_1} \times e^{-\omega\xi\frac{2\pi}{\omega_d}}} = e^{\omega\xi\frac{2\pi}{\omega_d}}$$

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \qquad \delta = \ln(e^{\omega\xi\frac{2\pi}{\omega_d}}) = \omega\xi\frac{2\pi}{\omega_d} = 2\pi\xi\frac{\omega}{\omega_d} = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

Logarithmic Decrement (Cont..)

- For many practical cases:

$\zeta < 0.2$ _therefore_ $1 - \zeta^2 \approx 1$

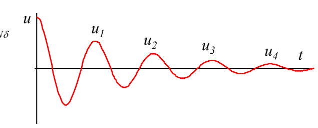
$\delta = 2\pi\zeta \Rightarrow \zeta = \frac{\delta}{2\pi}$

- For lightly damped system, the decay of motion is slow, it is desirable to relate amplitudes several cycles apart:

$$\frac{u_1}{u_{N+1}} = \frac{u_1}{u_2}\frac{u_2}{u_3}\frac{u_3}{u_4}........\frac{u_N}{u_{N+1}} = e^{N\delta}$$

$$\ln\frac{u_1}{u_{N+1}} = N\delta = N2\pi\zeta$$

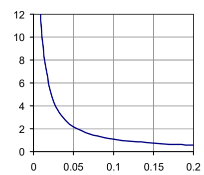
$$\zeta = \frac{1}{2\pi N}\ln\frac{u_1}{u_{N+1}}$$



Logarithmic Decrement (Cont..)

- The number of cycles in which the amplitude will decay to half its value at he beginning is given by:

$$N_{50\%} = \frac{1}{\delta}\ln 2 \approx \frac{1}{2\pi\zeta}\ln 2 \approx \frac{0.11}{\zeta}$$



- Since acceleration is easy to measure and also acceleration is proportional to the displacement in case of free vibration, the damping ratio is therefore determined from:

$$\zeta = \frac{1}{2\pi N}\ln\frac{\ddot{u}_1}{\ddot{u}_{N+1}}$$