DYNAMICS OF STRUCTURES

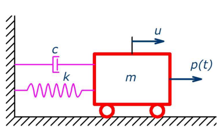
Single Degree of Freedom System:
Free Vibration Response

Solution of Equation of Motion

- For a single degree of freedom system the equation of motion is a linear second order differential equation:

$$m\ddot{u} + c\dot{u} + ku = p(t)$$

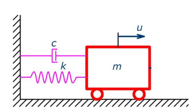
- Various methods for the solution of equation of motion are:
  - Classical Solution
  - Duhamel's Integral
  - Frequency Domain Method
  - Numerical Methods



Free Vibration

- The vibration of a system excided by initial disturbances (initial displacement or velocity) without any external force is called free vibration

$$m\ddot{u} + c\dot{u} + ku = 0$$

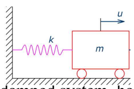


- The equation of motion becomes a homogenous 2nd order linear differential equation
- The study of free vibration response is important because:
  - Many practical systems are excited by initial disturbances
  - The complete solution of force vibration also include free vibration component.

Undamped Free Vibration

- The free vibration response an undamped system is governed by:

$$m\ddot{u} + ku = 0$$



- In reality there is no existence of undamped system, however the its response gives an insight into the nature of damped system.
- A possible solution of the above equation is:

$$u = Ge^{\lambda t}$$

- Substituting the solution into equation of motion:

$$G\lambda^2 me^{\lambda t} + Gke^{\lambda t} = 0 \Rightarrow \lambda^2 m + k = 0$$

$$\Rightarrow \lambda^2 + \omega^2 = 0 \Rightarrow \lambda = \pm i\omega$$

$where\_\omega = \sqrt{\frac{k}{m}}$ called *circular frequency* of system

Undamped Free Vibration

- The general solution of the equation of motion now becomes:

$$u = G_1 e^{i\omega t} + G_2 e^{-i\omega t} = A\cos\omega t + B\sin\omega t$$

- The arbitrary constants A and B are determined from the initial conditions i.e. At t = 0, $u(0) = u_o\_and\_\dot{u}(0) = v_o$

$$A = u_o\_and\_B = \frac{v_o}{\omega}$$

- The solution of undamped free vibration response of single degree of freedom system is, therefore:

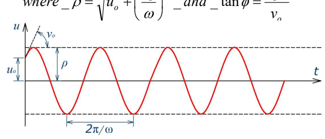
$$u = u_o\cos\omega t + \frac{v_o}{\omega}\sin\omega t$$

Undamped Free Vibration

- The solution may be written as:

$$u = \rho\sin(\omega t + \phi)$$

$$where\_\rho = \sqrt{u_o^2 + \left(\frac{v_o}{\omega}\right)^2}\_and\_\tan\phi = \frac{u_o\omega}{v_o}$$



- It can be shown that: $u(t_1) = u\left(t_1 + \frac{2\pi}{\omega}\right)$
- Thus the vibration repeats itself after $T = \frac{2\pi}{\omega}$ and is the *natural period* of system
- The reciprocal of T is call the *natural frequency* of system; $f = \frac{\omega}{2\pi}$

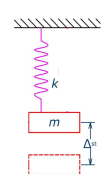
Undamped Free Vibration

$$W = k\Delta_{st} \Rightarrow mg = k\Delta_{st} \Rightarrow k = \frac{mg}{\Delta_{st}}$$

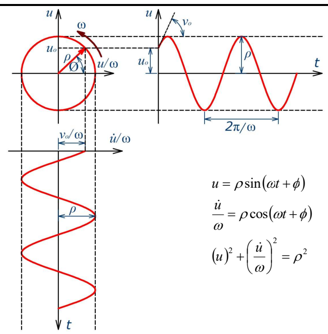
$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/\Delta_{st}}{m}} = \sqrt{\frac{g}{\Delta_{st}}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\Delta_{st}}{g}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta_{st}}}$$



Phase Plan Diagram

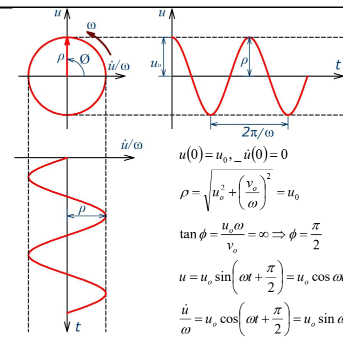


$$u = \rho \sin(\omega t + \phi)$$

$$\frac{\dot{u}}{\omega} = \rho \cos(\omega t + \phi)$$

$$(u)^2 + \left(\frac{\dot{u}}{\omega}\right)^2 = \rho^2$$

Phase Plan Diagram (Initial Displacement)



$$u(0) = u_0, \_ \dot{u}(0) = 0$$

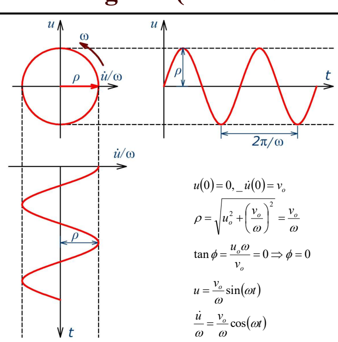
$$\rho = \sqrt{u_o^2 + \left(\frac{v_o}{\omega}\right)^2} = u_0$$

$$\tan\phi = \frac{u_o \omega}{v_o} = \infty \Rightarrow \phi = \frac{\pi}{2}$$

$$u = u_o \sin\left(\omega t + \frac{\pi}{2}\right) = u_o \cos \omega t$$

$$\frac{\dot{u}}{\omega} = u_o \cos\left(\omega t + \frac{\pi}{2}\right) = u_o \sin \omega t$$

Phase Plan Diagram (Initial Velocity)



$$u(0) = 0, \_ \dot{u}(0) = v_o$$

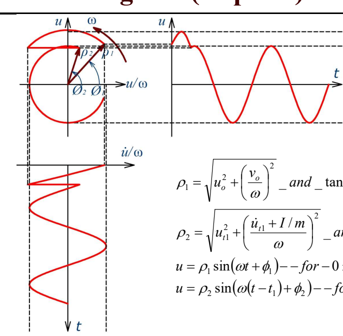
$$\rho = \sqrt{u_o^2 + \left(\frac{v_o}{\omega}\right)^2} = \frac{v_o}{\omega}$$

$$\tan\phi = \frac{u_o \omega}{v_o} = 0 \Rightarrow \phi = 0$$

$$u = \frac{v_o}{\omega} \sin(\omega t)$$

$$\frac{\dot{u}}{\omega} = \frac{v_o}{\omega} \cos(\omega t)$$

Phase Plan Diagram (Impulse)



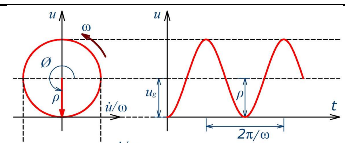
$$\rho_1 = \sqrt{u_o^2 + \left(\frac{v_o}{\omega}\right)^2} \_ and \_ \tan\phi_1 = \frac{u_o \omega}{v_o}$$

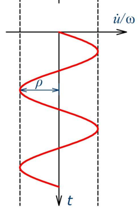
$$\rho_2 = \sqrt{u_{t1}^2 + \left(\frac{\dot{u}_{t1} + I/m}{\omega}\right)^2} \_ and \_ \tan\phi_1 = \frac{u_{t1} \omega}{\dot{u}_{t1} + I/m}$$

$$u = \rho_1 \sin(\omega t + \phi_1) - - for - 0 \le t \le t_1$$

$$u = \rho_2 \sin(\omega(t - t_1) + \phi_2) - - for - t \ge t_1$$

Phase Plan Diagram (Ground Motion)





$$\rho = \sqrt{u_g^2 + \left(\frac{0}{\omega}\right)^2} = u_g$$

$$\tan\phi = -\infty \_ \phi = \frac{3\pi}{2}$$

$$u = u_g + u_g \sin\left(\omega t + \frac{3\pi}{2}\right)$$

$$u = u_g(1 - \cos\omega t)$$