Chapter 10 Method of Virtual Work

10.1 Introduction

equations of equilibrium

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M = 0 \qquad \text{etc.}$$

a different method is now proposed, more effective for solving certain types of equilibrium problems, it is called method of virtual work

consider a particle or a rigid body, if it is given an arbitrary displacement from the position of equilibrium, the total work done of the external forces during the displacement is zero

method of virtual work may extend to the concept of potential energy

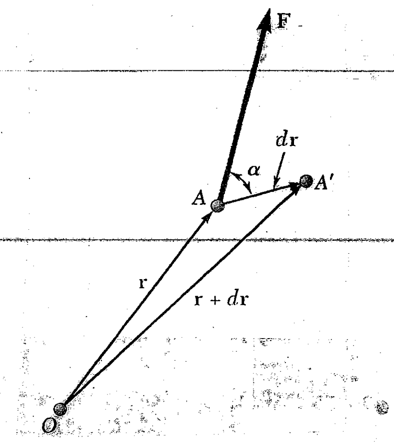
10.2 Work of a Force

consider a particle which moves from A to A'

dr is called the displacement of the particle

F is the force acting on the particle

the work of the force ***F*** corresponding to the displacement ***dr*** is defined as



$$dU \equiv \mathbf{F} \cdot \mathbf{dr} = F\, ds \cos \alpha$$

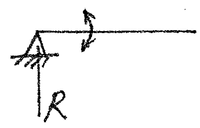
where $\quad |\mathbf{dr}| = ds \qquad |\mathbf{F}| = F$

work is a scalar quantity, has a magnitude and sign, but no direction

the unit of work is : N-m = J (joule)

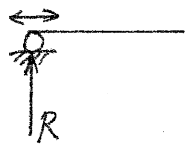
$$dU = F\, ds \cos \alpha = \begin{cases} F\, ds & (\alpha = 0^\circ) \quad \text{same direction} \\ 0 & (\alpha = 90^\circ) \quad \text{perpendicular} \\ -F\, ds & (\alpha = 180^\circ) \quad \text{opposite direction} \end{cases}$$

force acting on a fixed point (reactions at frictionless pin), it is no work done



forces acting in the direction perpendicular to displacement, it is no work done

[if the roller has friction with surface, then the friction force has work done]

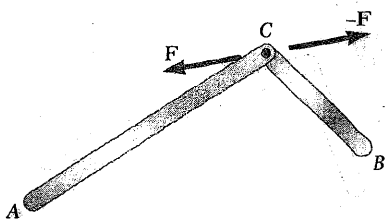


the sum of work done by internal forces is zero

F is the force acting on AC exerted by BC,

- ***F*** is the force acting on BC exerted by AC

F and - ***F*** has same magnitude but opposite direction, thus the total work of two internal forces at C cancels out

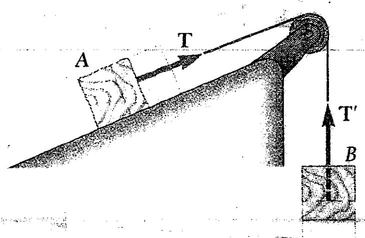


work done of force due to an inextensible cord AB

work at A : $T\,dS$ (same direction)

work at B : - $T\,dS$ (opposite direction)

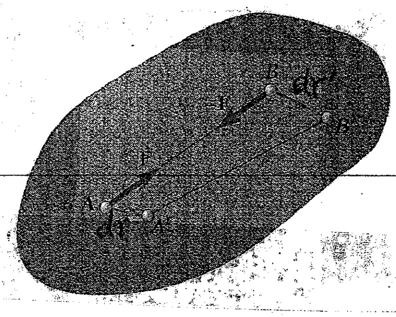
thus the work of the internal forces again cancels out



work done due to internal force in a rigid body

consider points A and B are two particles of a rigid body, ***F*** and - ***F*** are the internal forces holding together the particles

dr and ***dr'*** might different, but the components along AB must be equal, thus the sum of the work done is zero

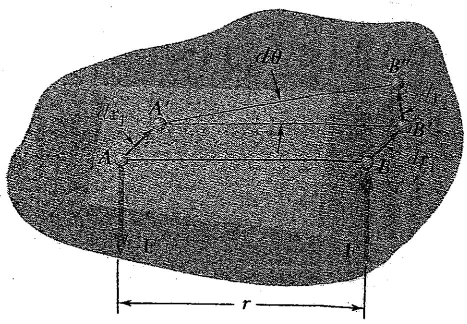


consider two forces $\boldsymbol{F}$ and $-\boldsymbol{F}$ forming a couple $\boldsymbol{M}$ acting on a rigid body

displacement at A is $\boldsymbol{dr}_1$

displacement at B is $\boldsymbol{dr}' = \boldsymbol{dr}_1 + \boldsymbol{dr}_2$

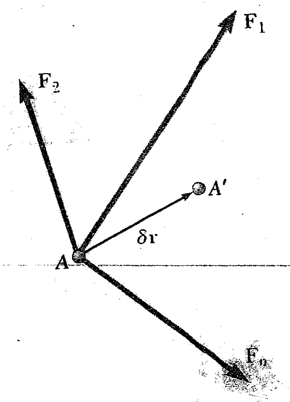
total work done is



$$dU = -\boldsymbol{F} \cdot \boldsymbol{dr}_1 + \boldsymbol{F} \cdot \boldsymbol{dr}'$$
$$= -\boldsymbol{F} \cdot \boldsymbol{dr}_1 + \boldsymbol{F} \cdot \boldsymbol{dr}_1 + \boldsymbol{F} \cdot \boldsymbol{dr}_2$$
$$= F\, dS_2 = F\, r\, d\theta$$
$$dU = M\, d\theta$$

10.3 Principle of Virtual Work

consider a particle A with force acted, let it has a displacement δr, and the particle moves form A to A'



this displacement is possible, but not necessarily take place, the forces may be balanced and the particle at rest

the displacement considered is therefore an imaginary displacement, is called virtual displacement, the work done of each force during the virtual displacement is called virtual work

$$\delta U = \boldsymbol{F}_1 \cdot \boldsymbol{\delta r} + \boldsymbol{F}_2 \cdot \boldsymbol{\delta r} + \boldsymbol{F}_3 \cdot \boldsymbol{\delta r} + \cdots$$
$$= (\boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \cdots) \cdot \boldsymbol{\delta r}$$
$$= \boldsymbol{R} \cdot \boldsymbol{\delta r}$$

principle of virtual work for a particle :

if a particle is in equilibrium, the total virtual work of the forces acting on the particle is zero for any virtual displacement of the particle [in this case, $\boldsymbol{R}$ must be zero, then $\delta U = 0$]

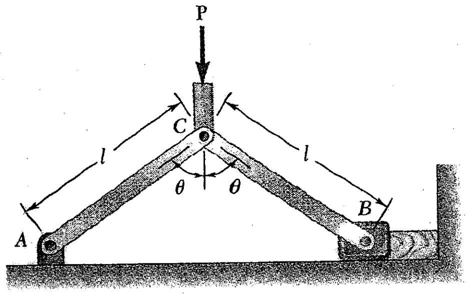
principle of virtual work of a rigid body :

if a rigid body is in equilibrium, the total virtual δU due to the external forces acting on the rigid body is zero for any $\delta \mathbf{r}$ [since total δU due to internal forces are zero]

for connected rigid body, only the work due to external forces of the system need be considered

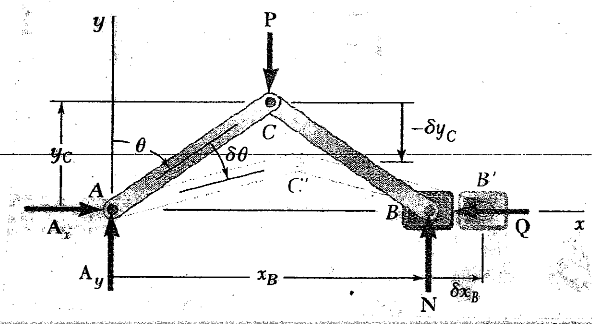
10.4 Application of the Principle of Virtual Work

consider the toggle vise *ABC*



choosing a system of coordinate axes with origin at *A*

expressing the virtual displacement in terms of $\delta\theta$



$$y_C = l\cos\theta \qquad \delta y_C = -\,l\sin\theta\,\delta\theta$$

$$x_B = 2l\sin\theta \qquad \delta x_B = 2l\cos\theta\,\delta\theta$$

$$\delta U = \delta U_P + \delta U_Q$$

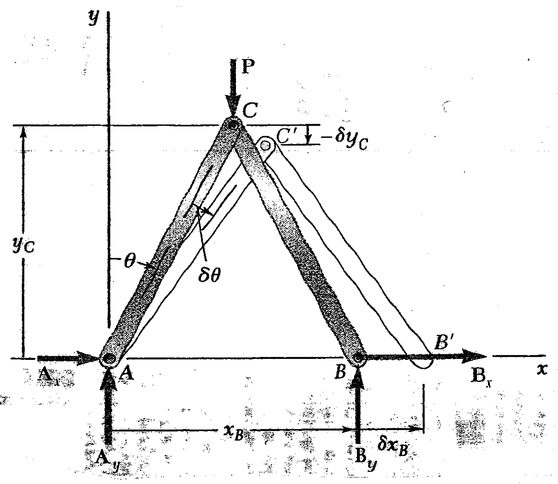
$$= -\,P\,\delta y_C - Q\,\delta x_B$$

$$= P\,l\sin\theta\,\delta\theta - Q\,2l\cos\theta\,\delta\theta$$

for equilibrium $\delta U = 0$

$$2\,Q\,l\cos\theta\,\delta\theta = P\,l\sin\theta\,\delta\theta$$

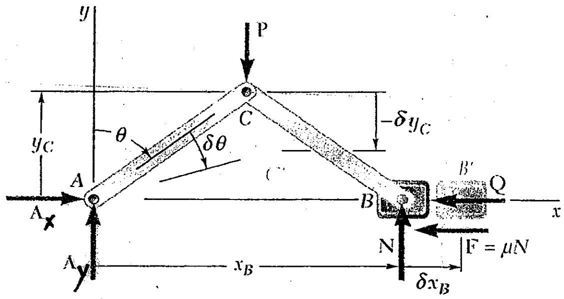
$$Q = \tfrac{1}{2}P\tan\theta$$



for a frame *ACB*, assume *A* fixed and *B* has a virtual displacement δx_B, the it is obtained

$$B_x = -\,\tfrac{1}{2}P\tan\theta$$

10.5 Real Machines, Mechanical Efficiency



consider the toggle vise ACB

input work done $\quad \boldsymbol{P}\,\delta y_C = P\,l \sin\theta\,\delta\theta$

output work done $\quad \boldsymbol{Q}\,\delta x_B = 2\,Q\,l\cos\theta\,\delta\theta$

if $U_i = U_o$, the machine is said to be ideal, but in real machine, friction force will always do some work, thus $U_o < U_i$

$$N = P/2 \qquad\qquad F = \mu\,P/2$$

$$\delta U = -\boldsymbol{Q}\,\delta x_B - \boldsymbol{P}\,\delta y_C - F\,\delta x_B$$

$$= -2\,Q\,l\cos\theta\,\delta\theta + P\,l\sin\theta\,\delta\theta - \mu\,P\,l\cos\theta\,\delta\theta$$

for $\delta U = 0$

$$Q = \tfrac{1}{2}\,P\,(\tan\theta - \mu)$$

when $\tan\theta = \mu \quad \Rightarrow Q = 0$

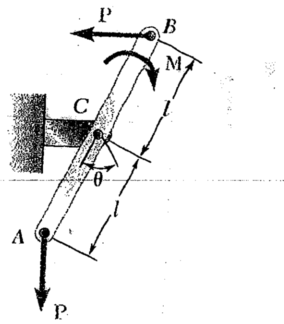
[what happen when $\tan\theta < \mu$, $Q < 0$?, NO, in this case, $F < \mu\,N$]

the mechanical efficiency η is defined

$$\eta = \frac{\text{output work}}{\text{input work}}$$

$\eta = 1$ for ideal machine, in general $\eta < 1$

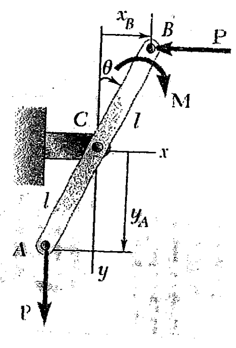
Sample Problem 10.1



determine M required to maintain equilibrium

$$x_B = l\sin\theta \qquad\qquad \delta x_B = l\cos\theta\,\delta\theta$$

$$y_A = l\cos\theta \qquad\qquad \delta y_A = -\,l\sin\theta\,\delta\theta$$



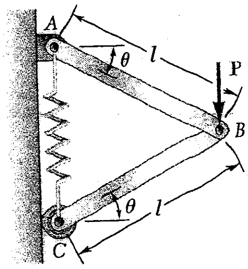
$$\delta U = 0 \qquad M\,\delta\theta - P\,\delta x_B + P\,\delta y_A = 0$$

$$M\,\delta\theta - P\,l\cos\theta\,\delta\theta + P\,(-\,l\sin\theta\,\delta\theta) = 0$$

$$M = P\,l\,(\sin\theta + \cos\theta)$$

Sample Problem 10.2

determine θ and the spring force for equilibrium



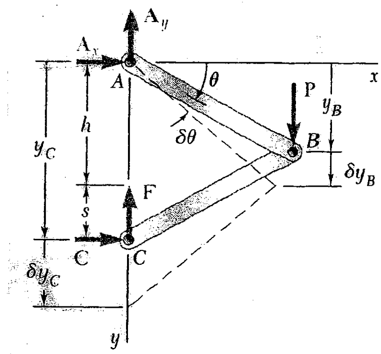
spring constant $= k$, $F = 0$ for $y_C = h$

$$y_B = l \sin\theta \qquad \delta y_B = l \cos\theta\, \delta\theta$$

$$y_C = 2\, l \sin\theta \qquad \delta y_C = 2\, l \cos\theta\, \delta\theta$$

$$s = y_C - h = 2\, l \sin\theta - h$$

$$F = k\, s = k\,(2\, l \sin\theta - h)$$



$$\delta U = 0 \qquad P\, \delta y_B - F\, \delta y_C = 0$$

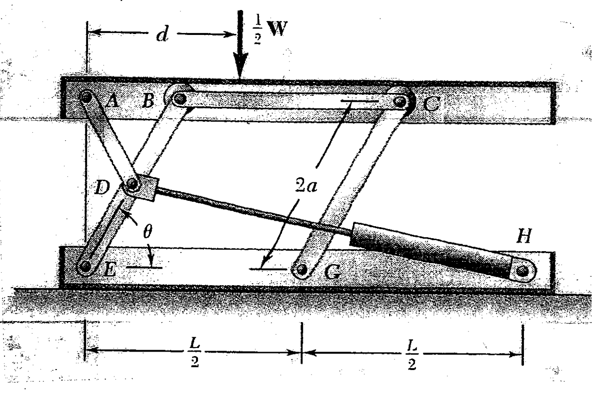
$$P\,(l \cos\theta\, \delta\theta) = [k\,(2\, l \sin\theta - h)]\,(2\, l \cos\theta\, \delta\theta)$$

$$\sin\theta = \frac{P + 2\, k\, h}{4\, k\, l}$$

and $$F = k\,(2\, l\, \frac{P + 2\, k\, h}{4\, k\, l} - h) = \frac{P}{2}$$

Sample Problem 10.3

determine F_{DH} of the hydraulic-lift

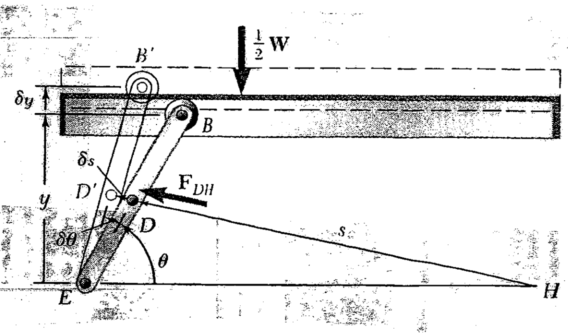


$$y = 2\, a \sin\theta \qquad \delta y = 2\, a \cos\theta\, \delta\theta$$

$$s^2 = a^2 + L^2 - 2\, a\, L \cos\theta$$

$$2\, s\, \delta s = -\, 2\, a\, L\, (-\sin\theta)\, \delta\theta$$

$$\delta s = \frac{a\, L \sin\theta}{s}\, \delta\theta$$



$$\delta U = 0 \qquad -\tfrac{1}{2}\, W\, \delta y + F_{DH}\, \delta s = 0$$

$$-\tfrac{1}{2}\, W\, (2\, a \cos\theta\, \delta\theta) + F_{DH}\, \frac{a\, L \sin\theta}{s}\, \delta\theta = 0$$

$$F_{DH} = W\, \frac{s}{L} \cot\theta$$

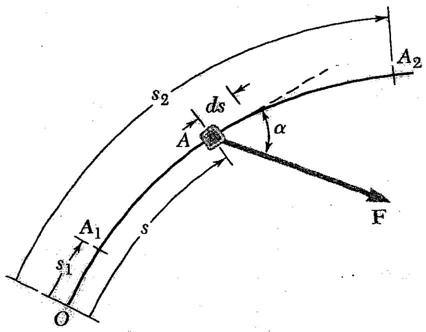
for $\theta = 60^\circ$ $a = 0.7\text{ m}$ $L = 3.2\text{ m}$

$s = 2.91\text{ m}$ $W = 9.81\text{ kN}$

$F_{DH} = 5.15\text{ kN}$

10.6 Work of a Force Due a Finite Displacement

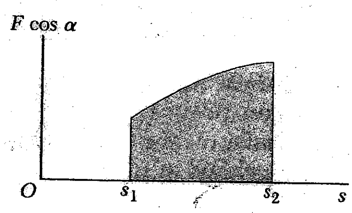
consider a force $\boldsymbol{F}$ acting on a particle



$$dU = \boldsymbol{F} \cdot \boldsymbol{dr}$$

the work of $\boldsymbol{F}$ due to a finite displacement of the particle form A_1 to A_2 is

$$U_{1\to2} = \int_{A_1}^{A_2} \boldsymbol{F} \cdot \boldsymbol{dr}$$



also $dU = F\,dS \cos a$ $\qquad a$ may vary

$$U_{1\to2} = \int_{S_1}^{S_2} F\,dS \cos a$$

if $\boldsymbol{F}$ and $\boldsymbol{dr}$ in the same direction, then $a = 0$ and $\cos a = 1$

i.e. $\qquad U_{1\to2} = F\,(S_2 - S_1)$

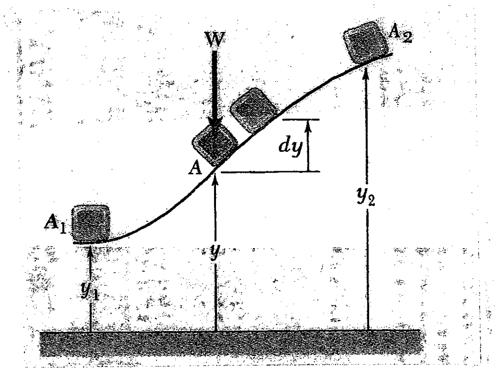
if the work is done by moment

$$dU = M\,d\theta$$

$$U_{1\to2} = \int_{\theta_1}^{\theta_2} M\,d\theta$$

if M = constant, then $U_{1\to2} = M\,(\theta_2 - \theta_1)$

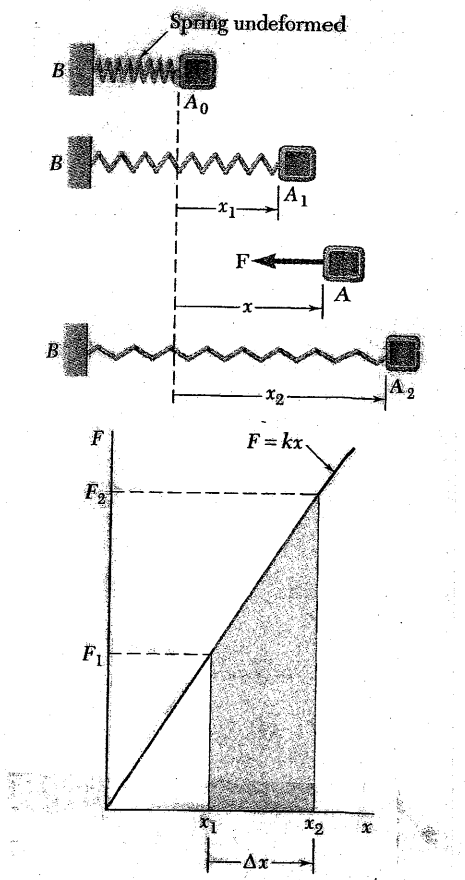
work of weight



$$dU = -\,W\,dy$$

$$U_{1\to2} = -\int_{y_1}^{y_2} W\,dy$$

$$= -\,W\,(y_2 - y_1) = -\,W\,\Delta y$$



work of force exerted by a spring

$$F = k\,x \qquad k : \text{spring constant}$$

$$dU = -F\,dx$$

$$U_{1\to2} = -\int_{x_1}^{x_2} F\,dx = -\int_{x_1}^{x_2} k\,x\,dx$$

$$= \tfrac{1}{2} k\,x_1^2 - \tfrac{1}{2} k\,x_2^2$$

$U_{1\to2}$ is positive when $x_2 < x_1$, i.e. spring is running back to its undeformed position

$$U_{1\to2} = -\int_{x_1}^{x_2} F\,dx = -\tfrac{1}{2}(F_1 + F_2)\,\Delta x$$

10.7 Potential Energy

work done by weight $= W y_1 - W y_2$

define V_g : the potential energy with respect to force of gravity W

$$V_g = W y$$

then $\quad U_{1\to2} = (V_g)_1 - (V_g)_2$

work done due to spring force $= \tfrac{1}{2} k\,x_1^2 - \tfrac{1}{2} k\,x_2^2$

define V_e : the potential energy due to spring force

$$V_e = \tfrac{1}{2} k\,x^2$$

then $\quad U_{1\to2} = (V_e)_1 - (V_e)_2$

the concept of potential energy may be used when forces other than gravity force and spring force

if it is possible to find a function V, called potential energy, such that

$$d\,U = -\,d\,V$$

integrating over a finite displacement, then

$$U_{1\to2} = V_1 - V_2$$

a force which satisfied the above equation is said to be conservative force

10.8 Potential Energy and Equilibrium

for a virtual displacement $\delta\theta$

$\delta U = -\,\delta V$ is a function of $\delta\theta$

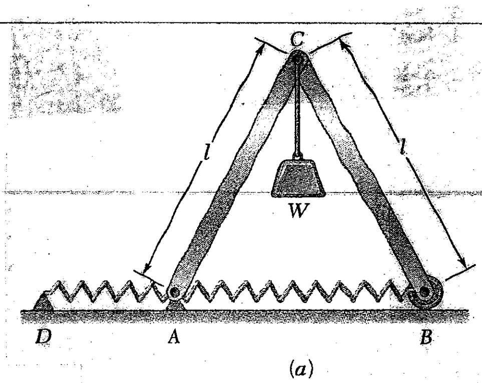
if the position of the system is defined by a single independent variable θ, we may write

$$\delta U = (\partial V / \partial\theta)\,\delta\theta$$

$\because$ $\delta\theta$ must be different from zero

thus the equilibrium condition is

$$\delta U = 0 \quad \Rightarrow \quad \partial V / \partial\theta = 0$$

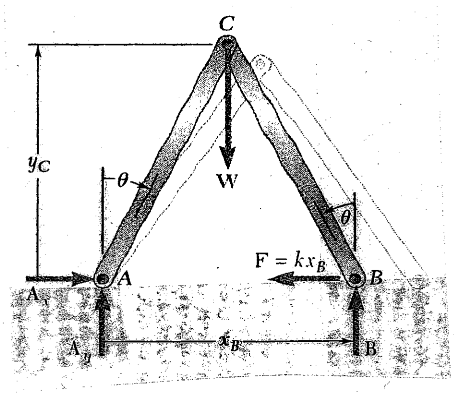


consider the example

DA : length of undeformed spring

$V_e = \frac{1}{2}\,k\,x_B^2$

$V_g = W\,y_C$



$x_B = 2\,l\sin\theta \qquad V_e = \frac{1}{2}\,k\,(2\,l\sin\theta)^2$

$y_C = l\cos\theta \qquad V_g = W\,l\cos\theta$

$V = V_e + V_g = 2\,k\,l^2\sin^2\theta + W\,l\cos\theta$

$dV / d\theta = 4\,k\,l^2\sin\theta\cos\theta - W\,l\sin\theta = 0$

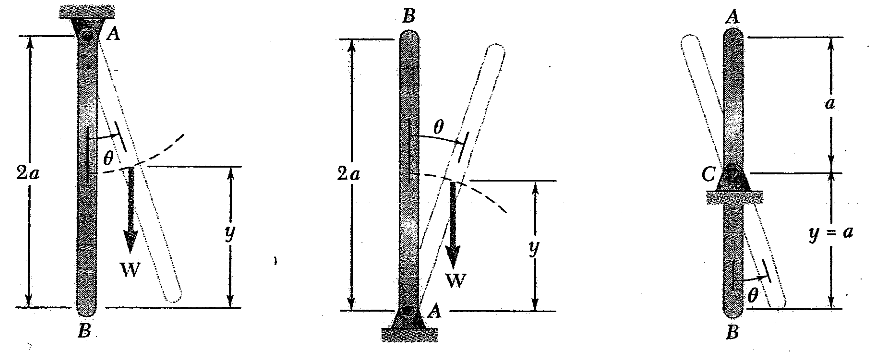
i.e. $\sin\theta = 0$ or $4\,k\,l\cos\theta = W$

$\theta = 0$

and $\theta = \cos^{-1}(W / 4\,kl)$ this position does not exist if $W > 4\,kl$

10.9 Stability of Equilibrium

consider the three uniform rods of length $2a$ and weight W



(a) Stable equilibrium (b) Unstable equilibrium (c) Neutral equilibrium

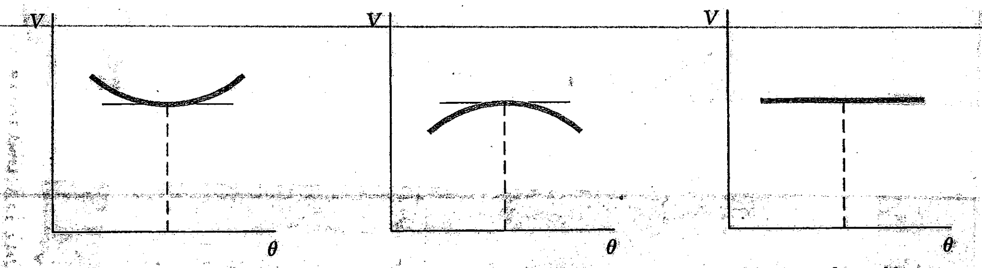
$$V = Wy$$

at the equilibrium position $\theta = 0$

(a) : V = minimum

(b) : V = maximum

(c) : V = constant



(a) Stable equilibrium (b) Unstable equilibrium (c) Neutral equilibrium

$dV/d\theta = 0$ for equilibrium

$d^2V/d\theta^2 > 0$ i.e. $V = V_{min}$ stable equilibrium

$d^2V/d\theta^2 < 0$ i.e. $V = V_{max}$ unstable equilibrium

$d^2V/d\theta^2 = d^3V/d\theta^3 = \cdots = 0$

i.e. V = constant neutral equilibrium

if the system considered possesses several degree of freedom, say $V = V(\theta_1, \theta_2)$, it will be minimum if the following relations are satisfied simultaneously

$$\frac{\partial V}{\partial \theta_1} = \frac{\partial V}{\partial \theta_2} = 0$$

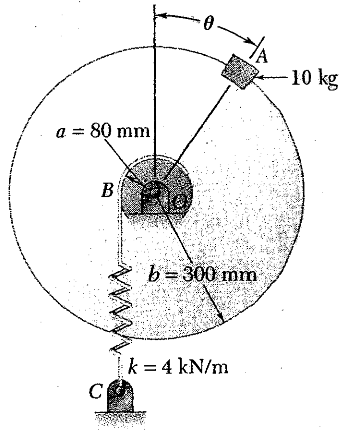
$$\left(\frac{\partial^2 V}{\partial \theta_1 \, \partial \theta_2}\right)^2 - \frac{\partial^2 V}{\partial \theta_1^2}\frac{\partial^2 V}{\partial \theta_2^2} < 0$$

$$\frac{\partial^2 V}{\partial \theta_1^2} > 0 \quad \text{or} \quad \frac{\partial^2 V}{\partial \theta_2^2} > 0$$

Sample Problem 10.4

the spring is unstretched when $\theta = 0°$

determine the positions of equilibrium



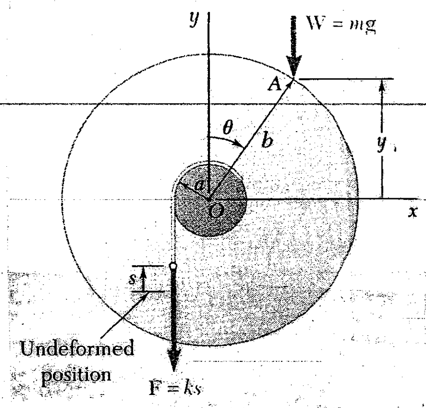
$$V_e = ½\, k\, s^2 = ½\, k\, (a\,\theta)^2$$

$$V_g = W\, y = mg\, b \cos\theta$$

$$V = V_e + V_g$$

$$= ½\, k\, a^2\, \theta^2 + mg\, b \cos\theta$$

for equilibrium $\quad dV / d\theta = 0$



$$k\, a^2\, \theta - mg\, b \sin\theta = 0$$

$$\sin\theta = \frac{k\, a^2}{mg\, b}\,\theta = \frac{4 \times 0.08^2}{10 \times 9.81 \times 0.3}\,\theta$$

$$\sin\theta = 0.8699\,\theta$$

$\theta = 0°$ $\quad$ or $\quad$ $\theta = 0.902 \text{ rad} = 51.7°$

stability for equilibrium

$$d^2V / d\theta^2 = k\, a^2 - mg\, b \cos\theta$$

$$= 4 \times 0.08^2 - 10 \times 9.81 \times 0.3 \cos\theta$$

$$= 25.6 - 29.43 \cos\theta$$

for $\theta = 0°$ $\quad d^2V / d\theta^2 = -3.83 < 0$ $\quad$ unstable

for $\theta = 51.7°$ $\quad d^2V / d\theta^2 = 7.36 > 0$ $\quad$ stable