Chapter 2
Three-Hinged Arches

This chapter is devoted to the analysis of statically determinate three-hinged arches, subjected to fixed and moving loads. Analysis of an arch in the case of fixed loads implies determination of reactions of supports and construction of internal force diagrams. Analysis of an arch in the case of moving load implies construction of influence lines for reactions, thrust, and internal forces.

Some important concepts are discussed. Among them are a reference beam, thrust, nil points of influence lines, etc. Analytical formulas for computation of internal forces as well as for construction of influence lines for reactions and internal forces are developed. Special types of arches are considered; among them are arches with simple and complex ties, arches with support points on different levels. Analysis of the multispan arched structure and truss enforced by arched chain are discussed.

Fundamental investigation in the area of static analysis of arches is attributed to Bresse [Bre59], Kirchhoff [Kir76], and Winkler [Tim53] to name a few.

2.1 General

Idealized design diagram of the arch without overarched members is shown in Fig. 2.1a. This diagram contains two curvilinear members which are hinged together at the crown; connections of curvilinear members with abutment are also hinged. These three hinges are distinguishing features of the three-hinged arch. Design diagram also contains information about the shape of the neutral line of the arch. Usually, this shape is given by an expression of the form $y = f(x)$. Expressions for some characteristic shapes are presented in Tables A.1 and A.2.

Degrees of freedom of the arch in Fig. 2.1a, according to Chebushev formula [Kar10], are determined by the formula

$$W = 3D - 2H_0 - S_0 = 3 \times 2 - 2 \times 1 - 4 = 0, \tag{2.1}$$

I.A. Karnovsky, *Theory of Arched Structures: Strength, Stability, Vibration*,
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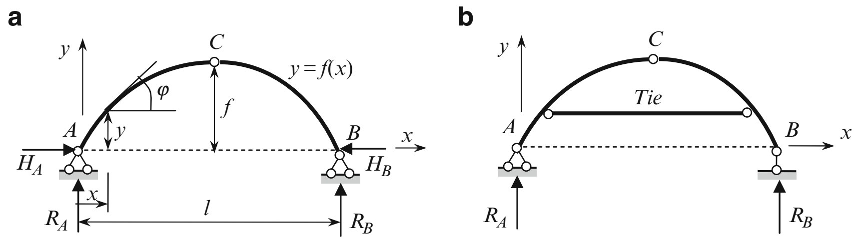

Fig. 2.1 (**a**, **b**) Design diagram of three-hinged arch without tie and with elevated tie

where D, H_0, and S_0 are the number of rigid discs, the number of simple hinges, and the number of constraints of support, respectively. Since $W = 0$, this structure does not have redundant constraints, while all existing constraints constitute the geometrically unchangeability. Indeed, two rigid discs AC and BC are connected with the ground by two hinges A and B and line AB does not pass through the intermediate hinge C.

This structure has four unknown reactions, i.e., two vertical reactions R_A, R_B and two horizontal reactions H_A, H_B. For their determination, three equilibrium equations can be formulated considering the structure in whole. Since bending moment at the hinge C is zero, this provides additional equilibrium equation. It means that the sum of the moments of all external forces, which are located on the right (or on the left) part of the structure with respect to hinge C is zero

$$\sum_{\text{left}} M_C = 0 \quad \text{or} \quad \sum_{\text{right}} M_C = 0 \tag{2.2}$$

These four equations of equilibrium determine all four reactions at the supports. Therefore, three-hinged arch is a geometrically unchangeable and statically determinate structure.

The fundamental feature of arched structure is that horizontal reactions appear even if the structure is subjected to vertical load only. These horizontal reactions $H_A = H_B = H$ called as a *thrust*; such types of structures are often called as thrusted structures.

It will be shown later that at any cross section of the arch, the bending moments, shear, and axial forces arise. However, the bending moments and shear forces are considerably smaller than corresponding internal forces in a simply supported beam covering the same span and subjected to the same load. This is the fundamental property of the arch thanks to thrust. Thrusts in both supports are oriented toward each other and reduce the bending moments that would arise in beams of the same span and load. Therefore, the height of the cross section of the arch can be much less then the height of a beam to resist the same loading. So the three-hinged arch is more economical than simply supported beam, especially for large-span structures.

Introducing a tie into the system increases the number of constraints by one and therefore, in order for the arch with a tie to remain statically determinate, one of the

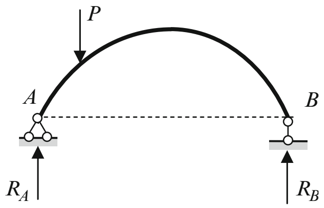

Fig. 2.2 Simply supported thrustless curvilinear member

pinned support must be replaced by a rolled support. A tie changes the distribution of internal forces in arch. The tie may be located at the level of the supports or above them. Arch with an elevated tie is shown in Fig. 2.1b. If tie is connected with arch by means of hinges, then the tie is subjected only to a tensile internal force.

In the case of vertical loads, which act on the arch with a tie, the horizontal reactions of supports equals zero while an extended force (thrust) arises in a tie.

Let us have a quick look at the structure shown in Fig. 2.2. Is this an arch? The arch is characterized by two fundamental markers such as a curvilinear axis and appearance of the thrust. Therefore, the structure in Fig. 2.2 presents the curvilinear trustless simply supported element, i.e., this is just a member with a curvilinear axis, but not an arch.

It is obvious that, unlike the beam, in this structure the axial compressed forces arise; however, the distribution of bending moments for this structure and for a beam of the same span and load will not differ, while the shear forces are less in this structure than that in beam. Thus, the fundamental feature of the arch (decreasing of the bending moments due to appearance of the thrust) for structure in Fig. 2.2 is not observed.

2.2 Reactions of Supports and Internal Forces

Let us consider a three-hinged symmetrical arch with intermediate hinge C at the highest point of the arch and with supports A and B at one elevation. Design diagram of the corresponding three-hinged arch is presented in Fig. 2.3; the span and rise of the arch are labeled as l and f, respectively. Equation of central line of the arch is $y = y(x)$.

Reactions of Supports

The stress analysis, and especially, construction of influence lines for internal forces of the three-hinged arch may be easily and elegantly performed if the conception of the "reference (or substitute) beam" is introduced. The reference

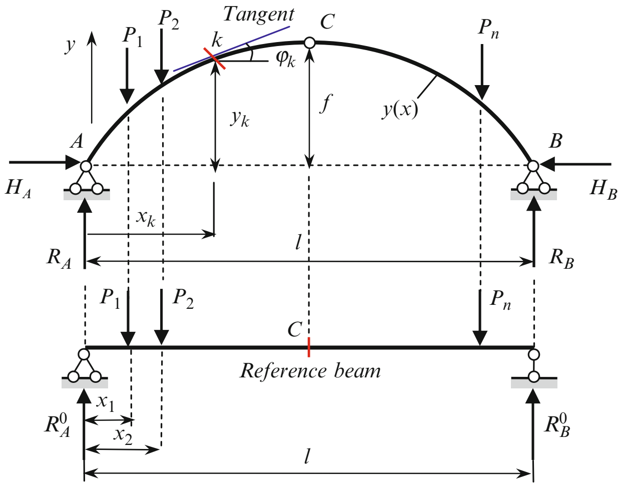

Fig. 2.3 Three-hinged arch. Design diagram and reference beam

beam is a simply supported beam of the same span as the given arch and subjected to the same loads, which act on the arch (Fig. 2.3).

The following reactions arise in arch: $R_A,\ R_B,\ H_A,\ H_B$. The vertical reactions of three-hinged arches carrying the vertical loads have same values as the reactions of the reference beam

$$R_A = R_A^0; \quad R_B = R_B^0. \tag{2.3}$$

The horizontal reactions (thrust) at both supports of three-hinged arches subjected to the vertical loads are equal in magnitude and opposite in direction

$$H_A = H_B = H. \tag{2.4}$$

Bending moment at the hinge C of the arch is zero. Therefore, by definition of the bending moment

$$M_C = \underbrace{R_A\frac{l}{2} - P_1\left(\frac{l}{2} - x_1\right) - P_2\left(\frac{l}{2} - x_2\right)}_{M_C^0} - H_A \times f = 0.$$

Underlined set of terms is the bending moment acting over section C of the reference beam (this section is located under the hinge of the arch). Therefore, last equation may be rewritten in the form

$$M_C^0 - H_A \times f = 0,$$

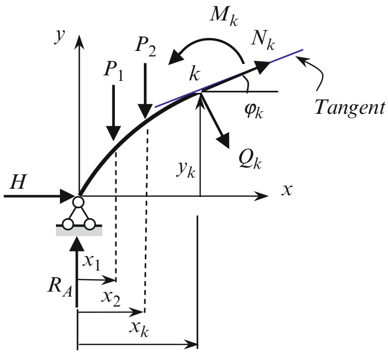

Fig. 2.4 Positive internal forces at any section k

which immediately allows us to calculate the thrust

$$H = \frac{M_C^0}{f}. \tag{2.5}$$

Thus, the thrust of the arch equals to bending moment at section C of the reference beam divided by the rise of the arch.

Internal Forces

In any section k of the arch, the following internal forces arise: the bending moment M_k, shear Q_k, and axial force N_k. The positive directions of internal forces are shown in Fig. 2.4.

Internal forces acting over a cross section k may be obtained considering the equilibrium of free body diagram of the left or right part of the arch. It is convenient to use the left part of the arch. By definition

$$M_k = R_A x_k - \sum_{\text{left}} P_i (x_k - x_i) - H y_k,$$

$$Q_k = \left(R_A - \sum_{\text{left}} P \right) \cos \varphi_k - H \sin \varphi_k,$$

$$N_k = -\left(R_A - \sum_{\text{left}} P \right) \sin \varphi_k - H \cos \varphi_k,$$

where P_i are forces which are located at the left side of section k; x_i are corresponding abscises of the points of application; x_k and y_k are coordinates of point k; and φ_k is the angle between the tangent to the center line of the arch at point k and a horizontal.

These equations may be represented in the following convenient form

$$\begin{aligned} M_k &= M_k^0 - Hy_k, \\ Q_k &= Q_k^0 \cos\varphi_k - H \sin\varphi_k, \\ N_k &= -Q_k^0 \sin\varphi_k - H \cos\varphi_k, \end{aligned} \tag{2.6}$$

where expressions

$$M_k^0 = R_A x - \sum_{\text{left}} P_i(x - x_i), \text{ and } Q_k^0 = R_A - \sum_{\text{left}} P,$$

represent the bending moment and shear force at section k for the reference beam (beam's bending moment and beam's shear).

Analysis of (2.5) and (2.6)

1. Thrust of the arch is inversely proportional to the rise of the arch.
2. In order to calculate the bending moment in any cross section of the three-hinged arch, the bending moment at the same section of the reference beam should be decreased by the value Hy_k. Therefore, the bending moment in the arch less than that of in the reference beam. This is the reason why the three-hinged arch is more economical than simply supported beam, especially for large-span structures.

   In order to calculate shear force in any cross section of the three-hinged arch, the shear force at the same section of the reference beam should be multiplied by $\cos\varphi_k$ and this value should be decreased by $H \sin\varphi_k$.
3. Unlike beams loaded by vertical loads only, there are axial forces, which arise in arches loaded by vertical loads only. These axial forces are always compressed.

Example 2.1. Design diagram of the three-hinged circular arch subjected to fixed loads is presented in Fig. 2.5a. The forces $P_1 = 10$ kN, $P_2 = 8$ kN, $q = 2$ kN/m. It is necessary to construct the internal force diagrams M, Q, N.

Solution. Reference beam. The reactions are determined from the equilibrium equations of all the external forces:

$$\sum M_B = 0 \rightarrow -R_A^0 \times 32 + P_1 \times 24 + q \times 8 \times 12 + P_2 \times 4 = 0 \rightarrow R_A^0 = 14.5\,\text{kN},$$
$$\sum M_A = 0 \rightarrow R_B^0 \times 32 - P_1 \times 8 - q \times 8 \times 20 - P_2 \times 28 = 0 \rightarrow R_B^0 = 19.5\,\text{kN}.$$

The bending moment M^0 and shear Q^0 diagrams for reference beam are presented in Fig. 2.5b. At point C ($x = 16$ m), the bending moment is $M_C^0 = 152$ kN m.

Three-hinged arch. The vertical reactions and thrust of the arch are

$$R_A = R_A^0 = 14.5\,\text{kN}, \qquad R_B = R_B^0 = 19.5\,\text{kN}, \quad H = \frac{M_C^0}{f} = \frac{152}{8} = 19\,\text{kN}.$$

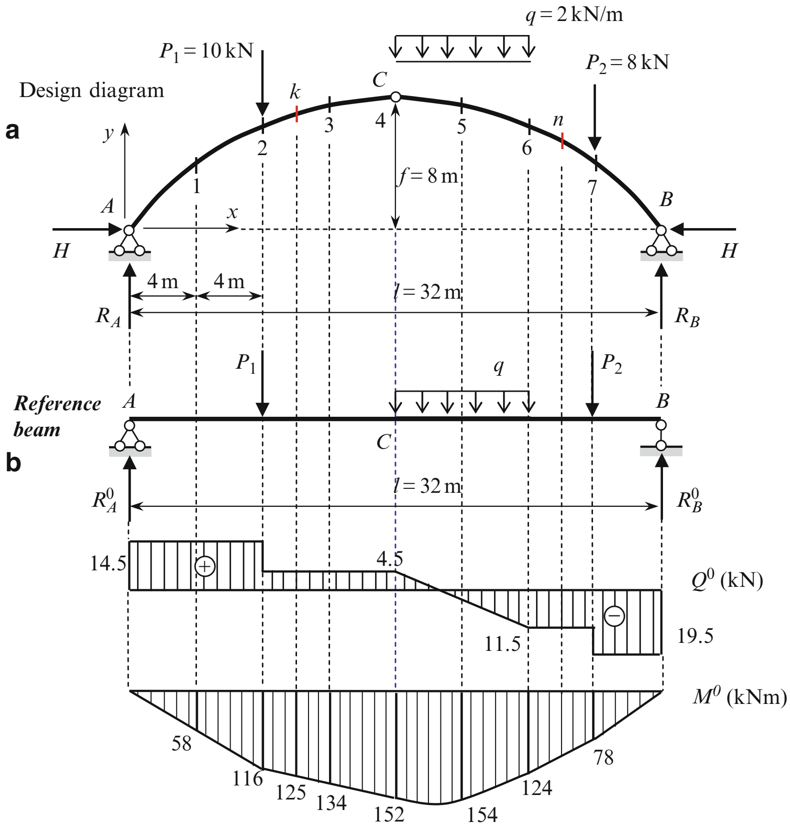

Fig. 2.5 (**a**) Design diagram of three-hinged circular arch and (**b**) reference beam and corresponding internal forces diagrams

For construction of internal forces diagrams of the arch, a set of sections has to be considered and for each section internal forces should be calculated. All computations concerning geometrical parameters and internal forces of the arch are presented in Table 2.1. The column 0 contains the numbers of sections. For specified sections A, 1–7, and B, the abscissa x and corresponding ordinate y (in meters) are presented in columns 1 and 2, respectively. Radius of curvature of the arch is

$$R = \frac{f}{2} + \frac{l^2}{8f} = \frac{8}{2} + \frac{32^2}{8 \times 8} = 20\,\text{m}.$$

Coordinates y are calculated using the following expression

$$y(x) = \sqrt{R^2 - \left(\frac{l}{2} - x\right)^2} - R + f = \sqrt{400 - (16 - x)^2} - 12\,(\text{m}).$$

Table 2.1 Internal forces in three-hinged circular arch (Fig. 2.6) ($R_A = 14.5\,\text{kN}$; $R_B = 19.5\,\text{kN}$; $H = 19\,\text{kN}$)

Section	x (m)	y (m)	$\sin\varphi$	$\cos\varphi$	M_x^O (kNm)	Hy (kNm)	M_x (kNm)	Q_x^O (kN)	Q_x (kN)	N_x (kN)
0	1	2	3	4	5	5′	6	7	8	9
A	0.0	0.0	0.8	0.6	0	0.0	0	14.5	−6.5	−23
1	4.0	4.0	0.6	0.8	58	76	−18	14.5	0.2	−23.9
2	8.0	6.330	0.4	0.9165	116	120.27	−4.27	14.5/4.5[a]	5.6892/−3.4757	−23.213/−19.213
k	10	7.0788	0.3	0.9539	125	134.497	−9.497	4.5	−1.4074	−19.474
3	12	7.596	0.2	0.9798	134	144.324	−10.324	4.5	0.6091	−19.516
4 (***C***)	16	8.0	0.0	1.0	152	152	0.0	4.5	4.5	−19.00
5	20	7.596	−0.2	0.9798	154	144.324	9.676	−3.5	0.3707	−19.316
6	24	6.330	−0.4	0.9165	124	120.27	3.73	−11.5	−2.9397	−22.013
n	26	5.3205	−0.5	0.8660	101	101.089	−0.089	−11.5	−0.459	−22.204
7	28	4	−0.6	0.8	78	76	2	−11.5/−19.5	2.2/−4.2	−22.1/−26.9
B	32	0.0	−0.8	0.6	0	0.0	0	−19.5	3.5	−27

Note the values of discontinuity due to concentrated load equal to $P\cos\varphi$ and $P\sin\varphi$ in shear and normal force diagrams, respectively

[a] Values in numerator and denominator (columns 7–9) mean value of the force to the left and to the right of corresponding section

Columns 3 and 4 contain values of sin φ and cos φ, which are calculated by the formula

$$\sin\varphi = \frac{l-2x}{2R} = \frac{32-2x}{40}, \quad \cos\varphi = \frac{y+R-f}{R} = \frac{y+12}{20}.$$

Values of bending moment and shear for reference beam, which are presented in columns 5 and 7, are taken directly from the corresponding diagrams in Fig. 2.5b. Values for Hy are contained in column 5′. Columns containing separate terms for $Q^0\cos\varphi, \quad Q^0\sin\varphi, \quad H\cos\varphi, \quad H\sin\varphi$ are not presented. Values of bending moment, shear, and normal forces for three-hinged arch are tabulated in columns 6, 8, and 9. They have been computed using (2.6). For example, for section A we have

$$\begin{aligned} Q_A &= Q_A^0\cos\varphi_A - H\sin\varphi_A = 14.5\times 0.6 - 19\times 0.8 = -6.5\,\text{kN},\\ N_A &= -Q_A^0\sin\varphi_A - H\cos\varphi_A = -14.5\times 0.8 - 19\times 0.6 = -23\,\text{kN}. \end{aligned}$$

The final internal force diagrams for the arch are presented in Fig. 2.6. Bending moment diagram is shown on the side of the extended fibers, thus the signs of bending moments are omitted. As for beam, the bending moment and shear diagrams satisfy to Schwedler's differential relationships. In particularly, if at any point a shear changes its sign, then a slope of the bending moment diagram equals zero, i.e., at this point the bending moment has local extreme (e.g., points 2, 7, etc.). It can be seen that the bending moments which arise in cross sections of the arch are much less than that of in a reference beam.

It is obvious that for supports $R_A^2 + H^2 = Q_A^2 + N_A^2$ and $R_B^2 + H^2 = Q_B^2 + N_B^2$.

2.3 Rational Shape of the Arch

The shape of the arch, which is subjected to a given fixed load, is called rational if the bending moments in the cross section of the arch equal to zero. An example of a rational arch could be in the form of a circular arch which is loaded by uniform radial (hydrostatic) load [Rzh82].

2.3.1 Vertical Load Does Not Depend on the Shape of the Arch

In this case, the reactions of the arch and bending moments for reference beam do not depend on the shape of the arch. Thus, for a rational arch, we have the condition

$$M_k = M_k^0 - Hy_k = 0, \tag{2.7}$$

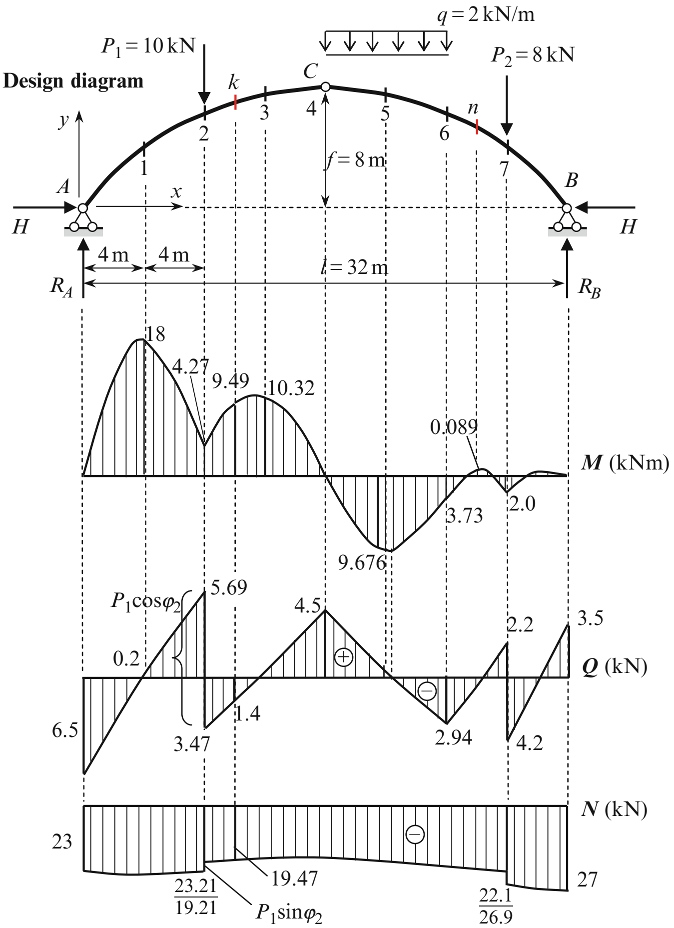

Fig. 2.6 Design diagram of three-hinged circular arch. Internal forces diagrams

where M_k^0 is a bending moment in the reference simply supported beam; H is a thrust of the arch; y_k is a vertical coordinate of the point on the axis of the arch. Therefore, the shape of the rational arch is determined by its y coordinate

$$y_k = \frac{M_k^o}{H}. \tag{2.8}$$

It is easy to prove the following statement: if a three-hinged arch is subjected to a vertical load and the vertical ordinates y of the arch, measured from the support line AB, are proportional to corresponding ordinates of the bending moment diagram of the reference beam, then the bending moments at all sections of the arch are equal to zero. This statement is true for any position of the intermediate hinge C [Rab60].

Indeed, let for any section k of the arch, the y-ordinate of the axis and bending moment of the reference beam be related by the formula $y_k = nM_k^0$, where n is an arbitrary number. Bending moment at section k is

$$M_k = M_k^0 - Hy_k = M_k^0 - HnM_k^0 = M_k^0(1 - nH).$$

For crown hinge C, the bending moment $M_C = M_C^0(1 - nH) = 0$. Since $M_C^0 \neq 0$, then $(1 - nH) = 0$.

Thus, the bending moment at any section equals to zero.

Example 2.2. Three-hinged symmetric arch of span l and rise f is loaded by a uniformly distributed load q within the entire span. Origin is placed on the left support and the axis x is directed to right. Expression for bending moment of the reference beam is $M_x^0 = qx(l - x)/2$.

The thrust of the arch is $H = M_C^0/f = ql^2/(8f)$. Therefore, the required equation of the axis of the arch becomes

$$y(x) = \frac{M_x^0}{H} = \frac{4f}{l^2}x(l - x).$$

Thus, if a uniformly distributed vertical load acts within the entire span of the three-hinged parabolic arch, then the bending moments do not arise in the arch.

Note, if a given load is governed by the law $q(x) = q_0 + kx$, then the bending moment diagram and the rational axis of the arch are characterized by third-order polynomials [Kis60].

2.3.2 Vertical Load Depends on Arch Shape

Let us consider a three-hinged arch load as shown in Fig. 2.7. We can see that a shape of the arch determine the value of load. According to the definition, in the case of a rational arch, only axial forces arise in the cross sections.

Free body diagram for infinitesimal element i–j is shown in Fig. 2.7; horizontal projection of this element is dx. Equilibrium equation

$$\sum X = N\cos\varphi - (N + \mathrm{d}N)\cos(\varphi + \mathrm{d}\varphi) = 0,$$

leads to $\mathrm{d}(N\cos\varphi) = 0$. It means that

$$N\cos\varphi = \mathrm{const} = H, \tag{2.9}$$

where H is the thrust of the arch.

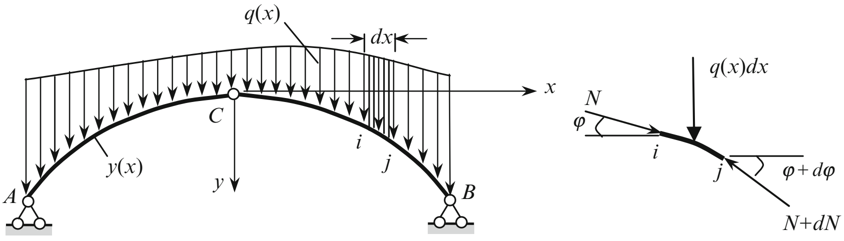

Fig. 2.7 Three-hinged arch subjected to load which depends on the shape of the arch

Equilibrium equation

$$\sum Y = N\ \sin\varphi + q(x)\mathrm{d}x - (N + \mathrm{d}N)\ \sin(\varphi + \mathrm{d}\varphi) = 0 \text{ leads to } \frac{\mathrm{d}}{\mathrm{d}x}(N\ \sin\varphi) = q(x). \tag{2.9a}$$

Since $N = H/\cos\varphi$, (2.9a) can be rewritten as follows

$$\frac{\mathrm{d}}{\mathrm{d}x}(H\tan\varphi) = q(x) \text{ or } \frac{\mathrm{d}}{\mathrm{d}x}\left(H\frac{\mathrm{d}y}{\mathrm{d}x}\right) = H\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = q(x).$$

Thus, the equation of the rational axis of the arch in the case of a load, which depends on the shape of the arch obeys the differential equation [Kis60]

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{q(x)}{H}. \tag{2.10}$$

For each specified load, the problem of determining the rational shape of the arch comes down to integration of (2.10).

Example 2.3. Symmetrical three-hinged arch of span l and rise f is subjected to vertical load $q(x)$, which consists of two parts. One part of load, q_0, is uniformly distributed within the entire span of the arch. The second part of load depends on the shape of the arch. Assume that this part of the load is proportional to coordinate y. Thus, the total load becomes $q(x) = q_0 + \gamma \times y$. Design diagram of right-hand part of the arch and location of the x and y axis are shown in Fig. 2.8.

Differential equation (2.10) becomes

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{q_0 + \gamma \times y}{H} \text{ or } \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - k^2y = \frac{q_0}{\gamma}k^2, \quad k^2 = \frac{\gamma}{H}.$$

Its solution and first derivative are

$$y = A\ \sinh\ kx + B\ \cosh\ kx - \frac{q_0}{\gamma};\ \frac{\mathrm{d}y}{\mathrm{d}x} = Ak\ \cosh\ kx + Bk\ \sinh\ kx.$$

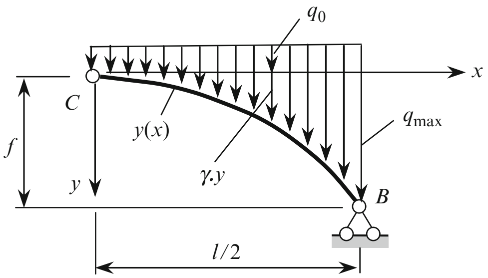

Fig. 2.8 Load change according to the shape of the arch, $q(x) = q_0 + \gamma y$

Constants of integration are found from the boundary conditions for symmetrical arch:

1. At $x = 0$ (indeterminate hinge C), $\mathrm{d}y/\mathrm{d}x = 0$. This condition leads to $A = 0$.
2. At $x = 0$ $y = 0$, so $B = q_0/\gamma$.

Equation of the axis of the rational shape of the arch becomes

$$y(x) = \frac{q_0}{\gamma}(\cosh kx - 1).$$

This curve is called a catenary [Kis80]. Some data for catenary arch with the given span l and rise f and parameter of the load $\delta = q_{max}/q_0$ are presented below.

Equation of the shape of the arch is

$$y = \frac{f}{\delta - 1}(\cosh kx - 1),$$

where relationship between parameters k and δ is

$$\delta = \cosh\frac{kl}{2}, \text{ so } k = \frac{2}{l}\operatorname{arc}\cosh\delta.$$

The slope of the axis of the arch is

$$\tan\varphi = \frac{f}{\delta - 1}k\sinh kx.$$

The thrust H of the arch, axial force N in any cross section of the arch, and maximum axial force N_{max} are:

$$H = \frac{q_0(\delta - 1)}{fk^2}, \quad N = H\sqrt{1 + \tan^2\varphi}, \quad N_{max} = H\sqrt{1 + k^2 f^2\frac{\delta + 1}{\delta - 1}}.$$

Vertical component of the reaction of support is

$$V = H\frac{f}{\delta - 1}k\sinh\frac{kl}{2}.$$

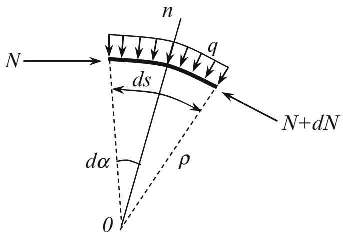

Fig. 2.9 Infinitesimal element subjected to radial load and axial force

2.3.3 Radial Load

Let us consider an arch with arbitrary equation for the central axis. The arch is loaded by a radial load. It means that the load is directed along the radius of curvature at each infinitesimal element of the arch. Design diagram of such element of length ds, central angle $2d\alpha$, and radius of curvature ρ is shown in Fig. 2.9. The load q is directed to the center of curvature; the load q should be treated as uniformly distributed within the portion ds. Since the arch is rational, then bending moments are absent.

From the equilibrium equations

$$\sum M_O = N\rho - (N + \mathrm{d}N)\rho = 0,$$

we get d$N = 0$. It means that in the case of a radial load, the axial force in arch is constant.

Since $\sin d\alpha \cong d\alpha$ and $\mathrm{d}s = \rho \times 2d\alpha$, then the equilibrium equation in projection of all forces onto the normal axis

$$\sum n = N \sin \mathrm{d}a + (N + \mathrm{d}N) \sin \mathrm{d}a - q\mathrm{d}s = 0,$$

leads to the following expression for the radius of curvature $\rho = N/q$. Curvature of the axis of the rational arch is proportional to the intensity q of the external load. In the case of a uniformly distributed radial load ($q =$ const), the axis of the rational arch presents a circle [Kis60].

The simplest problems of optimal three-hinged and redundant uniform arches are presented in [Gol80]: in these problems, it is necessary to find the shape of the arch which minimize its volume. Different types of loading are considered. Among them are fixed, moving, and wind loads.

2.4 Influence Lines for Reactions and Internal Forces

This section is devoted to construction of influence line for reactions, thrust, and internal forces. Three precise approaches are considered. They are the analytical approach, the nil points of influence lines, and fictitious beam methods. Influence lines method for structural analysis was developed by Winkler (1835–1888) and independently by Mohr (1835–1918) in 1868.

2.4.1 Analytical Approach

Equations (2.3), (2.5), and (2.6) can be used for deriving the equations for influence lines. The equations for influence lines for vertical reactions of the arch are derived from (2.3). Therefore, the equations for influence lines become

$$\mathrm{IL}(R_A) = \mathrm{IL}\left(R_A^O\right); \qquad \mathrm{IL}(R_B) = \mathrm{IL}\left(R_B^O\right). \tag{2.11}$$

The equation of influence lines for thrust is derived from (2.5). Since for a given arch, a rise f is a *fixed* number, then the equations for influence lines becomes

$$\mathrm{IL}(H) = \frac{1}{f} \times \mathrm{IL}\left(M_C^O\right). \tag{2.12}$$

Thus, influence line for trust H may be obtained from the influence line for bending moment at section C of the reference beam, if all ordinates of the latter will be divided by parameter f.

The equations for influence lines for internal forces at any section k may be derived from (2.6). Since for a given section k, the parameters y_k, sin φ_k, and cos φ_k are *fixed* numbers, then the equations for influence lines become

$$\begin{aligned}
\mathrm{IL}(M_k) &= \mathrm{IL}\left(M_k^O\right) - y_k \times \mathrm{IL}(H),\\
\mathrm{IL}(Q_k) &= \cos\varphi_k \times \mathrm{IL}\left(Q_k^O\right) - \sin\varphi_k \times \mathrm{IL}(H),\\
\mathrm{IL}(N_k) &= -\sin\varphi_k \times \mathrm{IL}\left(Q_k^O\right) - \cos\varphi_k \times \mathrm{IL}(H).
\end{aligned} \tag{2.13}$$

In order to construct the influence line for bending moment at section k, it is necessary to sum two graphs: one of them is influence line for bending moment at section k for reference beam and second is influence line for thrust H with all ordinates of which have been multiplied by a constant factor $(-y_k)$.

Equation of influence lines for shear also has two terms. The first term presents influence line for shear at section k in the reference beam, all the ordinates of which have been multiplied by a constant factor $\cos\varphi_k$. The second term presents the influence line of the thrust of the arch, all the ordinates of which have been multiplied by a constant factor $(-\sin\varphi_k)$. Summation of these two graphs leads to the required influence line for shear force at section k. Similar procedure should be applied for the construction of influence line for axial force. Note that both terms for axial force are negative.

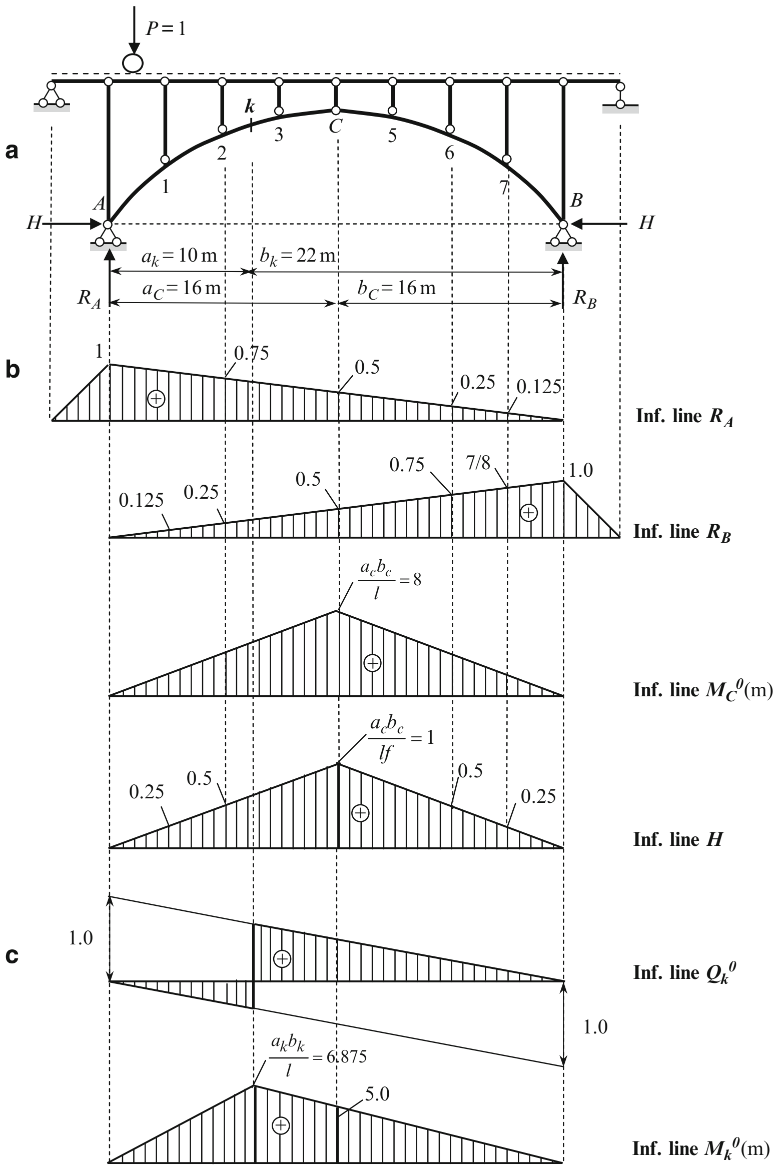

Fig. 2.10 Three-hinged arch. (**a**) Design diagram; (**b**) influence lines for reactions of the arch; and (**c**) influence lines for internal forces at section k for reference beam

Figure 2.10a presents the arched structure consists of the arch itself and overarched construction, which includes the set of simply supported beams and vertical posts with hinged ends. Unit load, which moves along the horizontal beams, is transmitted over the posts on the arch at discrete points. Thus, this design diagram corresponds to indirect load application. Parameters of the arch are same as in Fig. 2.5a.

It is required to construct the influence lines for vertical reactions, thrust and for bending moment M_k, shear Q_k, and normal force N_k for section k.

Influence Lines for Reactions

According to (2.11), influence lines for vertical reactions R_A and R_B of the arch do not differ from influence lines for reaction of supports of a simply supported beam.

Influence line for thrust may be constructed according to (2.12); the maximum ordinate of influence line for bending moment at section C of the reference beam equals to $a_c b_c / l = 8\,\mathrm{m}$. Therefore, the maximum ordinate of influence line for thrust H of the arch becomes $(1/f) \times (a_c b_c / l) = l/4f = 32/4 \times 8 = 1$. Influence lines for reactions of supports of the arch and internal forces for reference beam are shown in Fig. 2.10b, c. Indirect load application is taken into account [Kar10].

Influence Lines for Internal Forces at Section *k*

Section k is characterized by the following parameters: $a_k = 10\,\mathrm{m}$, $b_k = 22\,\mathrm{m}$, $y_k = 7.0788\,\mathrm{m}$, $\sin\varphi = 0.30$, $\cos\varphi = 0.9539$ (Table 2.1). Algorithms for the construction of influence lines of internal forces for arch are described in Sect. 2.4.1.

Bending moment. Influence line for M at section k may be constructed according to (2.13).

$$\mathrm{IL}(M_k) = \mathrm{IL}\left(M_k^0\right) - y_k \times \mathrm{IL}(H). \tag{2.13a}$$

Step 1. Influence line for bending moment at section k of reference beam M_k^0 presents the triangle with maximum ordinate $a_k b_k / l = 10 \times 22/32 = 6.875\,\mathrm{m}$ at sections k and 5.0 m at section C (Figs. 2.10 and 2.11).

Step 2. Influence line for thrust H presents triangle with maximum ordinate $l/(4f) = 1$ at section C. Term $y_k \times \mathrm{IL}(H)$ presents the similar graph; the maximum ordinate is $y_k \times 1 = 7.0788\,\mathrm{m}$. So the specified ordinates of graph $y_k \times \mathrm{IL}(H)$ at section k and C are 4.42425 and 7.0788 m, respectively (Fig. 2.11).

Step 3. Procedure (2.13a) is presented in Fig. 2.11, construction of influence line M_k. Since both terms in (2.13a) has *different* signs, then both graphs, $\mathrm{IL}\left(M_k^0\right)$ and $y_k \times \mathrm{IL}(H)$ should be plotted on the *one side* on the basic line. The ordinates of required $\mathrm{IL}(M_k)$ will be located *between* these both graphs. Specified ordinates of final influence line (2.13a) at section k and C are

$$6.875 - 4.42425 = 2.45075\,\mathrm{m} \text{ and } 5.0 - 7.0788 = -2.0788\,\mathrm{m}.$$

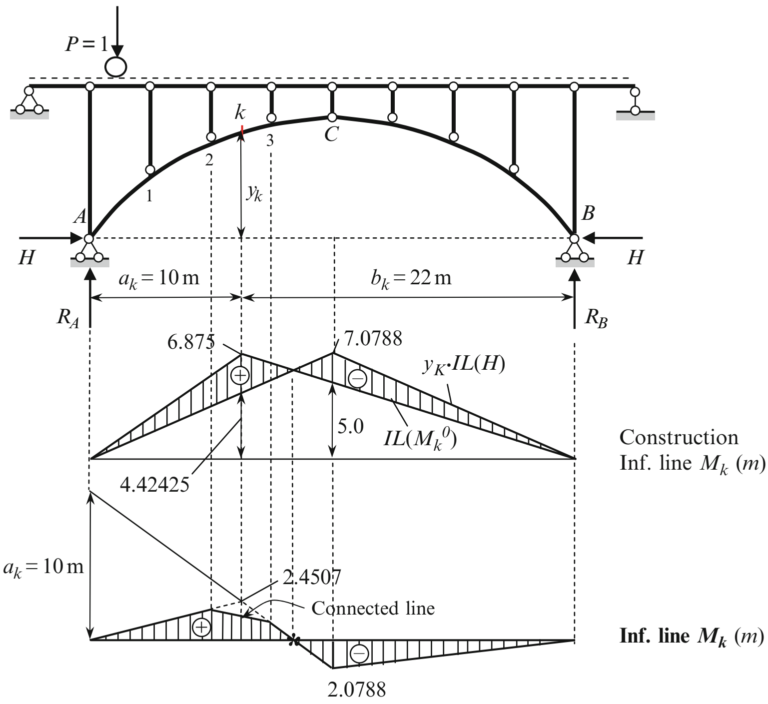

Fig. 2.11 Three-hinged arch. Design diagram and construction of influence line for bending moment at section k of the arch

Step 4. Influence line between joins 2 and 3 presents a straight line because of indirect load application [Kar10]; this connected line is shown by solid line. Final influence line $\mathrm{IL}(M_k)$ is shown in Fig. 2.11.

Shear force. This influence line may be constructed according to equation

$$\mathrm{IL}(Q_k) = \cos\varphi_k \times \mathrm{IL}\left(Q_k^0\right) - \sin\varphi_k \times \mathrm{IL}(H). \tag{2.13b}$$

Step 1. Influence line for shear at section k for the reference beam is shown in Fig. 2.10c; the specified ordinates at supports A and B equal to 1.0. The first term $\cos\varphi_k \times \mathrm{IL}\left(Q_k^0\right)$ of (2.13b) presents a similar graph with specified ordinates $\cos\phi_k = 0.954$ at supports A and B, so ordinates at the left and right of section k are -0.298 and 0.656, while at crown C is 0.477.

Step 2. Influence line for thrust is shown in Fig. 2.10b; the specified ordinates at crown C equals to 1.0. The second term $\sin\varphi_k \times \mathrm{IL}(H)$ of (2.13b) presents a similar graph with specified ordinates $0.3 \times 1.0 = 0.3$ at crown C. Specified ordinate at section k is 0.1875.

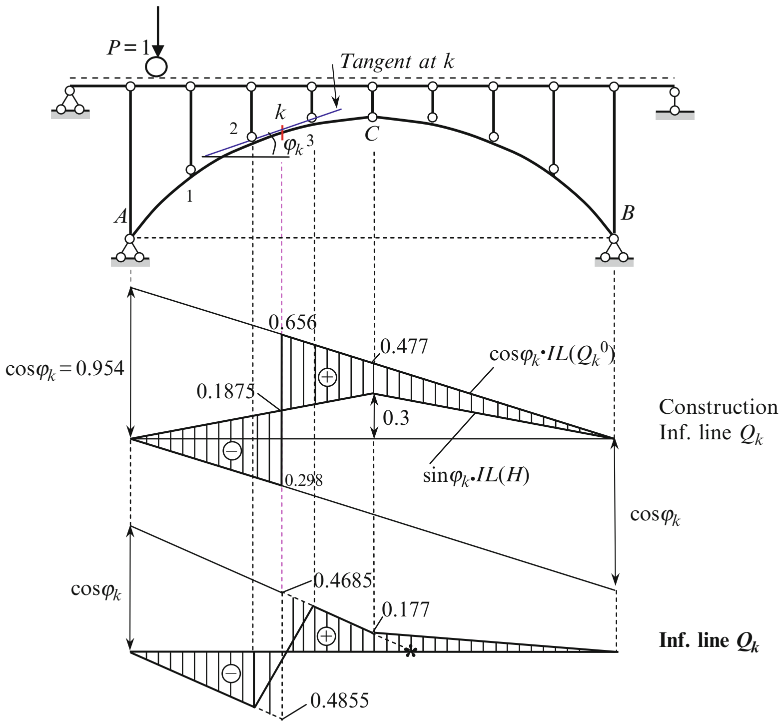

Fig. 2.12 Three-hinged arch. Design diagram and construction of influence line for shear at section k of the arch

Step 3. Procedure (2.13b) is presented in Fig. 2.12. As in case for bending moment, both terms in (2.13b) has *different* signs, therefore both graphs $\cos\varphi_k \times \text{IL}\left(Q_k^0\right)$ and $\sin\varphi_k \times \text{IL}(H)$ should be plotted on the *one side* on the basic line. Ordinates *between* both graphs present the required ordinates for influence line for shear. Specified ordinates of final influence line (2.13b) at left and right of section k are

$$0.298 + 0.1875 = 0.4855 \text{ and } 0.656 - 0.1875 = 0.4685.$$

At crown C, ordinate of influence line Q_k is $0.477 - 0.3 = 0.177$.

Step 4. Influence line between joins 2 and 3 presents a straight line; this connected line is shown by a solid line. Final influence line $\text{IL}(Q_k)$ is shown in Fig. 2.12.

Axial force. This influence line may be constructed according to the following equation

$$\text{IL}(N_k) = -\sin\varphi_k \times \text{IL}\left(Q_k^0\right) - \cos\varphi_k \times \text{IL}(H). \tag{2.13c}$$

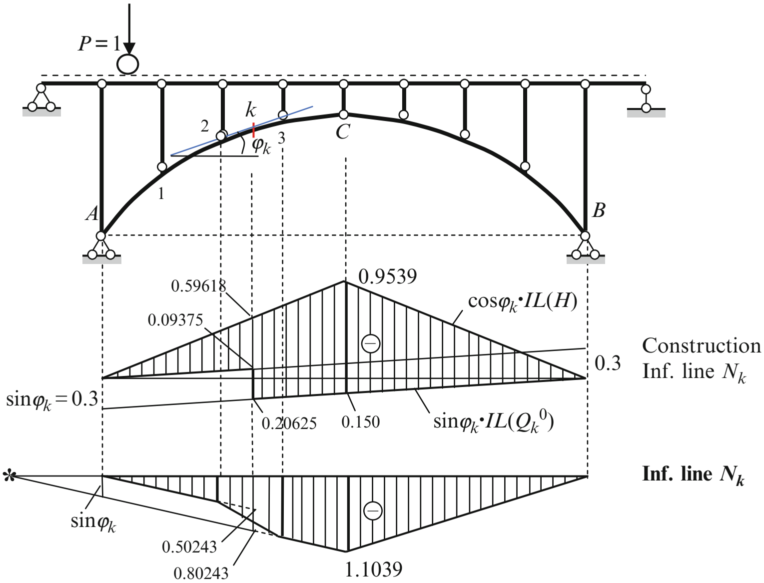

Fig. 2.13 Three-hinged arch. Design diagram and construction of influence lines for axial force at section k of the arch

Step 1. Influence line for shear at section k for the reference beam is shown in Fig. 2.10c. The first term $\sin\varphi_k \times \mathrm{IL}\left(Q_k^0\right)$ of (2.13c) presents a similar graph with specified ordinates $\sin\varphi_k = 0.30$ at supports A and B, so at the left and right of section k ordinates are 0.09375 and -0.20625, while at crown C is -0.15.

Step 2. Influence line for thrust is shown in Fig. 2.10b; the specified ordinates at crown C equals to 1.0. The second term $\cos\varphi_k \times \mathrm{IL}(H)$ of (2.13c) presents a similar graph with specified ordinates $0.9539 \times 1.0 = 0.9539$ at crown C. Specified ordinate at section k is 0.59618.

Step 3. Procedure (2.13c) is presented in Fig. 2.13. Both terms in (2.13c) has *same* signs; therefore, both graphs, $\sin\varphi_k \times \mathrm{IL}\left(Q_k^0\right)$ and $\cos\varphi_k \times \mathrm{IL}(H)$, should be plotted on the *different sides* on the basic line. Ordinates for required $\mathrm{IL}(N_k)$ are located *between* these both graphs. Specified ordinates of final influence line (2.13c) at left and right of section k are $-(0.59618 - 0.09375) = -0.50243$ and $-(0.59618 + 0.20625) = -0.80843$.
At crown C, ordinate of influence line N_k is $-(0.9539 + 0.15) = -1.1039$.

Step 4. Influence line between joins 2 and 3 presents a straight line; this connected line is shown by a solid line. Final influence line $\mathrm{IL}(N_k)$ is shown in Fig. 2.13.

Properties of the Influence Lines for Internal Forces

1. Influence line for bending moment has significantly less ordinates than for reference beam. This influence line contains the positive and negative ordinates. It means that at section k, extended fibers can be located below or above the neutral line depending on where the load is placed.

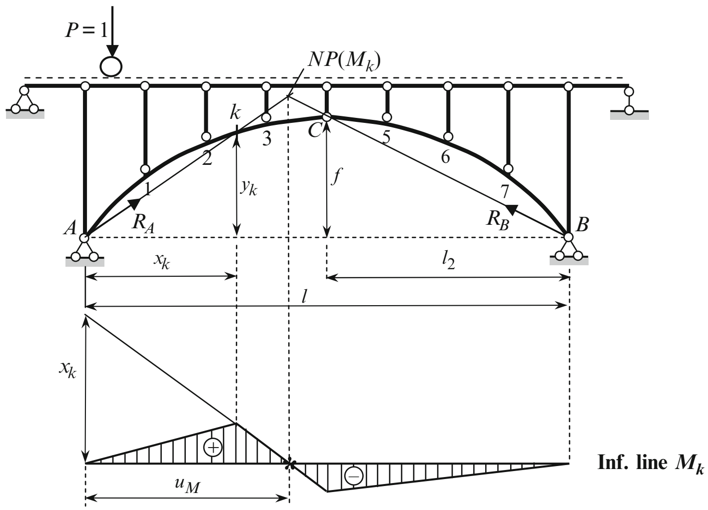

Fig. 2.14 Construction of influence line M_k using the nil point method

2. Influence line for shear, as in the case of reference beam, has two portions with positive and negative ordinates; all ordinates are significantly less than that of in the reference beam.
3. Influence line for axial force has only negative ordinates. So in case of arbitrary load, the axial forces in arch are always compressed.

2.4.2 Nil Points Method

Each influence lines shown in Figs. 2.11–2.13 has the specified point labeled as (*). These points are called as nil (or neutral) point of corresponding influence line. Such points of influence lines indicate a position of the concentrated load on the arch, so internal forces M, Q, and N in the given section k would be zero. Nil points may be used as simple procedure for the construction of influence lines for internal forces and checking the influence lines which were constructed by the analytical approach. This procedure for three-hinged arch of span l is discussed below.

Bending Moment

Step 1. Find nil point (NP) of influence line M_k. If load P is located on the left half of the arch, then reaction of the support B pass through crown C. Bending moment at section k equals zero, if reaction of support A pass through point k. Therefore, NP (M_k) is the point of intersection of line BC and Ak (theorem about three concurrent forces). The nil point (*) is always located between the crown C and section k (Fig. 2.14).

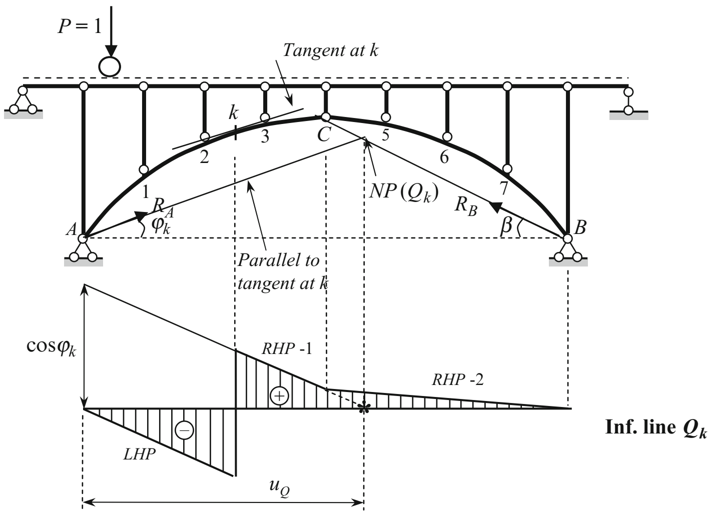

Fig. 2.15 Symmetrical three-hinged arch. Construction of influence line Q_k using the nil point method. The case of fictitious nil point

Step 2. Lay off along the vertical passing through the support A, the abscissa of section k, i.e., x_k.

Step 3. Connect this ordinate with nil point and continue this line till a vertical passing through crown C and then connect this point with support B.

Step 4. Take into account indirect load application; connecting line between joints 2 and 3 is not shown.

Location of NP(M_k) may be computed by the formula

$$u_M = \frac{lf\,x_k}{y_k l_2 + x_k f}. \tag{2.14}$$

Shear Force

Step 1. Find nil point (NP) of influence line Q_k. If load P is located on the left half of the arch, then reaction of the support B pass through crown C. Shear force at section k equals zero, if reaction of support A will be parallel to tangent at point k. Therefore, NP(Q_k) is point of intersection of line BC and line which is parallel to tangent at point k. For a given design diagram and specified section k, the nil point (*) is *fictitious* one (Fig. 2.15).

Step 2. Lay off along the vertical passing through the support A, the value $\cos\varphi_k$.

Step 3. Connect this ordinate with nil point. A working zone of influence line is portion between section k and vertical passing through crown C – right-hand portion

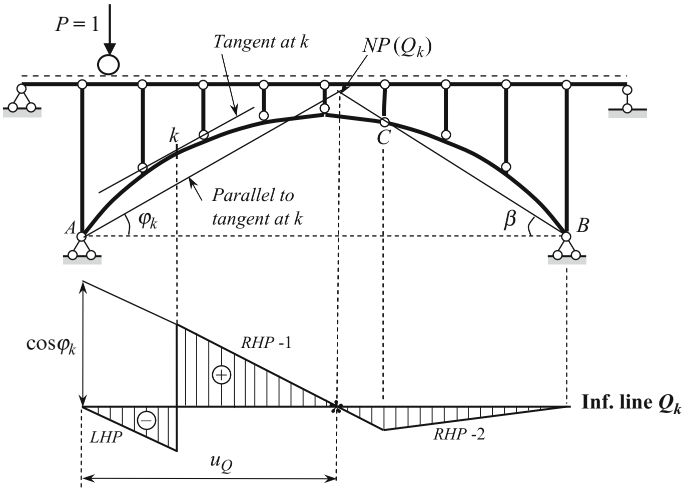

Fig 2.16 Nonsymmetrical three-hinged arch. Construction of influence line Q_k using the nil point method. The case of real nil point

1 (RHP-1). Then connect the point under crown C with support *B* – right-hand portion 2 (RHP-2).

Step 4. Left-hand portion (LHP) is parallel to right-hand portion 1 and connects two points: zero ordinate at support *A* and point under section *k*.

Figure 2.16 presents a nonsymmetrical three-hinged arch with *real* nil point for influence line Q_k; this point is located within the span of the arch. Therefore, we have one portion with positive shear and two portions with negative shear.

Location of NP(Q_k) for cases in Figs. 2.15 and 2.16 may be computed by the formula

$$u_Q = \frac{l \tan\beta}{\tan\beta + \tan\varphi_k}. \tag{2.15}$$

Axial Force

The nil point of influence line N_k is point of intersection of line *BC* and line passing through support *A* perpendicular to tangent at section *k*.

Step 1. Find nil point (NP) of influence line N_k. If load *P* is located on the left half of the arch, then reaction of the support *B* pass through crown C. Axial force at section *k* equals zero, if reaction of support *A* will be perpendicular to tangent at point *k*. The nil point (*) is located beyond the arch span (Fig. 2.17).

Step 2. Lay off along the vertical passing through the support *A*, the value $\sin\varphi_k$.

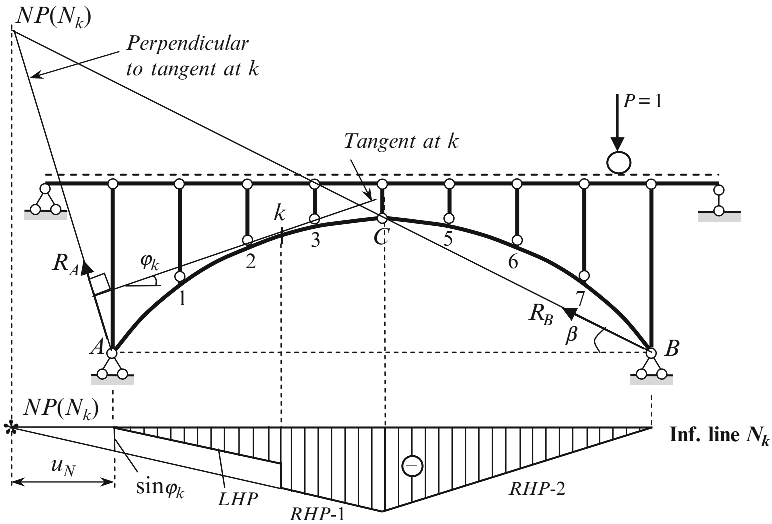

Fig. 2.17 Construction of influence line N_k using the nil point method

Step 3. Connect this ordinate with nil point and continue this line till vertical passes through crown C. A working zone is portion between section k and vertical passing through crown C (first right-hand portion RHP-1). Then connect the point under crown C with support B (second right-hand portion – RHP-2).

Step 4. LHP is parallel to RHP-1 and connects two points: zero ordinate at support A and point under section k.

Location of NP(N_k) may be computed by the formula

$$u_N = \frac{l \tan\beta}{\tan\beta - \cot\varphi_k}. \tag{2.16}$$

2.4.3 Fictitious Beam Method

Influence lines for internal forces of the three-hinged arch may be constructed as the bending moment diagram for the fictitious beam subjected to the special type of loads [Uma72-73].

Influence Line for M_k

Fictitious beam is loaded by two forces $P_k^f = 1$ at section k and $V_C^f = y_k/f$ at section C (Fig. 2.18). For arch in Fig. 2.5a and Table 2.1, we get $V_C^f = y_k/f = 7.0788/8 = 0.88485$.

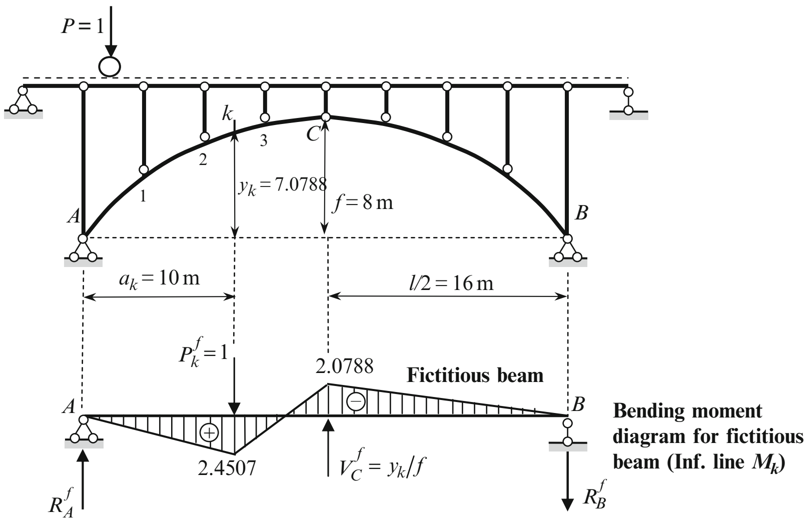

Fig. 2.18 Three-hinged arch. Fictitious beam for M_k is loaded by two forces $P_k^f = 1$ at section k and $V_C^f = y_k/f$; the bending moment diagram presents the influence line for M_k for the entire arch

Reactions of fictitious beam are

$$R_A^f = \frac{1 \times 22 - 0.88485 \times 16}{32} = 0.245075\,(\uparrow), \qquad R_B^f = 0.129925\,(\downarrow).$$

All forces and reactions are dimensionless. Bending moment diagram is shown on the extended fibers (positive ordinates are placed below the neutral line). Bending moments at specified points of the fictitious beam are

$$M_k^f = R_A^f \times a_k = 0.245075 \times 10 = 2.4507\,\text{m},$$

$$M_C^f = -R_B^f \times \frac{l}{2} = -0.129925 \times 16 = -2.0788\,\text{m}.$$

These ordinates of influence line for M_k have been obtained earlier and presented in Fig. 2.11.

Influence Line for Q_k

Fictitious beam is loaded by the couple $M_k^f = \cos\varphi_k = 0.9539$ (clockwise) at section k and force $V_C^f = \sin\varphi_k/f(1/\text{m})$ (upwards) at section C (Fig. 2.19). For arch in Fig. 2.5a and Table 2.1, we get $V_C^f = \sin\varphi_k/f = 0.3/8 = 0.0375\,(1/\text{m})$.

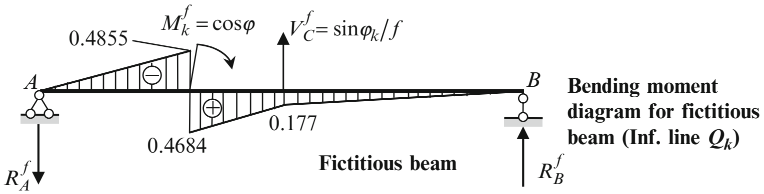

Fig. 2.19 Fictitious beam for Q_k. Bending moment diagram for fictitious beam presents the influence line Q_k for the entire arch

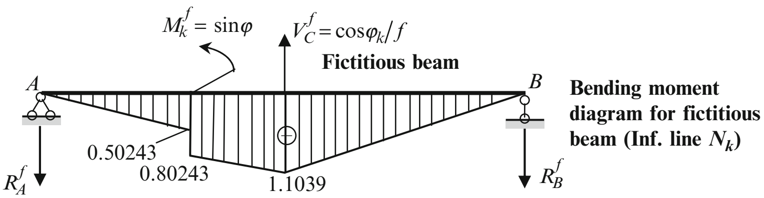

Fig. 2.20 Fictitious beam for N_k. Bending moment diagram for the fictitious beam presents the influence line N_k for the entire arch

Reactions of fictitious beam are

$$R_A^f = 0.048559\ (1/\text{m})(\downarrow) \text{ and } R_B^f = 0.011059\,(1/\text{m})(\uparrow).$$

Bending moments at specified points of the fictitious beam are

$$M_k^{f,\text{left}} = -R_A^f \times a_k = -0.048559 \times 10 = -0.4855,$$

$$M_k^{f,\text{right}} = -R_A^f \times a_k + 0.9539 = -0.048559 \times 10 + 0.9539 = 0.4684,$$

$$M_C^f = R_B^f \times \frac{l}{2} = 0.011059 \times 16 = 0.177.$$

Fictitious bending moments are dimensionless. These ordinates of influence line for Q_k have been obtained earlier and presented in Fig. 2.12.

Influence Line for N_k

Fictitious beam is loaded by the couple $M_k^f = \sin\varphi_k = 0.30$(counterclockwise) at section k and force $V_C^f = \cos\varphi_k/f = 0.9539/8 = 0.11924\,(1/\text{m})$ (upwards) at section C (Fig. 2.20).

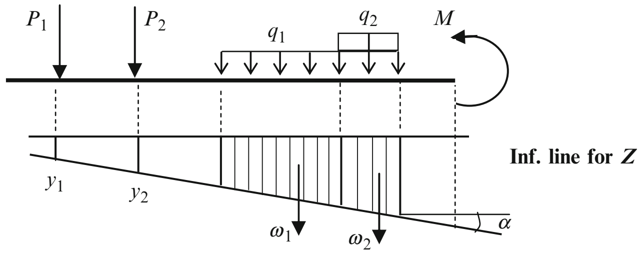

Fig. 2.21 Application of influence line for fixed loads

Reactions of fictitious beam are

$$R_A^f = 0.05024\,(1/\mathrm{m})(\downarrow) \text{ and } R_B^f = 0.06899\,(1/\mathrm{m})(\downarrow).$$

Bending moments at specified points of the fictitious beam are

$$M_k^{f,\mathrm{left}} = -R_A^f \times a_k = -0.050243 \times 10 = -0.50243,$$

$$M_k^{f,\mathrm{right}} = -R_A^f \times a_k - \sin\varphi_k = -0.50243 - 0.30 = -0.80243,$$

$$M_C^f = -R_B^f \times \frac{l}{2} = -0.06899 \times 16 = -1.1039.$$

Fictitious bending moments are dimensionless. These ordinates of influence line for N_k have been obtained earlier and presented in Fig. 2.13.

2.4.4 Application of Influence Lines

Influence lines, which describe the variation of any function Z (reaction, bending moment, shear, etc.) in the fixed section due to moving concentrated unit load $P = 1$ may be effectively used for calculation of this function Z due to *arbitrary fixed* and *moving* loads [Dar89], [Kar10].

Fixed load. Three types of fixed loads will be considered: concentrated loads P_i, uniformly distributed loads q_j, and couples M_k (Fig. 2.21).

Any function Z as a result of application of these loads may be calculated by the formula

$$Z = \pm\sum P_i y_i \pm \sum q_j \omega_j \pm \sum M_k \tan\alpha_k, \tag{2.17}$$

where y is the ordinates of influence line for function Z at the point where force P is applied; ω is the area of influence line graph for function Z within the portion where load q is applied; α_k is the angle between the x-axis and the portion of influence line for function Z within which M is applied.

The sign of Z due by load P depends on the sign of ordinate y of influence line. The sign of the area ω coincides with sign of ordinates of influence line; if the influence line within the load limits has different signs, then the areas must be taken with appropriate signs. If couple tends to rotate influence line toward base line through an angle less than 90°, then the sign is positive.

Formula (2.17) reflects the superposition principle and may be applied for any type of statically determinate and redundant structures.

Example 2.4. Assume that arch is subjected to fixed loads as shown in Fig. 2.5a. Calculate the reactions and internal forces of the arch at section k using influence lines.

Solution. *Reactions of supports*. Ordinates of influence line for R_A at the points of application the loads P_1 and P_2 are 0.75 and 0.125, respectively (Fig. 2.10b). The area of the influence line under the uniformly distributed load is

$$\omega = \frac{0.5 + 0.25}{2} \times 8 = 3.0\,(\mathrm{m}).$$

Therefore, the reaction $R_A = P_1 \times 0.75 + q \times 3 + P_2 \times 0.125 = 14.5\,\mathrm{kN}$.

The thrust H of the arch, using influence line (Fig. 2.10b) equals

$$H = P_1 \times 0.5 + q\frac{1 + 0.5}{2} \times 8 + P_2 \times 0.25 = 19\,\mathrm{kN}.$$

Internal forces in section k. The internal forces can be found in a similar way, using the relevant influence lines (Figs. 2.11–2.13). They are following:

$$M_k = P_1 \times 1.96 - q\frac{2.0788 + 1.0394}{2} \times 8 - P_2 \times 0.5194 = -9.500\,\mathrm{kN\,m}$$

$$Q_k = -P_1 \times 0.3883 + q\frac{0.177 + 0.0885}{2} \times 8 + P_2 \times 0.04425 = -1.405\,\mathrm{kN},$$

$$N_k = -P_1 \times 0.40194 - q\frac{1.1039 + 0.5519}{2} \times 8 - P_2 \times 0.2759 = -19.473\,\mathrm{kN}.$$

The magnitudes of just found internal forces M_k, Q_k, and N_k coincide with those computed in Example 2.1 and presented in Table 2.1.

These values of reactions coincide with those computed previously (Example 2.1).

Moving loads. Influence line for any function Z allows us to calculate Z for *any position* of a moving load, and that is very important, the most unfavorable position of the moving loads and corresponding value of the relevant function. Unfavorable (or dangerous) position of a moving load is such position, which leads to the maximum (positive or negative) value of the function Z. The following types of moving loads will be considered: one concentrated load, a set of loads, and a distributed load.

The set of connected moving loads may be considered as a model of moving truck. Specifications for truck loading may be found in various references, for example, in the American Association of State and Highway Transportation Officials (AASHTO). This code presents the size of the standard truck and the

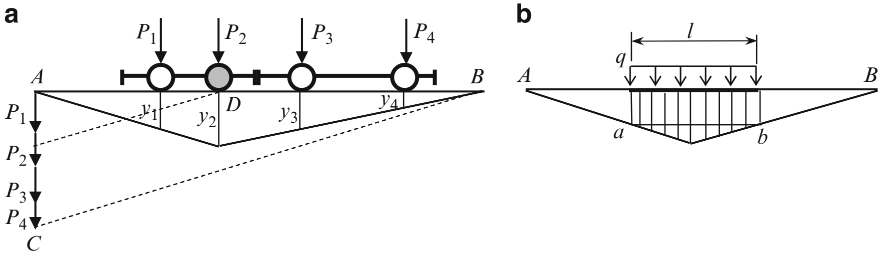

Fig. 2.22 Graphical definition of the unfavorable position of load for triangular influence line. (**a**) Set of concentrated load and (**b**) uniformly distributed load of fixed length l

distribution of its weight on each axle. The moving distributed load may be considered as a model of a set of containers which may be placed along the loading counter of the arch at arbitrary position.

Note that the term "moving load" with respect to influence line concept implies only that position of the load is arbitrary, i.e. this is a *static* load, which may have different positions along the beam. The time, velocity of the moving load, and any dynamic effects are not taken into account. Thus, for convenience, in this section we will use notion of "moving" or "traveling" load for static load, which may have *different* position along the structure.

The most unfavorable position of a single concentrated load is its position at a section with maximum ordinate of influence line. If influence line has positive and negative signs, then it is necessary to calculate corresponding maximum of the function Z using the largest positive and negative ordinates of influence line.

In case of set of concentrated moving loads, we assume that some of loads may be connected. This case may be applicable for moving cars, bridge cranes, etc. We will consider different forms of influence line.

Influence Line Forms a Triangle

A dangerous position occurs when one of the loads is located *over the vertex* of an influence line; this load is called a critical load. (The term "critical load" for problems of elastic stability, Chaps. 4 and 5, has a different meaning.) The problem is to determine *which* load among the group of moving loads is critical. After a critical load is known, all other loads are located according to the given distances between them.

The critical load may be easily defined by a graphical approach. Let the moving load be a model of two cars, with loads P_i on the each axle (Fig. 2.22). All distance between forces are given.

Step 1. Trace the influence line for function Z. Plot all forces P_1, P_2, P_3, P_4 in order using arbitrary scale from the left-most point A of influence line; the last point is denoted as C.

Step 2. Connect the right most point B with point C.

Step 3. On the base line show point D, which corresponds to the vertex of influence line and from this point draw a line, which is parallel to the line CB until it intersect with the vertical line AC.

Step 4. The intersected force (in our case P_2) presents a critical load; unfavorable location of moving cars presented in Fig. 2.22a.

Step 5. Maximum (or minimum) value of relative function is $Z = \sum P_i \times y_i$.

Influence Line Forms a Polygon

A dangerous position of the set of moving concentrated loads occurs when one or more loads stand over vertex of the influence line. Both the load and the apex of the influence line over which this load must stand to induce a maximum (or minimum) of the function under consideration are called critical. The critical apex of the influence line must be convex.

In case of uniformly distributed moving load, the maximum value of the function Z corresponds to the location of a distributed load q, which covers maximum one-sign area of influence line. The negative and positive portions of influence line must be considered in order to obtain minimum and maximum of function Z.

The special case of uniformly distributed moving load happens, if load is distributed within the *fixed length* l. In case of triangular influence line, the most unfavorable location of such load occurs when the portion $ab = l$ and base AB will be parallel (Fig. 2.22b).

Example 2.5. Simply supported beam with two overhangs is presented in Fig. 2.23. Determine the most unfavorable position of load, which leads to maximum (positive and negative) values of the bending moment and shear at section k. Calculate corresponding values of these functions. Consider the following loads: uniformly distributed load q and two connected loads P_1 and P_2 (a twin-axle cart with different wheel loads).

Solution. Influence lines for required functions Z are presented in Fig. 2.23.

Action of a uniformly distributed load $q = 1.6$ kNm. Distributed load leads to maximum value of the function if the area of influence lines within the distributed load is maximum. For example, the positive shear at section k is peaked if load q covers all portions of influence line with positive ordinates; for minimum shear in the same section the load q must be applied within portion with negative ordinates.

$$Q_{k(\max +)} = 1.6 \times \frac{1}{2}(0.3 \times 3 + 0.4 \times 4) = 2\,\text{kN};$$
$$Q_{k(\max -)} = -1.6 \times \frac{1}{2}(0.6 \times 6 + 0.3 \times 3) = -3.6\,\text{kN};$$
$$M_{k(\max +)} = 1.6 \times \frac{1}{2}10 \times 2.4 = 19.2\,\text{kNm};$$
$$M_{k(\max -)} = -1.6 \times \frac{1}{2}(1.2 \times 3 + 1.8 \times 3) = -7.2\,\text{kNm}.$$

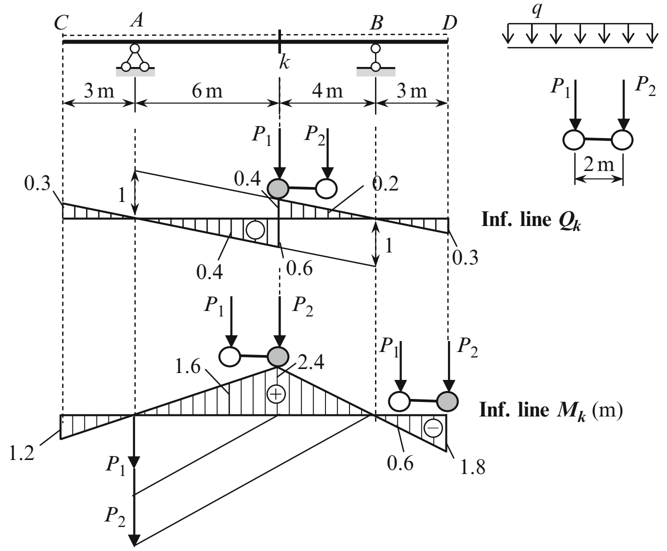

Fig. 2.23 Design diagram of the beam, influence lines, and most unfavorable positions of two connected loads

Positive value of $M_{k\ \max}$ means that if load is located between AB, the tensile fibers of the beam at section k are located below longitudinal axis of the beam. If load is located within the overhangs, then bending moment at section k is negative, i.e., the tensile fibers at section k are located above the longitudinal axis of the beam.

Action of the set of loads $P_1 = 5$ kN and $P_2 = 8$ kN. Unfavorable locations of two connected loads are shown in Fig. 2.23. Critical load for bending moment at section k (triangular influence line) is defined by the graphical method; the load P_2 is a critical one and it should be placed over the vertex of influence line.

$$
\begin{aligned}
Q_{k(\max +)} &= 5 \times 0.4 + 8 \times 0.2 = 3.6\,\text{kN},\\
Q_{k(\max -)} &= -(5 \times 0.4 + 8 \times 0.6) = -6.8\,\text{kN},\\
M_{k(\max)+} &= 5 \times 1.6 + 8 \times 2.4 = 27.2\,\text{kNm},\\
M_{k(\max -)} &= -(5 \times 0.6 + 8 \times 1.8) = -17.4\,\text{kNm}.
\end{aligned}
$$

If a set of loads P_1 and P_2 modeling a crane bridge, then the *order* of loads is fixed and cannot be changed. If a set of loads P_1 and P_2 is a model of a moving car, then we need to consider the case when a car moves in opposite direction. In this case, the order of forces from left to right becomes P_2 and P_1.

2.5 Core Moments and Normal Stresses

This paragraph is devoted to simplifying the procedure to calculate the normal stresses caused by the simultaneous action of M and N. The concept of the "core moments" is introduced and their influence lines are constructed. We discuss the most unfavorable loading of the influence line.

2.5.1 Normal Stresses

Let us consider an arbitrary section nm of the arch. Assume that the load acts in the one of the main planes of the cross section. The point of application of the resultant R is shifted from the axial line of the arch by a length e; magnitude of this force, its direction, and point of application may be determined using a concept "curve of pressure" as explained in Appendix "Pressure curve". This force is resolved into the normal N and shear force Q (Fig. 2.24a).

In the case of an eccentrically loaded bar, the maximum normal stresses, caused by the bending moment M and compressed force N, arises at the extreme fibers of the cross section

$$\sigma = -\frac{N}{A} \pm \frac{M}{W}, \tag{2.18}$$

where N is the normal component of a force R and the bending moment $M = Ne$; A, W, I_x are the area, elastic section modulus, and moment of inertia of the cross section of the arch, respectively. In the case of a nonsymmetrical section, the elastic section moduli are $W_n = I_x/a_1$ and $W_m = I_x/a_2$, where a_1 and a_2 are the distances from the neutral line to an extreme fibers.

For determining the maximum normal stresses due to moving load, it is necessary to load the influence lines for M and N. These influence lines have different shapes and the influence lines for M can alternate in sign. Therefore, this procedure becomes cumbersome. However, the two-termed formula (2.18) may be simplified.

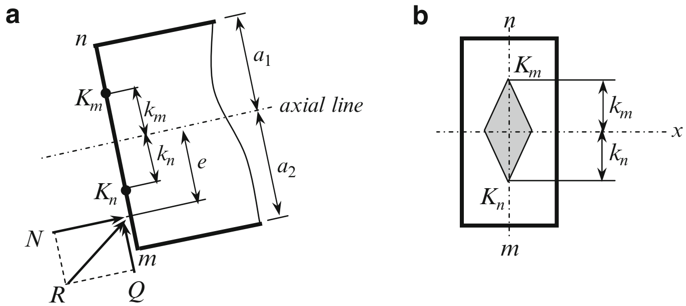

Fig. 2.24 (**a**) Internal forces at section n–m and (**b**) core of the cross section

Figure 2.24b presents the core (kern) for rectangular cross section; determination of its shapes and dimensions for arbitrary cross section may be found in the strength of materials textbooks. The concept of the core of the cross section was introduced by Bresse [Bre54], [Tim53], [Tod60]. The top and bottom points of the core are denoted by K_m and K_n.

If a force is applied at the bottom point K_n of the core, then $M = N \times k_n$ and normal stresses at the top fibers n equals zero

$$\sigma_n = -\frac{N}{A} + \frac{M}{W_n} = -\frac{N}{A} + \frac{Nk_n}{W_n} = 0. \tag{2.18a}$$

This equation leads to the formula $k_n = W_n/A$. Similarly, if a force is applied at the top point K_m of the core, then normal stresses at the bottom fibers m equals to zero and we get $k_m = W_m/A$.

If the compressed force N is applied as shown in Fig. 2.24a, then the normal stress at the bottom point m is

$$\sigma_m = -\frac{N}{A} - \frac{M}{W_m} = -\frac{N}{A} - \frac{Ne}{W_m} = -\frac{N}{W_m}\left(\frac{W_m}{A} + e\right) = -\frac{N}{W_m}(k_m + e).$$

The core moment presents the moment of the force N about the top core point K_m

$$M_{K_m}^{\text{core}} = N(e + k_m). \tag{2.19}$$

This moment differs from the usual bending moment by a term Nk_m. Finally, for normal stress in the bottom fibers of the cross section, we get the formula

$$\sigma_m = -\frac{M_{K_m}^{\text{core}}}{W_m}. \tag{2.20}$$

This formula shows that the maximal normal stresses caused by the moment M and force N equal to the normal stress caused by the core moment only. Similarly, the normal stress at the upper fibers n may be calculated by the formula $\sigma_n = M_{K_n}^{\text{core}}/W_n$, where core moment

$$M_{K_n}^{\text{core}} = N(e - k_n),$$

presents the moment of the force N about the bottom core point K_n.

2.5.2 *Influence Lines for Core Moments*

For construction of the influence line for core moments at section k, we will use the nil point method. This procedure will be the same as for construction of influence

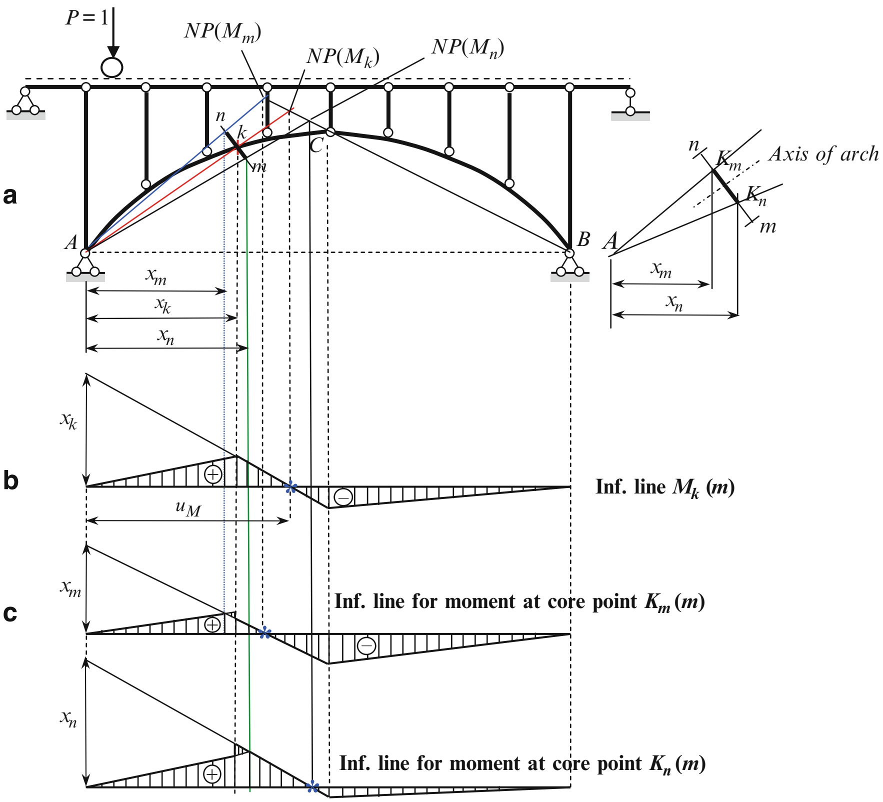

Fig. 2.25 Three-hinged arch. (**a**) Design diagram; (**b**) influence lines for bending moment M_k; and (**c**) influence lines for core moments at section k

line for bending moment at section k (Figs. 2.14 and 2.25a, b); indirect load application is not taken into account.

We show the top and bottom fiber points n and m at section k and denote the top and bottom core points by K_m and K_n (Fig. 2.25). These core points have coordinates x_m and x_n. Influence lines for core moments contain additional areas which are placed between two vertical lines; one of these lines passes through point k laying on the axis of the arch, and other vertical line passes over the core point (Fig. 2.25c). Additional areas of influence lines arise because in this section of the arch the influence line of axial force has a jump. Ordinates of this additional area of influence line are small and they may be neglected [Dar89]. However, it is important that the location of the nil points for core moments do not coincides with nil point for bending moment.

Influence lines for core moments allow us to answer the following question: which part of influence lines should be loaded by a uniformly distributed load (or any live load) in order for the tensile normal stresses at extrados (top) fibers of section k to be maximum.

The stresses at the top fibers n will be *tensile* if a resultant of all external left-hand (or right-hand) forces will passes below the bottom core point K_n. Given this, the moment about the core point K_n will be negative. Therefore, the load should be placed over the *negative* ordinates of the influence line for bending moment at core point K_n. If load will be placed over the positive ordinates of the same influence line, then a compressed stresses at extrados fibers n of section k will arise.

2.6 Special Types of Three-Hinged Arches

This paragraph contains analysis of the special types of three-hinged arch subjected to fixed and moving loads. Among them are the circular arch with elevated simple tie, parabolic arch with complex tie, and askew arch.

2.6.1 Arch with Elevated Simple Tie

Three-hinged arch with tie may be obtained from an ordinary three-hinged arch without a tie, if the horizontal constraint at support B (or A), which prevents horizontal displacement of the abutment hinge, is replaced by a tie. The tie may be located on the level of the supported points (Fig. 2.26a) or above them (elevated tie) (Fig. 2.26b). Application of complex tie is also possible. One type of an arch with a complex tie is shown in Fig. 2.26c. Three-hinged arches with ties represent geometrically unchangeable statically determinate structures and have certain peculiarities of their analysis, which are presented below.

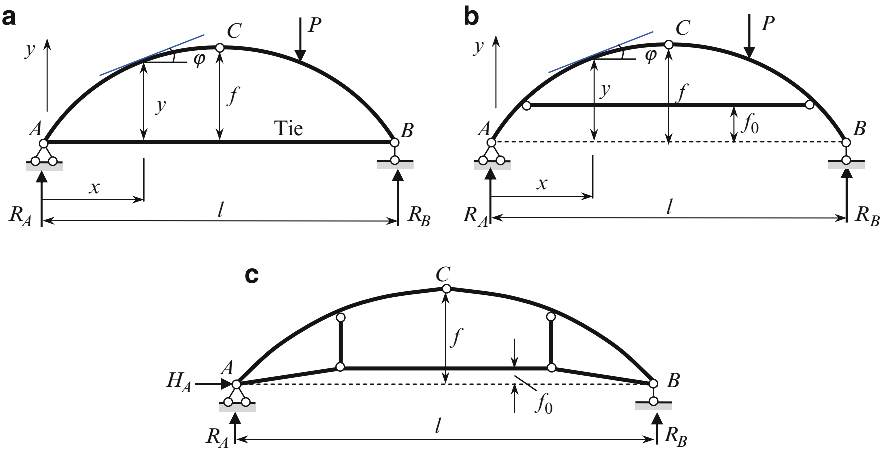

Fig. 2.26 Design diagrams of three-hinged arches with tie. (**a**) Simple tie on support level; (**b**) arch with elevated simple tie; and (**c**) arch with complex tie

In case (2.26a), the tensile force in the tie (thrust) is $H = M_C^O/f$, where M_C^O represent the bending moment at section C for the reference beam. Two forces H act at points A and B, as for an ordinary three-hinged arch without tie. Therefore, internal forces in cross sections of the given arch will be exactly the same as for arch without tie and may be calculated using (2.6). However, support B of the arch with tie has a horizontal displacement due to the elastic properties of the tie, while a three-hinged arch without tie has no a horizontal displacement.

In case (2.26b), the thrust in the tie is

$$H = \frac{M_C^O}{f - f_0}. \tag{2.21}$$

Two forces H act above points A and B. Internal forces in cross sections of the arch are obtained from modified (2.6); they depend on location of the section on the arch (below or above the tie). If sections are located below the tie level then

$$M_x = M_x^0, \quad Q_x = Q_x^0 \cos\varphi, \quad N_x = -Q_x^0 \sin\varphi. \tag{2.22}$$

If sections are located above the tie level, then

$$\begin{aligned} M_x &= M_x^0 - H(y - f_0), \\ Q_x &= Q_x^0 \cos\varphi - H \sin\varphi, \\ N_x &= -Q_x^0 \sin\varphi - H \cos\varphi, \end{aligned} \tag{2.23}$$

where M_x^0, Q_x^0 are bending moment and shear force at section x for the reference beam.

In the case of a complex tie, it is necessary to determine a thrust in the tie, then internal forces in all the members of the tie and finally, internal forces in the arch itself. The complex tie of the arch allows us not only to increase the strength of the arch structure but also to distribute internal forces in the arch as required.

Example 2.6. Design diagram of three-hinged circular arch with elevated tie is presented in Fig. 2.27. Geometrical parameters of the arch and loads are the same as for a three-hinged arch without tie (Fig. 2.5a). We need to compute the internal forces in the arch and compare results obtained for the same arch without tie.

Solution. The vertical reactions of supports, as in Example 2.1, are $R_A = R_A^0 = 14.5\,\text{kN}$, $\quad R_B = R_B^0 = 19.5\,\text{kN}$.

Horizontal reaction H_A at the support A may be calculated from the equation $\sum X = 0 \rightarrow H_A = 0$.

The force H in the tie may be determined using equilibrium condition for left (or right) part of the arch (section 1–1)

$$H \rightarrow \sum M_C^{\text{left}} = 0 \rightarrow R_A \times 16 - P_1 \times 8 - H(f - f_0) = 0 \rightarrow H = \frac{M_C^0}{f - f_0} = \frac{152}{8 - 2} = 25.33\,\text{kN}.$$

Computations of the geometrical parameters and internal forces of the arch are presented in Table 2.2.

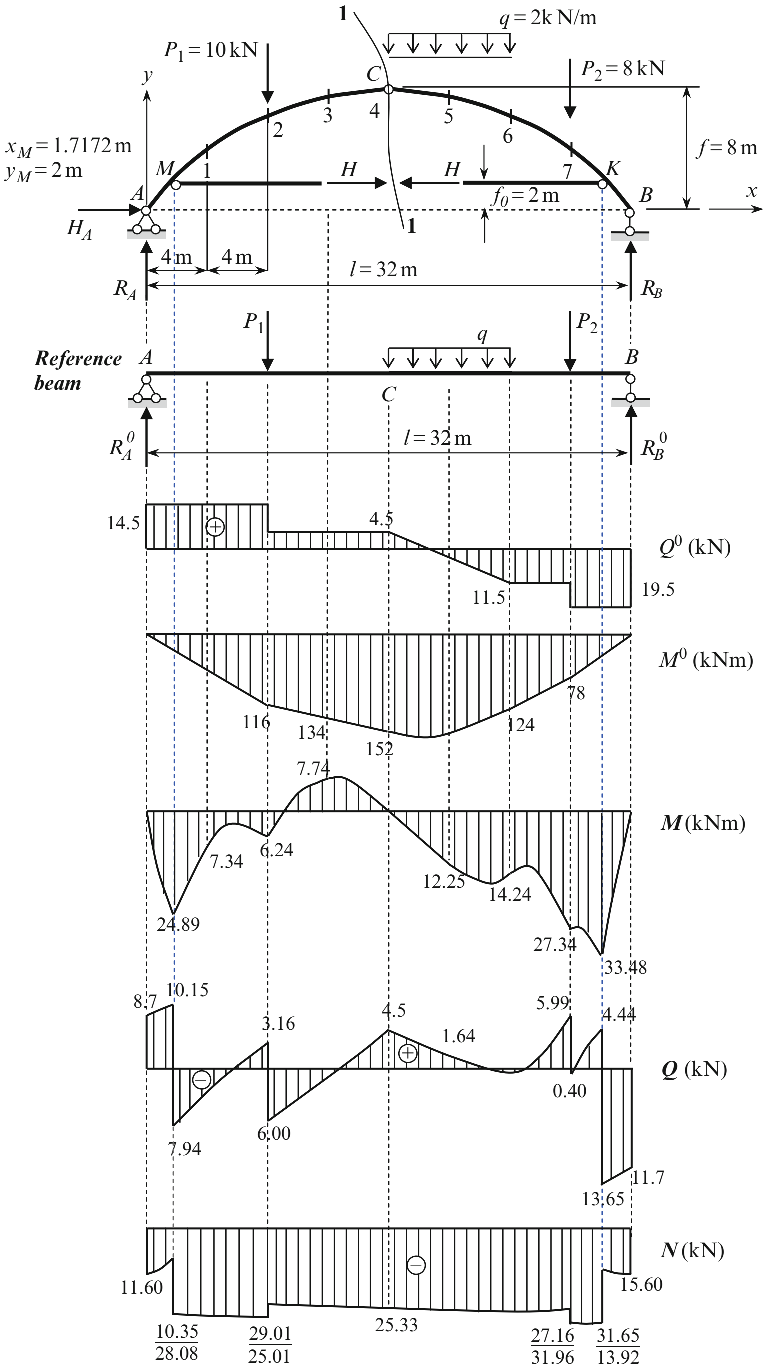

Fig. 2.27 Three-hinged arch with simple elevated tie. Design diagram, reference beam, and internal force diagrams

Table 2.2 Internal forces in three-hinged circular arch with simple elevated tie ($R_A = 14.5\,\text{kN}$; $R_B = 19.5\,\text{kN}$; $H = 25.33\,\text{kN}$)

Section	x (m)	y (m)	$\sin\varphi$	$\cos\varphi$	M_x^O (kNm)	$H(y - f_0)$ (kNm)	M_x (kNm)	Q_x^O (kN)	Q_x (kN)	N_x (kN)
0	1	2	3	4	5	5′	6	7	8	9
A	0.0	0.0	0.8	0.6	0	–	0	14.5	8.7	−11.6
M	1.7171	2.0	0.7141	0.70	24.8975	0.0	24.8975	14.5	10.15/−7.938	−10.3544/−28.0854
1	4	4.0	0.6	0.8	58	50.66	7.34	14.5	−3.598	−28.964
2	8	6.333	0.4	0.9165	116	109.755	6.245	14.5/4.5[a]	3.1572/−6.0	−29.0149/−25.0149
3	12	7.596	0.2	0.9798	134	141.747	−7.7467	4.5	−0.6569	−25.718
4 (C)	16	8.0	0.0	1.0	152	152	0.0	4.5	4.5	−25.33
5	20	7.596	−0.2	0.9798	154	141.746	12.2533	−3.5	1.6367	−25.5183
6	24	6.333	−0.4	0.9165	124	109.755	14.245	−11.5	−0.4077	−27.8149
7	28	4	−0.6	0.8	78	50.66	27.34	−11.5/−19.5	5.998/−0.402	−27.164/−31.964
K	30.2828	2.0	−0.7141	0.70	33.4834	0.0	33.4834	−19.5	4.4381/−13.65	−31.6559/−13.9249
B	32	0.0	−0.8	0.6	0	–	0	−19.5	−11.7	−15.6

Columns containing separate terms $Q^0\cos\varphi$, $Q^0\sin\varphi$, $H\cos\varphi$, $H\sin\varphi$ are not presented

Note: Discontinuity of shear and axial force diagrams at sections 2 and 7 is caused by vertical forces P_1 and P_2, while discontinuity of same diagrams at sections M and K is caused by thrust H

[a] Values in numerator and denominator (columns 7–9) are the mean value of the force to the left and to the right of corresponding section

Radius of the circle, according to (1) is $R = (f/2) + l^2/8f = (8/2) + [32^2/(8 \times 8)] = 20\,\text{m}$. Columns 1 and 2 contain ordinate x and corresponding ordinate y (in meters) for specified sections. Ordinate $y(x)$ is

$$y = \sqrt{R^2 - \left(\frac{l}{2} - x\right)^2} - R + f = \sqrt{400 - (16 - x)^2} - 12\,(\text{m}).$$

Columns 3 and 4 contain values of $\sin\varphi = (l - 2x)/2R = (32 - 2x)/40$ and $\cos\varphi = (\mathrm{y} + \mathrm{R} - \mathrm{f})/\mathrm{R} = (\mathrm{y} + 12)/20$.

Values of bending moment and shear for reference beam are tabulated in columns 5 and 7 and taken directly from corresponding diagrams, which are presented in Fig. 2.27. Values of $H(y - f_0)$ are given in column 5′. Sections A and B have no entries for column 5′, which means that force in the tie does not effect on the bending moment at corresponding section of the arch. Values of bending moment, shear, and normal forces for three-hinged arch are tabulated in columns 6, 8, and 9. They have been computed using (2.22) for sections which are located below the tie. For example, for section A, we get

$$\begin{aligned} Q_A &= Q_A^0 \cos\varphi_A = 14.5 \times 0.6 = 8.7\,\text{kN} \\ N_A &= -Q_A^0 \sin\varphi_A = -14.5 \times 0.8 = -11.6\,\text{kN}. \end{aligned}$$

For sections above the tie, we need to use (2.23). For example, for section 3, we get

$$\begin{aligned} M_x &= M_x^0 - H(y - f_0) = 134 - 25.33 \times (7.596 - 2) = -7.7467\,\text{kNm}, \\ Q_x &= Q_x^0 \cos\varphi - H \sin\varphi = 4.5 \times 0.9798 - 25.33 \times 0.2 = -0.6569\,\text{kN}, \\ N_x &= -Q_x^0 \sin\varphi - H \cos\varphi = -4.5 \times 0.2 - 25.33 \times 0.9798 = -25.718\,\text{kN}. \end{aligned}$$

Corresponding diagrams are presented in Fig. 2.27. Bending moment diagrams for beam and arch are shown on the extended fibers; therefore, the signs of bending moments are omitted. For convenience, different scales have been adopted for different diagrams.

Verification. The vertical concentrated force P leads to value of discontinuity $P\cos\varphi$ and $P\sin\varphi$ for diagram Q and N, respectively; the horizontal force H leads to value of discontinuity $H\sin\varphi$ and $H\cos\varphi$ for same diagrams Q and N.

Values of discontinuity on shear and normal force diagrams due to concentrated forces H and P_i are:

Shear force diagram point at M: $10.15 - (-7.938) = 18.088 = H \sin\varphi$,
Point 2: $3.1572 - (-6.0) = 9.1572 = P_1 \cos\varphi$.

Normal force diagram point at M: $-10.35 - (-28.08) = 17.73 = H \cos\varphi$,
Point 2: $-25.01 - (-29.01) = 4.0 = P_1 \sin\varphi$.

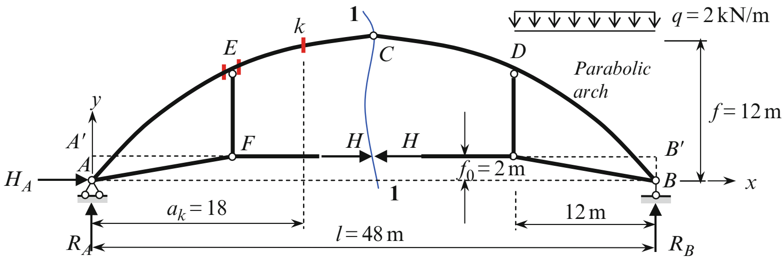

Fig. 2.28 Design diagram of the arch with complex tie

Values of discontinuity on shear and normal force diagrams for points 7 and *K* are verified in a similar manner.

Now we will compare the internal force diagrams for the arch without tie (Fig. 2.6) and the arch with the elevated tie (Fig. 2.27). Unlike the arch without tie, two horizontal forces *H* act at points *M* and *K*. Therefore, the shear and axial force diagrams at points *M* and *K* have abrupt changes $H \sin\varphi$ for the *Q* diagram and $H \cos\varphi$ for the *N* diagram. The axial force *N* for both arches remains compressed.

The fundamental change occurs in the distribution of bending moments. For example, for all sections of the left part of the arch without tie, the extended fibers are located above the neutral line (Fig. 2.6), while for arch with the tie, the extended fibers are located below the neutral line (Fig. 2.27) (portion *A*-2 and slightly further). For the right part of the arch without tie, the bending moment diagram changes the sign three times: in the neighborhood of point *n* and 7, the extended fibers are located above and below the neutral line, respectively, while for arch with tie, the entire right part of the arch has extended fibers below the neutral line.

2.6.2 Arch with Complex Tie

Analysis of such structure subjected to fixed and moving load has some features. Design diagram of the symmetrical parabolic arch with complex tie is presented in Fig. 2.28. The arch is loaded by vertical uniformly distributed load $q = 2$ kN/m. We need to determine the reactions of the supports, thrust, and internal forces at section *k* ($a_k = 18$ m, $y_k = 11.25$ m, $\tan\varphi_k = 0.25$, $\cos\varphi_k = 0.970$, $\sin\varphi_k = 0.2425$) as well as to construct the influence line for above-mentioned factors.

Reactions and Internal Forces at Section *k*

The vertical reactions are determined from the equilibrium equations of all the external forces acting on the arch

$$R_A \to \sum M_B = 0: \quad -R_A \times 48 + q \times 12 \times 6 = 0 \to R_A = 3\,\text{kN},$$
$$R_B \to \sum M_A = 0: \quad R_B \times 48 - q \times 12 \times 42 = 0 \to R_B = 21\,\text{kN}.$$

Horizontal reaction at support A is $H_A = 0$.

The thrust H in the tie (section 1–1) is determined from the following equation

$$H \to \sum M_C^{\text{left}} = 0: \quad -R_A \frac{l}{2} + H(f - f_0) = 0 \to H = M_C^0/(f - f_0) = 7.2\,\text{kN}. \tag{2.24a}$$

Equilibrium equations of joint F lead to the axial forces at the members of AF and EF of the tie.

Internal forces at section k for a reference simply supported beam are as follows:

$$M_k^0 = R_A \times x_k = 3 \times 18 = 54\,\text{kNm},$$
$$Q_K^0 = R_A = 3\,\text{kN}.$$

Internal forces at point k for three-hinged arch are determined as follows

$$\begin{aligned} M_k &= M_k^0 - H(y_k - f_0) = 54 - 7.2 \times (11.25 - 2) = -12.6\,\text{kNm}, \\ Q_k &= Q_k^0 \cos\varphi_k - H \sin\varphi_k = 3 \times 0.970 - 7.2 \times 0.2425 = 1.164\,\text{kN}, \\ N_k &= -\left(Q_k^0 \sin\varphi_k + H \cos\varphi_k\right) = -(3 \times 0.2425 + 7.2 \times 0.970) = -7.711\,\text{kN}. \end{aligned} \tag{2.24b}$$

Note, that the discontinuity of the shear and normal forces at section E left and right of the vertical member EF is $N_{EF} \times \cos\varphi$ and $N_{EF} \times \sin\varphi$, respectively.

Influence Lines for Thrust and Internal Forces (M, Q, N) at Section k

Influence lines for vertical reactions R_A and R_B for arch and for reference simply supported beam coincide, i.e.,

$$\text{IL}(R_A) = \text{IL}\left(R_A^0\right), \qquad \text{IL}(R_B) = \text{IL}\left(R_B^0\right).$$

According to (2.24a), the equation of influence line for thrust becomes

$$\text{IL}(H) = \frac{1}{f - f_0} \times \text{IL}\left(M_C^0\right).$$

The maximum ordinate of influence line for H at crown C is $1/(f - f_0) \times (l/4) = 48/[4 \times (12 - 2)] = 1.2$. Influence line for thrust H may be considered as a key influence line (Fig. 2.29).

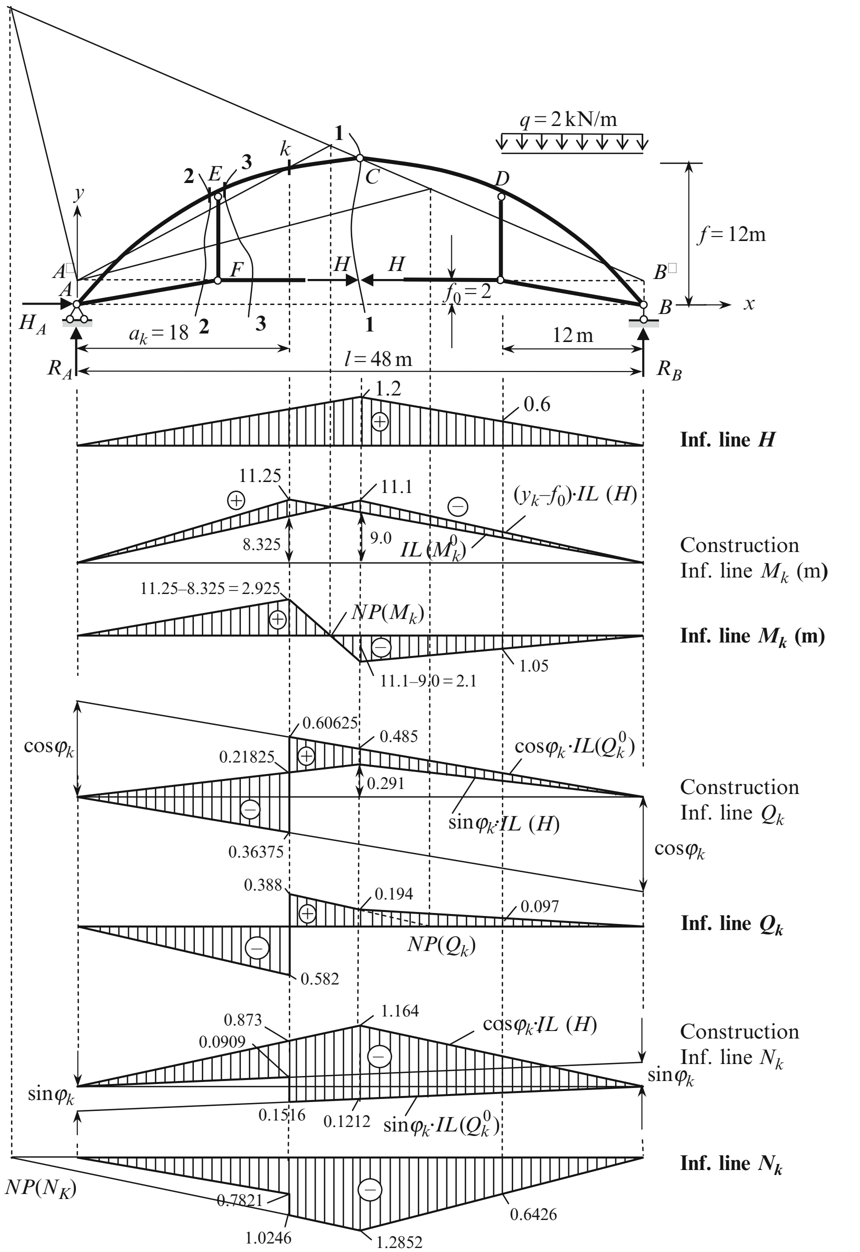

Fig. 2.29 Three-hinged arch with complex tie. Design diagram and influence lines

Bending Moment

According to (2.24b) for bending moment at any section, the equation of influence line for bending moment at section k is

$$\mathrm{IL}(M_k) = \mathrm{IL}\left(M_k^0\right) - (y_k - f_0) \times \mathrm{IL}(H) = \mathrm{IL}\left(M_k^0\right) - 9.25 \times \mathrm{IL}(H). \qquad (2.24c)$$

Influence line M_K^0 presents a triangle with maximum ordinate $a_k b_k / l = 18 \times 30/48 = 11.25\,\mathrm{m}$ at section k, so the ordinate at crown C equals to 9 m. Influence line for thrust H presents the triangle with maximum ordinate 1.2 at crown C. Ordinate of the graph $(y_k - f_0) \times \mathrm{IL}(H)$ at crown C equals $(11.25 - 2) \times 1.2 = 11.1\,\mathrm{m}$, so ordinate at section k equals 8.325 m. Detailed construction of influence line M_k is shown in Fig. 2.29. Since both terms in (2.24c) has *different* signs, they should be plotted on the one side on the basic line and the final ordinates of influence line are located *between* two graphs $\mathrm{IL}\left(M_k^0\right)$ and $9.25 \times \mathrm{IL}(H)$.

Shear Force

According to (2.24b) for shear at any section, the equation of influence line for shear at section k is

$$\begin{aligned}\mathrm{IL}(Q_k) &= \cos\varphi_k \times \mathrm{IL}\left(Q_k^0\right) - \sin\varphi_k \times \mathrm{IL}(H) \\ &= 0.970 \times \mathrm{IL}\left(Q_k^0\right) - 0.2425 \times \mathrm{IL}(H). \qquad (2.24d)\end{aligned}$$

Ordinates of the graph $0.970 \times \mathrm{IL}\left(Q_k^0\right)$ are 0.36375 and 0.60625 to the left and to the right at section k, so ordinate at crown C is 0.485. Maximum ordinate of the graph $0.2425 \times \mathrm{IL}(H) = 0.2425 \times 1.2 = 0.291$ is located at crown C, so ordinate at section k is 0.21825.

Ordinate of influence line for shear at crown C equals 0.485–0.291 $= 0.194$; the left and the right of section k ordinates of influence line become

$$-(0.36375 + 0.21825) = -0.582 \text{ and } 0.60625 - 0.21825 = 0.388.$$

Detailed construction of influence line Q_k is shown in Fig. 2.29.

Normal Force

According to (2.24b) for normal force at any section, the equation of influence line for normal force at section k is

$$\begin{aligned}\mathrm{IL}(N_k) &= -\sin\varphi_k \times \mathrm{IL}\left(Q_k^0\right) - \cos\varphi_k \times \mathrm{IL}(H) \\ &= -\left[0.2425 \times \mathrm{IL}\left(Q_k^0\right) + 0.970 \times \mathrm{IL}(H)\right] \qquad (2.24e)\end{aligned}$$

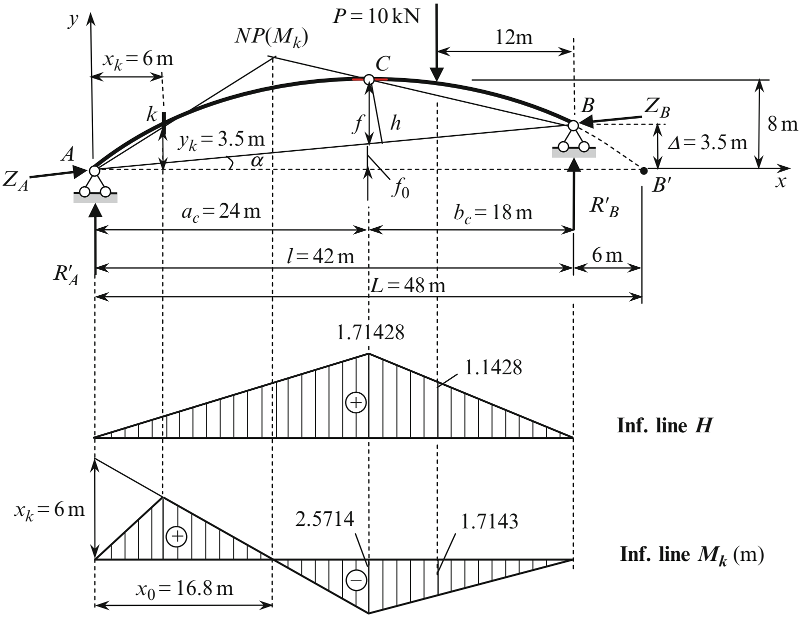

Fig. 2.30 Three-hinged askew arch. Design diagram and influence lines

Maximum ordinate of the graph $0.970 \times \text{IL}(H)$ is $0.970 \times 1.2 = 1.164$; this ordinate is located at crown C. Specific ordinates of the graph $0.2425 \times \text{IL}(Q_k^{\,0})$ are 0.09094 and 0.1516 and located to the left and to the right of section k.

Detailed construction of influence line N_k is shown in Fig. 2.29. This figure also represents the construction of influence lines using nil point method; note that construction of the nil points must be done on the basis of conventional supports A' and B'.

2.6.3 Askew Arch

The arch with support points located on the different levels is called askew (or rising) arch. Three-hinged askew arch is geometrically unchangeable and statically determinate structure. Analysis of askew arch subjected to fixed and moving loads has some features.

Design diagram of three-hinged askew arch is presented in Fig. 2.30. Let the shape of the arch is parabola, span of the arch $l = 42$ m and support B is $\Delta = 3.5$ m higher than support A. The total height of the arch at hinge C is 8 m. The arch is loaded by force $P = 10$ kN. It is necessary to calculate the reactions and bending moment at section k, construct the influence lines for thrust and bending moment M_k, and apply influence lines for calculation of bending moment and reactions due to fixed load.

Equation of the axis of parabolic arch is

$$y = 4(f + f_0) \times (L - x) \times \frac{x}{L^2},$$

where span for arch A–C–B' with support points on the same level is $L = 48$ m. For $x = 42$ m (support B), ordinate $y = 3.5$ m, so

$$\tan\alpha = \frac{\Delta}{l} = \frac{3.5}{42} = 0.0833 \rightarrow \cos\alpha = 0.9965 \rightarrow \sin\alpha = 0.08304.$$

Other geometrical parameters are

$$f_0 = 24\tan\alpha = 2.0\,\text{m} \rightarrow f = 8 - 2 = 6\,\text{m} \rightarrow h = f\,\cos\alpha = 6 \times 0.9965 = 5.979\,\text{m}.$$

For $x = 6$ m (section k), the ordinate $y_k = 3.5$ m.

Reactions and Bending Moment at Section *k*

It is convenient to resolve total reaction at point A into two components. One of them, R'_A, has vertical direction and other, Z_A, is directed along the line AB. Similarly resolve the reaction at support B. These components are R'_B and Z_B. The vertical forces R'_A and R'_B represent a *part* of the total vertical reactions. These vertical forces may be computed as for the reference beam

$$\begin{aligned} R'_A &\rightarrow \sum M_B = 0: \qquad -R'_A \times 42 + P \times 12 = 0 \rightarrow R'_A = 2.857\,\text{kN},\\ R'_B &\rightarrow \sum M_A = 0: \qquad R'_B \times 42 - P \times 30 = 0 \rightarrow R'_B = 7.143\,\text{kN}. \end{aligned}$$

Since a bending moment at crown C is zero then

$$\begin{aligned} &Z_A \rightarrow \sum M_C^{\text{left}} = 0: \quad Z_A \times h - M_C^0 = 0 \rightarrow Z_A = \frac{M_C^0}{h} = \frac{2.857 \times 24}{5.979} = 11.468\,\text{kN},\\ &Z_A = Z_B = Z, \end{aligned}$$

where M_C^0 is a bending moment at section C for the reference beam.

Thrust H represents the horizontal component of the Z, i.e., $H = Z\cos\alpha = 11.468 \times 0.9965 = 11.428\,\text{kN}$.

The total vertical reactions may be defined as follows

$$\begin{aligned} R_A &= R'_A + Z\sin\alpha = 2.857 + 11.468 \times 0.08304 = 3.809\,\text{kN},\\ R_B &= R'_B - Z\sin\alpha = 7.143 - 11.468 \times 0.08304 = 6.191\,\text{kN}. \end{aligned}$$

Bending moment at section k:

$$M_k = M_k^0 - Hy = 3.809 \times 6 - 11.428 \times 3.5 = -17.144\,\text{kN}.$$

Influence Lines for Thrust and Bending Moment M_k

Thrust. Since $H = Z\cos\alpha = \left(M_C^0/h\right)\cos\alpha$, then equation of influence line for thrust becomes

$$\text{IL}(H) = \frac{\cos\alpha}{h} \times \text{IL}\left(M_C^0\right).$$

The maximum ordinate of influence line occurs at crown C and equals

$$\frac{\cos\alpha}{h} \times \frac{a_C b_C}{l} = \frac{0.9965}{5.979} \times \frac{24 \times 18}{42} = 1.71428.$$

Bending moment M_k. Since $M_k = M_k^0 - Hy_k$, then equation of influence line for bending moment at section k becomes

$$\text{IL}(M_k) = \text{IL}\left(M_k^0\right) - y_k \times \text{IL}(H).$$

Influence line may be easily constructed using the nil point method. Equation of the line Ak is

$$y = \frac{3.5}{6}x = 0.5833x.$$

Equation of the line BC is

$$y - y_C = m(x - x_C) \rightarrow y - 8 = -\frac{4.5}{18}(x - 24) \rightarrow y = 14 - 0.25x,$$

where m is a slope of the line BC.

The nil point NP(M_k) of influence line for M_k is the point of intersection of lines Ak and BC. Solving these equations leads to $x_0 = 16.8$ m. Influence lines for H and M_k are presented in Fig. 2.30. Maximum positive and negative bending moment at section k occurs if load P is located at section k and hinge C, respectively. If load P is located within portion x_0, then extended fibers at section k are located below the neutral line of the arch.

The thrust and bending moment at section k may be calculated using the relevant influence lines

$$H = Py = 10 \times 1.1428 = 11.428\,\text{kN}$$
$$M_k = Py = 10 \times (-1.7143) = -17.143\,\text{kNm}.$$

These values coincide exactly with those calculated previously.

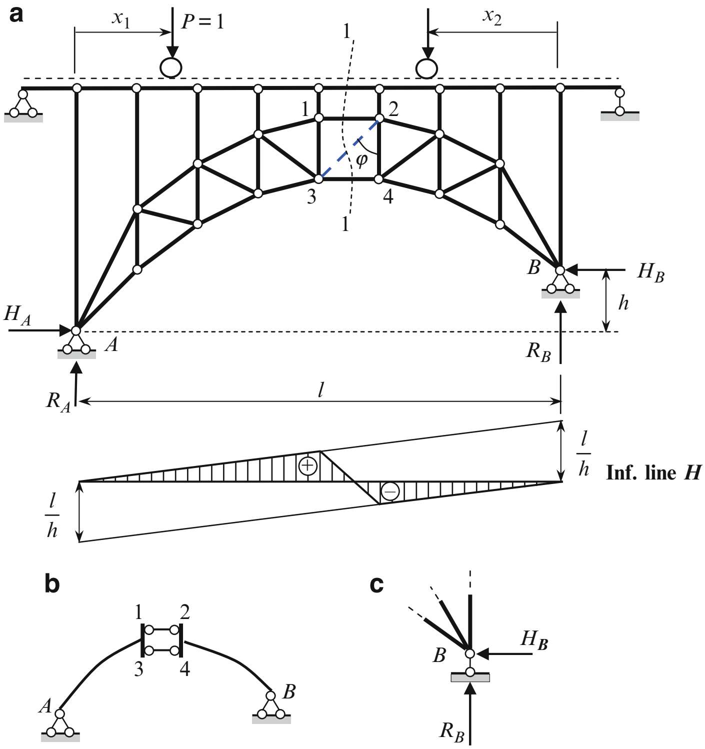

Fig. 2.31 Modified askew arched structure

As before, the influence line for thrust constructed *once* may be used for its computation for different cases of arbitrary loads. Then, knowing the vertical reactions and thrust, the internal forces at any point of the arch may be calculated by definition without using influence line for that particular internal force.

2.6.4 Latticed Askew Arch

Design diagram of the modified askew arched structure with over-arch construction is presented in Fig. 2.31a. Pinned supports A and B are located at different elevations. Each half-arch itself (A-1–3 and B-2–4) represents the structure with webbed members. Panel block 1–2–3–4 has no diagonal member, thus both half-arches are connected by means of two parallel rods 1–2 and 3–4. Therefore, the vertical relative displacement of two half-arches is possible (Fig. 2.31b), while in the classic three-hinged arch only angular relative displacement of two half-arches is possible. The vertical posts are used only to transmit loads directly to the upper chord of the structure.

Degree of freedom equals

$$W = 2J - S - S_0 = 2 \times 27 - 47 - 7 = 0,$$

where J, S, and S_0 are the number of hinged joints, members of structure and constraints of supports, respectively [Kar10]. Though the both part of arch represent the simplest truss (or rigid disc), they are connected in a specific way, mainly by members 1–2 and 3–4 as well as an imagine member AB (ground). These members are not parallel. The structure is statically determinate and geometrically unchangeable.

For analysis of this structure, we will apply the following procedure:

1. Replace the constraint of the support B, which prevents horizontal displacement, by a diagonal member 2–3 (dotted line in Fig. 2.31a) and apply external force H_B at point B (Fig. 2.31a, c). Such a substitution does not change the number of degree of freedom.
2. Consider two positions x_1 and x_2 of a moving load P and determine thrust $H_A = H_B = H$ in terms of x, l, and h, when the internal force in the substitute member 2–3 is zero.

Force P = 1 is located at the left part of the structure. Thrust $H \rightarrow \sum M_A = 0$:

$$R_B l + Hh - Px = 0 \rightarrow R_B = \frac{1}{l}(Px - Hh).$$

Internal force in substitute member D_{23} section 1–1 is determined as follows

$$\sum Y^{\text{right}} = 0: \quad R_B - D_{23} \cos\varphi = 0.$$

Taking into account the previous result for reaction R_B, internal force in diagonal member becomes

$$\frac{1}{l}(Px - Hh) - D_{23} \cos\varphi = 0.$$

However, diagonal member is absent, therefore $D_{23} = 0$ and the expression for thrust is

$$H = \frac{Px}{h}, \text{ so IL}(H) = \frac{x}{h}.$$

Force P = 1 is located at the right part of the structure. Thrust $H \to \sum M_A = 0$:

$$R_B l + Hh - P(l - x) = 0 \to R_B = \frac{P(l - x) - Hh}{l}$$

Internal force in the substitute member $D_{23} \to \sum Y^{\text{right}} = 0 : R_B - D_{23} \times \cos\varphi - P = 0$.

Taking into account the previous result for reaction R_B, equation for internal force in diagonal member becomes

$$-D_{23}\cos\varphi - \frac{Px}{l} - \frac{Hh}{l} = 0.$$

However, $D_{23} = 0$ so the expression for thrust becomes

$$H = -\frac{Px}{h}, \text{ so } \mathrm{IL}(H) = -\frac{x}{h}.$$

Influence line for H represents two parallel lines with ordinates l/h at the support points and connecting line within the panels 1–2. The sign of thrust H depends on location of the moving load (unlike previously considered arched structures).

Influence line H is a fundamental characteristic of the system. Knowing the influence line H allows us to calculate this reaction for any type of loadings. Calculation of all other reactions and internal forces in any members presents no difficulties.

Note if supports A and B will be located at the same level, then the system becomes instantaneously changeable. Indeed, in this case, two rigid discs (the left and right parts of the structure) are connected by three parallel members, mainly 1–2, 3–4 and AB [Kar10].

2.7 Complex Arched Structures

This paragraph contains analysis of the complex arched structures subjected to fixed and moving load. Among them are the multispan three-hinged arched structure and trusses with arched hinged chain.

2.7.1 *Multispan Three-Hinged Arched Structure*

Multispan three-hinged arched structure is a geometrically unchangeable structure, which consists of three-hinged arches connected together by means of hinges. Figure 2.32a presents the multispan arched structure which contains three-hinged arch ACB with overhang BG, arch DIF with overhang HD, and central three-hinged arch GEH, which is connected with left and right arches by means of hinges G and H.

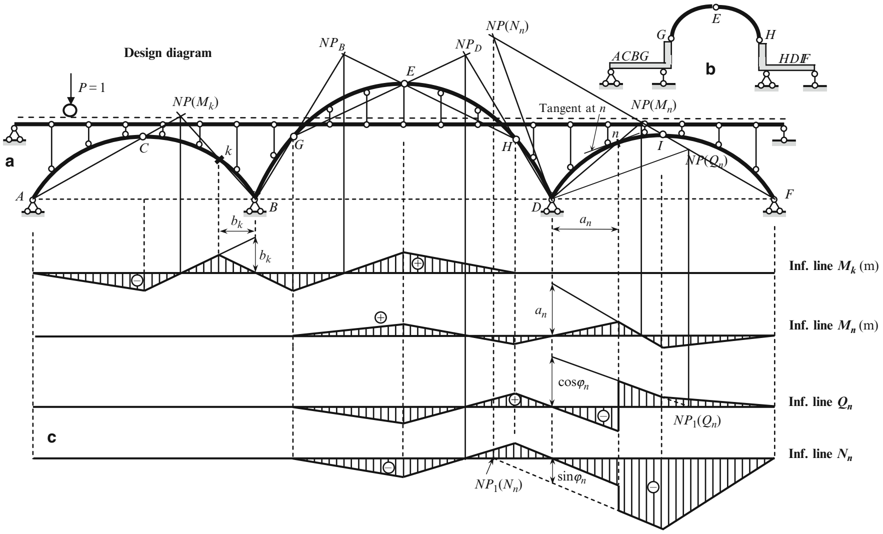

Fig. 2.32 Multispan arched structure. (**a**) Design diagram, (**b**) interaction scheme, and (**c**) influence lines for internal forces at sections k and n

It is necessary to construct the influence lines for bending moment, shear, and normal forces at sections k and n, using the nil point method. Indirect application of the load on the arch system should not be taken into account.

Kinematical Analysis

Degrees of freedom of this arch structure, according to Chebushev formula, are determined as $W = 3D - 2H_0 - S_0 = 3 \times 6 - 2 \times 5 - 8 = 0$, where D, H_0, and S_0 are number of rigid discs, number of simple hinges, and number of constraints of support, respectively [Kar10].

The whole structure may be presented as two main arched structures *ACBG* and *HDIF* and a suspended arch *GEH*; corresponding interaction diagram is shown in Fig. 2.32b. Each arched structures *ACBG* and *HDIF* present two rigid discs, connected with the ground. Two curvilinear members *GE* and *EH* are connected by hinge *E* and supported by two unmovable rigid discs, which can be considered as ground. Thus, the entire structure is statically determinate and geometrically unchangeable.

Influence line for bending moment M_k. There exist two nil points of influence line for M_k as the points of intersection of two lines:

1. Lines *AC* and *Bk*: their intersection point is $NP(M_k)$.
2. Lines *BG* and *HE*: their intersection point is NP_B.

The nil point NP_B possesses interesting properties. If moving load is traveling along the horizontal portion *GE*, then reaction at *H* is passing through hinge *E* and reaction at G might have various directions in accordance to the theorem about three concurrent forces R_H, $P = 1$, and R_G. The last reaction R_G is transformed as active force on the arch $ACBG$, in which reactions R_A and R_B arise. Reaction R_G is an active force for arch $ACBG$, passing through support B. This force is perceived by support B and reaction R_A is zero. Therefore, at all sections of the arch ACB, all internal forces are zero. Thus, if load P is located at NP_B, then *all* internal forces of the arch ACB are zero.

Since arch GEH is suspended, the bending moment M_k does not arise if load P is traveling along the portion HF.

Influence line for M_n. There exist two nil points of IL(M_n) as the points of intersection of two lines:

1. Lines FI and Dn: their intersection point is $NP(M_n)$.
2. Lines DH and GE: their intersection point is NP_D.

It is evident that point NP_D possesses the same properties for arch DIF as point NP_B for arch ACB: if moving load is located on the vertical passing through point NP_D, then at all sections of the arch DIF all internal forces are zero.

Influence line for shear force Q_n. There exist two nil points of influence line for Q_n. They are the point of intersection of line FI and the line which is parallel to tangent at section n and point NP_D.

Influence line for axial force N_n. There exist two nil points of influence line for N_n. They are the point of intersection of line FI and line which is perpendicular to tangent at section n and point NP_D. Specific ordinates and positions of the nil points allow us to easily construct the influence lines. Some of them are presented in Fig. 2.32c.

Note that the nil points $NP_1(Q_n)$ and $NP_1(N_n)$ are not real; they only facilitate the construction of influence lines Q_n and N_n, respectively.

It is left to reader to construct influence lines of shear and normal force in section k; construction of influence lines for internal forces for any section of central arch GEH should present no challenge.

2.7.2 Arched Combined Structures

Some examples of arches combined structures are presented in Fig. 2.33. In all cases, these systems consist of two trusses, AC and CB, connected by hinge C and stiffened by additional structures called a hinged (or arched) chain. The hinged chain may be located above or below the trusses. Vertical members connect the hinged chain with the trusses. The connections between the members of the arched chain and the hangers or posts are hinged. In case (c), all the hinges of the hinged chain are located on one line. In cases (a) and (b), a load is applied to the truss directly, while in case (c), the load is applied to the joint of the hinged chain and then transmitted to the truss.

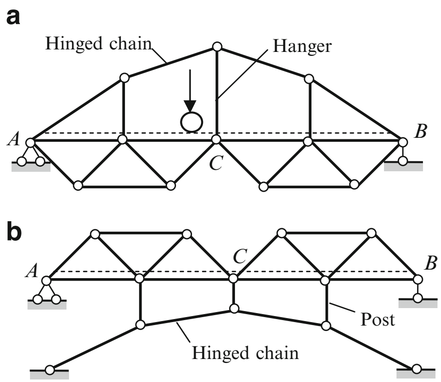

Fig. 2.33 Trusses with hinged chain

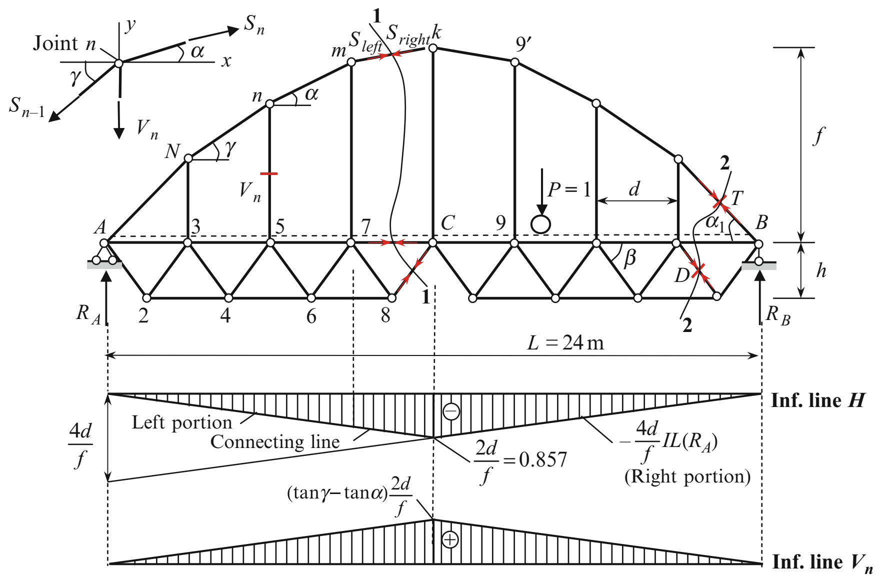

Fig. 2.34 Truss with over-truss arched chain. Design diagram and influence lines

Truss with Over-Truss Arched Chain

The typical truss with a hinged chain located above the truss is shown in Fig. 2.34. Assume that the parameters of the structure are as follows: $d = 3$ m, $h = 2$ m, $f = 7$ m, $L = 24$ m. We need to construct the influence lines for the reactions and the internal forces in hanger, V_{n-1}.

As usual we start with the kinematical analysis of the structure. Since the structure consist only members with hinges at the ends, then degrees of freedom of this complex arched structure is determined as $W = 2J - S - S_0 = 2 \times 24 - 45 - 3 = 0$, so the structure is geometrically unchangeable and statically determinate.

Reaction of Supports and Internal Forces

Reactions R_A and R_B for any load can be calculated using following equilibrium conditions:

$$R_A \to \sum M_B = 0: \quad R_B \to \sum M_A = 0.$$

For calculation of the internal forces that arise in the members of the hinged chain, we need to show the free body diagram for any joint n (Fig. 2.34). The equilibrium condition $\sum X = 0$ leads to relationship

$$S_n \cos \alpha = S_{n-1} \cos \gamma = H. \tag{2.25}$$

Thus, for any vertical load acting on the given structure, the horizontal component of the forces, which arise in all the members of the hinged chain, is equal. The horizontal component of the forces S_n, S_{n-1} is called a thrust.

Now we will provide an analysis for the case of a moving load. The influence lines for reactions R_A and R_B are the same as for a simply supported beam. However, the construction of an influence line for thrust H has some special features. Let us consider them.

Thrust H (section 1–1, the sectioned panel of the load contour – SPLC – is panel 7-C; Ritter's point is C). Internal force S, which arises in the element m–k of the hinged chain, is denoted as S_{left} and S_{right}. The meaning of the subscript notation is clear from Fig. 2.34.

If load $P = 1$ is located to the left of joint 7, then thrust H can be calculated by considering the *right* part of the structure. The active forces are reaction R_B and internal forces $S_{7\text{-}C}$, $S_{8\text{-}C}$, and S_{right}. The last force S_{right} can be resolved into two components: a horizontal component, which is the required thrust H, and a vertical component, which acts along the vertical line C–k. Now we form the sum of the moment of all forces acting on the right part of the structure around point C, i.e., $H \to \sum M_C^{\text{right}} = 0$. In this case, the vertical component of force S_{right} produces no moment, while the thrust produces moment Hf.

If load $P = 1$ is located right at joint C, then thrust H can be calculated by considering the *left* part of the structure. The active forces are reaction R_A and internal forces $S_{7\text{-}C}$, $S_{8\text{-}C}$, and S_{left}. The force S_{left}, which is applied at joint m, can be resolved into a horizontal component H and a vertical component. The latter component acts along vertical line m–7. Now we find the sum of the moment of all the forces, which act on the *left* part of the truss, around point C. In this case, the vertical component of force S_{right} produce the nonzero moment around joint C and thrust H has a new arm (m–7) around the center of moments C. In order to avoid these difficulties, we translate the force S_{left} along the line of its action from joint m into joint k. After that we resolve this force into its vertical and horizontal components. This procedure allows us to eliminate the moment due to the vertical component of S, while the moment due to the horizontal component of S is easily calculated as Hf.

Construction of the influence line for H is presented in the table below.

$P=1$ left at SPLC	$P=1$ right at SPLC
$H \to \sum M_C^{\text{right}} = 0: \quad R_B 4d + Hf = 0$	$H \to \sum M_C^{\text{left}} = 0: \quad R_A 4d + Hf = 0$
$H = -\dfrac{4d}{f} R_B \to \text{IL}(H) = -\dfrac{4d}{f} \text{IL}(R_B)$	$H = -\dfrac{4d}{f} R_A \to \text{IL}(H) = -\dfrac{4d}{f} \text{IL}(R_A)$

The left portion of the influence line for H (portion A-7) presents the influence line for R_B multiplied by coefficient $-4d/f$ and the right-hand portion (portion C–B) presents the influence line for R_A multiplied by the same coefficient. The connecting line is between points 7 and C (Fig. 2.34). The negative sign for thrust indicates that all members of the arched chain are in compression.

Force V_n. Equilibrium condition for joint n leads to the following result:

$$\sum Y = 0: \quad -V_n + S_n \sin\alpha - S_{n-1} \sin\gamma = 0 \to V_n = H(\tan\alpha - \tan\gamma).$$

Therefore,

$$\text{IL}(V_n) = (\tan\alpha - \tan\gamma) \times \text{IL}(H).$$

Since $\alpha < \gamma$ and H is negative, then all hangers are in tension. The corresponding influence line is shown in Fig. 2.34.

The influence line for thrust H can be considered as the key influence line, since thrust H always appears in any cut-section for the entire structure. This influence line allows us to calculate thrust for an arbitrary load. After that, the internal force in any member can be calculated simply by considering all the external loads, the reactions, and the thrust as an additional external force.

Discussion

For any location of a load, the hangers are in tension and all members of the chain are compressed. The maximum internal force at *any* hanger occurs if load P is placed at joint C.

To calculate the internal forces in different members caused by an arbitrary fixed load, the following procedure is recommended:

1. Construct the influence line for the thrust.
2. Calculate the thrust caused by a fixed load.
3. Calculate the required internal force considering thrust as an additional external force.

This algorithm combines both approaches: the methods of fixed and of moving loads and so provides a very powerful tool for the analysis of complex structures.

Example 2.7. The structure in Fig. 2.34 is subjected to a uniformly distributed load q within the entire span L. Calculate the internal forces T and D in the indicated elements.

Solution. The thrust of the arch chain equals $H = q\omega_H = -q(1/2)L(2d/f) = -(qLd/f)$, where ω_H is area of the influence line for H under the load q. After that, the required force T according to (a) is

$$T = \frac{H}{\cos\alpha_1} = -\frac{qLd}{f\cos\alpha_1}.$$

We can see that in order to decrease the force T, we must increase the height f and/or decrease the angle α_1.

To calculate force D, we can use section 2–2 and consider the equilibrium of the right part of the structure:

$$D \to \sum Y = 0: \quad D\sin\beta + R_B + T\sin\alpha_1 = 0 \to$$
$$D = -\frac{1}{\sin\beta}\left(\frac{qL}{2} - \frac{qLd}{f}\tan\alpha_1\right).$$

Thus, this problem is solved using the fixed and moving load approaches: thrust H is determined using corresponding influence lines, while internal forces D and T are computed using H and the classical method of through sections.

Arched Chain with Over-Arch Trussed Structure

The typical arched chain with a truss located above the arched chain is shown in Fig. 2.35. Assume that the parameters of the structure are as follows: $d = 2$ m, $h = 2$ m, $f = 8$ m, $l = 12d = 24$ m, $a_K = 6$ m. We need to construct the influence lines for the reactions, thrust, and the internal forces in indicated members U_4 and D_4.

Kinematical analysis shows that degree of freedom is $W = 2J - S - S_0 = 2 \times 34 - 61 - 7 = 0$, so the structure is statically determinate and geometrically unchangeable. The structure has the four support points: A_1, A_2, B_1, and B_2 and the following reactions: $R_{A1},\ R_{A2}, R_{B1},\ R_{B2},\ H_A,\ H_B$.

Reactions of Support and Internal Forces

Total vertical reactions *of a structure as a whole are* $R_A = R_{A1} + R_{A2}$, $R_B = R_{B1} + R_{B2}$, *where*

$$R_A \to \sum M_B = 0: \quad -R_A l + P(l-x) = 0 \to R_A = \frac{P(l-x)}{l} \to \mathrm{IL}(R_A) = \frac{l-x}{l},$$
$$R_B \to \sum M_A = 0: \quad R_B l - Px = 0 \to R_B = \frac{Px}{l} \to \mathrm{IL}(R_B) = \frac{x}{l}.$$

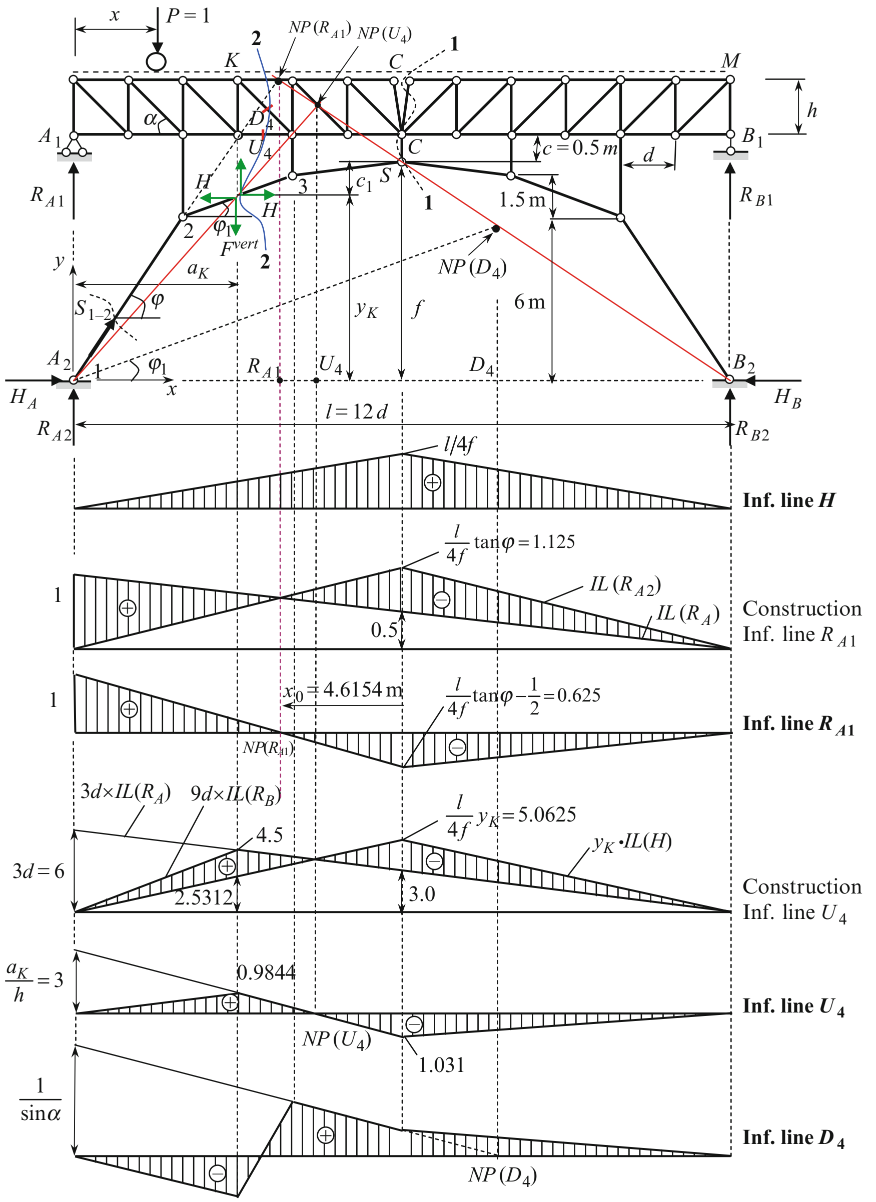

Fig. 2.35 Arched chain with over-arch trussed structure. Design diagram and influence lines

Influence lines for total vertical reactions of support are the same as for a simply supported beam.

Thrust. For the entire structure, the equilibrium condition $\sum X = 0$ leads to relationship $H_A = H_B = H$. Section 1–1 passes through joints S and C.

$P = 1$ left at joint C'	$P = 1$ right at joint C'
$H \to \sum M_S^{\text{right}} = 0: \quad -Hf + R_B\dfrac{l}{2} = 0$	$H \to \sum M_S^{\text{left}} = 0: \quad Hf - R_A\dfrac{l}{2} = 0$
$H = \dfrac{l}{2f}R_B \to \text{IL}(H) = \dfrac{l}{2f} \times \text{IL}(R_B)$	$H = \dfrac{l}{2f}R_A \to \text{IL}(H) = \dfrac{l}{2f} \times \text{IL}(R_A)$

The maximum ordinate under the joint C is equal to $l/4f$.

Vertical components of reactions. Equilibrium conditions for joint A_2 are

$$\begin{aligned}&\sum X = 0: \quad S_{1-2}\cos\varphi + H = 0 \to S_{1-2} = -H/\cos\varphi,\\&\sum Y = 0: \quad S_{1-2}\sin\varphi + R_{A2} = 0 \to R_{A2} = -S_{1-2}\sin\varphi.\end{aligned}$$

So the vertical component of reaction at point A_2 becomes $R_{A2} = H\tan\varphi$; corresponding influence line is

$$\text{IL}(R_{A2}) = \tan\varphi \times \text{IL}(H).$$

Similarly, $\text{IL}(R_{B2}) = \tan\varphi \times \text{IL}(H)$.

Influence lines for R_{A2} and R_{B2} may be obtained by multiplying all ordinates of influence line for H by a constant factor tan φ. The maximum ordinate under joint C is equal to $(l/4f)\tan\varphi$.

Reaction at point A_1. Since total reaction $R_A = R_{A1} + R_{A2}$, then

$$R_{A1} = R_A - R_{A2} \to \text{IL}(R_{A1}) = \text{IL}(R_A) - \text{IL}(R_{A2}) = \text{IL}(R_A) - \tan\varphi \times \text{IL}(H).$$

Construction and final influence line for vertical component R_{A1} is presented in Fig. 2.35. The nil point of influence line for R_{A1} is point of intersection of lines B_2–S and 1–2. The location of this nil point is defined by the formula $x_0 = (l/2) \times [(l\tan\varphi - 2f)/(l\tan\varphi + 2f)]$. For the entire structure, we get $\tan\varphi = (3/2)$ and $x_0 = 4.6154\,\text{m}$.

Ordinate of influence line R_{A1} at point C equals to $(l/4f)\tan\varphi - (1/2) = [24/(4 \times 8)] \times (3/2) - (1/2) = 0.625$.

Note that reaction R_{A1} may be directed upward and downward as well.

Force U_4. Section 2–2 passes across the fourth panel of the truss and arch member 2–3 just under joint K; the vertical line passing through joint K intersects the member 2–3 at $y_K = 6.75$ m. The internal force $F_{2\text{–}3}$ in the arch chain is resolved into vertical F^{vert} and horizontal H components. Obviously, the horizontal component equals to thrust H.

$P = 1$ left at SPLC	$P = 1$ right at SPLC
$U_4 \to \sum M_K^{\text{right}} = 0$	$U_4 \to \sum M_K^{\text{left}} = 0$
$-U_4h + R_B9d - H(f + c + h) + H(c_1 + c + h) = 0$	$U_4h - R_A3d + H(f + c + h) - H(c_1 + c + h) = 0$
$-U_4h + R_B9d - H \times y_K = 0$	$U_4h - R_A3d + H \times y_K = 0$
$U_4 = \dfrac{1}{h}(R_B9d - H \times y_K) \to$	$U_4 = \dfrac{1}{h}(R_A3d - H \times y_K) \to$
$\text{IL}(U_4) = \dfrac{1}{h}[9d \times \text{IL}(R_B) - y_K \times \text{IL}(H)]$	$\text{IL}(U_4) = \dfrac{1}{h}[3d \times \text{IL}(R_A) - y_K \times \text{IL}(H)]$

The term $H(c_1 + c + h)$ presents the moment with respect to point K due to thrust, which arise in member 2–3; the moment with respect to the same point K due to F^{vert} (vertical component of force $F_{2\text{–}3}$) is zero.

The nil point of influence line for U_4 is the point of intersection of lines B_2–S and the line which originates from joint 1 and passes through member 2–3 under point K. This point is real.

Ordinate of influence line U_4 at point C equals to $1/h\,(5.0625 - 3.0) = 1.031$.

Force D_4 (section 1–1). Assume that internal force $S_{2\text{–}3}$ is tensile.

$P = 1$ left at SPLC	$P = 1$ right at SPLC
$D_4 \to \sum Y^{\text{right}} = 0$	$D_4 \to \sum Y^{\text{right}} = 0$
$D_4\ \sin\alpha + R_B - S_{2-3}\ \sin\varphi_1 = 0, S_{2-3} = -\dfrac{H}{\cos\varphi_1}$	$R_A - D_4 \sin\alpha + S_{2-3} \sin\varphi_1 = 0, S_{2-3} = -\dfrac{H}{\cos\varphi_1}$
$D_4\ \sin\alpha + R_B + H\ \tan\varphi_1 = 0$	$R_A - D_4 \sin\alpha - H \tan\varphi_1 = 0$
$D_4 = -\dfrac{1}{\sin\alpha}(R_B + H \times \tan\varphi_1) \to$	$D_4 = \dfrac{1}{\sin\alpha}(R_A - H \times \tan\varphi_1) \to$
$\text{IL}(D_4) = -\dfrac{1}{\sin\alpha}[\text{IL}(R_B) + \tan\varphi_1 \times \text{IL}(H)]$	$\text{IL}(D_4) = \dfrac{1}{\sin\alpha}[\text{IL}(R_A) - \tan\varphi_1 \times \text{IL}(H)]$

Construction and final influence line for D_4 is presented in Fig. 2.35. The nil point of influence line for D_4 is point of intersection of lines B_2–S and the line which originates from joint 1 and passes parallel to the member 2–3. This point for given φ_1 is fictitious.

Influence line for thrust H of the structure is very useful for the calculation of internal force in any member of the truss. Let the structure be subjected to uniformly distributed load q along the entire span l of the truss. In this case, the thrust of the arch chain equals to $H = q\omega_H = q(1/2)l(l/4f) = (ql^2/8f)$, where ω_H is the area of influence line for H under the load q. Positive sign indicates that shown direction for the thrust at points A_2 and B_2 coincides with actual direction of thrust. Knowing the thrust allows us to perform an analysis of the structure. For example, the force $S_{1-2} = -(H/\cos\varphi) = -(ql^2/8f\cos\varphi)$. Negative sign indicates that the member 1–2 is compressed.

In the case of a fixed concentrated force P at joint C' and uniformly distributed load q within C'–M, we get:

$$U_4 = -1.031P - \frac{1}{2} \times 1.031 \times 6d \times q = -1.031P - 6.186q\,(\text{kN}).$$

2.8 Deflection of Three-Hinged Arches Due to External Loads

This section presents computation of displacement of three-hinged arch. Different approaches are applied: Maxwell–Mohr integral and graph multiplication method using Simpson–Kornouhkov rule.

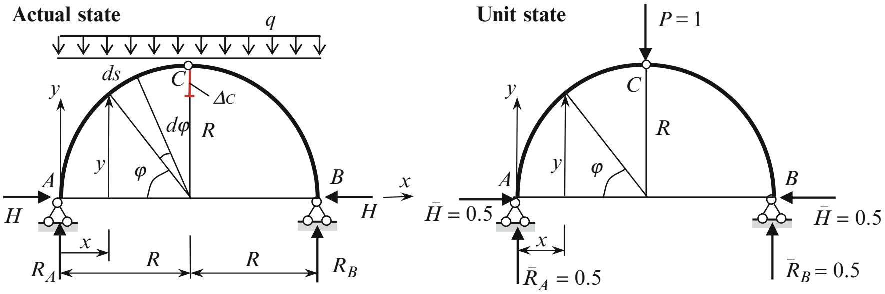

Fig. 2.36 Design diagram of the arch and unit state

2.8.1 Uniform Circular Arch: Exact Solution

Three-hinged semicircular uniform arch of radius R carrying uniformly distributed load q is shown in Fig. 2.36. The flexural stiffness is EI. For calculating the vertical displacement of the hinge C, we assume that influence of axial and shear forces on displacement is negligible. The expression for displacement for this problem takes into account only the bending moments

$$\Delta_C = \int_0^s \frac{M_P \bar{M}}{EI} \mathrm{ds},$$

where M_P denotes the bending moment due to actual load. Now we will consider two states, the actual and unit ones, and form the expressions for bending moments for both of them.

Actual State

The vertical reactions of supports and thrust are:

$$R_A = R_B = \frac{ql}{2} = qR; \quad H = \frac{M_C^0}{f} = \frac{q(2R)^2}{8R} = \frac{qR}{2},$$

where $l = 2R$ is the span of the arch; M_C^0 is the bending moment at point C for reference beam; f is the rise of the arch, $f = R$. The magnitude of the bending moment induced at any section by the given load q is

$$M_P = R_A x - Hy - \frac{qx^2}{2} = q\left(Rx - \frac{Ry}{2} - \frac{x^2}{2}\right).$$

Unit State

This state presents the same arch subjected to unit vertical force P at hinge C. The vertical reactions of supports and thrust are:

$$\bar{R}_A = \bar{R}_B = \frac{1}{2}; \quad \bar{H} = \frac{M_C^0}{f} = \frac{1 \times l}{4R} = \frac{1}{2}.$$

The magnitude of the bending moment induced at any section by the unit load P is

$$\bar{M} = \bar{R}_A x - \bar{H} y = \frac{1}{2} x - \frac{1}{2} y.$$

Now, the vertical displacement at point C may be presented as:

$$\Delta_C = 2 \int_0^{\pi R/2} \frac{M_P \bar{M}}{EI} \mathrm{d}s = \frac{2q}{EI} \int_0^{\pi R/2} \left(Rx - \frac{Ry}{2} - \frac{x^2}{2} \right) \times \left(\frac{x}{2} - \frac{y}{2} \right) \mathrm{d}s \tag{2.26a}$$

Let us change to polar coordinates: $\mathrm{d}s = R\mathrm{d}\varphi, y = R \sin \varphi, x = R - R \cos \varphi = R(1 - \cos \varphi)$. The upper limit $s = \pi R/2$ should be changed to $\varphi = \pi/2$. In this case, (2.26a) becomes

$$\begin{aligned} \Delta_C =& \frac{q}{EI} \int_0^{\pi/2} \left[R^2 (1 - \cos\varphi) - \frac{R^2}{2} \sin\varphi - \frac{R^2}{2} (1 - \cos\varphi)^2 \right] \\ &\times [R(1 - \cos\varphi) - R \sin\varphi] R\mathrm{d}\varphi \\ =& \frac{qR^4}{EI} \int_0^{\pi/2} \left[1 - \cos\varphi - \frac{1}{2} \sin\varphi - \frac{1}{2} (1 - \cos\varphi)^2 \right] \times (1 - \cos\varphi - \sin\varphi) \mathrm{d}\varphi. \end{aligned}$$

Integrating procedure is cumbersome, but elementary. On rearrangement, the final result for vertical displacement at C can be written as

$$\Delta_C = \frac{qR^4}{4EI} (\pi - 3). \tag{2.26b}$$

In case of concentrated force P at point C, the vertical displacement at C is $\Delta_C = (PR^3/2EI)(\pi - 3)$.

2.8.2 *Nonuniform Arch of Arbitrary Shape: Approximate Solution*

In general case of the nonuniform arch and arbitrary shape, the general idea for computation of displacement remains the same – it is necessary to “multiply” the

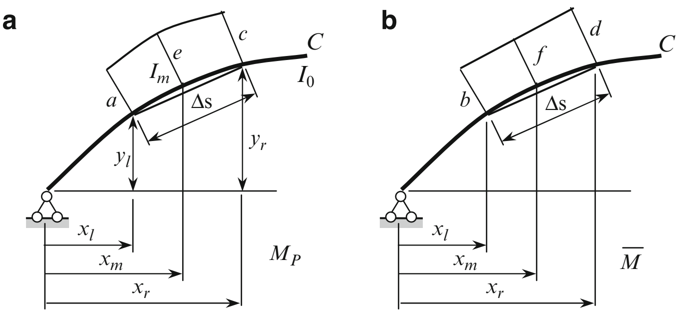

Fig. 2.37 Notation of ordinates of the bending moment diagrams within the one straight segment; M_P and $\overline{M}$ are bending moment diagrams in the actual and unit states

bending moments diagrams in the entire and the unit states. However, the Vereshchagin rule becomes none applicable, since the basic line of both diagrams is curvilinear. Therefore, it is only possible to determine the displacement in the general case of the arch numerically. For this, a curvilinear axis of the arch should be presented as a set of straight elements (usually 8–10), followed by a multiplication procedure of two bending moment diagrams. As before, the normal and shear forces will be neglected.

Let us subdivide the arch into segments with equal horizontal projections. The length of the ith chord between two nodal points equals $\Delta s = \sqrt{(x_r - x_l)^2 + (y_r - y_l)^2}$. Ordinates of the left and right ends of the portion, x_l, y_l and x_r, y_r, should be calculated according to the equation $y = f(x)$ of the axis of the arch. Now Mohr integral may be presented in approximate form

$$\Delta_{iP} = \int \frac{\overline{M}_i M_P}{EI} \mathrm{d}s \cong \frac{1}{EI_0} \sum_n \overline{M}_i M_P \times \frac{I_0}{I_m} \Delta s, \tag{2.27}$$

where M_P and $\overline{M}_i$ are bending moment diagrams in the entire and unit states, respectively; n is the total number of segments, I_0 and I_m are the moment of inertia of the cross section at the crown C and at the middle of the segment Δs. The moment of inertia I_m should be calculated according to the law $I = I(x)$, or as half-sum of the moments of inertia at the ends of a segment. Simpson's formula [Dar89]

$$EI_0 \Delta_{iP} = \sum_n \frac{\Delta s'}{6} (ab + 4ef + cd), \Delta s' = \Delta s \frac{I_0}{I_m} \tag{2.28}$$

is applied to each straight segment and is subsequently summed over all the segments. Ordinates a, e, and c of the bending moment diagram M_P in the loading state relate to the left end, the middle point, and the right end of the ith segment (Fig. 2.37a); ordinates b, f, d of the bending moment diagram $\bar{M}$ in the unit state relate to the same points (Fig. 2.37b).

Figure 2.38 presents a nonuniform parabolic arch and its approximate model.

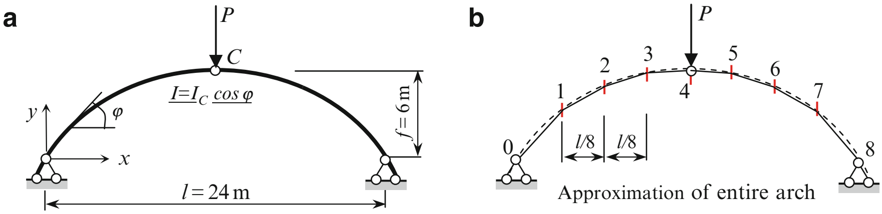

Fig. 2.38 Parabolic arch. Design diagram and approximation of entire arch

Table 2.3 Geometrical parameters of parabolic arch

Points	Coordinates (m) x	y	$\tan\varphi$	$\cos\varphi$
0	0	0.0	1.00	0.7070
1	3	2.625	0.75	0.800
2	6	4.500	0.50	0.8944
3	9	5.625	0.25	0.9701
4	12	6.000	0.0	1.0
5	15	5.625	−0.25	0.9701
6	18	4.500	−0.5	0.8944
7	21	2.625	−0.75	0.800
8	24	0.0	−1.00	0.7070

Table 2.4 The chord lengths of each straight segment

Portion	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8
Length (m)	3.9863	3.5377	3.2040	3.0233	3.0233	3.2040	3.5377	3.9863

Table 2.5 Geometrical parameters at specified sections of nonuniform arch ($I = I_C\cos\varphi$)

Portion	Geometrical parameters (m) x_l	x_m	x_r	y_l	y_m	y_r	Δs	I_m (factor I_C)	$\Delta s' = \Delta s\frac{I_C}{I_m}$
0–1	0	1.5	3	0	1.4062	2.625	3.9863	0.7535	5.2904
1–2	3	4.5	6	2.625	3.6562	4.500	3.5377	0.8475	4.1743
2–3	6	7.5	9	4.500	5.1562	5.625	3.2040	0.9323	3.4366
3–4	9	10.5	12	5.625	5.9062	6.000	3.0233	0.9850	3.0693

Specified Points of the Arch

The *span* of the arch is divided into eight equal parts; the specified points are labeled 0–8. Parameters of the arch for these sections are presented in Table 2.3; the following formulas for calculation of trigonometric functions of the angle φ between the tangent to the arch and x-axis have been used: $\tan\varphi = y' = [4f(l-2x)]/l^2, \quad \cos\varphi = 1/\sqrt{1+\tan^2\varphi}$.

The lengths of each straight segment are presented in Table 2.4. Table 2.5 presents the geometrical parameters at specified sections of the arch, and computation of the conventional length $\Delta s'$ for each segment.

Table 2.6 Bending moments at specified sections and computation of deflection

		M_P, factor $(-P)$			$\overline{M}$ factor (-1)			$\frac{\Delta s'}{6}(ab+4ef+cd)$
Portion	$\Delta s'$	a	e	c	b	f	d	
0–1	5.2904	0.0	0.65625	1.125	0.0	0.65625	1.125	2.6348P
1–2	4.1743	1.125	1.40625	1.50	1.125	1.40625	1.50	7.9491P
2–3	3.4366	1.50	1.40625	1.125	1.50	1.40625	1.125	6.5443P
3–4	3.0693	1.125	0.65625	0.0	1.125	0.65625	0.0	1.5286P

The moment of inertia $I_m = 0.5(I_l + I_r)$. For example, for segment 0–1 we get

$$I_{0-1} = 0.5\,(0.707 + 0.800)\,I_C = 0.7535 I_C.$$

Table 2.6 contains the bending moments at specified sections for loaded and unit states. These moments are calculated by the formula $M_k = M_k^0 - Hy_k$, where $H = M_C^0/f = Pl/4f = 1 \times P$. For each segment, the section at the left end has ordinates a and b, at the middle section the ordinates are e and f and at the right end ordinates are c and d.

For example, in P-condition $R_A = P/2$ and $H = P$, so for points 1 and 2 - (portion 1–2) we get

$$M_1 = \frac{P}{2} \times 3 - P \times 2.625 = -1.125P, \quad M_2 = \frac{P}{2} \times 6 - P \times 4.5 = -1.5P.$$

Data for the right half-arch is not presented due to the symmetry of structure.

Required displacement of point C is equals to twice the sum of the members of the last column. In our case,

$$\Delta_{iP} = 2(2.6348 + 7.9491 + 6.5443 + 1.5286)\frac{P}{EI_C} = 37.3136\frac{P}{EI_C}. \qquad (2.26c)$$

The above-discussed procedure is very effective for computation of displacement of any nonsymmetrical three-hinged arches. If it is necessary to take into account the shear and axial forces, the corresponding terms of Maxwell–Mohr integral (1.8) should be included and Table 2.6 to be expanded [Rab54a].

2.9 Displacement Due to Settlement of Supports and Errors of Fabrication

Settlement of supports and errors of fabrication often occur in engineering practice. If this happens in a statically determinate structure, the internal stresses in the members of the structures are not induced. So computation of displacement of any point of statically determinate structures reflects the kinematical nature of a problem.

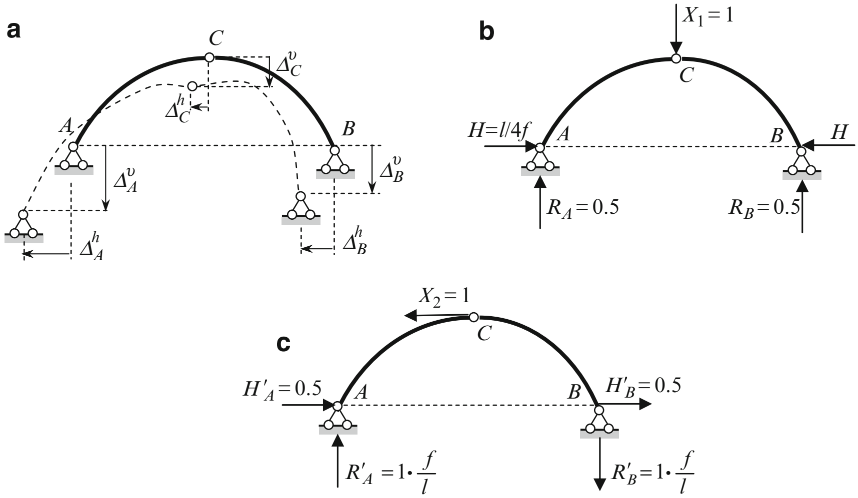

Fig. 2.39 (**a**) Settlement of supports A and B; (**b**) unit state for calculation of Δ_C^{υ}; and (**c**) unit state for calculation of Δ_C^{h}

2.9.1 Settlements of Supports

Let us consider a three-hinged arch of span l and rise f; supports A and B settles in vertical and horizontal directions as shown in Fig. 2.39a. The new position of the arch, in an exaggeration scale, is shown by a dotted line. It is necessary to calculate the vertical Δ_C^{υ} and horizontal Δ_C^{h} displacements of the hinge C. Unit state presents the same structure subjected to unit force X, which corresponds to the required displacement.

An effective method for solution of this type of problem is the principle of virtual displacements

$$\sum \delta W_{\text{act}} = 0. \tag{2.29}$$

According to this principle, *the elementary work done by all active forces on any virtual displacements, which are compatible with constraints, is zero*.

Procedure for Computation of Displacement Caused by the Settlement of Support

1. At point K where displacement should be determined, we need to apply a unit generalized force $X = 1$, corresponding to the required displacement.
2. Show reactions R at the settled support, caused by unit generalized force $X = 1$, and compute these reactions.

3. Calculate the total work (2.29) done by unit force and all reactions on the displacements of the supports.
4. Solve this equation with respect to required displacement.

Vertical displacement of the hinge C. Let us apply $X_1 = 1$ in vertical direction; this force corresponds to the required vertical displacement Δ_C^{υ}. Reactions at the supports A and B are shown in Fig. 2.39b. These reactive forces should be considered as active, and (2.29) becomes

$$X_1 \times \Delta_C^{\upsilon} - R_A \times \Delta_A^{\upsilon} - R_B \times \Delta_B^{\upsilon} - H \times \Delta_A^{\mathrm{h}} + H \times \Delta_B^{\mathrm{h}} = 0.$$

Since $X = 1$, then

$$\Delta_C^{\upsilon} = -\sum R \times \Delta = R_A \times \Delta_A^{\upsilon} + R_B \times \Delta_B^{\upsilon} + H \times \Delta_A^{\mathrm{h}} - H \times \Delta_B^{\mathrm{h}}. \qquad (2.29\text{a})$$

Formula (2.29a) may be generalized for the case of displacements caused by settlements of *several* supports

$$\Delta_{ks} = -\sum R\Delta, \qquad (2.30)$$

where Δ_{ks} is the displacement in kth direction due to settlement of supports, Δ is the given settlement of support; R are the reactions in the support which is settled; this reaction caused by unit load which corresponds to the required displacement. Summation covers all supports.

Horizontal displacement of the hinge C. Horizontal force $X_2 = 1$ corresponds to the required horizontal displacement Δ_C^{h}. Reactions at the supports A and B are shown in Fig. 2.39b. Equation (2.30) leads to the following result:

$$\Delta_C^{\mathrm{h}} = -\sum R \times \Delta = R'_A \times \Delta_A^{\upsilon} - R'_B \times \Delta_B^{\upsilon} + H'_A \times \Delta_A^{\mathrm{h}} + H'_B \times \Delta_B^{\mathrm{h}}.$$

Discussion

1. Equation (2.30) reflects a *kinematical* nature of problem; it means that displacements of any point of a statically determinate structure are determined by the geometrical parameters of a structure without taking into account the deformations of its elements. Any settlement of support of such structure does not depend on the stiffness of the structure, and therefore leads to displacement of its separate parts as rigid discs.
2. The positive results for required Δ_i means that unit load X on the displacement Δ_i performs positive work.
3. Assume that $\Delta_A^{\upsilon} \neq 0$, while all other displacements are zero. Thus, in case of vertical displacement of only one of the support, the crown hinge C has the vertical and horizontal displacements.

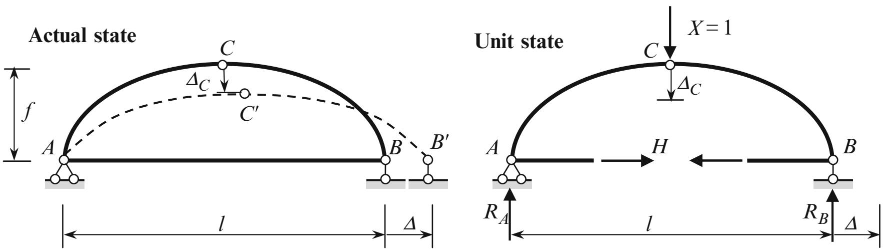

Fig. 2.40 Design diagram of the arch (error fabrication) and unit state

2.9.2 Errors of Fabrication

Deflections of the structural members may occur as a result of the geometric misfit. This topic is sometimes referred to as geometric incompatibility.

The following procedure may be applied for this type of problems:

1. At point K, where displacement should be determined, we need to apply a unit generalized force $X = 1$ corresponding to the required displacement.
2. Compute all reactions caused by the unit generalized force $X = 1$.
3. Calculate the work done by these reactions on the displacements.

Example 2.8. The tie AB of the arch ACB in Fig. 2.40 is $\Delta = 0.02$ m longer then required length l. Find the vertical displacement at point C, if $l = 48$ m, $f = 6$ m.

Solution. The actual position of the tie is AB' instead of required AB position. For computation of the vertical displacement Δ_C we have to apply a unit vertical force at C. Reactions of the three-hinged arch and thrust in tie caused by the force $P = 1$ equal

$$R_A = R_B = 0.5, H = M_C^0/f = l/(4f) = 2.$$

Application of principle of virtual displacements leads to the following expression

$$X \times \Delta_C - H \times \Delta = 0.$$

Since $X = 1$, then the required displacement becomes

$$\Delta_C = +H \times \Delta = +0.04\,\text{m (downward)}$$

It is obvious that the effect of geometric incompatibility may be useful for the regulation of stresses in the structure. Let us consider a three-hinged arch which is loaded by any fixed load. The bending moments are $M(x) = M^0 - Hy$, where M^0 is the bending moment in the reference beam. If a tie is fabricated longer than is required, then the thrust becomes $H = H_1 + H_2$ where H_1 and H_2 are thrust due to fixed load and errors of fabrication, respectively.

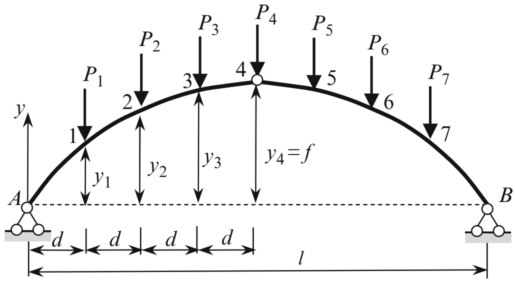

Fig. 2.41 Design diagram of three-hinged parabolic arch

Discussion

For computation of displacement due to the settlement of supports and errors of fabrication, we use the principle of virtual work. This principle and the Maxwell–Mohr integral method have the general concept of generalized coordinate and corresponding generalized unit force in common.

2.10 Matrix Form Analysis of Arches Subjected to Fixed and Moving Load

This paragraph presents the matrix analysis of three-hinged arch subjected to fixed and moving load.

Design diagram of three-hinged parabolic arch subjected to fixed load is shown in Fig. 2.41. The span of the arch is divided into n equal portions, so $d = l/n$; in the case of Fig. 2.41, $n = 8$.

Let the span and rise of the arch be $l = 16$ m and $f = 4$ m, respectively.

If the equation of the arch obeys formula (3), then

$$y_1 = y_7 = 1.75\,\text{m}, \quad y_2 = y_6 = 3.0\,\text{m}, \quad y_3 = y_5 = 3.75\,\text{m}, \quad y_4 = f = 4.0\,\text{m}.$$

Vector of bending moments at the nodal points 1–7 is

$$\vec{\mathbf{M}} = \mathbf{L}_m^* \mathbf{L}_m \vec{\mathbf{P}}, \tag{2.31}$$

where the influence matrix of bending moments is

$$\mathbf{L}_m^* = \begin{bmatrix} 1 & 0 & 0 & m_1^* & 0 & 0 & 0 \\ 0 & 1 & 0 & m_2^* & 0 & 0 & 0 \\ 0 & 0 & 1 & m_3^* & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_5^* & 1 & 0 & 0 \\ 0 & 0 & 0 & m_6^* & 0 & 1 & 0 \\ 0 & 0 & 0 & m_7^* & 0 & 0 & 1 \end{bmatrix}$$

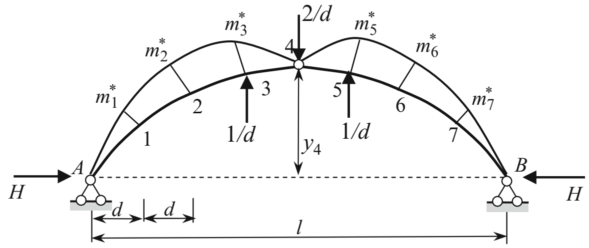

Fig. 2.42 Bending moment diagram due to self-balanced load $1/d$–$2/d$–$1/d$

To find the entries m_i^* we need to construct a bending moment diagram for three-hinged arch subjected to self-balanced load, which acts as shown in Fig. 2.42. The trust is $H = 1/f$, so the bending moments at the nodal points are

$$m_1^* = m_7^* = -\frac{y_1}{y_4} = -\frac{1.75}{4}; \quad m_2^* = m_6^* = -\frac{y_2}{y_4} = -\frac{3}{4};$$
$$m_3^* = m_5^* = -\frac{y_3}{y_4} = -\frac{3.75}{4}.$$

Therefore, matrix L_m^* becomes

$$\mathbf{L}_m^* = \begin{bmatrix} 1 & 0 & 0 & -0.4375 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.75 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.9375 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.9375 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.75 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0.4375 & 0 & 0 & 1 \end{bmatrix}.$$

Influence matrix of bending moments for the arch coincides with influence matrix for bending moments for simply supported beam of the same span

$$\mathbf{L}_m = \frac{d}{n}\mathbf{I}_{(n-1)},$$

where $\mathrm{I}_{(n-1)}$ is a matrix of order $n-1$ and has the following special form

$$\mathbf{I}_{(n-1)} = \begin{bmatrix} n-1 & n-2 & n-3 & \dots & 1 \\ n-2 & \dots & \dots & \dots & 2 \\ \dots & \dots & \dots & \dots & \dots \\ 2 & 4 & 6 & \dots & n-2 \\ 1 & 2 & 3 & \dots & n-1 \end{bmatrix}.$$

If $n = 8$ then L_m becomes

$$\mathbf{L}_m = \frac{d}{8}\begin{bmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 12 & 10 & 8 & 6 & 4 & 2 \\ 5 & 10 & 15 & 12 & 9 & 6 & 3 \\ 4 & 8 & 12 & 16 & 12 & 8 & 4 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 \\ 2 & 4 & 6 & 8 & 10 & 12 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix}.$$

This matrix is symmetric with respect to both diagonals. The entries of the last row and last column, as well as the entries of the first column (from bottom to top) and first row (from right to left) present the natural numbers 1,2,…, $(n-1)$. Any entry m_{ki}, which is located on the main diagonal or above, is determined as a product of the k-th entry of the very first row and the number i of the row .

Vector of bending moments is the result of multiplication of the following matrices

$$\vec{\mathbf{M}} = \begin{bmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \\ M_5 \\ M_6 \\ M_7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -0.4375 & 0 & 0 & 0 \\ 0 & 1 & 0 & -0.75 & 0 & 0 & 0 \\ 0 & 0 & 1 & -0.9375 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.9375 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.75 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0.4375 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \frac{2}{8}\begin{bmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 6 & 12 & 10 & 8 & 6 & 4 & 2 \\ 5 & 10 & 15 & 12 & 9 & 6 & 3 \\ 4 & 8 & 12 & 16 & 12 & 8 & 4 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 \\ 2 & 4 & 6 & 8 & 10 & 12 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix} \times \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \end{bmatrix}.$$

If we assume that the vector of external loads is $\vec{P} = \lfloor 1 \ \ 4 \ \ 2 \ \ 0 \ \ 0 \ \ 2.5 \ \ 0 \rfloor^T$ [Kle80], then the vector of bending moments at the nodal points 1–7 becomes

$$\vec{\mathbf{M}} = \lfloor 2.75 \quad 6 \quad 3.75 \quad 0 \quad -1.25 \quad 0 \quad -1.25 \rfloor^{\mathrm{T}}.$$

This matrix approach may be effectively used for the construction of influence lines for bending moments. If force $P = 1$ is placed only at joint 1, then the vector of external load becomes

$$\overrightarrow{P} = \lfloor 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \rfloor^{\mathrm{T}}$$

and procedure (2.31) gives us the bending moments at the nodal points 1–7.

In order to calculate all ordinates of influence lines for bending moments at sections 1–7, the vector of loads $\overrightarrow{P}$ should be replaced by an identity matrix $\mathbf{P}$; if $n = 8$, then this matrix is of order 7.

The final result of multiplication of the three squared matrices is

$$\mathbf{M} = \begin{bmatrix} \mathrm{IL}(M_1) \\ \mathrm{IL}(M_2) \\ \mathrm{IL}(M_3) \\ \mathrm{IL}(M_4) \\ \mathrm{IL}(M_5) \\ \mathrm{IL}(M_6) \\ \mathrm{IL}(M_7) \end{bmatrix}$$

$$= \begin{bmatrix} 1.3125 & 0.6250 & -0.0625 & -0.7500 & -0.5625 & -0.3750 & -0.1875 \\ 0.7500 & 1.5000 & 0.2500 & -1.0000 & -0.7500 & -0.5000 & -0.2500 \\ 0.3125 & 0.6250 & 0.9375 & -0.7500 & -0.5625 & -0.3750 & -0.1875 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.1875 & -0.3750 & -0.5625 & -0.7500 & 0.9375 & 0.6250 & 0.3125 \\ -0.2500 & -0.5000 & -0.7500 & -1.0000 & 0.2500 & 1.5000 & 0.7500 \\ -1.1875 & -0.3750 & -0.5625 & -0.7500 & -0.0625 & 0.6250 & 1.3125 \end{bmatrix}.$$

The ith row of this matrix represents the influence line of bending moment at the ith nodal point.

It is easy to verify that each influence line for bending moment consist of the strength portions; this means that the structure under consideration is indeed statically determined.