Module
2
Analysis of Statically Indeterminate Structures by the Matrix Force Method

Lesson
9
The Force Method of Analysis: Beams (Continued)

Instructional Objectives

After reading this chapter the student will be able to

1. Calculate additional stresses developed in statically indeterminate structures due to support settlements.
2. Analyse continuous beams which are supported on yielding supports.
3. Sketch the deflected shape of the member.
4. Draw banding moment and shear force diagrams for indeterminate beams undergoing support settlements.

9.1 Introduction

In the last lesson, the force method of analysis of statically indeterminate beams subjected to external loads was discussed. It is however, assumed in the analysis that the supports are unyielding and the temperature remains constant. In the design of indeterminate structure, it is required to make necessary provision for future unequal vertical settlement of supports or probable rotation of supports. It may be observed here that, in case of determinate structures no stresses are developed due to settlement of supports. The whole structure displaces as a rigid body (see Fig. 9.1). Hence, construction of determinate structures is easier than indeterminate structures.

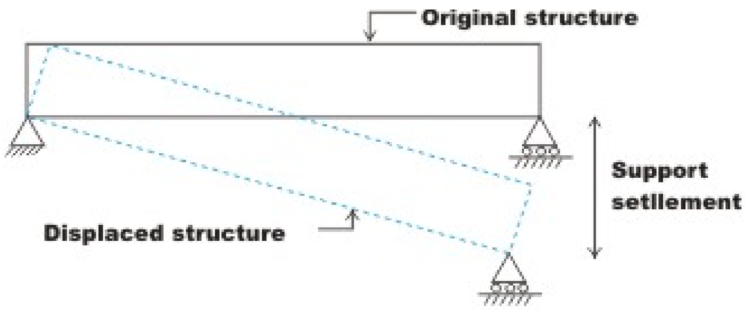


Fig. 9.1 Effect of support settlement on determinate structure

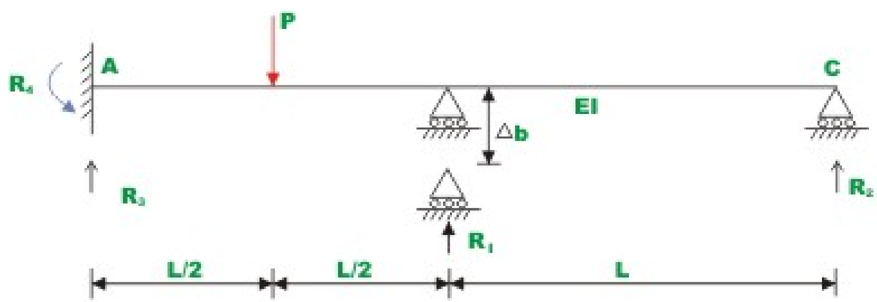


Fig. 9.2 Continuous beam with yielding of support

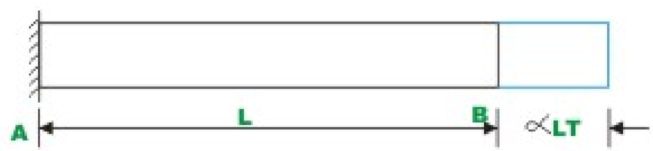


Fig. 9.3 Effect of temperature change in beam

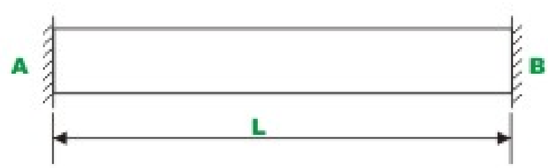
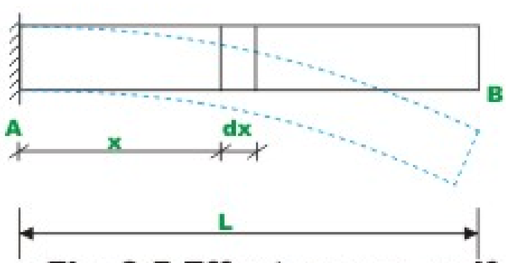


Fig. 9.4 Fixed - fixed beam



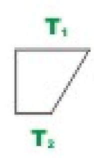


Fig. 9.5 (b)

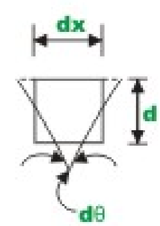


Fig. 9.5 (c)

Fig. 9.5 Effect on non- uniform temperature change across the depth of a beam

The statically determinate structure changes their shape due to support settlement and this would in turn induce reactions and stresses in the system. Since, there is no external force system acting on the structures, these forces form a balanced force system by themselves and the structure would be in equilibrium. The effect of temperature changes, support settlement can also be easily included in the force method of analysis. In this lesson few problems, concerning the effect of support settlement are solved to illustrate the procedure.

9.2 Support Displacements

Consider a two span continuous beam, which is statically indeterminate to second degree, as shown in Fig. 9.2. Assume the flexural rigidity of this beam to be constant throughout. In this example, the support B is assumed to have settled by an amount Δ_b as shown in the figure.

This problem was solved in the last lesson, when there was no support settlement (vide section 8.2). In section 8.2, choosing reaction at B and C as the redundant, the total deflection of the primary structure due to applied external loading and redundant R_1 and R_2 is written as,

$$\Delta_1 = (\Delta_L)_1 + a_{11}R_1 + a_{12}R_2 \quad (9.1a)$$

$$\Delta_2 = (\Delta_L)_2 + a_{21}R_1 + a_{22}R_2 \quad (9.1b)$$

wherein, R_1 and R_2 are the redundants at B and C respectively, and $(\Delta_L)_1$, and $(\Delta_L)_2$ are the deflections of the primary structure at B and C due to applied loading. In the present case, the support B settles by an amount Δ_b in the direction of the redundant R_1. This support movement can be readily incorporated in the force method of analysis. From the physics of the problem the total deflection at the support may be equal to the given amount of support movement. Hence, the compatibility condition may be written as,

$$\Delta_1 = -\Delta_b \quad (9.2a)$$

$$\Delta_2 = 0 \quad (9.2b)$$

It must be noted that, the support settlement Δ_b must be negative as it is displaces downwards. It is assumed that deflections and reactions are positive in the upward direction. The equation (9.1a) and (9.1b) may be written in compact form as,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} - \begin{Bmatrix} (\Delta_L)_1 \\ (\Delta_L)_2 \end{Bmatrix} \tag{9.3a}$$

$$[A]\{R\} = \{\Delta\} - \{(\Delta_L)\} \tag{9.3b}$$

Solving the above algebraic equations, one could evaluate redundants R_1 and R_2 due to external loading and support settlement.

9.3 Temperature Stresses

Internal stresses are also developed in the statically indeterminate structure if the free movement of the joint is prevented.

For example, consider a cantilever beam AB as shown in Fig. 9.3. Now, if the temperature of the member is increased uniformly throughout its length, then the length of the member is increased by an amount

$$\Delta_T = \alpha\ L\ T \tag{9.4}$$

In which, Δ_T is the change in the length of the member due to temperature change, α is the coefficient of thermal expansion of the material and T is the change in temperature. The elongation (the change in the length of the member) and increase in temperature are taken as positive. However if the end B is restrained to move as shown in Fig 9.4, then the beam deformation is prevented. This would develop an internal axial force and reactions in the indeterminate structure.

Next consider a cantilever beam AB, subjected to a different temperature, T_1 at the top and T_2 at the bottom as shown in Fig. 9.5(a) and (b). If the top temperature T_1 is higher than the bottom beam surface temperature T_2, then the beam will deform as shown by dotted lines. Consider a small element dx at a distance x from A. The deformation of this small element is shown in Fig. 9.5c. Due to rise in temperature $T_1 °C$ on the top surface, the top surface elongates by

$$\Delta_{T_1} = \alpha\ \ T_1\, dx \tag{9.5a}$$

Similarly due to rise in temperature T_2, the bottom fibers elongate by

$$\Delta_{T_2} = \alpha\ \ T_2\, dx \tag{9.5b}$$

As the cross section of the member remains plane, the relative angle of rotation $d\theta$ between two cross sections at a distance dx is given by

$$d\theta = \frac{\alpha(T_1 - T_2)dx}{d} \tag{9.6}$$

where, d is the depth of beam. If the end B is fixed as in Fig. 9.4, then the differential change in temperature would develop support bending moment and reactions.

The effect of temperature can also be included in the force method of analysis quite easily. This is done as follows. Calculate the deflection corresponding to redundant actions separately due to applied loading, due to rise in temperature (either uniform or differential change in temperature) and redundant forces. The deflection in the primary structure due to temperature changes is denoted by $(\Delta_T)_i$ which denotes the deflection corresponding to i^{th} redundant due to temperature change in the determinate structure. Now the compatibility equation for statically indeterminate structure of order two can be written as

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \begin{Bmatrix} \Delta_1 \\ \Delta_2 \end{Bmatrix} - \begin{Bmatrix} (\Delta_L)_1 \\ (\Delta_L)_2 \end{Bmatrix} - \begin{Bmatrix} (\Delta_T)_1 \\ (\Delta_T)_2 \end{Bmatrix}$$

$$[A]\{R\} = \{\Delta\} - \{(\Delta_L)\} - \{(\Delta_T)\} \tag{9.7}$$

wherein, $\{\Delta_L\}$ is the vector of displacements in the primary structure corresponding to redundant reactions due to external loads; $\{\Delta_T\}$ is the displacements in the primary structure corresponding to redundant reactions and due to temperature changes and $\{\Delta\}$ is the matrix of support displacements corresponding to redundant actions. Equation (9.7) can be solved to obtain the unknown redundants.

Example 9.1

Calculate the support reactions in the continuous beam ABC (see Fig. 9.6a) having constant flexural rigidity EI throughout, due to vertical settlement of the support B by 5 mm as shown in the figure. $E = 200 \text{ GPa}$ and $I = 4 \times 10^{-4} \text{m}^4$.

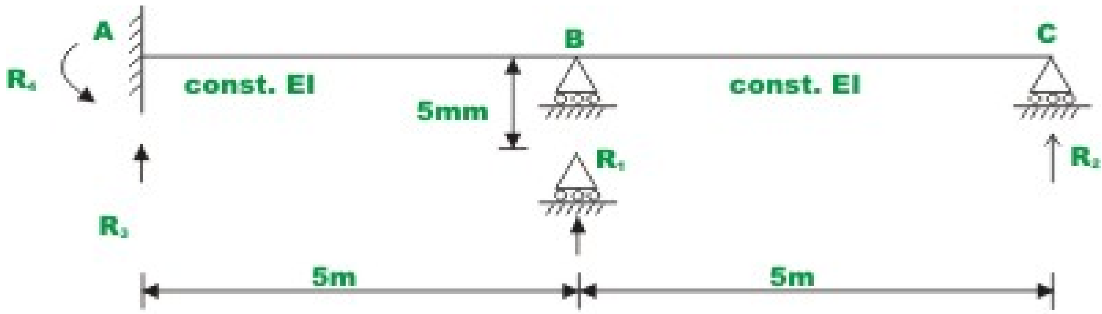


Fig. 9.6 (a) Continuous beam

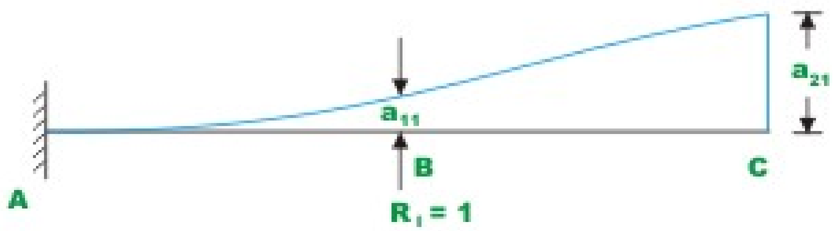


Fig. 9.6 (b) primary structure with unit along R_1

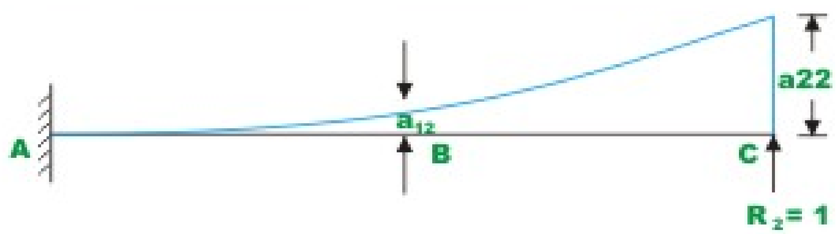


Fig. 9.6 (c) primary structure with unit load along R_2

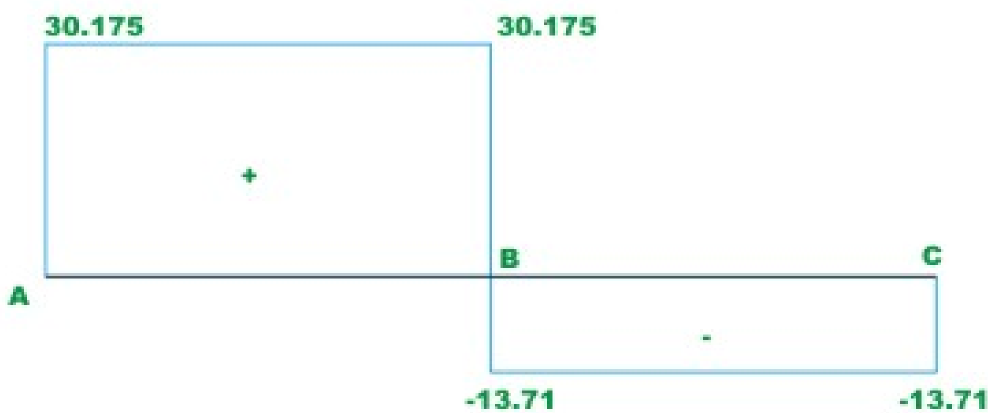


Fig. 9.6(d) Shear force diagram

As the given beam is statically indeterminate to second degree, choose reaction at $B\ (R_1)$ and $C\ (R_2)$ as the redundants. In this case the cantilever beam AC is the basic determinate beam (primary structure). On the determinate beam only redundant reactions are acting. The first column of flexibility matrix is evaluated by first applying unit load along the redundant R_1 and determining deflections a_{11} and a_{21} respectively as shown in Fig. 9.6b.

$$a_{11} = \frac{5^3}{3EI} = \frac{125}{3EI}$$

$$a_{21} = \frac{125}{3EI} + \frac{25}{2EI} \times 5 = \frac{625}{6EI} \qquad (1)$$

Simply by applying the unit load in the direction of redundant R_2, one could evaluate flexibility coefficients a_{12} and a_{22} (see Fig. 9.6c).

$$a_{12} = \frac{625}{6EI} \quad \text{and} \quad a_{22} = \frac{1000}{3EI} \qquad (2)$$

The compatibility condition for the problem may be written as,

$$\begin{aligned} a_{11}R_1 + a_{12}R_2 &= -5 \times 10^{-3} \\ a_{21}R_1 + a_{22}R_2 &= 0 \end{aligned} \qquad (3)$$

The redundant reactions are,

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} -5 \times 10^{-3} \\ 0 \end{Bmatrix} \qquad (4)$$

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{3EI}{27343.75} \begin{bmatrix} 1000 & -312.5 \\ -312.5 & 125 \end{bmatrix} \times \begin{Bmatrix} -5 \times 10^{-3} \\ 0 \end{Bmatrix} \qquad (5)$$

Substituting the values of E and I in the above equation, the redundant reactions are evaluated.

$$R_1 = -43.885 \text{ kN} \quad \text{and} \quad R_2 = 13.71 \text{ kN}$$

R_1 acts downwards and R_2 acts in the positive direction of the reaction *i.e.* upwards. The remaining two reactions R_3 and R_4 are evaluated by the equations of equilibrium.

$$\sum F_y = 0 \Rightarrow R_1 + R_2 + R_3 = 0$$

Hence $R_3 = 30.175 \text{ kN}$

$$\sum M_A = 0 \Rightarrow \quad R_4 + 5 \times R_1 + 10 \times R_2 = 0$$

Solving for R_4,

$R_4 = 82.325 \text{ kN.m} \quad \text{(counter clockwise)}$

The shear force and bending moment diagrams are shown in Figs. 9.6d and 9.6e respectively.

Example 9.2

Compute reactions and draw bending moment diagram for the continuous beam $ABCD$ loaded as shown in Fig. 9.7a, due to following support movements.

Support B, $0.005 m$ vertically downwards.

Support C, $0.01 m$ vertically downwards.

Assume, $E = 200 GPa$; $I = 1.35 \times 10^{-3} m^4$.

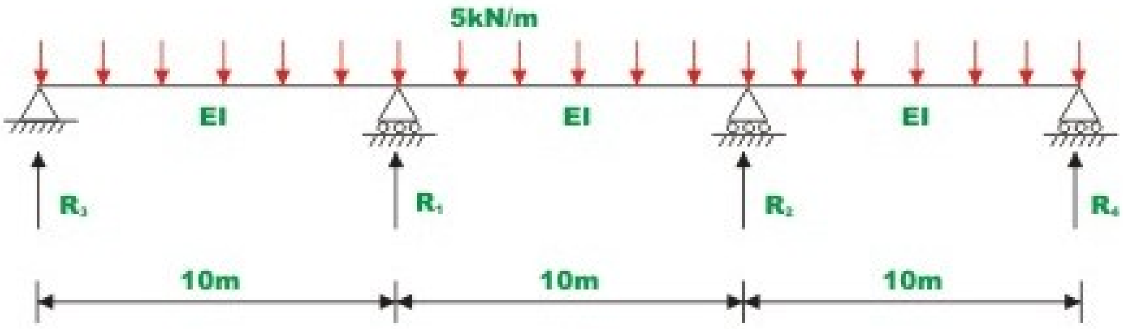


Fig. 9.7(a) Continuous beam

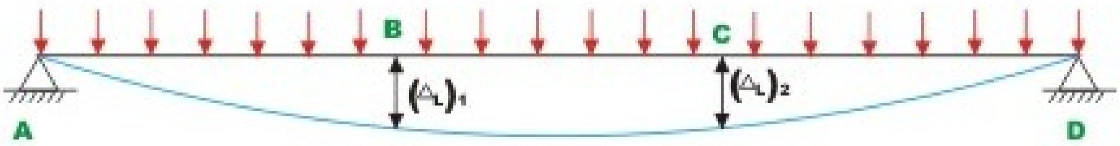


Fig. 9.7(b) Primary structure with external load

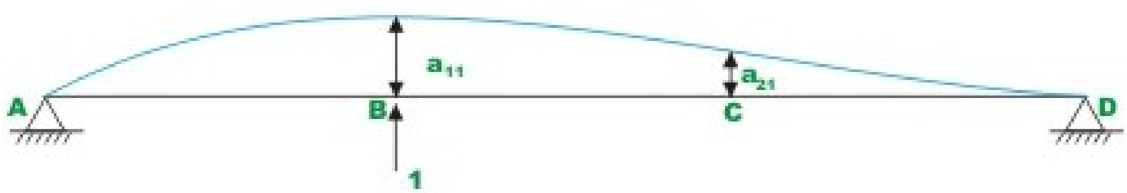


Fig. 9.7 (c) Primary structure with unit load along R

Fig. 9. 7 (d) Primary structure with unit load along R_2

The given beam is statically indeterminate to second degree. Select vertical reactions at $B(R_1)$ and $C(R_2)$ as redundants. The primary structure in this case is a simply supported beam AD as shown in Fig. 9.7b.

The deflection $(\Delta_L)_1$ and $(\Delta_L)_2$ of the released structure are evaluated from unit load method. Thus,

$$\left(\Delta_L\right)_1 = \frac{-45833.33\times10^3}{EI} = \frac{-45833.33\times10^3}{200\times10^9\times1.35\times10^{-3}} = -0.169\,m$$

$$\left(\Delta_L\right)_2 = \frac{-45833.33\times10^3}{EI} = -0.169\,m \qquad (1)$$

The flexibility matrix is evaluated as explained in the previous example, *i.e.* by first applying unit load corresponding to the redundant R_1 and determining deflections a_{11} and a_{21} respectively as shown in Fig. 9.7c. Thus,

$$a_{11} = \frac{444.44}{EI}$$

$$a_{21} = \frac{388.89}{EI} \qquad (2)$$

$$a_{22} = \frac{444.44}{EI}$$

$$a_{12} = \frac{388.89}{EI}$$

In this case the compatibility equations may be written as,

$$\begin{aligned} -0.169 + a_{11}R_1 + a_{12}R_2 &= -0.005 \\ -0.169 + a_{21}R_1 + a_{22}R_2 &= -0.01 \end{aligned} \qquad (3)$$

Solving for redundant reactions,

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{EI}{46291.48}\begin{bmatrix} 444.44 & -388.89 \\ -388.89 & 444.44 \end{bmatrix} \times \begin{Bmatrix} 0.164 \\ 0.159 \end{Bmatrix} \qquad (4)$$

Substituting the value of E and I in the above equation,

$R_1 = 64.48kN$ and $R_2 = 40.174kN$

Both R_1 and R_2 acts in the upward direction. The remaining two reactions R_3 and R_4 are evaluated by the equations of static equilibrium.

$$\sum M_A = 0 \qquad 10 \times R_1 + 20 \times R_2 + 30 \times R_4 - 5 \times 30 \times 15 = 0$$

Hence $\qquad R_4 = 26.724 \text{ kN}$

$$\sum F_y = 0 \qquad R_3 + R_1 + R_2 + R_4 - 5 \times 30 = 0$$

Hence $\qquad R_3 = 18.622 \text{ kN}$ (5)

The shear force and bending moment diagrams are now constructed and are shown in Figs. 9.7e and 9.7f respectively.

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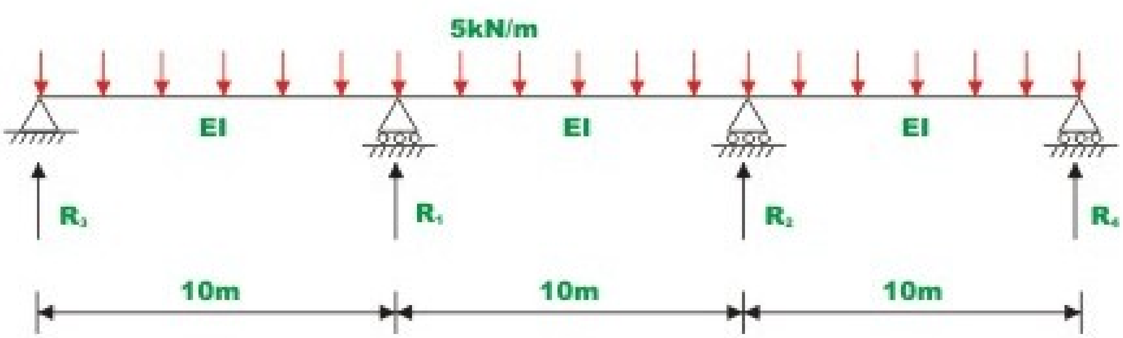
Fig. 9.6 (e) Bending moment diagram



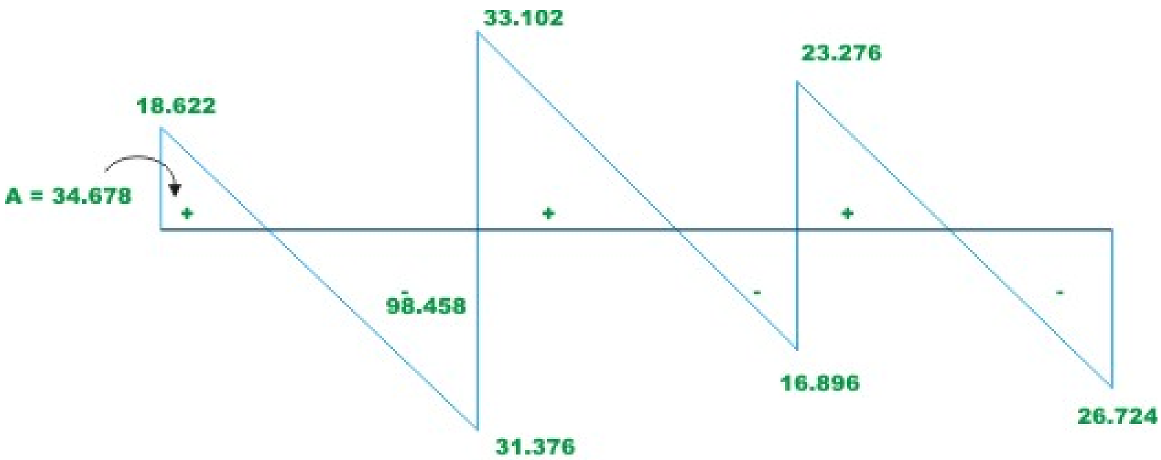


Fig. 9.7 (e) Shear force diagram

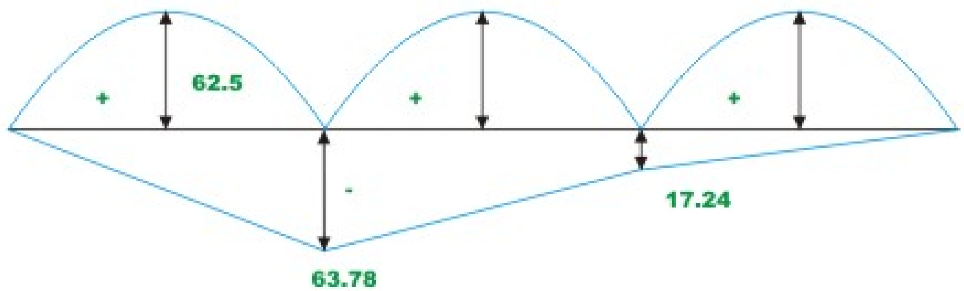


Fig. 9.7 (f) Bending moment diagram

Summary

In this lesson, the effect of support settlements on the reactions and stresses in the case of indeterminate structures is discussed. The procedure to calculate additional stresses caused due to yielding of supports is explained with the help of an example. A formula is derived for calculating stresses due to temperature changes in the case of statically indeterminate beams.