Module 2

Selection of Materials and Shapes

Lecture
5
Selection of Shapes

Instructional objectives

By the end of this lecture, the student will learn

(a) what is *shape factor* and how it can be used to enhance the mechanical efficiency of a material, and

(b) how to develop shape factors considering appropriate load and different cross section.

Selection of Shapes

So far we have learned how the combination of material properties can be used to develop a material index for the selection of a suitable material for a given application under different loading conditions. Similarly, the cross-sectional shape of a part can be used to enhance the load bearing capacity. An engineering material confirms to a modulus and strength, but it can be made stiffer and stronger when loaded under bending or twisting by shaping it into an I-beam or a hollow tube, respectively. It can be made less stiff by flattening it into a leaf or winding it, in the form of a wire, or into a helix. 'Shaped' sections (i.e. cross-section formed to a tube, a box-section, an I-section or the like) carry bending, torsional, and axial-compressive loads more 'efficiently' (i.e. for a given loading conditions, the section uses as little material as possible) than solid sections. The efficiency can be enhanced by introducing sandwich panels of the same or different materials. But when choosing shapes one has to be careful so the basic functional requirement is not violated.

Shape Factor (φ)

Shape Factor is a dimensionless number that characterizes the efficiency of the shape, regardless of its scale, for a given mode of loading, e.g. bending, torsion, twisting, etc. The four primary shape factors of our consideration are,

- ϕ_B^e Macro shape factor for elastic bending
- ϕ_T^e Macro shape factor for elastic torsion
- ϕ_B^f Macro shape factor for onset of failure in bending
- ϕ_T^f Macro shape factor for onset of plasticity or failure in torsion

All the shape factors are defined equal to 1 for a solid cylinder i.e. our reference cross sectional shape is circular. Shape Factor of all other cross section will be evaluated w.r.t this one.

Example 9: Selection of shape factor in *elastic bending of beam*

The bending stiffness *S* of a beam is proportional to the product *EI* and can be given as

$$S \propto EI \tag{1}$$

where *E* is Young's modulus and *I* is the second moment of area of the beam about the axis of bending (the x axis), which can be written as

$$I = \int y^2 dA \tag{2}$$

where *y* is measured normal to the bending axis and *dA* is the differential element of area at *y*. The values of the moment *I* and of the area *A* for the common sections are listed in the first two columns of Table 2.5.1. The second moment of area, I_0, for a reference beam of circular section with radius *r* is simply

$$I_0 = \frac{\pi r^4}{4} = \frac{A^2}{4\pi} \tag{3}$$

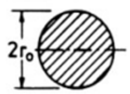
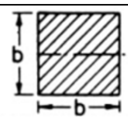
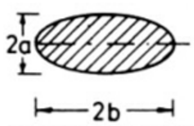
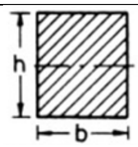
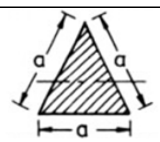
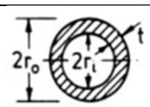
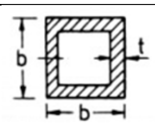
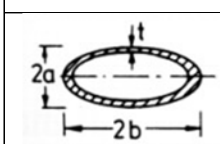
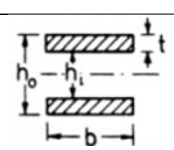
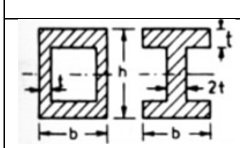
The bending stiffness of any shaped section differs from that of a circular one with the same area *A* by the factor ϕ_B^e where

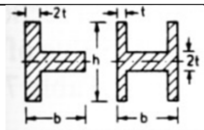
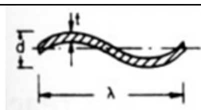
$$\phi_B^e = \frac{S}{S_0} = \frac{EI}{EI_0} = 4\pi \frac{I}{A^2} \tag{4}$$

Note that the factor ϕ_B^e is dimensionless and depends only on the shape. For example, the big and small beams have the same value of ϕ_B^e if their section shapes are the same. Figure 2.5.1 shows three different shapes with their corresponding values of shape factor to be considered for elastic bending of beams. It can be noted that the values of the shape factor do not change with the size of the shape.

Figure 2.5.1 Schematic pictures of a set of (a) rectangular sections with ϕ_B^e = 2; (b) I-sections with ϕ_B^e = 10; and (c) tubes section with ϕ_B^e = 12.

Table 2.5.1 Area (A), Second moment of Area (I), Torsional moment of Area (K), Section modulus in bending (Z), and in torsion (Q) for common engineering shapes

Section Shape	***A*** **(m^2)**	***I*** **(m^4)**	***K*** **(m^4)**	***Z*** ***(m3)***	***Q*** ***(m3)***
	πr^2	$\frac{\pi}{4} r^4$	$\frac{\pi}{2} r^4$	$\frac{\pi}{4} r^3$	$\frac{\pi}{2} r^3$
	b^2	$\frac{b^4}{12}$	$0.14 b^4$	$\frac{b^3}{6}$	$0.21 b^3$
	πab	$\frac{\pi}{4} a^3 b$	$\frac{\pi a^3 b^3}{(a^2+b^2)}$	$\frac{\pi}{4} a^2 b$	$\frac{\pi a^2 b}{2}$ $(a<b)$
	bh	$\frac{bh^3}{12}$	$\frac{b^3 h}{3}\left(1-0.58\frac{b}{h}\right)$ $(h>b)$	$\frac{bh^2}{6}$	$\frac{b^2 h^2}{3h+1.8b}$ $(h>b)$
	$\frac{\sqrt{3}}{4} a^2$	$\frac{a^4}{32\sqrt{3}}$	$\frac{a^4\sqrt{3}}{80}$	$\frac{a^3}{32}$	$\frac{a^3}{20}$
	$\pi(r_o^2-r_i^2)$ $\approx 2\pi r t$	$\frac{\pi}{4}(r_o^4-r_i^4)$ $\approx \pi r^3 t$	$\frac{\pi}{2}(r_o^4-r_i^4)$ $\approx 2\pi r^3 t$	$\frac{\pi}{4r_o}(r_o^4-r_i^4)$ $\approx \pi r^2 t$	$\frac{\pi}{2r_o}(r_o^4-r_i^4)$ $\approx 2\pi r^2 t$
	$4bt$	$\frac{2}{3} b^3 t$	$b^3 t\left(1-\frac{t}{b}\right)^4$	$\frac{4}{3} b^2 t$	$2b^2 t\left(1-\frac{t}{b}\right)^2$
	$\pi(a+b)t$	$\frac{\pi}{4} a^3 t\left(1+\frac{3b}{a}\right)$	$\frac{4\pi(ab)^{5/2} t}{a^2+b^2}$	$\frac{\pi a^2 t}{4}\left(1+\frac{3b}{a}\right)$	$2\pi t(a^3 b)^{1/2}$ $(b>a)$
	$b(h_o-h_i)$ $\approx 2bt$	$\frac{b}{12}(h_o^3-h_i^3)$ $\approx bth_o^2/2$	------	$\frac{b}{6h_o}(h_o^3-h_i^3)$ $\approx bth_o$	
	$2t(h+b$	$\frac{t}{6}(h^3+4bt^2)$	$\frac{2}{3} bt^3\left(1+\frac{4h}{b}\right)$	$\frac{h^2 t}{3}\left(1+\frac{3b}{h}\right)$	$\frac{2}{3} bt^2\left(1+\frac{4h}{b}\right)$

	$2t(h+b)$	$\frac{t}{6}(h^3+4bt^2)$	$\frac{2}{3}ht^3\left(1+\frac{4b}{h}\right)$	$\frac{t}{3h}(h^3+4bt^2)$	$\frac{2}{3}ht^2\left(1+\frac{4b}{h}\right)$
	$t\lambda\left(1+\frac{\pi^2 d^2}{4\lambda^2}\right)$	$\frac{t\lambda d^2}{8}$	------	$\frac{t\lambda d}{4}$	

Example 10: Selection of shape factor in *elastic twisting of shafts*

The shapes that can resist bending effectively may not be so good when loaded under torsion. The stiffness of a shaft in torsion i.e. the torque T divided by the angle of twist (θ) is proportional to GK, where G is the shear modulus and K is the torsional moment of area. For a typical circular sections K is identical with the polar moment of area, J, which can be given as

$$J = \int r^2 dA \qquad (5)$$

where dA is the differential element of area at the radial distance r and measured from the centre of the section. For typical non-circular sections, K is less than J and is defined such that the angle of twist is related to the torque T by

$$S_T = \frac{T}{\theta} = \frac{KG}{L} \qquad (6)$$

where L is length of the shaft and G the shear modulus of the material of the shaft. The approximate expressions for K for several common sections are listed in Table 2.5.1. The shape factor for a shaft under elastic twisting (ϕ_T^e) can therefore be given by

$$\phi_T^e = \frac{S_T}{S_{T0}} = \frac{K}{K_0} = \frac{K}{\pi r^4/2} = \frac{K}{A^2/2\pi} = \frac{2\pi K}{A^2} \qquad (7)$$

Example 11: Selection of shape factor for *failure in bending* and *twisting*

The bending stress, σ_b, is the largest at the point y_m on the surface of the beam [as shown in figure 2.5.2] that lies furthest from the neutral axis and can be given as

$$\sigma_b = \frac{My_m}{I} = \frac{M}{Z} \qquad (8)$$

where M is the bending moment and Z is the section modulus. Failure in bending can occur when σ_b exceeds the failure strength (yield strength or the ultimate tensile strength) of the material of the beam. The shape factor in this case is considered through the section modulus, *Z*, and is measured by the ratio, Z/Z_0, where Z_0 is the section modulus of a reference beam of circular section with the same cross-sectional area, *A*. Hence, the shape factor to be considered against failure in bending can be given as

$$\phi_B^f = \frac{Z}{Z_0} = \frac{Z}{\pi r^3/4} = \frac{4\sqrt{\pi}Z}{A^{3/2}} \qquad (9)$$

Similarly, in case of a circular rod subjected to a torque *T*, the maximum shear stress τ_{max} occurs at the maximum radial distance r_{max} from the axis of twisting and can be given as

$$\tau_{max} = \frac{Tr_{max}}{J} = \frac{T}{Q} \qquad (10)$$

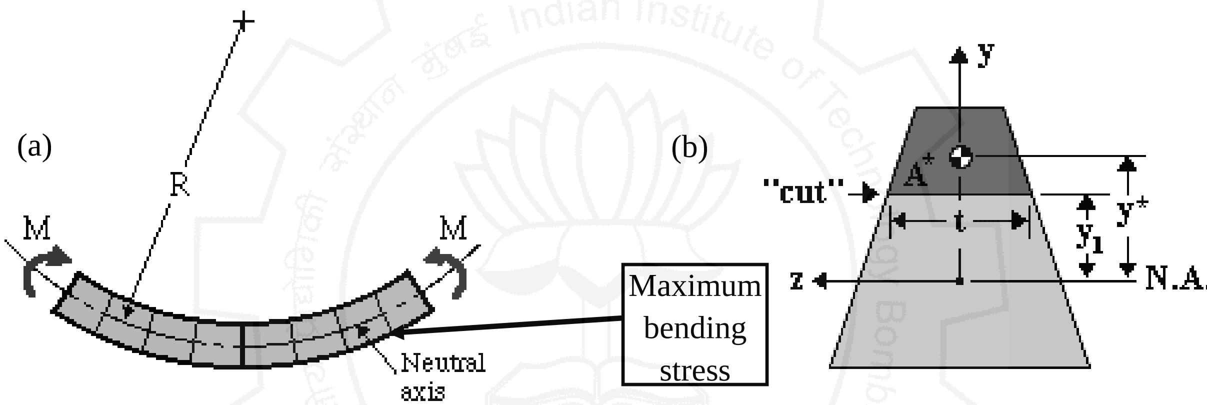


Figure 2.5.2 Schematic picture of (a) bending of beam, and (b) cross-section of beam

where *M* is the bending moment and *Z* is the section modulus. Failure in bending can occur when σ_b exceeds the failure strength (yield strength or the ultimate tensile strength) of the material of the beam. The shape factor in this case is considered through the section modulus, *Z*, and is measured by the ratio, Z/Z_0, where Z_0 is the section modulus of a reference beam of circular section with the same cross-sectional area, *A*. Hence, the shape factor to be considered against failure in bending can be given as

$$\phi_B^f = \frac{Z}{Z_0} = \frac{Z}{\pi r^3/4} = \frac{4\sqrt{\pi}Z}{A^{3/2}} \qquad (11)$$

Similarly, in case of a circular rod subjected to a torque *T*, the maximum shear stress τ_{max} occurs at the maximum radial distance r_{max} from the axis of twisting and can be given as

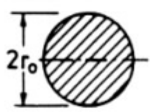
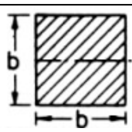
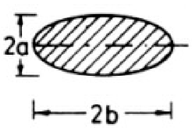
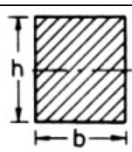
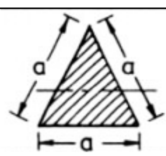
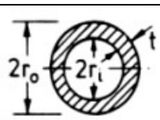
$$\tau_{max} = \frac{Tr_{max}}{J} = \frac{T}{Q} \qquad (12)$$

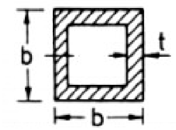
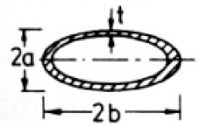
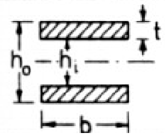
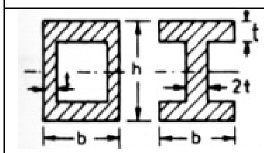
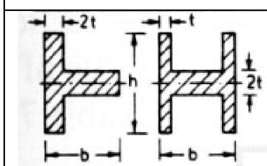
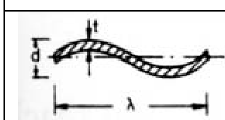
The quantity in J/r_{max} in equation (12) has the same character as I/y in bending. Hence, the shape factor in failure in twisting can be given by the ratio Q/Q_0 as

$$\phi_T^f = \frac{Q}{Q_0} = \frac{2\sqrt{\pi}Q}{A^{3/2}} \qquad (13)$$

The expression for the shape factors for different shapes is given in the Table 2.5.2.

Table 2.5.2 Shape factors of common engineering shapes in elastic bending and twisting, and in failure in bending and twisting

Section Shape	ϕ_B^e	ϕ_T^e	ϕ_B^f	ϕ_T^f
	1	1	1	1
	$\frac{\pi}{3} = 1.05$	0.88	$\frac{2\sqrt{\pi}}{3} = 1.18$	0.74
	$\frac{a}{b}$	$\frac{2ab}{a^2+b^2}$	$\sqrt{\frac{a}{b}}$	$\sqrt{\frac{a}{b}} \quad (a<b)$
	$\frac{\pi h}{3b}$	$\frac{2\pi b}{3h}\left(1-0.58\frac{h}{b}\right)$ $(h>b)$	$\frac{2\sqrt{\pi}}{3}\left(\frac{h}{b}\right)^{1/2}$	$\frac{2\sqrt{\pi}(b/h)^{1/2}}{3(1+0.6b/h)^2}$ $(h>b)$
	$\frac{2\pi}{3\sqrt{3}} = 1.21$	$\frac{2\pi}{5\sqrt{3}} = 0.73$	0.77	0.62
	$\frac{r}{t}$	$\frac{r}{t}$	$\left(\frac{2r}{t}\right)^{1/2}$	$\left(\frac{2r}{t}\right)^{1/2}$

	$\dfrac{\pi b}{6t}$	$\dfrac{\pi b}{8t}\left(1-\dfrac{t}{b}\right)^4$	$\dfrac{2\sqrt{\pi}}{3}\left(\dfrac{b}{t}\right)^{1/2}$	$\dfrac{\sqrt{\pi}}{2}\left(\dfrac{b}{t}\right)^{1/2}\left(1-\dfrac{t}{b}\right)^2$
	$\dfrac{a(1+3b/a)}{t(1+b/a)^2}$	$\dfrac{8(ab)^{5/2}}{t(a^2+b^2)(a+b)^2}$	$\left(\dfrac{a}{t}\right)^{1/2}\dfrac{(1+3b/a)}{(1+b/a)^{3/2}}$	$\dfrac{4a^{1/2}}{t^{1/2}(1+a/b)^{3/2}}$
	$\dfrac{\pi h^2}{2bt}$	-----	$\dfrac{\sqrt{2\pi}h}{(bt)^{1/2}}$	-----
	$\dfrac{\pi h(1+3b/h)}{6t(1+b/h)^2}$	$\dfrac{\pi b^2h^2}{t(h+b)^3}$	$\dfrac{\sqrt{2\pi}}{3}\left(\dfrac{h}{t}\right)^{1/2}\dfrac{(1+3b/h)}{(1+b/h)^{3/2}}$	$\dfrac{\sqrt{2\pi}h}{(bt)^{1/2}(1+h/b)^{3/2}}$
	$\dfrac{\pi h(1+4bt^2/h^3)}{6t(1+b/h)^2}$	$\dfrac{\pi t(1+8b/h)}{6h(1+b/h)^2}$	$\dfrac{\sqrt{\pi}}{2}\left(\dfrac{h}{t}\right)^{1/2}\dfrac{(1+4bt^2/h^3)}{(1+b/h)^{3/2}}$	$\left(\dfrac{\pi t}{18h}\right)^{1/2}\dfrac{(1+8b/h)}{(1+b/h)^{3/2}}$
	$\dfrac{\pi d^2}{2t\lambda}$	-----	$\dfrac{\sqrt{\pi}d}{(t\lambda)^{1/2}}$	-----

Limits to shape factor

From the above discussion it can be concluded that to make stiff and strong structures, efficient shape factors have to be made which is often limited by a number of factors as follows.

[1] The range of shape factor for a given material is limited either by manufacturing constraints or by local buckling.

[2] Steel, for example, can be drawn to thin walled tubing or formed (by rolling, folding or welding) into other efficient shapes; shape factors as high as 30 are common and they may reach 65.

[3] Wood cannot be shaped so easily and shapes with values greater than 3 are rare. However, *bamboo* is a gift of nature and is already shaped in tubular fashion which possesses a high value of shape factor. But it is very difficult to give it any other shape. Composites, too, can be limited by the present difficulty in making thin-walled shapes.

The maximum useful shape factor for simple shapes is related to the ratio E/σ_f of a given material. Table 2.5.3 outlines the maximum possible shape factors in common engineering materials based on various manufacturing technologies available today.

Table 2.5.3 Maximum values of shape factors in common engineering materials

Material	$(\phi_B^e)_{max}$	$(\phi_T^e)_{max}$	$(\phi_B^f)_{max}$	$(\phi_T^f)_{max}$
Structural steel	65	25	13	7
6061 aluminium alloy	44	31	10	8
GFRP and CFRP	39	26	9	7
Polymers (e.g. nylons)	12	8	5	4
Woods (solid sections)	5	1	3	1
Elastomers	<6	3	-	-

Exercise

Choose the correct answer

1. What will be the expression for shape factor, ϕ_B^e, when the reference cross section is a square of area A?

   (a) $\frac{12I}{A^2}$ (b) $\frac{\pi I}{A^2}$ (c) $\frac{4\sqrt{I}}{A^{3/2}}$ (d) $\frac{2\sqrt{\pi I}}{A^2}$

2. What will be the expression for shape factor, ϕ_T^e, when the reference cross section is a square of area A?

   (a) $\frac{2\pi K}{A^2}$ (b) $\frac{\pi\sqrt{K}}{A^{3/2}}$ (c) $\frac{7.14K}{A^2}$ (d) $\frac{4\sqrt{\pi K}}{A^2}$

3. What will be the expression for shape factor, ϕ_B^f, when the reference cross section is a square of area A?

   (a) $\frac{6Z}{A^{3/2}}$ (b) $\frac{\pi\sqrt{Z}}{A^{3/2}}$ (c) $\frac{\pi Z}{A^{3/2}}$ (d) $\frac{4\sqrt{Z}}{A^2}$

4. What will be the expression for shape factor, ϕ_T^f, when the reference cross section is a square of area *A*?

(a) $\frac{\pi\sqrt{Q}}{A^{3/2}}$ (b) $\frac{4.8Q}{A^{3/2}}$ (c) $\frac{4.8Q}{A^{2}}$ (d) $\frac{\pi\sqrt{Q}}{A^{2}}$

Answers: 1(a), 2(c), 3(a), 4(b)

References

1. G Dieter, Engineering Design - a materials and processing approach, McGraw Hill, NY, 2000.
2. M F Ashby, Material Selection in Mechanical Design, Butterworth-Heinemann, 1999.