## **2.5 Plastic analysis**

In plastic analysis and design of a structure, the ultimate load of the structure as a whole is regarded as the design criterion. The term *plastic* has occurred due to the fact that the ultimate load is found from the strength of steel in the plastic range. This method is rapid and provides a rational approach for the analysis of the structure. It also provides striking economy as regards the weight of steel since the sections required by this method are smaller in size than those required by the method of elastic analysis. Plastic analysis and design has its main application in the analysis and design of statically indeterminate framed structures.

## 2.5.1 Basics of plastic analysis

Plastic analysis is based on the idealization of the stress-strain curve as elastic-perfectly-plastic. It is further assumed that the width-thickness ratio of plate elements is small so that local buckling does not occur- in other words the sections will classify as plastic. With these assumptions, it can be said that the section will reach its plastic moment capacity and then undergo considerable rotation at this moment. With these assumptions, we will now look at the behaviour of a beam up to collapse.

Consider a simply supported beam subjected to a point load *W* at midspan. as shown in Fig. 2.14(*a*). The elastic bending moment at the ends is  $w\ell^2/12$ and at mid-span is  $w\ell^2/24$ , where  $\ell$  is the span. The stress distribution across any cross section is linear [Fig. 2.15(*a*)]. As *W* is increased gradually, the bending moment at every section increases and the stresses also increase. At a section close to the support where the bending moment is maximum, the stresses in the extreme fibers reach the yield stress. The moment corresponding to this state is called the *first yield moment*  $M_y$ , of the cross section. But this does not imply failure as the beam can continue to take additional load. As the load continues to increase, more and more fibers reach the yield stress and the stress distribution is as shown in Fig 2.15(*b*). Eventually the whole of the cross section reaches the yield stress and the corresponding to this state is known as the *plastic moment* of the cross section and is denoted by  $M_p$ . In order to find out the fully plastic moment of a yielded section of a beam, we employ the force equilibrium equation, namely the total force in compression and the total force in tension over that section are equal.



(a) at  $M_y$  (b)  $M_y < M < M_p$  (c) at  $M_p$ 



The ratio of the plastic moment to the yield moment is known as the shape factor since it depends on the shape of the cross section. The cross section is not capable of resisting any additional moment but may maintain this moment for some amount of rotation in which case it acts like a *plastic hinge*. If this is so, then for further loading, the beam, acts as if it is simply supported with two additional moments  $M_{p}$  on either side, and continues to carry additional loads until a third plastic hinge forms at mid-span when the bending moment at that section reaches  $M_p$ . The beam is then said to have developed a **collapse mechanism** and will collapse as shown in Fig 2.14(b). If the section is thinwalled, due to local buckling, it may not be able to sustain the moment for additional rotations and may collapse either before or soon after attaining the plastic moment. It may be noted that formation of a single plastic hinge gives a collapse mechanism for a simply supported beam. The ratio of the ultimate rotation to the yield rotation is called the *rotation capacity* of the section. The yield and the plastic moments together with the rotation capacity of the crosssection are used to classify the sections.

#### Shape factor

As described previously there will be two stress blocks, one in tension, the other in compression, both of which will be at yield stress. For equilibrium of the cross section, the areas in compression and tension must be equal. For a rectangular cross section, the elastic moment is given by,

$$M = \frac{bd^2}{6}f_y$$
 (2.21a)

The plastic moment is obtained from,

$$M_{p} = 2.b.\frac{d}{2}.\frac{d}{4}.f_{y} = \frac{bd^{2}}{4}f_{y}$$
 (2.21b)

Thus, for a rectangular section the plastic moment  $M_p$  is about 1.5 times greater than the elastic moment capacity. For an I-section the value of shape factor is about 1.12.

Theoretically, the plastic hinges are assumed to form at points at which plastic rotations occur. Thus the length of a plastic hinge is considered as zero. However, the values of moment, at the adjacent section of the yield zone are more than the yield moment upto a certain length  $\Delta L$ , of the structural member. This length  $\Delta L$ , is known as the hinged length. The hinged length depends upon the type of loading and the geometry of the cross-section of the structural member. The region of hinged length is known as **region of yield** or **plasticity**.

## **Rigid plastic analysis**



Fig. 2.16

In a simply supported beam (Fig. 2.16) with central concentrated load, the maximum bending moment occurs at the centre of the beam. As the load is

increased gradually, this moment reaches the fully plastic moment of the section  $M_p$  and a plastic hinge is formed at the centre.

Let  $x (= \Delta L)$  be the length of plasticity zone.

From the bending moment diagram shown in Fig. 2.16

$$(L-x)M_p=LM_y$$
  
x=L/3 (2.22)

Therefore the hinged length of the plasticity zone is equal to one-third of the span in this case.

$$M_{p} = \frac{Wl}{4}$$
$$= f_{y} \cdot \frac{bh^{2}}{4} \left( \because Z_{p} = \frac{bh^{2}}{4} \right)$$
$$M_{y} = f_{y} \cdot \frac{bh^{2}}{6} = \left( f_{y} \cdot \frac{bh^{2}}{4} \right) \frac{2}{3}$$
$$M_{y} = \frac{2}{3} M_{p}$$

# 2.5.2 Principles of plastic analysis

#### Fundamental conditions for plastic analysis

- (i) Mechanism condition: The ultimate or collapse load is reached when a mechanism is formed. The number of plastic hinges developed should be just sufficient to form a mechanism.
- (ii) Equilibrium condition :  $\Sigma F_x = 0$ ,  $\Sigma F_y = 0$ ,  $\Sigma M_{xy} = 0$
- (iii) **Plastic moment condition:** The bending moment at any section of the structure should not be more than the fully plastic moment of the section.

## **Collapse mechanisms**

When a system of loads is applied to an elastic body, it will deform and will show a resistance against deformation. Such a body is known as a *structure*. On the other hand if no resistance is set up against deformation in the body, then it is known as a *mechanism*.

Various types of independent mechanisms are identified to enable prediction of possible failure modes of a structure.

#### (i) Beam mechanism

Fig. 2.17 shows a simply supported and a fixed beam and the corresponding mechanisms.





## (ii) Panel or Sway mechanism

Fig. 2.18 (A) shows a panel or sway mechanism for a portal frame fixed at both ends.



# (iii) Gable mechanism

Fig. 2.18(B) shows the gable mechanism for a gable structure fixed at both the supports.

# (iv) Joint mechanism

Fig. 2.18(C) shows a joint mechanism. It occurs at a joint where more than two structural members meet.

## **Combined mechanism**

Various combinations of independent mechanisms can be made depending upon whether the frame is made of strong beam and weak column combination or strong column and weak beam combination. The one shown in Fig. 2.19 is a combination of a beam and sway mechanism. Failure is triggered by formation of hinges at the bases of the columns and the weak beam developing two hinges. This is illustrated by the right hinge being shown on the beam, in a position slightly away from the joint.



Fig. 2.19 Combined mechanism

From the above examples, it is seen that the number of hinges needed to form a mechanism equals the statical redundancy of the structure plus one.

#### Plastic load factor and theorems of plastic collapse

The plastic load factor at rigid plastic collapse  $(\lambda_p)$  is defined as the lowest multiple of the design loads which will cause the whole structure, or any part of it to become a mechanism.

In a limit state approach, the designer is seeking to ensure that at the appropriate factored loads the structure will not fail. Thus the rigid plastic load factor  $\lambda_{\rho}$  must not be less than unity.

The number of independent mechanisms (n) is related to the number of possible plastic hinge locations (h) and the number of degree of redundancy (r) of the frame by the equation.

$$n = h - r$$
 (2.23)

The three theorems of plastic collapse are given below.

#### Lower bound or Static theorem

A load factor ( $\lambda_s$ ) computed on the basis of an arbitrarily assumed bending moment diagram which is in equilibrium with the applied loads and where the fully plastic moment of resistance is nowhere exceeded will always be less than or at best equal to the load factor at rigid plastic collapse, ( $\lambda_p$ ). In other words,  $\lambda_p$  is the highest value of  $\lambda_s$  which can be found.

## Upper bound or Kinematic theorem

A load factor  $(\lambda_k)$  computed on the basis of an arbitrarily assumed mechanism will always be greater than, or at best equal to the load factor at rigid

plastic collapse ( $\lambda_p$ ). In other words,  $\lambda_p$  is the lowest value of  $\lambda_k$  which can be found.

#### Uniqueness theorem

If both the above criteria are satisfied, then the resulting load factor corresponds to its value at rigid plastic collapse ( $\lambda_p$ ).

#### Mechanism method

In the mechanism or kinematics method of plastic analysis, various plastic failure mechanisms are evaluated. The plastic collapse loads corresponding to various failure mechanisms are obtained by equating the internal work at the plastic hinges to the external work by loads during the virtual displacement. This requires evaluation of displacements and plastic hinge rotations.

As the plastic deformations at collapse are considerably larger than elastic ones, it is assumed that the frame remains rigid between supports and hinge positions i.e. all plastic rotation occurs at the plastic hinges.

Considering a simply supported beam subjected to a point load at midspan, the maximum strain will take place at the centre of the span where a plastic hinge will be formed at yield of full section. The remainder of the beam will remain straight, thus the entire energy will be absorbed by the rotation of the plastic hinge.

Considering a centrally loaded simply supported beam at the instant of plastic collapse (see Fig. 2.17)

Workdone at the plastic hinge = 
$$M_p 2\theta$$
 (2.24a)

Workdone by the displacement of the load =  $W\left(\frac{L}{2}.\theta\right)$  (2.24b)

At collapse, these two must be equal

$$2Mp.\theta = W\left(\frac{L}{2}.\theta\right)$$

$$M_{p} = \frac{WL}{4}$$
 (2.25)

The moment at collapse of an encastre beam with a uniform load is similarly worked out from Fig. 2.20. It should be noted that three hinges are required to be formed at *A*, *B* and *C* just before collapse.

Workdone at the three plastic hinges =  $M_p (\theta + 2\theta + \theta) = 4M_p \theta$  (2.26.a)

Workdone by the displacement of the load = W/L . L/2 . L/2 .  $\theta$  (2.26.b)



$$M_{p} = \frac{WL}{16}$$
 (2.28)

In other words the load causing plastic collapse of a section of known value of  $M_p$  is given by eqn. (2.28).

#### Rectangular portal framework and interaction diagrams

The same principle is applicable to frames as indicated in Fig. 2.21(a) where a portal frame with constant plastic moment of resistance  $M_p$  throughout is subjected to two independent loads *H* and *V*.



This frame may distort in more than one mode. There are basic independent modes for the portal frame, the pure sway of Fig. 2.21 (b) and a beam collapse as indicated in Fig. 2.21 (c). There is now however the possibility of the modes combining as shown in Fig. 2.21(d).

From Fig. 2.21(b)



Work done in hinges =  $4 M_p \theta$ Work done by loads =  $Ha\theta$ At incipient collapse  $Ha / M_p = 4$  (2.29)

From Fig. 2.21 (c) Work done in hinges =  $4 M_p \theta$ Work done by loads =  $Va\theta$ At incipient collapse =  $Va / M_p = 4$  (2.30)

From Fig. 2.21(d)

Work done in hinges =  $6 M_p \theta$ Work done by loads =  $Ha\theta + Va\theta$ At incipient collapse  $Ha / M_p + Va / M_p = 6$  (2.31)

The resulting equations, which represent the collapse criteria, are plotted on the interaction diagram of Fig. 2.22. Since any line radiating from the origin represents proportional loading, the first mechanism line intersected represents failure. The failure condition is therefore the line *ABCD* and any load condition within the area *OABCD* is therefore safe.

#### Stability

For plastically designed frames three stability criteria have to be considered for ensuring the safety of the frame. These are

- 1. General Frame Stability.
- 2. Local Buckling Criterion.
- 3. Restraints.

## Effect of axial load and shear

If a member is subjected to the combined action of bending moment and axial force, the plastic moment capacity will be reduced.

The presence of an axial load implies that the sum of the tension and compression forces in the section is not zero (Fig. 2.23). This means that the neutral axis moves away from the equal area axis providing an additional area in tension or compression depending on the type of axial load.

the interaction equation can be obtained:

$$M_x/M_p = 1 - P^2/P_y$$
 (2.32)

The presence of shear forces will also reduce the moment capacity.



## Plastic analysis for more than one condition of loading

When more than one condition of loading can be applied to a beam or structure, it may not always be obvious which is critical. It is necessary then to perform separate calculations, one for each loading condition, the section being determined by the solution requiring the largest plastic moment.

Unlike the elastic method of design in which moments produced by different loading systems can be added together, plastic moments obtained by different loading systems cannot be combined, i.e. the plastic moment calculated for a given set of loads is only valid for that loading condition. This is because the 'Principle of Superposition' becomes invalid when parts of the structure have yielded.

