<u>CHAPTER SEVEN</u>

PLASTIC COLLAPSE and LIMIT ANALYSIS

7.1 Introduction and bars with axial loading

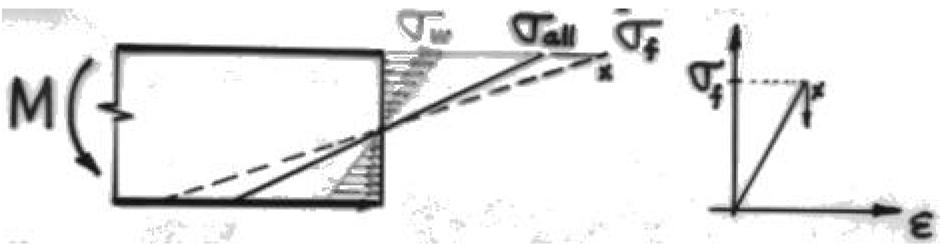
7.2 Plastic action in bending and plastic hinges

7.3 Collapse loads in beams

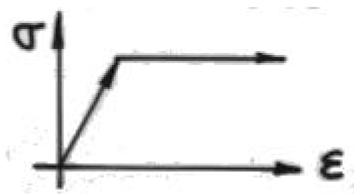
Review and Summary

- Structural design under static load

(1) Elastic analysis ---- Allowable stress concept and safety factors($\boldsymbol{s}_r / \boldsymbol{s}_{all}$)



Limit analysis ---- For ductile materials and asks for the load that places a structure on the verge of plastic collapse.



- Advantages of limit analysis

  (1) Leads to a design that uses less material to carry a given load

  (2) Simple in calculation and More realistic

- Goal of this chapter ---- calcualation of the plastic collapse loads

- Basic assumption about the material:

  (1) yield rather than fracture

  (2) The material has a yield point Y and is elastic perfectly plastic

- Single bar under axial loading

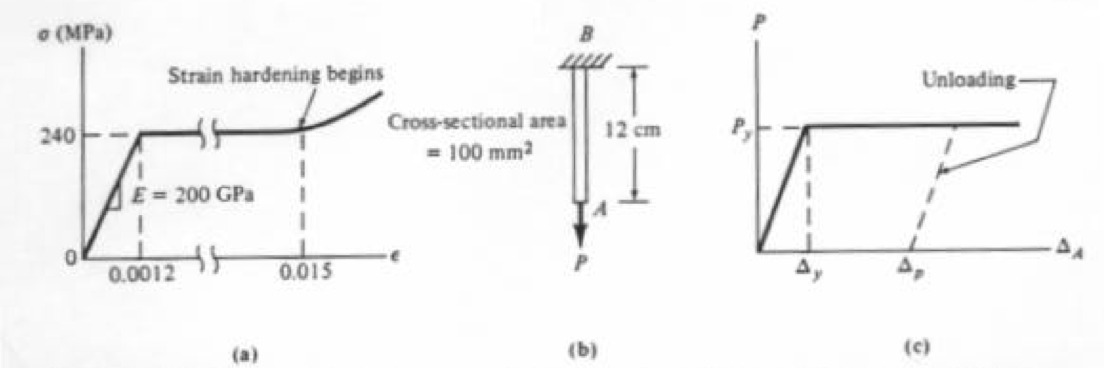


FIGURE 11.2.1. (a) Typical stress–strain behavior of a ductile steel. The yield point is $Y = 240$ MPa. (b) Prismatic bar AB under axial load P. (c) Load–deflection behavior of bar AB.

(1) The yield load $P_y = YA = 240 \times 10^2 \times 10^{-6} = 24kN$

(2) The elongation at yield $\Delta_y = P_y L / AE = 144 \times 10^{-6} m$

(3) The plastic yielding does not change the collapse load P_y (always equals to the initial yielding load)

- Load-carrying capacity

(1) Statically determinate structure: no change

(2) Statically indeterninate structure: The collapse load is larger than the load that initiates yielding

- Three-bar truss (statically indeterminate) : **an example**

(1) The collapse load P_p

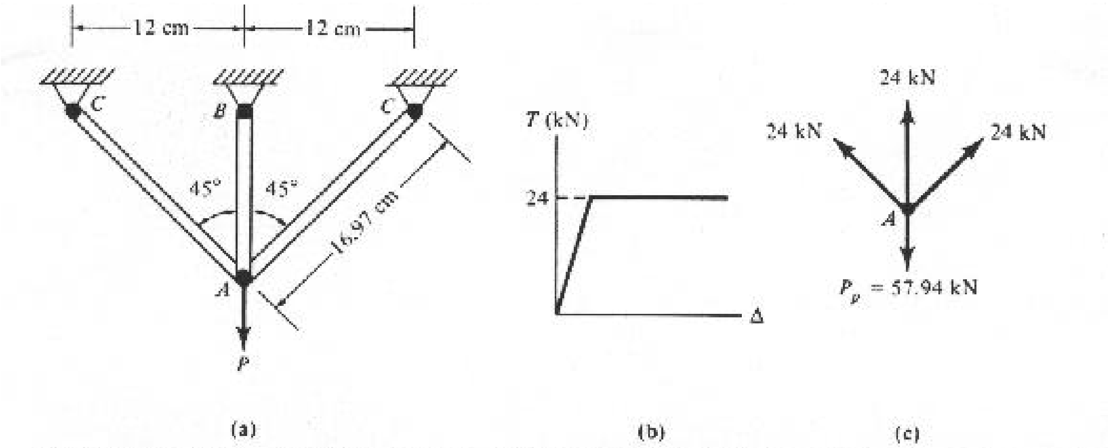


FIGURE 11.2.2. (a) Three-bar truss. (b) Tension versus deflection relation for each separate bar. (c) Forces that act on point A at the collapse load P_p.

$$P_p = 24 + (24\sqrt{2}) = 57.94kN$$

(2) Elastic analysis (more tedious)

(a) bar AB begin to yield (statically indeterminate)

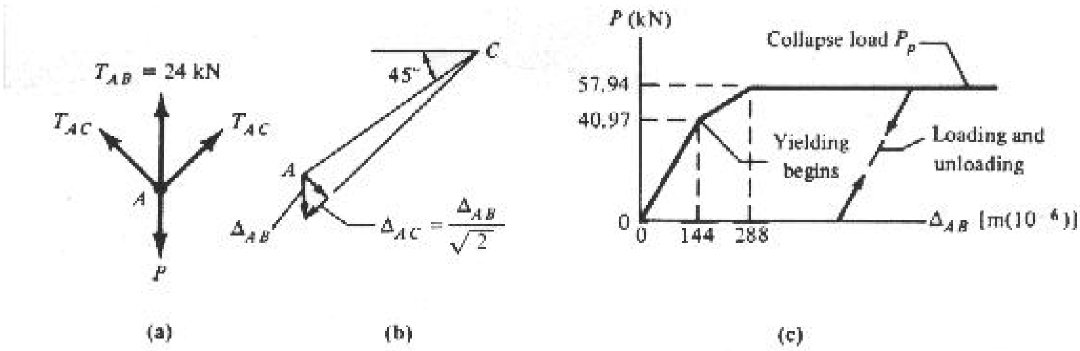


FIGURE 11.2.3. (a) Forces that act at point A in the truss of Fig. 11.2.2a when the most highly stressed bar has yielded. (b) Relations among the (small) deflections at point A. (c) Load–deflection response of the three-bar truss.

Equilibrium: $P = 24 + 2\dfrac{T_{AC}}{\sqrt{2}}$

Deformation : $\Delta_{AC} = \dfrac{\Delta_{AB}}{\sqrt{2}} \rightarrow \dfrac{T_{AC}L_{AC}}{AE} = \dfrac{1}{\sqrt{2}}\dfrac{24L_{AB}}{AE}$

With E=200Gpa, we have

T_{AC}=12kN , P=40.97 kN , $\Delta_{AB} = 144 \times 10^{-6} m$

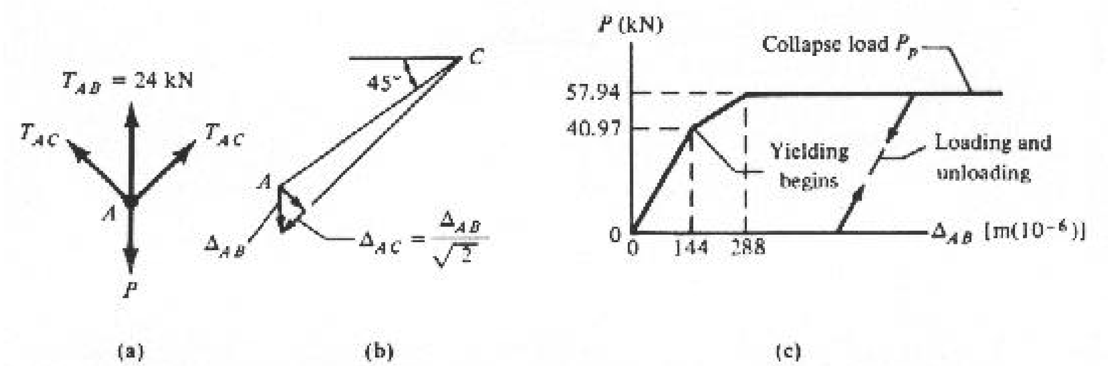


FIGURE 11.2.3. (a) Forces that act at point A in the truss of Fig. 11.2.2a when the most highly stressed bar has yielded. (b) Relations among the (small) deflections at point A. (c) Load–deflection response of the three-bar truss.

* The collpase load P_p is 41% greater than the load that initiate yielding

(b) The deflection Δ_{AB} at the collpase load

At the instant when collapse load P_p is reached, Bars AC are at the end of their elastic response, i.e.

$$\Delta_{AB} = \sqrt{2}\Delta_{AC} = \sqrt{2}\frac{24L_{AC}}{AE} = 288\times10^{-6}\,m$$

This deflection is small

If safety factor is 2.0, the $\begin{cases} P_W = 40.97/2 = 20.49kN(\substack{elastic \\ analysis}) \\ P_W = 57.94/2 = 28.97kN(\substack{\lim itc \\ analysis}) \end{cases}$

(c) Unloading the structure after collapse load P_p has been reached

After unloading, we can find the residual forces in bars of the unloading structure by superposition two cases:

(1) the structure loaded by $+P_p$ (downward)

(2) the structure loaded by -P P_p (upward)

because the bar response elastically, the case (2) is:

$$T_{AB}^{(2)} = 24\left(\frac{-57.94}{40.97}\right) = 33.94kN$$

$$T_{AC}^{(2)} = 12\left(\frac{-57.94}{40.97}\right) = -16.97kN$$

The total internal forces(residual forces) are

$$T_{AB} = T_{AB}^{(1)} + T_{AB}^{(2)} = 24 - 33.94 = -9.94kN$$

$$T_{AC} = T_{AC}^{(1)} + T_{AC}^{(2)} = 24 - 16.97 = 7.03kN$$

(3) Some discussion

- After the load P_p is released, if we continue loading in the opposite direction (upward), then what value of upward P produces renewed yielding in compression? Assume that bars will yield at 24kN in compression as in tension. In this case the upward load P must be large enough to produce an elastic force of 24-9.94=14.06kN (compressive) in bar AB, we find

$$P = 14.06(\frac{40.97}{24}) = 24.00kN(upward)$$

The range of elastic response is from 57.94kN (downward) to 24kN (upward), a total range of 57.94+24=81.94kN. Before any yielding, the elastic range was 2 × 40.97=81.94kN, the same value. Thus we further have:

- Residual stress have not increased the total elastic range, but have only shifted it. This is true in general: increase the limit of elastic action if further loads act in the same direction but decrease the limit of elastic action if further loads act in the opposite direction

7.2 Plastic action in bending and plastic hinges

- **Stress and strain distribution change as M increases.**

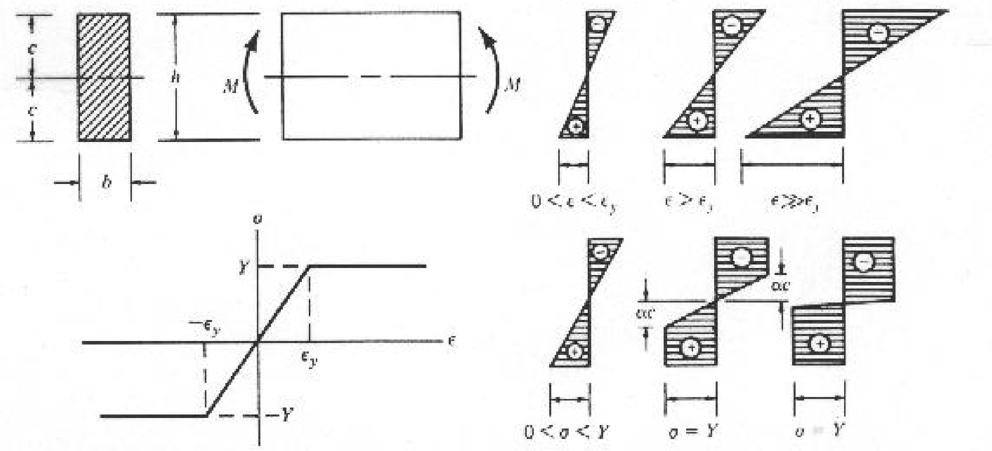


FIGURE 11.3.1. Beam of rectangular cross section b by $h = 2c$, loaded by bending moment M. With a flat-topped stress–strain relation and increasing curvature, the strain distribution remains linear while distance αc between the centroid and the elastic–plastic boundary becomes small. At the onset of yielding, $\alpha = 1$; for greater bending moment, $0 < \alpha < 1$.

(1) $$M = \frac{4\alpha c}{3} F_1 + (1+\alpha) c F_2 = \frac{3-\alpha^2}{3} bc^2 Y$$

(2) when yield just begins, $\alpha = 1 \rightarrow M_y = \frac{2}{3} bc^2 Y$

(3) when all material has yield, $\alpha = 1, \rightarrow M_y = bc^2 Y$

M_p ---- fully plastic moment. Shape factor = M_p/M_y

- Curvature κ of the beam in the elastic plastic state

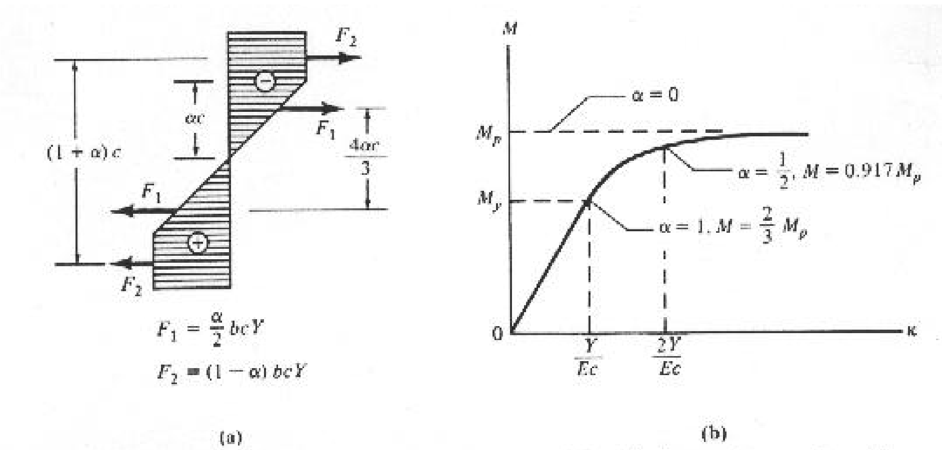


FIGURE 11.3.2. (a) Forces F_1 (elastic part) and F_2 (plastic part) are produced by stress that acts on the cross section of a partially yielded beam of rectangular cross section. (b) Bending moment versus curvature relation for a beam of rectangular cross section.

Using relation $\kappa = M/EI$ to the central elastic core of depth 2αc, we have

$$\kappa = Y/E\alpha c \quad (0 < \alpha < 1)$$

$$M = [c^2 - \frac{1}{3}\left(\frac{Y}{E\kappa}\right)^2] bY \quad \text{(valid for } \kappa \geq Y/Ec\text{)}$$

- The lateral extent of the yield zone

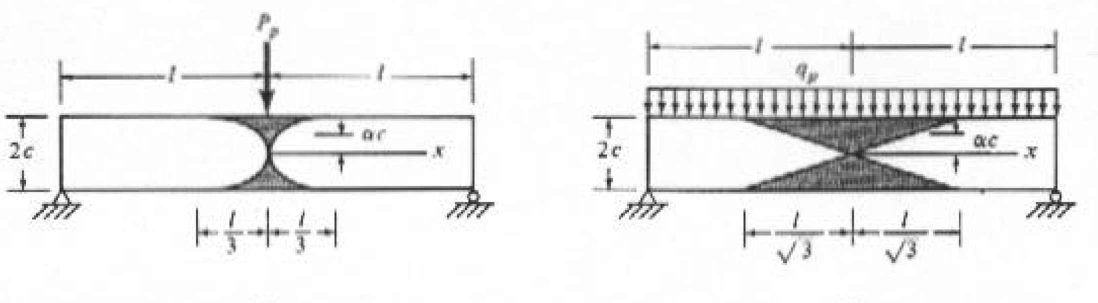


FIGURE 11.3.3. Yielded zones when the fully plastic moment M_p prevails in simply supported beams of rectangular cross section with (a) concentrated center load P_p, and (b) uniformly distributed load q_p.

(a) The bending moment is M_p at x=0 and drops linearly to zero at the ends $x = \pm l$, thus we have

$$M = M_p\left(\frac{l-x}{l}\right) = bc^2Y\left(\frac{l-x}{l}\right) = P_p(l-x)/2$$

by using $M = \frac{3-a^2}{3}bc^2Y$, we find $a = \sqrt{3x/l} \rightarrow x = l/3, a = 1$

(b) $M = bc^2Y(l^2 - Y^2)/l^2, \quad a = \sqrt{3}x/l \rightarrow x = l/\sqrt{3}, a = 1$

- **Plastic hinge**: Although yield material extends an appreciable distance along the beam, the fully plastic zone spans an infinitesimal length dx at x=0. This zone os called a **plastic hinge** Further deformation takes place entirely within the hinges as its material strains at constant Y (without increase of the load)
- **Residual stress after unloading**

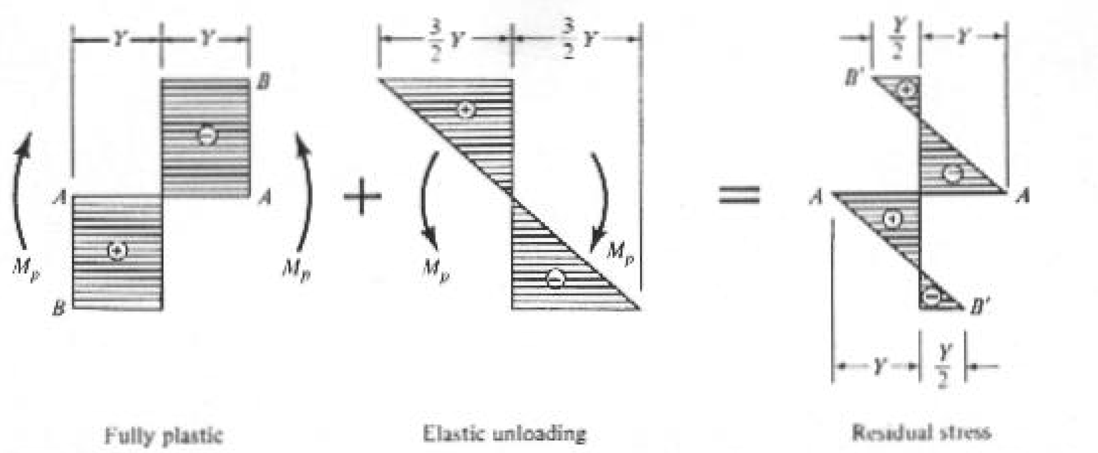
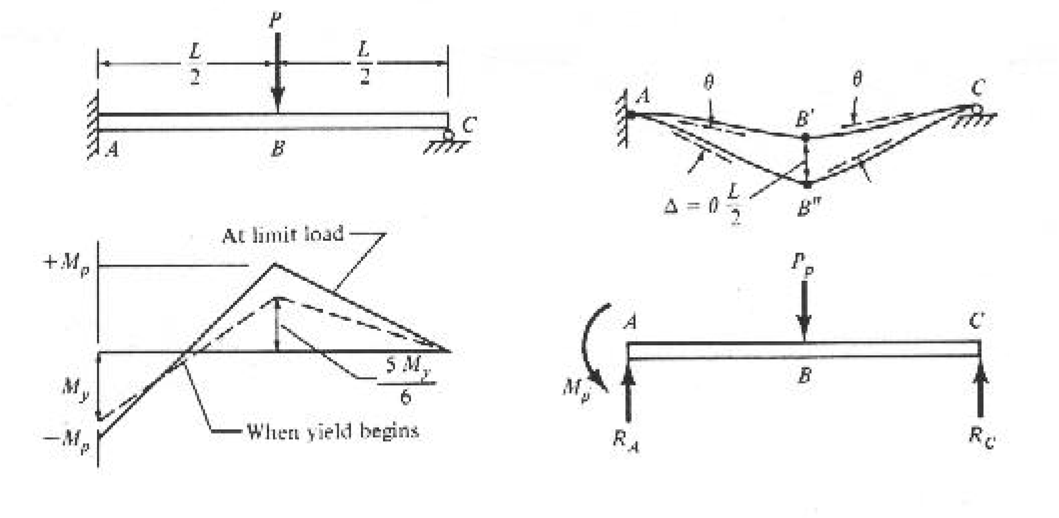


FIGURE 11.3.4. Calculation of the residual stress pattern after unloading a plastic hinge in a rectangular cross section.

7.3 Collapse loads in beams

- **Calculate the collapse load on a beam** by a given loading pattern, support condition and fully plastic moment M_p



(a) (b)

FIGURE 11.4.1. (a) Uniform propped cantilever beam, showing bending moment diagrams at the onset of yielding and at the collapse condition. (b) Deflection pattern after the collapse P_p is reached, and the corresponding support forces on the beam. Deflections in the sketch are greatly exaggerated.

(1) Elastic solution: M_A=-3PL/16, M_B=5PL/32, so yield will begin first at point A with $M_A=M_y$

(2) The corresponding load at initial yield is $P_y = \dfrac{16M_y}{3L}$.

(3) Further increase in load ($P>P_y$),→ plastic action will spread at A and then begin at B, until $M_A=M_B=M_P$, at this time the collapse load P_p is reached.

- At the time the collapse load P_p is reached, an attempt to increase P_p further result in point B moving further downward as P_p remains constant. The structure has become what is called a mechanism or a **kinematic mechanism**: it is a **linkage of bars**. Its configuration can found by strictly geometric considerations. In collapse analysis, only **small motions** are needed to develop plastic hinges.

- The calculation of collapse load P_p by virtual-work argument:

(a) Uniform propped cantilever beam with concentrated load P ---- As comstant load P_p moves through virtual through virtual distance $\Delta = q(L/2)$, the work it does is absorbed by the hinges as they rotate with constant moment M_p through the respective virtual angles θ at hinges A and 2θ at hinges B. This represents a small additional deflection imagined to take place after a collapse mechanism is formed. Loads, hinge moment and elastic distortions all remain constant as the hinges rotate.

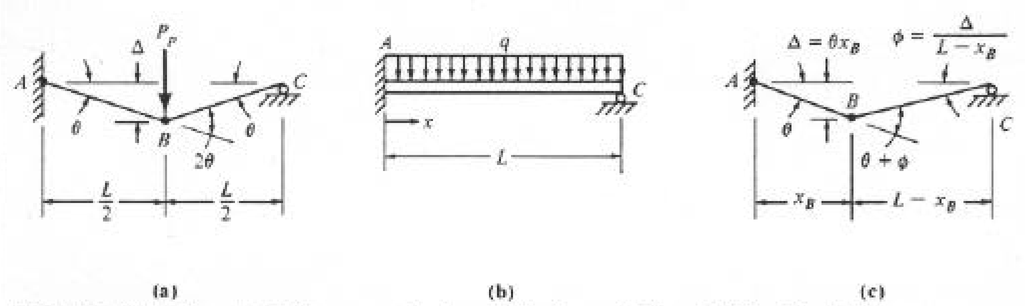


FIGURE 11.4.2. (a) Collapse mechanism of the beam in Fig. 11.4.1a. Plastic hinges are indicated by dots. Deflections are exaggerated. (b) Uniform propped cantilever beam with a uniformly distributed load q. (c) Collapse mechanism of the beam of part (b). Plastic hinges are at points A and B. Deflections are exaggerated.

(1) $p_p(q\frac{L}{2}) = M_p \cdot (q) + M_p \cdot (2q) \rightarrow P_p = \frac{6M_p}{L}$

(2) If end A were simply supported, then it is statically determinate and a single hinges will form at B, we have $P_0 = 4M_p/L$.

(3) If both ends were fixed, hinges will form at A. B. C, and by

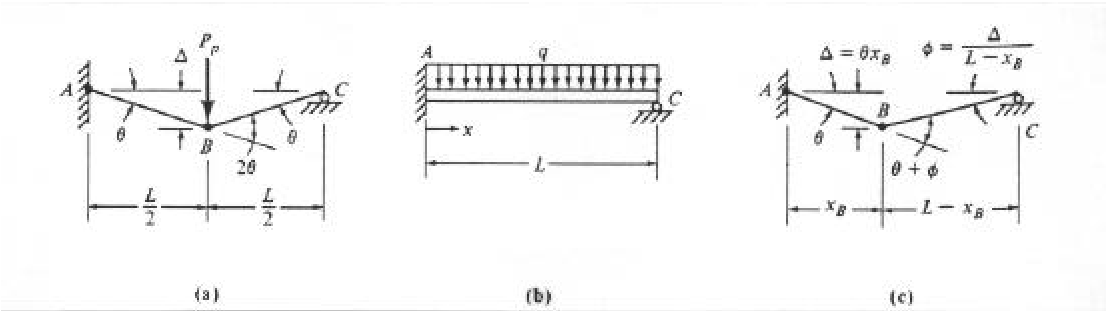
$P_p(q\frac{L}{2}) = 2M_p \cdot (q) + M_p \cdot (2q)$, we have $P_p = \frac{8M_p}{L}$.

- We see that collapse load increase as the degree of static indeterminacy increases.

$P_p / P_y = M_p / M_y$ for statically determinate

$P_p / P_y > M_p / M_y$ for statically indeterminate

(c) Uniform propped cantilever beam with a uniformly distributed load P ----



(1) Two hinges are needed to produce a mechanism, the location of hinge B is not known ----x_B

(2) Loads that act on sections AB and BC are, respectively, $q_p x_B$ and $q_p(L\text{-}x_b)$. They can be regarded as two equivalent concentrated forces. The virtual work equation is

$$q_p x_B (q \frac{x_q}{2}) + q_p (L - x_B)(f \frac{L - x_B}{2}) = M_p \cdot (q) + M_p (q + f)$$

where $f = q x_B / (L - x_B)$, solving for q_p, we have

$$q_p = \frac{2M_p}{L^2} \frac{2 - x_B / L}{\frac{x_B}{L}(1 - x_B / L)}$$

If we assume $x_B = L/2$, then $q_p = \frac{12M_p}{L^2}$ (approximate)

If we assume B is at $x_B = \frac{5}{8}L$ (where elastic theory shows a maximum moment), then

$$q_p = \frac{11.733M_p}{L^2} \quad \text{(approximate)}$$

the correct hinges location is such that q_p is a minimum, thus $dq_p / dx_B = 0 \rightarrow q_p = \frac{11.657M_p}{L^2}$ (exact)

We see that approximate values ar always higher than exact value.

- Upper and Lower Bound Theorems

Upper Bound Theorem. A load computed on the basis of an assumed collapse mechanism is greater than the actual collapse load or at best equal to it. The **lowest upper bound** is the correct collapse load. The corresponding mechanism is the one that actually occurs.

Lower Bound Theorem. A load computed on the basis of an assumed equilibrium moment diagram in which the magnitude of lending moment never exceeds M_p is **less than the actual collapse load P_p** or at best equal to it.

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The lower bound theorem is attractive because ***it gives a conservative result.***
Uniqueness: If the equilibrium moment distribution in the lower bound theorem is associated with a mechanism, then the applied load is the collapse load.

- <u>Example 1</u>

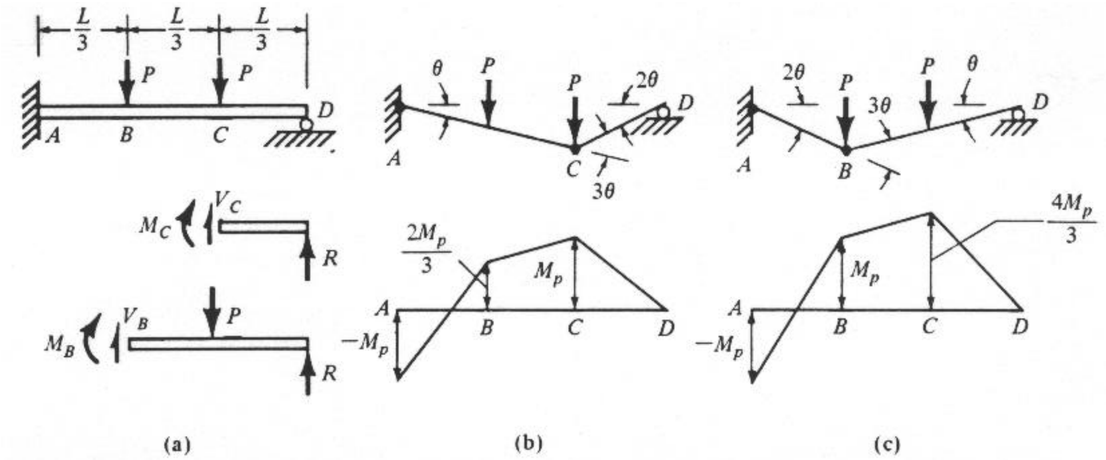

FIGURE 11.6.1. (a) Uniform propped cantilever beam with two equal loads *P*, and two free-body diagrams. (b, c) Assumed collapse mechanisms and the bending moment diagrams associated with each. Deflections sketched are greatly exaggerated.

Find the collapse load P_p for the beam shown whose bending moment capacity is M_p throughout.

(1) For the mechanism shown in (b), the virtual work equation:

$$P(q\frac{L}{3})+P(q\frac{2L}{3})=M_p(q)+M_p(3q)$$

We have $P=4M_p/L$. We see the bending moment is everywhere $\leq M_p$, so the mechanism is correct and $P=P_p=4M_p/L$ (by the uniqueness theorem)

(2) For the mechanism shown in (c), the virtual work equation is $P(2q\frac{L}{3})+P(q\frac{L}{3})=M_p(2q)+M_p(3q)$

This equation yields $P=5M_p/L$, since this load is larger than the load in (b), so from the upper bound theorem it is not correct. Also from the bending moment we know that

$$M_p=\frac{4M_p}{3}>M_p$$

i.e. the mechanism assumed does not in fact occur.

- The principal of superposition does not apply to plastic problems!

- Example 2

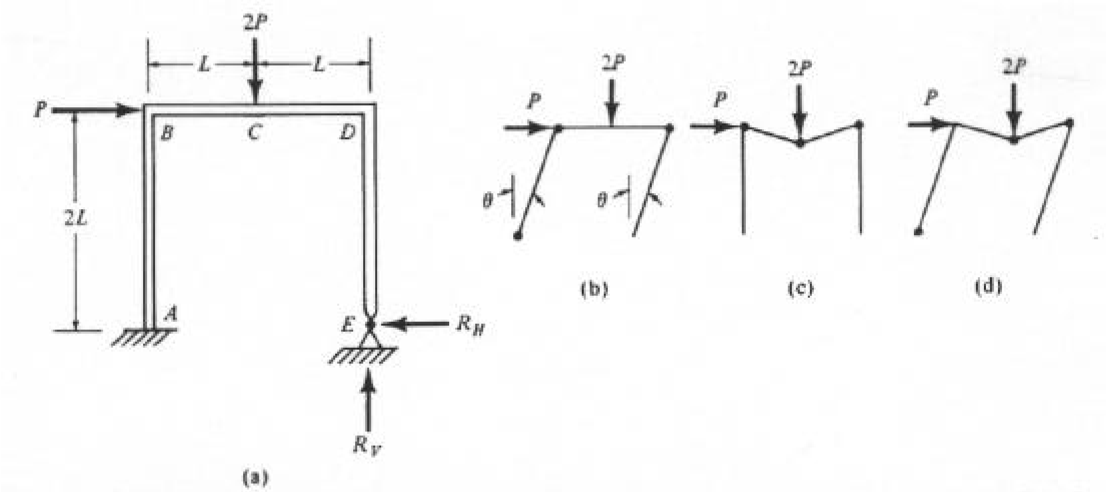

FIGURE 11.6.2. (a) Plane frame with loads *P* and 2*P*, one leg fixed and the other pinned. (b–d) Assumed collapse mechanisms, with deflections exaggerated.

The fully plastic bending moment capacity is M_p throughout the frame. Find the collapse load associate with the mechanism of (b), and see if the associated moment is acceptable:

(1) For the case (b), only horizontal force does work during virtual displacement θ, the virtual work eq. is

$$P(2Lq) = M_p(3q) \quad \text{hence} \quad P = \frac{3M_p}{2L}$$

(2) Calculation of the bending moment at each of the critical sections (in terms of R_H and R_v)

$$M_A = -2R_V L + 4PL \quad , \quad M_B = -2R_V L + 2PL + 2R_H L$$

$$M_C = -R_V L + 2R_H L \quad , \quad M_D = 2R_H L$$

eliminating R_V and R_H, we have

$$M_A - 2M_C + 2M_D = 4PL$$

$$M_A - 2M_C + M_D = PL$$

For he mechanism shown in (b), M_A=M_p, M_B= $-M_p$, M_D=M_p, thus with P=3M_p/2L, we finally have M_c= $-1.5M_p$ (positive M represents tensile stress outside the frame). The mechanism of (b) does not in fact occur.