MODULE
3

ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES BY THE DISPLACEMENT METHOD

LESSON 19

THE MOMENT-DISTRIBUTION METHOD: STATICALLY INDETERMINATE BEAMS WITH SUPPORT SETTLEMENTS

Instructional Objectives

After reading this chapter the student will be able to

1. Solve continuous beam with support settlements by the moment-distribution method.
2. Compute reactions at the supports.
3. Draw bending moment and shear force diagrams.
4. Draw the deflected shape of the continuous beam.

19.1 Introduction

In the previous lesson, moment-distribution method was discussed in the context of statically indeterminate beams with unyielding supports. It is very well known that support may settle by unequal amount during the lifetime of the structure. Such support settlements induce fixed end moments in the beams so as to hold the end slopes of the members as zero (see Fig. 19.1).

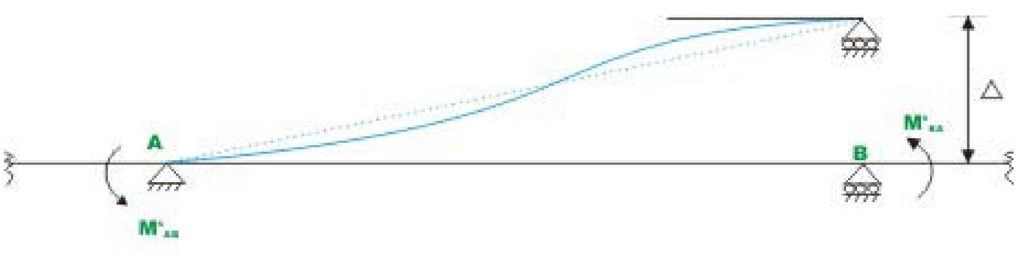

Fig . 19.1 Support settlement without ratation

In lesson 15, an expression (equation 15.5) for beam end moments were derived by superposing the end moments developed due to

1. Externally applied loads on beams
2. Due to displacements θ_A, θ_B and Δ (settlements).

The required equations are,

$$M_{AB} = M_{AB}^F + \frac{2EI_{AB}}{L_{AB}}\left[2\theta_A + \theta_B - \frac{3\Delta}{L_{AB}}\right] \quad (19.1a)$$

$$M_{BA} = M_{BA}^{F} + \frac{2EI_{AB}}{L_{AB}}\left[2\theta_B + \theta_A - \frac{3\Delta}{L_{AB}}\right] \tag{19.1b}$$

This may be written as,

$$M_{AB} = M_{AB}^{F} + 2K_{AB}\left[2\theta_A + \theta_B\right] + M_{AB}^{S} \tag{19.2a}$$

$$M_{BA} = M_{BA}^{F} + 2K_{AB}\left[2\theta_B + \theta_A\right] + M_{BA}^{S} \tag{19.2b}$$

where $K_{AB} = \frac{EI_{AB}}{L_{AB}}$ is the stiffness factor for the beam *AB*. The coefficient 4 has been dropped since only relative values are required in calculating distribution factors.

Note that $M_{AB}^{S} = M_{BA}^{S} = -\frac{6EI_{AB}\Delta}{L_{AB}^{2}}$ (19.3)

M_{AB}^{S} is the beam end moments due to support settlement and is negative (clockwise) for positive support settlements (upwards). In the moment-distribution method, the support moments M_{AB}^{S} and M_{BA}^{S} due to uneven support settlements are distributed in a similar manner as the fixed end moments, which were described in details in lesson 18.

It is important to follow consistent sign convention. Here counterclockwise beam end moments are taken as positive and counterclockwise chord rotation $\left(\frac{\Delta}{L}\right)$ is taken as positive. The moment-distribution method as applied to statically indeterminate beams undergoing uneven support settlements is illustrated with a few examples.

Example 19.1

Calculate the support moments of the continuous beam ABC (Fig. 19.2a) having constant flexural rigidity EI throughout, due to vertical settlement of support B by 5mm. Assume $E = 200$ GPa ; and $I = 4\times10^{-4}\,\text{m}^4$.

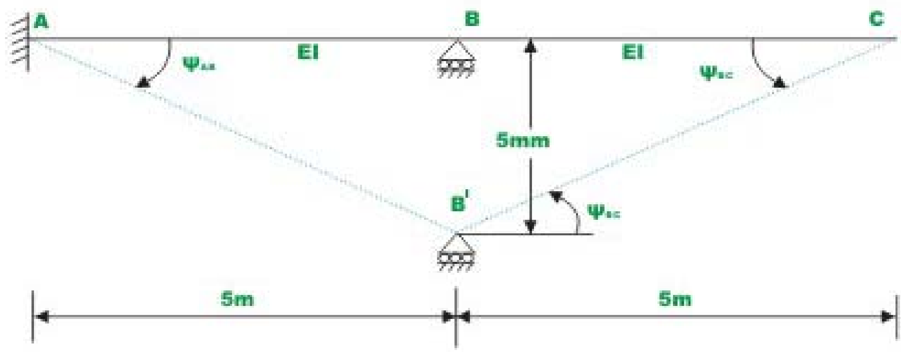

Fig . 19.2a Chord rotation due to support settlement (Example 19.1)

Solution

There is no load on the beam and hence fixed end moments are zero. However, fixed end moments are developed due to support settlement of B by 5mm. In the span AB, the chord rotates by ψ_{AB} in clockwise direction. Thus,

$$\psi_{AB} = -\frac{5\times10^{-3}}{5}$$

$$M^S_{AB} = M^S_{BA} = -\frac{6EI_{AB}}{L_{AB}}\psi_{AB} = -\frac{6\times200\times10^9\times4\times10^{-4}}{5}\left(-\frac{5\times10^{-3}}{5}\right)$$

$$= 96000 \quad \text{Nm} = 96 \quad \text{kNm.} \qquad (1)$$

In the span BC, the chord rotates by ψ_{BC} in the counterclockwise direction and hence taken as positive.

$$\psi_{BC} = \frac{5\times10^{-3}}{5}$$

$$M^S_{BC} = M^S_{CB} = -\frac{6EI_{BC}}{L_{BC}}\psi_{BC} = -\frac{6\times 200\times 10^9 \times 4\times 10^{-4}}{5}\left(\frac{5\times 10^{-3}}{5}\right)$$

$$= -96000\ Nm = -96\ kNm. \qquad (2)$$

Now calculate stiffness and distribution factors.

$$K_{BA} = \frac{EI_{AB}}{L_{AB}} = 0.2EI \quad \text{and} \quad K_{BC} = \frac{3}{4}\frac{EI_{BC}}{L_{BC}} = 0.15EI \qquad (3)$$

Note that, while calculating stiffness factor, the coefficient 4 has been dropped since only relative values are required in calculating the distribution factors. For span BC, reduced stiffness factor has been taken as support C is hinged.

At B:

$$\sum K = 0.35EI$$

$$DF_{BA} = \frac{0.2EI}{0.35EI} = 0.571$$

$$DF_{BC} = \frac{0.15EI}{0.35EI} = 0.429 \qquad (4)$$

At support C:

$$\sum K = 0.15EI\ ; \qquad DF_{CB} = 1.0\ .$$

Now joint moments are balanced as discussed previously by unlocking and locking each joint in succession and distributing the unbalanced moments till the joints have rotated to their final positions. The complete procedure is shown in Fig. 19.2b and also in Table 19.1.

Fig. 19.2b Computation

Table 19.1 Moment-distribution for continuous beam ABC

Joint	A	B		C
Member		BA	BC	CB
Stiffness factor		0.2EI	0.15EI	0.15EI
Distribution Factor		0.571	0.429	1.000
Fixd End Moments (kN.m)	96.000	96.000	-96.000	-96.000
Balance joint C and C.O. to B			48.00	96.000
Balance joint B and C.O. to A	-13,704	-27.408	-20.592	
Final Moments (kN.m)	82.296	68.592	-68.592	0.000

Note that there is no carry over to joint C as it was left unlocked.

Example 19.2

A continuous beam $ABCD$ is carrying uniformly distributed load $5kN/m$ as shown in Fig. 19.3a. Compute reactions and draw shear force and bending moment diagram due to following support settlements.

Support B, 0.005m vertically downwards.

Support C, .0100m vertically downwards.

Assume $E = 200GPa$; $I = 1.35 \times 10^{-3} m^4$.

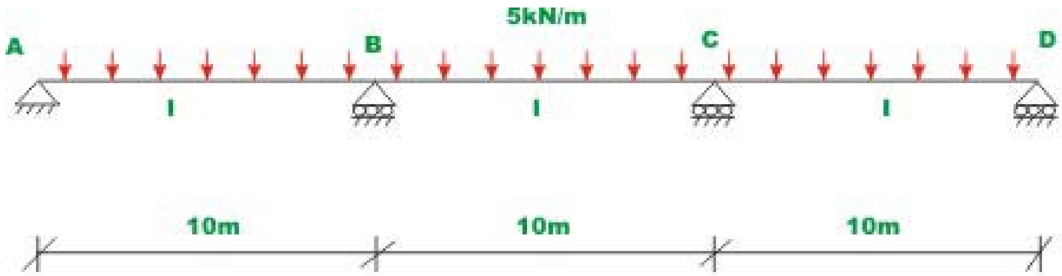

Fig .19.3a Continuous beam of Example 19.2

Solution:

Assume that supports A, B, C and D are locked and calculate fixed end moments due to externally applied load and support settlements. The fixed end beam moments due to externally applied loads are,

$$M_{AB}^F = \frac{5 \times 100}{12} = 41.67 \text{ kN.m}; \; M_{BA}^F = -41.67 \text{ kN.m}$$
$$M_{BC}^F = +41.67 \text{ kN.m}; \qquad M_{BC}^F = -41.67 \text{ kN.m}$$
$$M_{CD}^F = +41.67 \text{ kN.m}; \qquad M_{DC}^F = -41.67 \text{ kN.m} \qquad (1)$$

In the span AB, the chord joining joints A and B rotates in the clockwise direction as B moves vertical downwards with respect to A (see Fig. 19.3b).

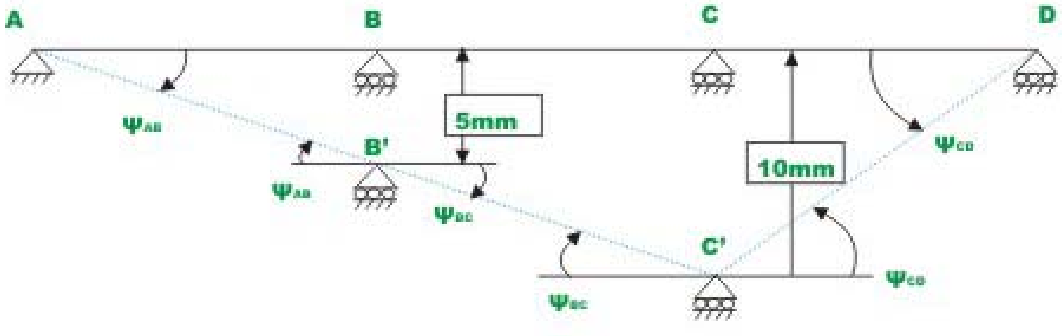

Fig. 19.3b Member rotation due to support settlement

$\psi_{AB} = -0.0005$ radians (negative as chord AB' rotates in the clockwise direction from its original position)

$\psi_{BC} = -0.0005$ radians

$\psi_{CD} = 0.001$ radians (positive as chord $C'D$ rotates in the counterclockwise direction).

Now the fixed end beam moments due to support settlements are,

$$M_{AB}^{S} = -\frac{6EI_{AB}}{L_{AB}}\psi_{AB} = -\frac{6\times 200\times 10^{9}\times 1.35\times 10^{-3}}{10}(-0.0005)$$

$$= 81000 \text{ N.m} = 81.00 \text{ kN.m}$$

$$M_{BA}^{S} = 81.00 \text{ kN.m}$$

$$M_{BC}^{S} = M_{CB}^{S} = 81.00 \text{ kN.m}$$

$$M_{CD}^{S} = M_{DC}^{S} = -162.00 \text{ kN.m} \quad (3)$$

In the next step, calculate stiffness and distribution factors. For span AB and CD modified stiffness factors are used as supports A and D are hinged. Stiffness factors are,

$$K_{BA} = \frac{3}{4}\frac{EI}{10} = 0.075EI\,; \qquad K_{BC} = \frac{EI}{10} = 0.10EI$$

$$K_{CB} = \frac{EI}{10} = 0.10EI\,; \qquad K_{CD} = \frac{3}{4}\frac{EI}{10} = 0.075EI \qquad (4)$$

At joint A : $\sum K = 0.075EI$; $\quad DF_{AB} = 1.0$

At joint B : $\sum K = 0.175EI$; $\quad DF_{BA} = 0.429$; $\quad DF_{BC} = 0.571$

At joint C : $\sum K = 0.175EI$; $\quad DF_{CB} = 0.571$; $\quad DF_{CD} = 0.429$

At joint D : $\sum K = 0.075EI$; $\quad DF_{DC} = 1.0$

The complete procedure of successively unlocking the joints, balancing them and locking them is shown in a working diagram in Fig.19.3c. In the first row, the distribution factors are entered. Then fixed end moments due to applied loads and support settlements are entered. In the first step, release joints A and D. The unbalanced moments at A and D are 122.67 kN.m, -203.67 kN.m respectively. Hence balancing moments at A and D are -122.67 kN.m, 203.67 kN.m respectively. (Note that we are dealing with beam end moments and not joint moments). The joint moments are negative of the beam end moments. Further leave A and D unlocked as they are hinged joints. Now carry over moments -61.34 kN.m and 101.84 kN.m to joint B and C respectively. In the next cycle, balance joints B and C. The unbalanced moment at joint B is 100.66 kN.m. Hence balancing moment for beam BA is -43.19 (-100.66×0.429) and for BC is -57.48 kN.m (-100.66 x 0.571). The balancing moment on BC gives a carry over moment of -26.74 kN.m to joint C. The whole procedure is shown in Fig. 19.3c and in Table 19.2. It must be noted that there is no carryover to joints A and D as they were left unlocked.

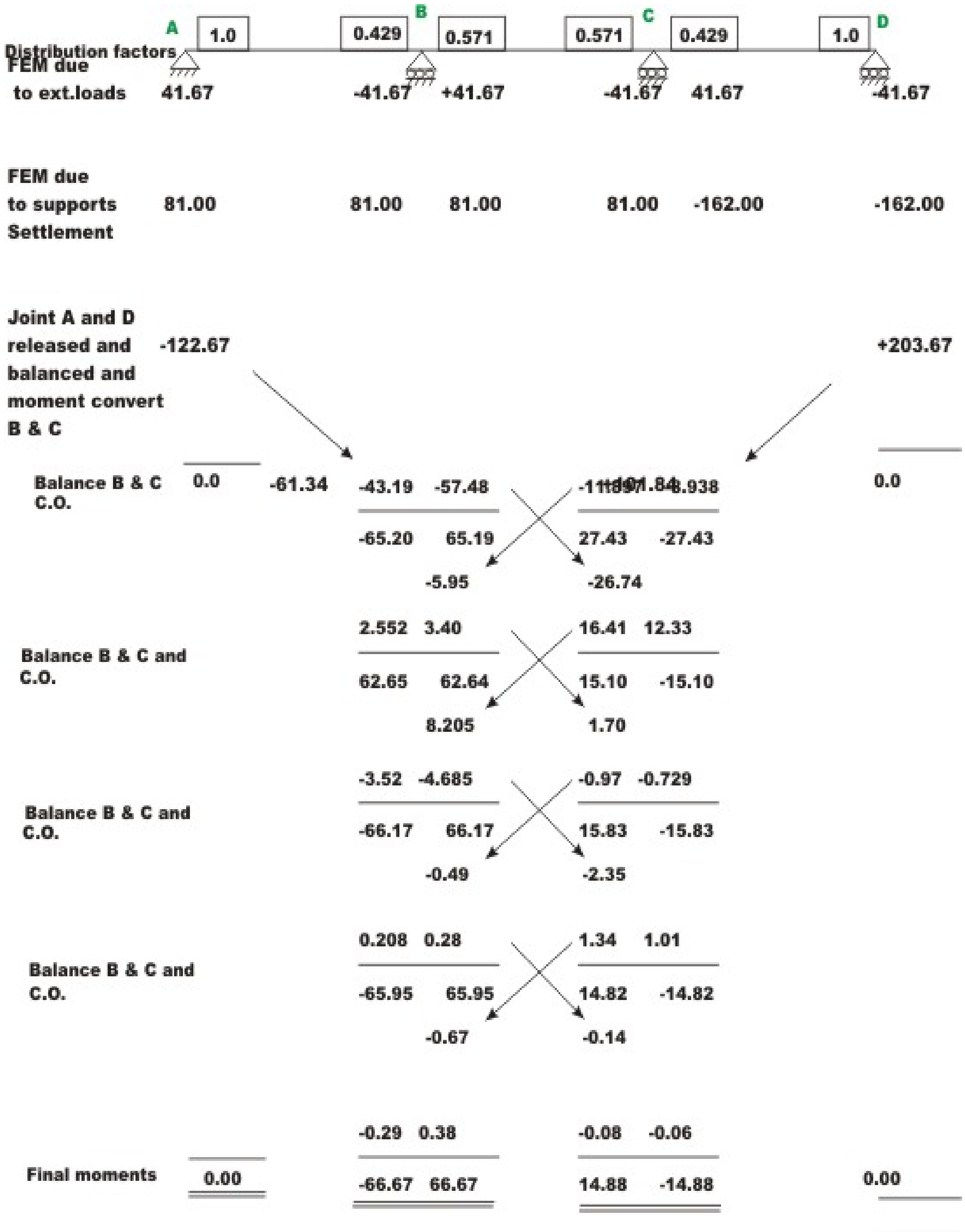

Fig. 19.3 © Computation

Table 19.2 Moment-distribution for continuous beam ABCD

Joint	A	B		C		D
Members	AB	BA	BC	CB	CD	DC
Stiffness factors	0.075 EI	0.075 EI	0.1 EI	0.1 EI	0.075 EI	0.075 EI
Distribution Factors	1.000	0.429	0.571	0.571	0.429	1.000
FEM due to externally applied loads	41.670	-41.670	41.670	-41.670	41.670	-41.670
FEM due to support settlements	81.000	81.000	81.000	81.000	-162.000	-162.000
Total	122.670	39.330	122.670	39.330	-120.330	-203.670
Balance A and D released	-122.670					203.670
Carry over		-61.335			101.835	
Balance B and C		-43.185	-57.480	-11.897	-8.94	
Carry over			-5.95	-26.740		
Balance B and C		2.552	3.40	16.410	12.33	
Carry over to B and C			8.21	1.70		
Balance B and C		-3.52	-4.69	-0.97	-0.73	
C.O. to B and C			-0.49	-2.33		
Balance B and C		0.21	0.28	1.34	1.01	
Carry over			0.67	0.14		
Balance B and C		-0.29	-0.38	-0.08	-0.06	
Final Moments	0.000	-66.67	66.67	14.88	-14.88	0.000

Example 19.3

Analyse the continuous beam ABC shown in Fig. 19.4a by moment-distribution method. The support B settles by $5mm$ below A and C. Assume EI to be constant for all members $E = 200GPa$; and $I = 8 \times 10^6 mm^4$.

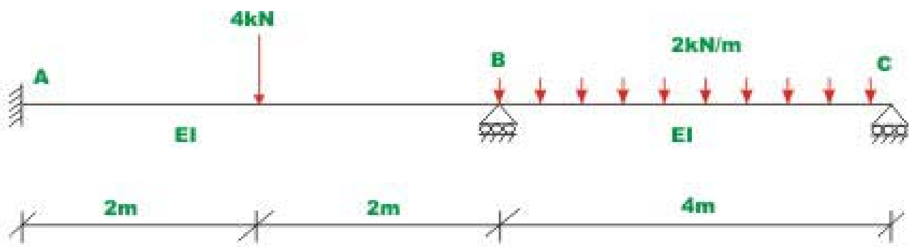

Fig. 19.4 (a) Example 19.4a

Solution:

Calculate fixed end beam moments due to externally applied loads assuming that support B and C are locked.

$$M_{AB}^{F} = +2\ kN.m; \qquad M_{BA}^{F} = -2\ kN.m$$
$$M_{BC}^{F} = +2.67\ kN.m; \qquad M_{CB}^{F} = -2.67\ kN.m \qquad (1)$$

In the next step calculate fixed end moments due to support settlements. In the span AB, the chord AB' rotates in the clockwise direction and in span BC, the chord $B'C$ rotates in the counterclockwise direction (Fig. 19.4b).

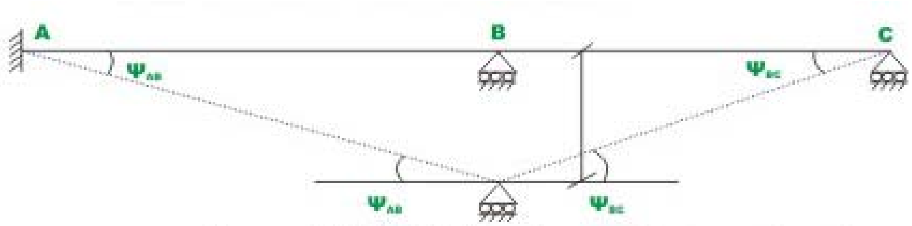

Fig. 19.4 (b) Member rotation due to support settlement

$$\psi_{AB} = -\frac{5\times 10^{-3}}{4} = -1.25\times 10^{-3}\,radians$$

$$\psi_{BC} = \frac{5\times 10^{-3}}{4} = 1.25\times 10^{-3}\,radians \qquad (2)$$

$$M_{AB}^{S} = M_{BA}^{S} = -\frac{6EI_{AB}}{L_{AB}}\psi_{AB} = -\frac{6\times 200\times 10^{9}\times 8\times 10^{-6}}{4}\left(-\frac{5\times 10^{-3}}{4}\right)$$

$$= 3000\ Nm = 3\,kNm. \qquad (3)$$

$$M_{BC}^{S} = M_{CB}^{S} = -3.0 kN.m$$

In the next step, calculate stiffness and distribution factors.

$$K_{AB} = K_{BA} = 0.25EI$$
$$K_{BC} = \frac{3}{4}0.25EI = 0.1875EI \qquad (4)$$

At joint B : $\sum K = 0.4375EI$; $DF_{BA} = 0.571$; $DF_{BC} = 0.429$

At joint C : $\sum K = 0.1875EI$; $DF_{CB} = 1.0$

At fixed joint, the joint does not rotate and hence no distribution moments are developed and consequently distribution factor is equal to zero. The complete moment-distribution procedure is shown in Fig. 19.4c and Table 19.3. The diagram is self explanatory. In this particular case results are obtained in two cycles. In the first cycle joint C is balanced and carry over moment is taken to joint B. In the next cycle , joint B is balanced and carry over moment is taken to joint A. The bending moment diagram is shown in fig. 19.4d.

Table 19.3 Moment-distribution for continuous beam ABC

Joints	A	B		C
Member	AB	BA	BC	CB
Stiffness factor	0.25 EI	0.25 EI	0.1875 EI	0.1875 EI
Distribution Factor		0.571	0.429	1.000
Fixed End Moments due to applied loads (kN.m)	2.000	-2.000	2.667	-2.667
Fixed End Moments due to support settlements (kN.m)	3.000	3.000	-3.000	-3.000
Total	5.000	1.000	-0.333	-5.667
Balance joint C and C.O.			2.835	5.667
Total	5.000	1.000	2.502	0.000
Balance joint B and C.O. to A	-1.00	-2.000	-1.502	
Final Moments (kN.m)	4.000	-1.000	1.000	0.000

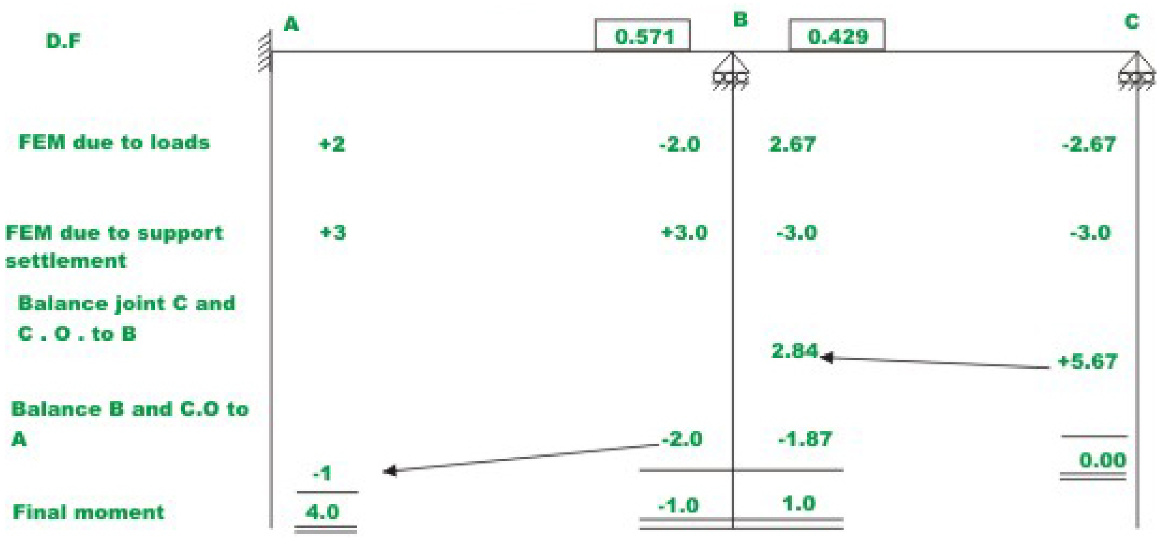

Fig. 19.4 (c) Computation

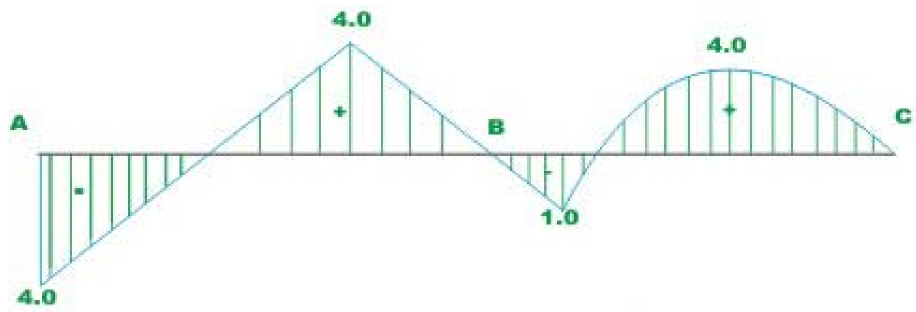

FIG. 19.4 (d) B.M.D

Summary

The moment-distribution method is applied to analyse continuous beam having support settlements. Each step in the numerical example is explained in detail. All calculations are shown at appropriate locations. The deflected shape of the continuous beam is sketched. Also, wherever required, the bending moment diagram is drawn. The numerical examples are explained with the help of free-body diagrams.