11 Continuous Beam

Objectives:

Derive the Clapeyron's theorem of three moments Analyze continuous beam with different moment of inertia with unyielding supports Analyze the continuous beam with different moment of inertia in different spans along with support settlements using three moment equation.

11.0 INTRODUCTION

A beam is generally supported on a hinge at one end and a roller bearing at the other end. The reactions are determined by using static equilibrium equations. Such as beam is a statically determinate structure. If the ends of the beam are restrained/clamped/encastre/fixed then the moments are included at the ends by these restraints and this moments make the structural element to be a statically indeterminate structure or a redundant structure. These restraints make the slopes at the ends zero and hence in a fixed beam, the deflection and slopes are zero at the supports.

A continuous beam is one having more than one span and it is carried by several supports (minimum of three supports). Continuous beams are widely used in bridge construction. Consider a three bay of a building which carries the loads W_1, W_2 and W_3 in two ways.

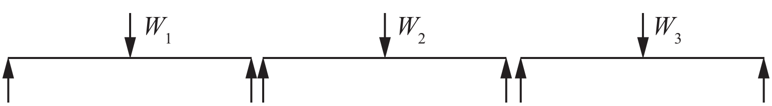


FIG. 11a Simply supported beam

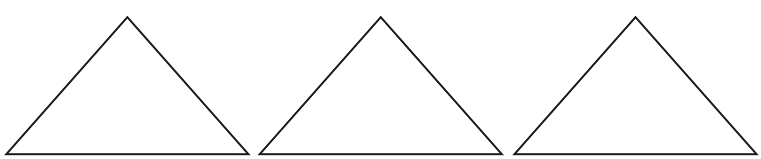


FIG. 11b Bending moment diagrams

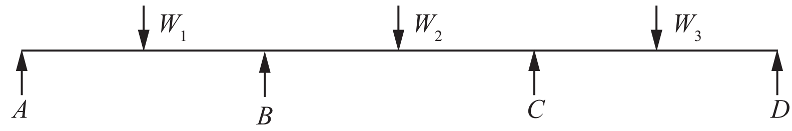


FIG. 11c Continuous beam

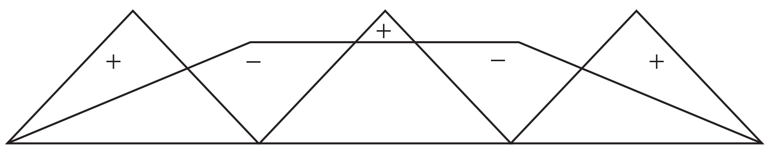


FIG. 11d Bending moment diagram

If the load is carried by the first case then the reactions of individual beams can be obtained by equilibrium equations alone. The beam deflects in the respective span and does not depend on the influence of adjacent spans.

In the second case, the equilibrium equations alone would not be sufficient to determine the end moments. The slope at an interior support *B* must be same on either side of the support. The magnitude of the slope can be influenced by not only the load on the spans either side of it but the entire loads on the span of the continuous beam. The redundants could be the reactions or the bending moments over the support. Clapeyron (1857) obtained the compatibility equation in term of the end slopes of the adjacent spans. This equation is called theorem of three moments which contain three of the unknowns. It gives the relationship between the loading and the moments over three adjacent supports at the same level.

11.1 DERIVATION OF CLAPEYRON'S THEOREM (THEOREM OF THREE MOMENTS)

Figure 11e shows two adjacent spans *AB* and *BC* of a continuous beam with two spans. The settlement of the supports are Δ_A, Δ_B and Δ_C and the deflected shape of the beam is shown in $A'B'C'$ (Fig. 11f).

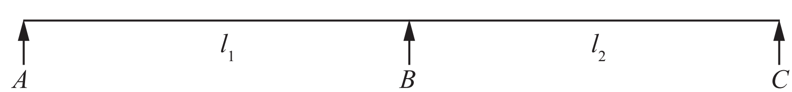


FIG. 11e

The primary structure is consisting of simply supported beams with imaginery hinges over each support (Fig 11g). Fig 11h shows the simply beam bending moment diagrams and Fig 11i shows the support moment diagram for the supports.

A compatibility equation is derived based on the fact that the end slopes of adjacent spans are equal in magnitude but opposite in sign. Using Fig 11f and the property similar triangles

$$\frac{GD}{DB'} = \frac{HF}{B'F}$$

$$\frac{\Delta_B - \Delta_A + \delta_A^B}{l_1} = \frac{\Delta_C - \Delta_B + \delta_C^B}{l_2}$$

$$\text{i.e.} \quad \frac{\delta_A^B}{l_1} + \frac{\delta_C^B}{l_2} = \frac{\Delta_A - \Delta_B}{l_1} + \frac{\Delta_C - \Delta_B}{l_2} \qquad \text{(i)}$$

The displacements are obtained as follows.

$$\delta_A^B = \frac{1}{E_1 I_1}\left\{A_1 \bar{x}_1 + \frac{1}{2} M_A l_1 \cdot \frac{l_1}{3} + \frac{1}{2} M_B \cdot l_1 \cdot 2l_1/3\right\} \qquad \text{(ii)}$$

$$\delta_C^B = \frac{1}{E_2 I_2}\left\{A_2 \bar{x}_2 + \frac{1}{2} M_C l_2 \cdot \frac{l_2}{3} + \frac{1}{2} M_B l_2 \cdot 2l_2/3\right\}$$

Combining the equations (i) and (ii)

$$\frac{M_A l_1}{E_1 I_1} + 2M_B\left(\frac{l_1}{E_1 I_1} + \frac{l_2}{E_2 I_2}\right) + M_C \frac{l_2}{E_2 I_2} + 6\left\{\frac{A_1 \bar{x}_1}{E_1 I_1 l_1} + \frac{A_2 \bar{x}_2}{E_2 I_2 l_1}\right\}$$

$$= 6\left\{\frac{\Delta_A - \Delta_B}{l_1} + \frac{\Delta_C - \Delta_B}{l_2}\right\} \qquad \text{(iii)}$$

The above equation is called as Clapeyron's equation of three moments.

In a simplified form of an uniform beam section (EI = constant); when there are no settlement of supports

$$M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2 = -6\left(\frac{A_1 \bar{x}_1}{l_1} + \frac{A_2 \bar{x}_2}{l_2}\right) \qquad \text{(iv)}$$

It is to be mentioned here that $\bar{x}_1$ and $\bar{x}_2$ are measured outwards in each span from the loads to the ends.

11.1.1 Procedure for Analysing the Continuous Beams using Theorem of Three Moments

(1) Draw simple beam moment diagram for each span of the beam. Compute the area of the above diagrams viz, $A_1, A_2 \ldots A_n$ and locate the centroid of such diagrams $\bar{x}_1, \bar{x}_2 \ldots \bar{x}_n$. It must be remembered that the distances $\bar{x}_1$, $\bar{x}_2 \ldots \bar{x}_n$ are the centroidal distances measured towards the ends of each span as shown in Fig. 11j.

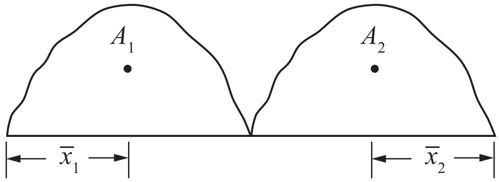


FIG. 11j Simple beam moment diagrams

(2) Identify the support moments which are to be determined viz, M_A, M_B and M_C

(3) Apply three moment equation for each pair of spans which results in an equation or equations which are to be solved simultaneously. If the beam is of uniform section (EI = constant) and no support settlements apply equation (iv) and in case the beam is non-uniform and the support settles/raises apply equation (iii).

(4) The solution of the equations gives the values of the support moments and the bending moment diagram can be drawn.

(5) The reactions at the supports and the shear force diagram can be obtained by using equilibrium equations.

11.2 APPLICATION OF THREE MOMENT EQUATION IN CASE OF BEAMS WHEN ONE OR BOTH OF THE ENDS ARE FIXED

11.2.1 Propped Cantilever Beam

Consider the propped cantilever beam of span AB, which is fixed at A and supported on a prop at B. It is subjected to uniformly distributed load over the entire span. The fixed end moment at the support A can be determined by using theorem of three moments.

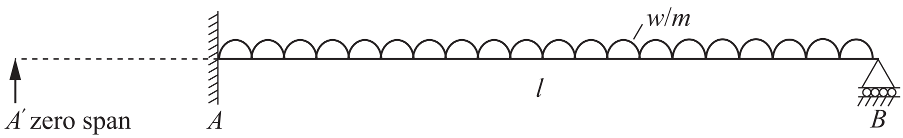


FIG. 11k Propped cantilever beam

As the A is fixed support, extend the beam form A to A' of span 'zero length' and A' is simply supported.

(1) The simple beam moment diagram is a parabola with a central ordinate of $(wl^2/8)$. The centroid of this bending moment diagram (symmetrical parabola) is at a distance '$l/2$' from the supports A and B.

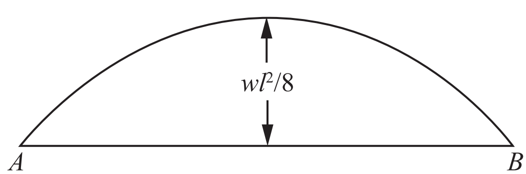


FIG. 11l Simple beam moment diagram

It's area is $A = \left(\frac{2}{3}\right)(l)\left(\frac{wl^2}{8}\right) = \frac{wl^3}{12}$.

(2) The support moment diagram is drawn as

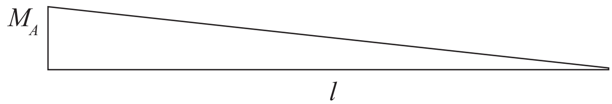


FIG. 11m Pure moment diagram

(3) Apply three moment theorem for the span *AB*.

$$M'_A(0) + 2M_A(0 + l) + 0 = -6\left(\frac{wl^3}{12}\right)\left(\frac{l}{2}\right)$$

$$\therefore \quad \boxed{M_A = -wl^2/8}$$

(4) The support reactions are computed by drawing the free body diagram as

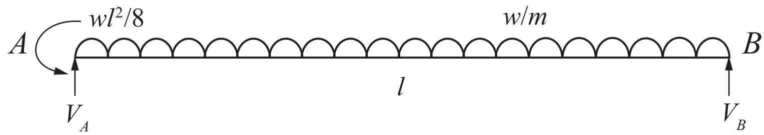


FIG. 11n Free body diagram

$$\sum V = 0; \qquad V_A + V_B = wl$$

$$\sum M_A = 0; \qquad \frac{-wl^2}{8} + \frac{wl^2}{2} - V_B l = 0$$

and hence

$$\boxed{\begin{aligned} V_B &= \frac{3wl}{8} \\ V_A &= \frac{5wl}{8} \end{aligned}}$$

(5) Using the reactions, the shear force diagram and bending moment diagrams are obtained as

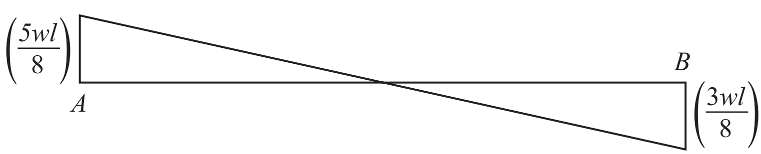


FIG. 11o Shear force diagram

The point of contraflexure is determined by equating the bending moment expression to zero and hence

$$\frac{5wl}{8}x - \frac{wx^2}{2} - \frac{wl^2}{8} = 0$$
$$l^2 + 4x^2 - 5lx = 0$$

Solving the above equation we get $x = l$ and

$$\boxed{x = 0.25l}$$

The location of maximum positive bending moment from support A is obtained by equating the shear force to zero.

$$\frac{5wl}{8} - wx = 0$$
$$\boxed{x = \frac{5l}{8}}$$

At this location, the maximum positive bending moment is obtained from

$$\text{Max} + \text{ve BM} = \frac{-wl^2}{8} + \left(\frac{5wl}{8}\right)\left(\frac{5l}{8}\right) - \frac{w(5l/8)^2}{2}$$
$$M_C = -\frac{wl^2}{8} + \frac{25wl^2}{64} - \frac{25wl^2}{128} = \frac{9wl^2}{128} = 0.07wl^2$$

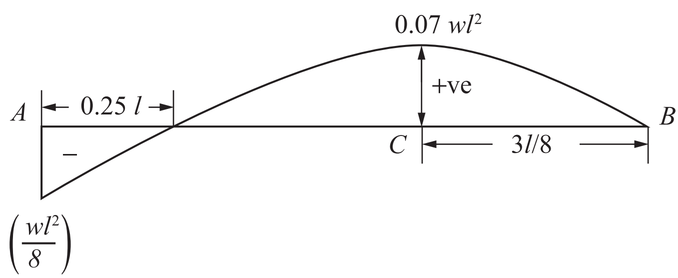


FIG. 11p Bending moment diagram

11.2.2 Beams with Both the Ends Fixed

Consider a beam AB of span l is fixed at both the ends. The beam is carrying a concentrated load of W at a distance of '$l/3$' from the fixed end A.

As the end A is a fixed support, extend this A to A' of span (l') of zero length and is also simply supported at A'. Likewise the end B is extended to B'.

The simply supported bending moment diagram is drawn with the maximum ordinate as $\frac{W \times (l/3) \times (2l/3)}{l} = 2Wl/9$.

The centroid of the unsymmetrical triangle is shown in Fig. 11.3j.

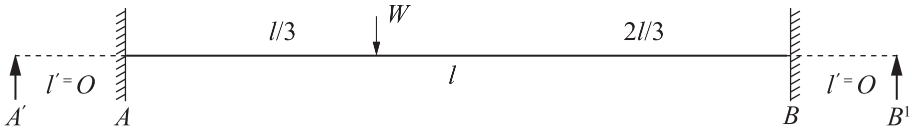


FIG. 11q Fixed beam

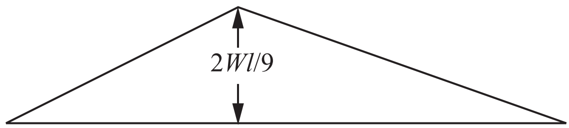


FIG. 11r Simple beam moment diagram

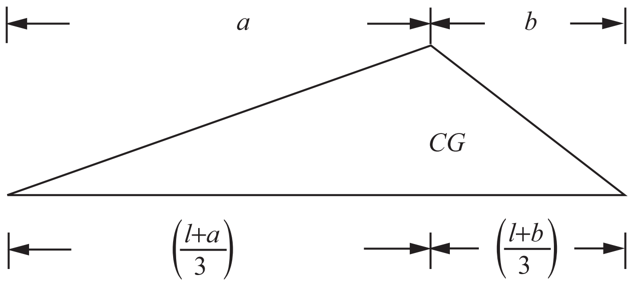


FIG. 11s Centroid of an unsymmetrical triangle

The centroid of the simply supported BMD is obtained using the above as $\left(\frac{4l}{9}\right)$ from A and $\left(\frac{5l}{9}\right)$ from B.

The area of the bending moment diagram is $\left(\frac{1}{2}\right)(l)\left(\frac{2Wl}{9}\right) = \frac{Wl^2}{9}$.

The support moment diagram can be drawn by identifying the support moments as M_A and M_B. Thus

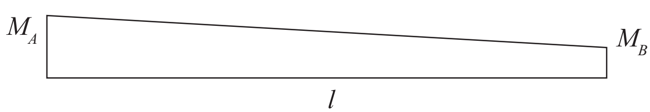


FIG. 11t Pure moment diagram

Applying three moment theorem for a pair of spans of $A'AB$ (Ref Eq (iv))

$$M'_A(0) + 2M_A(0+l) + M_B(l) = 0 - 6\left(\frac{Wl^2}{9}\right)\left(\frac{5l}{9}\right) \times 1/l$$

$$2M_A + M_B = -0.37\,Wl$$

Considering the next pair of spans ABB'

$$M_A l + 2M_B(l+0) + M'_B(0) = -6\left(\frac{Wl^2}{9}\right)\left(\frac{4l}{9}\right)$$

$$M_A + 2M_B = -0.296\,Wl$$

Thus the support moments are obtained by solving the above equations

$$\boxed{\begin{aligned} M_A &= -0.148\,Wl \\ M_B &= -0.074\,Wl \end{aligned}}$$

Free body diagram to determine the reactions

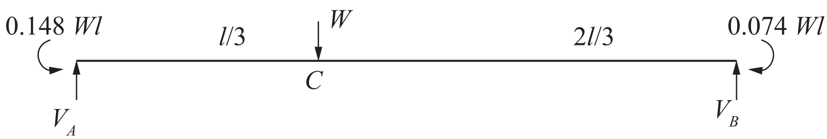


FIG. 11u

Using the static equilibrium;

$$\sum V = O; \qquad V_A + V_B = W$$

$$\sum M_A = O; \qquad -0.148\,Wl + W\left(\frac{l}{3}\right) - V_B l + 0.074\,Wl = O$$

$$\boxed{\begin{aligned} V_B &= 0.26W \\ V_A &= 0.74W \end{aligned}}$$

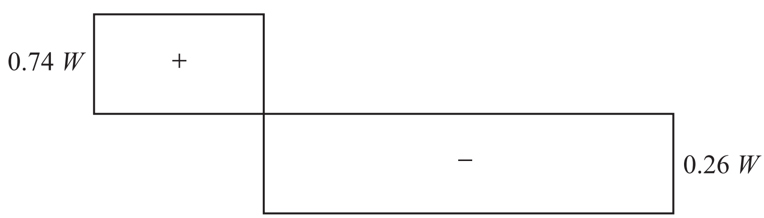


FIG. 11v Shearforce diagram

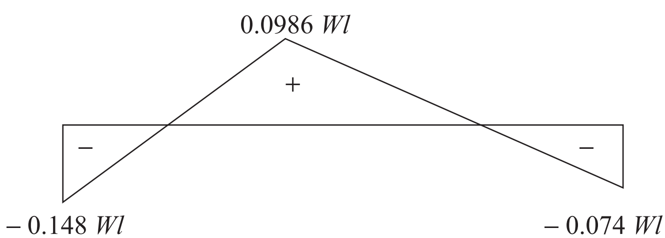


FIG. 11w Bending moment diagram

11.3 NUMERICAL EXAMPLES ON CONTINUOUS BEAMS

EXAMPLE 11.1: A continuous beam *ABC* is simply supported at *A* and *C* and continuous over support *B* with $AB = 4m$ and $BC = 5m$. A uniformly distributed load of 10 kN/m is acting over the beam. The moment of inertia is *I* throughout the span. Analyse the continuous beam and draw *SFD* and *BMD*.

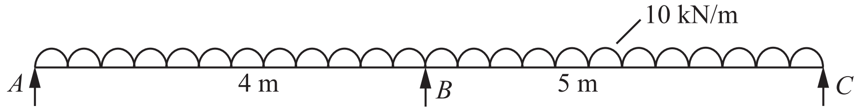


FIG. 11.1a

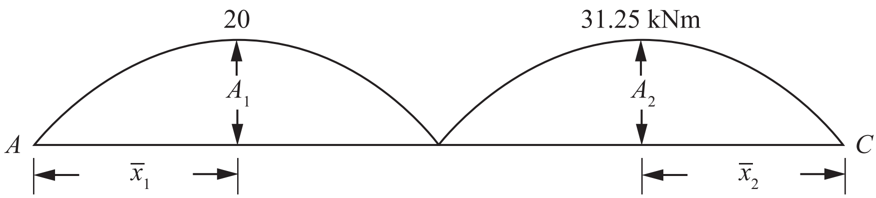


FIG. 11.1b Simple beam moment diagram

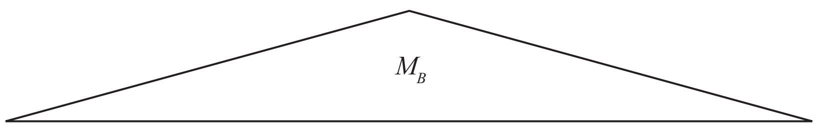


FIG. 11.1c Pure moment diagram

Properties of the simple beam BMD

$A_1 = \frac{2}{3} \times 4 \times 20 = 53.33 \text{ kNm}^2$	$A_2 = \frac{2}{3} \times 5 \times 31.25 = 104.17 \text{ kNm}^2$
$\bar{x}_1 = 2\text{m}$	$\bar{x}_2 = 2.5\text{m}$
$l_1 = 4\text{m}$	$l_2 = 5.0\text{m}$

Applying three moment equation for the span *ABC*

$$\cancel{M_A} l_1 + 2M_B(l_1 + l_2) + \cancel{M_C} l_2 = -6\left(\frac{A_1\bar{x}_1}{l_1} + \frac{A_2\bar{x}_2}{l_2}\right)$$

$$2M_B(4+5) = -6\left(\frac{53.33 \times 2}{4} + \frac{104.17 \times 2.5}{5}\right)$$

$$18M_B = -6(26.67 + 52.1)$$

$$M_B = -26.26 \text{ kNm.}$$

EXAMPLE 11.2: Analyse the continuous beam by three moment theorem. Draw *SFD* and *BMD*.

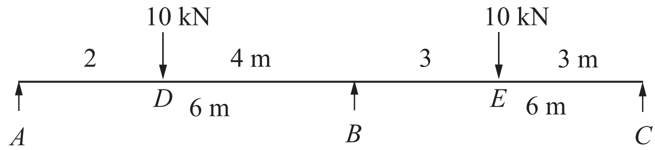


FIG. 11.2a

SOLUTION

The simple beam moment diagram is drawn as

$$M_D = Wab/l = \frac{10 \times 2 \times 4}{6} = 13.33 \text{ kNm}$$

$$M_E = Wl/4 = 10 \times \frac{6}{4} = 15 \text{ kNm}$$

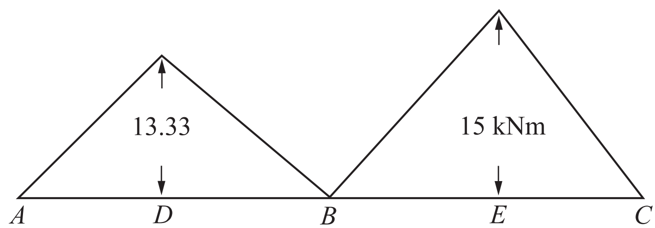


FIG. 11.2b Simple beam moment diagram

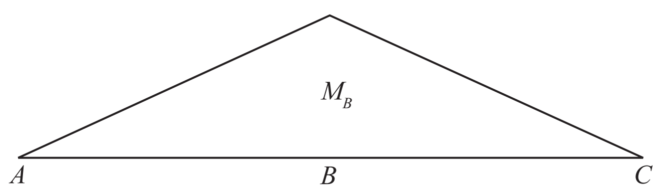


FIG. 11.2c Pure moment diagram

Properties of the simple beam BMD

$A_1 = \frac{1}{2}(6)13.33 = 40.0$	$A_2 = \frac{1}{2} \times 6 \times 15 = 45$
$\bar{x}_1 = \frac{6+2}{3} = 2.67$ m	$\bar{x}_2 = 3$ m
$l_1 = 6$ m	$l_2 = 6$ m

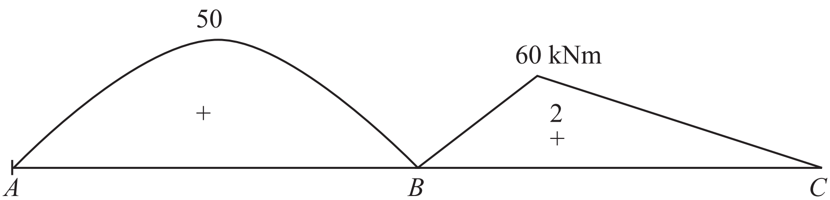


FIG. 11.4b Simple beam moment diagram

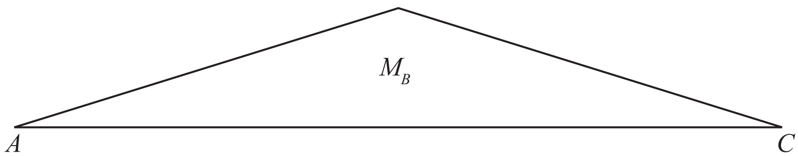


FIG. 11.4c Pure moment diagram

Properties of the simple beam BMD

$$A_1 = \frac{2}{3} \times 5 \times 50 = 167.5 \text{ kNm}^2$$

$$\bar{x}_1 = 2.5\text{m}$$

$$l_1 = 5.0\text{m}$$

$$A_2 = \frac{1}{2} \times 10 \times 60 = 300 \text{ kNm}^2$$

$$\bar{x}_2 = \frac{10+6}{3} = 5.33 \text{ m}$$

$$l_2 = 10 \text{ m}$$

$$\cancel{5M_A} + 2M_B(5+10) + \cancel{10M_C} = -6\left(\frac{167.5 \times 2.5}{5.0} + \frac{300 \times 5.33}{10}\right)$$

$$30\, M_B = -6\,(83.75 + 159.9)$$

$$\boxed{M_B = -48.73 \text{ kNm}}$$

Properties of the simple beam BMD

$$\sum V = 0; \quad V_A + V_{B1} = 80 \qquad \text{(i)}$$

$$\sum M = 0; \quad 5V_A + 49 - \frac{16(5)^2}{2} = 0 \qquad \text{(ii)}$$

$$V_A = 30.2 \text{ kN}$$

$$V_{B1} = 49.8 \text{ kN}$$

$$V_{B2} + V_C = 25 \qquad \text{(iii)}$$

$$10V_{B2} - 25(6) - 49 = 0 \qquad \text{(iv)}$$

$$V_{B2} = 19.9 \text{ kN}$$

$$V_C = 5.1 \text{ kN}$$

The simple beam moments are

$$M_D = 20 \times 10^2/8 = 250 \text{ kNm}$$

$$M_E = \frac{50 \times 6 \times 2}{8} = 75 \text{ kNm}$$

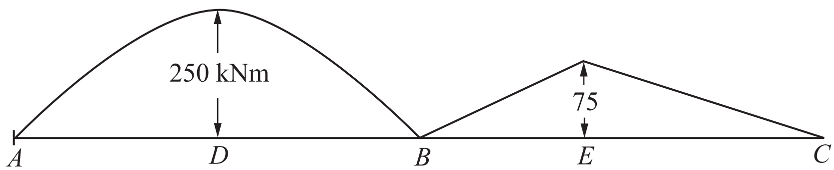


FIG. 11.5b Simple beam moment diagram

Properties of simple beam BMD

$A_1 = \frac{2}{3} \times 10 \times 250 = 1666.7 \text{ kNm}^2$	$A_2 = \frac{1}{2} \times 8 \times 75 = 300 \text{ kNm}^2$
$\bar{x}_1 = 5 \text{ m}$	$\bar{x}_2 = \frac{8+2}{3} = 3.33 \text{ m}$
$l_1 = 10 \text{ m}$	$l_2 = 8.0 \text{ m}$

Since A is fixed imagine a span $A'A$ of zero length and A' as simply supported. Apply three moment theorem for the spans $A'AB$.

$$M'_A(0) + 2M_A(0+10) + M_B(10) = -6\left(\frac{1666.7 \times 5}{10} + 0\right)$$

$$20M_A + 10M_B = -5000$$

$$2M_A + M_B = -500 \qquad \text{(i)}$$

Apply three moment theorem for the spans ABC.

$$M_A(10) + 2M_B(10+8) + \cancel{8M_C} = -6\left(\frac{1666.7 \times 5}{10} + \frac{300 \times 3.33}{8}\right)$$

$$10M_A + 36M_B = -6(833.35 + 124.875)$$

$$10M_A + 36M_B = -5749.35 \qquad \text{(ii)}$$

Solving equations (i) and (ii)

$$\boxed{\begin{aligned} M_A &= -197.6 \text{ kNm} \\ M_B &= -104.8 \text{ kNm} \end{aligned}}$$

Free body diagram of spans *AB* and *BC*

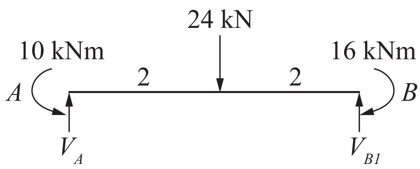


FIG. 11.7d

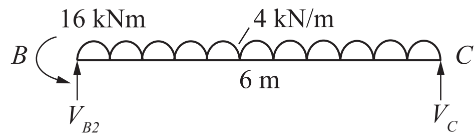


FIG. 11.7e

Static equilibrium of AB

$\sum V = 0; \quad V_A + V_{B1} = 24$ (i)

$\sum M = 0; \quad 4V_A + 16 - 10 - 48 = 0$ (ii)

$V_A = 10.5$ kN.

$V_{B1} = 13.5.$

Static equilibrium of BC

$V_{B2} + V_C = 24$ (iii)

$\sum M_B = 0;$

$$16 + 6V_C - 4 \times \frac{6^2}{2} = 0$$

$V_C = 9.3$ kN.

$V_{B2} = 14.7$ kN.

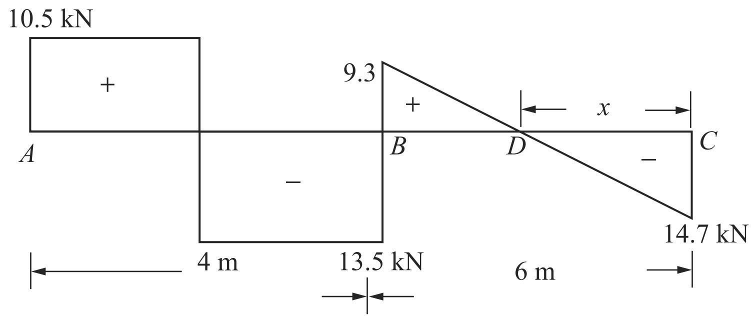


FIG. 11.7f Shear force diagram

The zero shear location in span *BC* is

$$14.7 - 4x = 0$$

$$x = 3.67 \text{ m.}$$

$$\therefore \quad \text{Maximum +ve BM} = 14.7(3.67) - 4(3.67)^2/2 - 16 = 11 \text{ kNm}$$

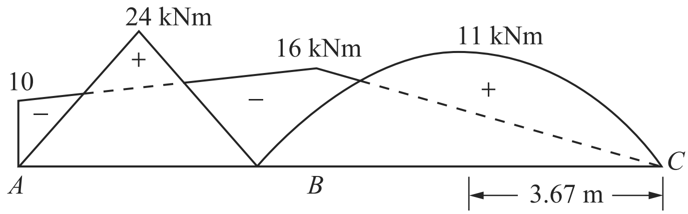


FIG. 11.7g Bending moment diagram

Solving (i) and (ii); From eq (i); $M_B = -2M_A$ and putting in eq (ii)

$$M_A - 12M_A = -60$$

$$\therefore \quad M_A = 5.45 \text{ kNm.} \qquad M_B = -10.9 \text{ kNm.}$$

Free body diagram of span *AB* and *BC*

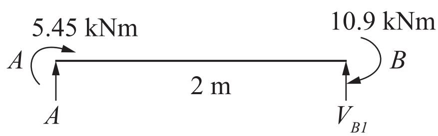


FIG 11.8d

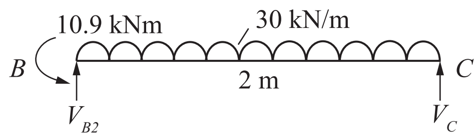


FIG 11.8e

Static equilibrium of span AB

$\sum V = 0;$

$V_A + V_{B_1} = 0$ (i)

$\sum M_B = 0;$

$5.45 + 10.9 + 2V_A = 0$ (ii)

$$V_A = -8.2 \text{ kN} \qquad V_{B_1} = +8.2 \text{ kN}$$

Static equilibrium of span BC

$\sum V = 0$

$V_{B2} + V_C = 60$ (iii)

$\sum M_B = 0;$

$$-10.9 + 2V_{B2} - \frac{30 \times 2^2}{2} = 0$$

$$V_{B2} = 35.5 \text{ kN} \qquad V_C = 24.5 \text{ kN}$$

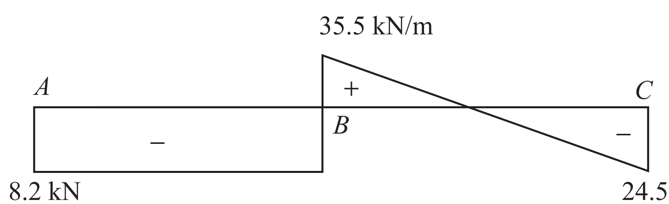


FIG. 11.8f Shear force diagram

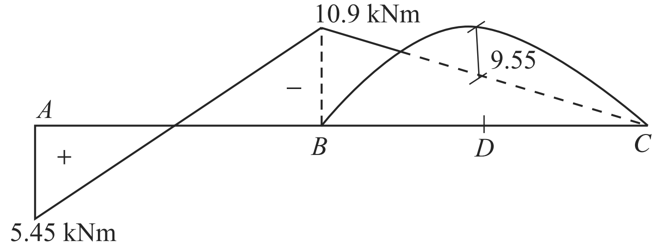


FIG. 11.8g Bending moment diagram

Applying three moment theorem for the span *ABC*

$$M_A\left(\frac{5}{I}\right)+2M_B\left(\frac{5}{I}+\frac{6}{1.5I}\right)-30\times\frac{6}{1.5I}$$
$$=-6\left(\frac{240\times 2.67}{5I}+\frac{360\times 3}{6\times 1.5I}\right)$$
$$5M_A+18M_B-120=-6(128.16+120)$$
$$5M_A+18M_B=-1488.96+120$$
$$5M_A+18M_B=-1368.96 \qquad \text{(ii)}$$

Solving equations (i) and (ii)

$$\boxed{\begin{aligned}M_A&=-33.76\text{ kNm.}\\ M_B&=-66.67\text{ kNm.}\end{aligned}}$$

Shear forces and moments in members *AB* and *BC*.

Member AB

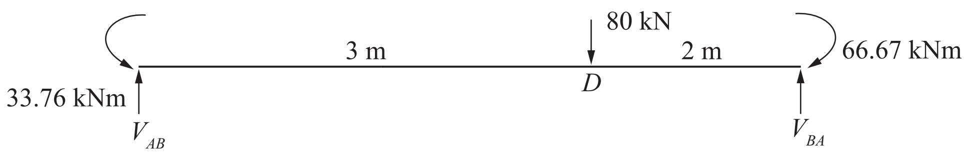


FIG. 11.9d

$$\sum V=0; \qquad V_{AB}+V_{BA}=80 \qquad \text{(i)}$$
$$\sum M_B=0; \qquad 5V_{AB}+66.67-33.76-80(2)=0 \qquad \text{(ii)}$$
$$V_{AB}=25.42$$
$$\therefore \quad V_{BA}=54.58$$
$$M_D=-33.76+25.42(3)=42.5\text{ kNm.}$$

Member BC

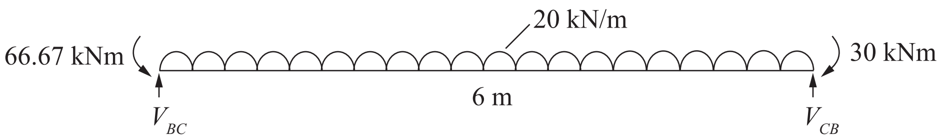


FIG. 11.9e

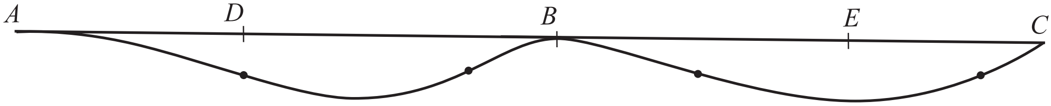


FIG. 11.9i Elastic curve

EXAMPLE 11.10: A continuous beam *ABCD* is simply supported at *A* and continuous over spans *B* and *C*. The span *AB* is 6 m and *BC* are of length 6 m respectively. An overhang *CD* is of 1 metre length. *A* concentrated load of 20 kN is acting at 4 m from support *A*. An uniformly distributed load of 10 kN/m is acting on the span *BC*. *A* concentrated load of 10 kN is acting at *D*.

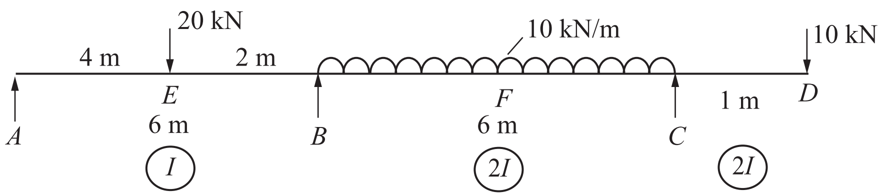


FIG. 11.10a

The simple beam moments are

$$M_E = \frac{20 \times 4 \times 2}{6} = 26.7 \text{ kNm}$$

$$M_F = 10 \times \frac{6^2}{8} = 45.0 \text{ kNm}$$

$$M_C = -10 \times 1 = -10 \text{ kNm}$$

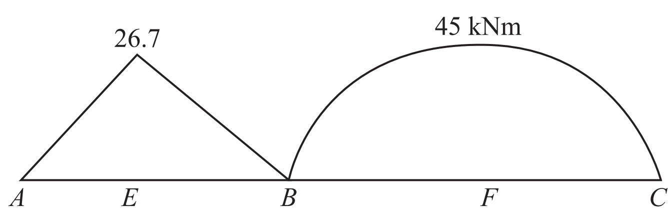


FIG. 11.10b Simply suppored BMD

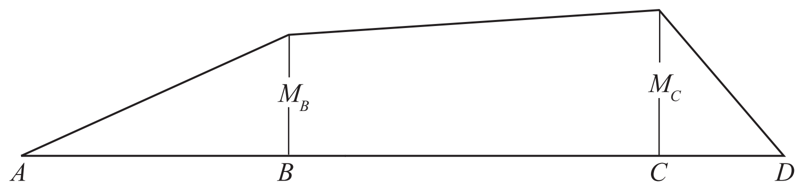


FIG. 11.10c Pure moment diagram

Considering spans ABC

Properties the simple beam BMD

$A_1 = \frac{1}{2} \times 6 \times 26.7 = 80.1\,\text{kNm}^2$	$A_2 = \frac{2}{3} \times 6 \times 45 = 180\,\text{kNm}^2$
$\bar{x}_1 = \frac{6+4}{3} = 3.33$ m.	$\bar{x}_2 = 3$ m.
$l_1 = 6$ m.	$l_2 = 6$ m.

$$\frac{6}{I}M_A + 2M_B\left(\frac{6}{I} + \frac{6}{2I}\right) + \frac{6M_C^{(-10)}}{2I} = -6\left(\frac{80.1 \times 3.33}{6} + \frac{180 \times 3}{6 \times 2}\right)$$

$$18M_B - 30 = -6\,(44.45 + 45)$$

$$M_B = 28.15 \text{ kNm.}$$

Shear force and bending moment values for the spans *AB* and *BC*

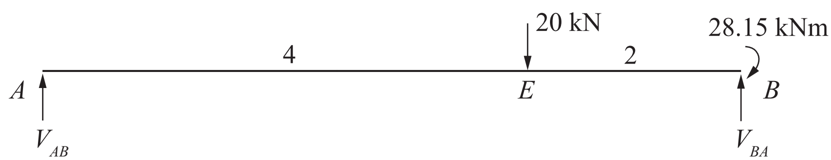


FIG. 11.10d

Using equilibrium conditions;

$$\sum V = 0; \quad V_{AB} + V_{BA} = 20 \qquad \text{(i)}$$

$$\sum M = 0; \quad 6V_{AB} + 28.15 - 20(2) = 0 \qquad \text{(ii)}$$

$$\boxed{\begin{aligned} V_{AB} &= 1.98 \\ V_{BA} &= 18.02 \end{aligned}}$$

$$\therefore \quad M_E = V_{AB}(4) = 7.9 \text{ kNm}$$

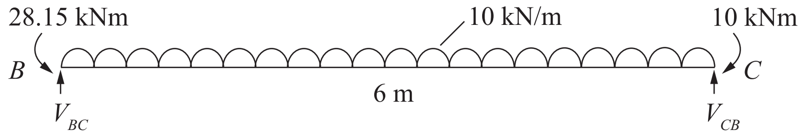


FIG. 11.10e

Using equilibrium conditions;

$$\sum V = 0; \quad V_{BC} + V_{CB} = 10(6) = 60 \tag{iii}$$

$$\sum M_C = 0 \quad 10 - 28.15 + 6\,V_{BC} - 10 \times \frac{6^2}{2} = 0 \tag{iv}$$

$$\boxed{\begin{aligned} V_{BC} &= 33 \text{ kN.} \\ V_{CB} &= 27 \text{ kN.} \end{aligned}}$$

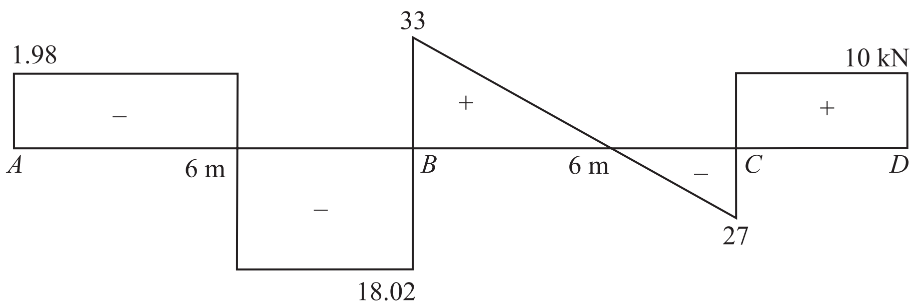


FIG. 11.10f Shear force diagram

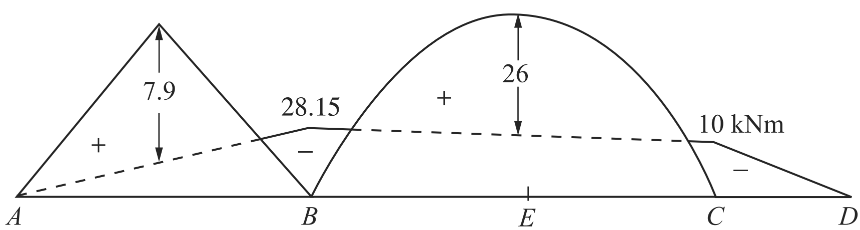


FIG. 11.10g Bending moment diagram

EXAMPLE 11.11: Analyse the continuous beam shown in figure by three moment theorem. Draw *SFD* & *BMD*.

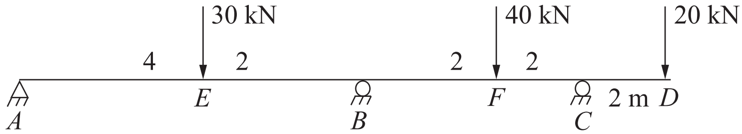


FIG. 11.11a

SOLUTION

The simple beam moments at E and F are

$$M_E = \frac{Wab}{l} = \frac{30 \times 4 \times 2}{6} = 40 \text{ kNm}$$

$$M_F = \frac{Wl}{4} = \frac{40 \times 4}{4} = 40 \text{ kNm}$$

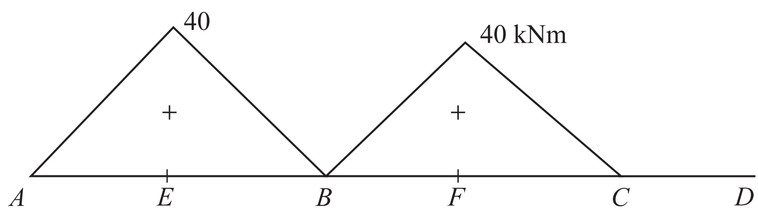


FIG. 11.11b Simply supported beam BMD

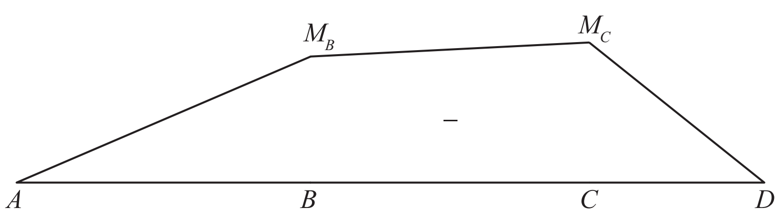


FIG. 11.11c Pure moment diagram

Properties of simply supported beam BMD

$$A_1 = \frac{1}{2} \times 6 \times 40 = 120 \text{ kNm}^2$$

$$\bar{x}_1 = \frac{6+4}{3} = 3.33$$

$$l_1 = 6.00$$

$$A_2 = \frac{1}{2} \times 4 \times 40 = 80 \text{ kNm}^2$$

$$\bar{x}_2 = 2 \text{ m.}$$

$$l_2 = 4.0 \text{ m.}$$

Applying three moment theorem for spans *AB* & *BC*

$$\cancel{6M_A} + 2M_B(6+4) + \cancel{4M_C} = -6\left(\frac{1}{2}\times 6\times 40\times\frac{3.33}{6.00}+\frac{80(2)}{4}\right)$$

$$20M_B - 160 = -6\ (66.6+40)$$

$$20M_B = -479.6$$

$$\boxed{M_B = -23.98\ \text{kNm}}$$

Free Body diagrams

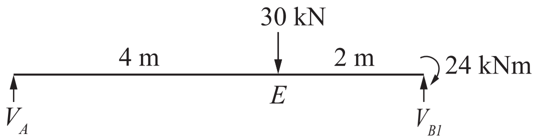


FIG. 11.11d

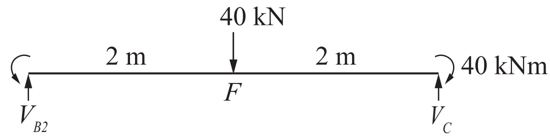


FIG. 11.11e

$$\sum V = 0; \quad V_A + V_{B1} = 30 \quad \text{(i)}$$

$$\sum M_B = 0; \quad 6V_A + 24 - 30(2) = 0 \quad \text{(ii)}$$

$$\boxed{\begin{aligned} V_A &= 6\ \text{kN} \\ V_{B1} &= 24\ \text{kN} \end{aligned}}$$

$$\sum V = 0; \quad V_{B2} + V_C = 40 \quad \text{(iii)}$$

$$\sum M_B = 0; \quad 40 + 40(2) - 24 - 4V_C = 0 \quad \text{(iv)}$$

$$\therefore \boxed{\begin{aligned} V_C &= 24\ \text{kN} \\ V_{B2} &= 16\ \text{kN} \end{aligned}}$$

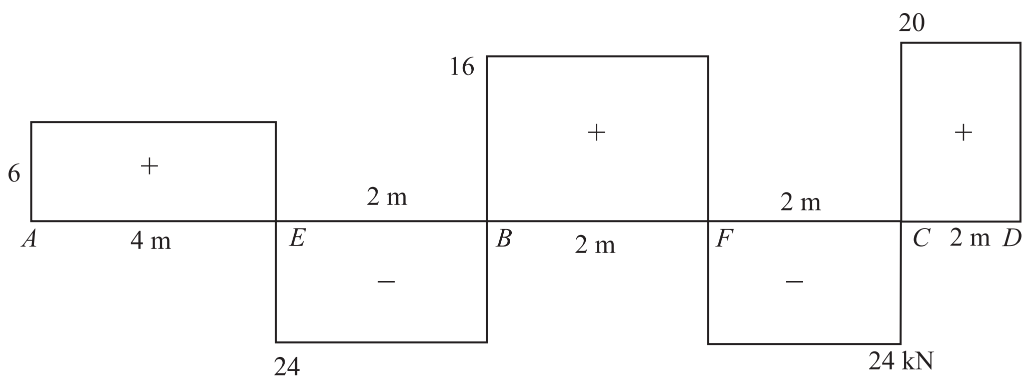


FIG. 11.11f Shear force diagram

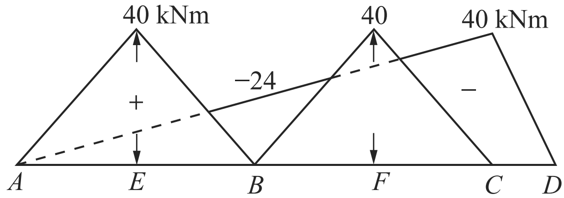


FIG. 11.11g Bending moment diagram

EXAMPLE 11.12: Draw the shear force diagram and bending moment diagram for the beam shown in figure.

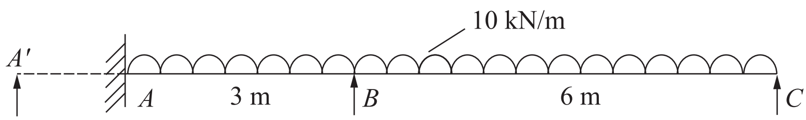


FIG. 11.12a

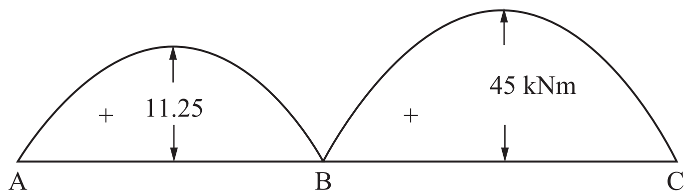


FIG. 11.12b Simply supported beam BMD

Solution

As the end A is fixed, imagine an imaginery span $A'A$ of zero length with no load and A' is simply supported.

Considering the span $A'AB$

$$\cancel{M'_A}(0) + 2M_A(0+3) + 3M_B = -6\left\{\frac{0+\left(\frac{2}{3}\times 3\times 11.25\right)1.5}{3}\right\} \quad \text{(i)}$$

$$6M_A + 3M_B = -67.5$$

Considering the span ABC

$$3M_A + 2M_B(3+6) + \cancel{6M_C} = -6\left(\frac{22.5\times 1.5}{3} + \frac{180\times 3}{6}\right)$$

$$3M_A + 18M_B = -6\ (11.25+90)$$

$$3M_A + 18M_B = -607.5 \quad \text{(ii)}$$

Solving (i) & (ii)

$$M_B = -34.77 \text{ kNm.}$$
$$M_A = 6.14 \text{ kNm.}$$

The *BMD* is drawn using the above end moments as

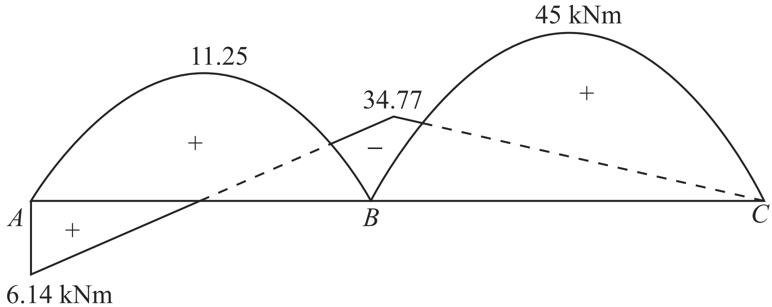


FIG. 11.12c Bending moment diagram

Shear force and BM values for spans *AB* and *BC*

Static equilibrium of *AB*

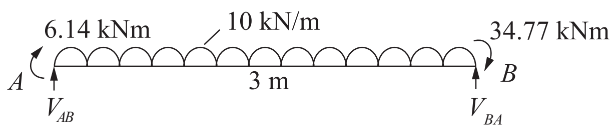


FIG. 11.12d

$$\sum V = 0; \quad V_{AB} + V_{BA} = 30 \qquad \text{(i)}$$

$$\sum M_B = 0; \quad 34.77 + 6.14 + 3V_{AB} - 10 \times \frac{3^2}{2} = 0 \qquad \text{(ii)}$$

$$V_{AB} = 1.36 \text{ kN.}$$
$$V_{BA} = 28.64 \text{ kN.}$$

Static equilibrium of *BC*

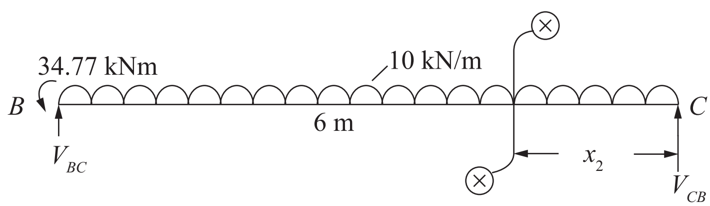


FIG. 11.12e

$$\sum V = 0; \qquad V_{BC} + V_{CB} = 10(6) = 60 \qquad \text{(iii)}$$

$$\sum M_C = 0; \qquad -34.77 + 6V_{BC} - 10 \times \frac{6^2}{2} = 0$$

$$\boxed{\begin{aligned} V_{BC} &= 35.8 \text{ kN.} \\ V_{CB} &= 24.2 \text{ kN.} \end{aligned}}$$

Maximum positive *BM* is span *AB*

The location of zero shear force in *AB* zone is

$$1.36 - 10x_1 = 0.$$
$$x_1 = 0.135 \text{ m}$$

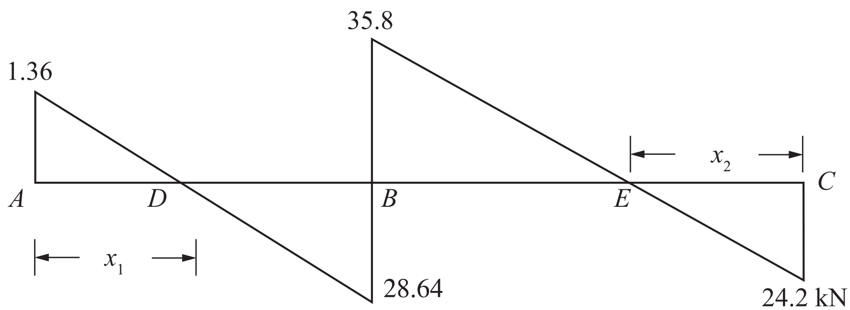


FIG. 11.12f Shear force diagram

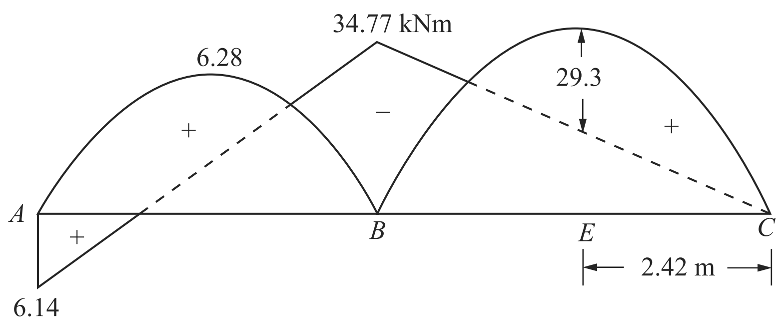


FIG. 11.12g Bending moment diagram

$$M_{X1X1} = 6.19 + 1.35\ (0.135) - 10\ (0.135)^2/2 = 6.28 \text{ kNm.}$$

Maximum positive *BM* in span *BC*

The location of zero shear force in *BC* zone is

$$24.2 - 10\,x_2 = 0$$
$$x_2 = 2.42 \text{ m}$$
$$M_{\times 2 \times 2} = 24.2(2.42) - 10(2.42)^2/2$$
$$= 29.3 \text{ kNm.}$$

EXAMPLE 11.13: A continuous beam *ABCD* is of uniform section. It is fixed at *A*, simply supported at *B* and *C* and *CD* is an overhang. $AB = BC = 5$ m and $CD = 2$ m. If a concentrated load of 30 kN acts at *D*, determine the moments and reactions at A, B and *C*. Sketch the shear force and bending moment diagram and mark in the salient values.

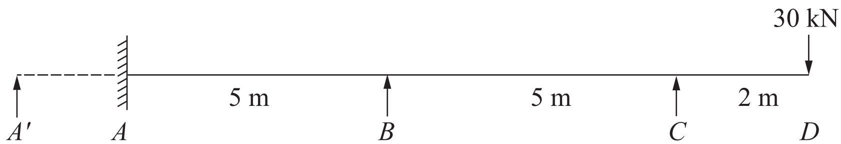


FIG. 11.13a

SOLUTION

As the end *A* is fixed imagine a imaginery span $A'A$ of zero length and A' is simply supported.
Apply three moment theorem for the spans $A'AB$

$$M_{A(0)} + 2M_A(0+5) + 5M_B = -6(0+0)$$
$$10M_A + 5M_B = 0$$
$$2M_A + M_B = 0 \qquad \text{(i)}$$

Apply three moment theorem for the spans *ABC*

$$5M_A + 2M_B(5+5) - 60(5) = -6(0+0)$$
$$5M_A + 20M_B = 300 \qquad \text{(ii)}$$

Solving (i) and (ii)

$$\boxed{\begin{aligned} M_A &= -8.57 \text{ kNm.} \\ M_B &= +17.14 \text{ kNm.} \end{aligned}}$$

Shear force and BM values for spans *AB* and *BC*

Span AB

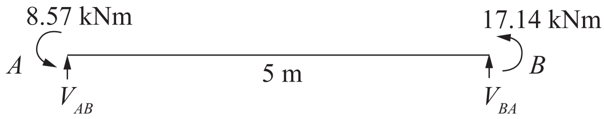


FIG. 11.13b

$$\sum V = 0; \quad V_{AB} + V_{BA} = 0 \tag{i}$$

$$\sum M_B = 0; \quad 5V_{AB} - 8.57 - 17.14 = 0$$

$$\boxed{\begin{aligned} V_{AB} &= 5.14 \text{ kN.} \\ V_{BA} &= -5.14 \text{ kN.} \end{aligned}}$$

Span BC

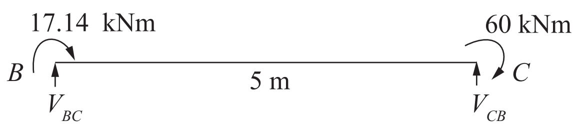


FIG. 11.13c

$$\sum V = 0; \quad V_{BC} + V_{CB} = 0$$

$$\sum M = 0; \quad 60.0 + 17.14 + 5V_{BC} = 0$$

$$\boxed{\begin{aligned} V_{BC} &= -15.43 \text{ kN.} \\ V_{CB} &= +15.43 \text{ kN.} \end{aligned}}$$

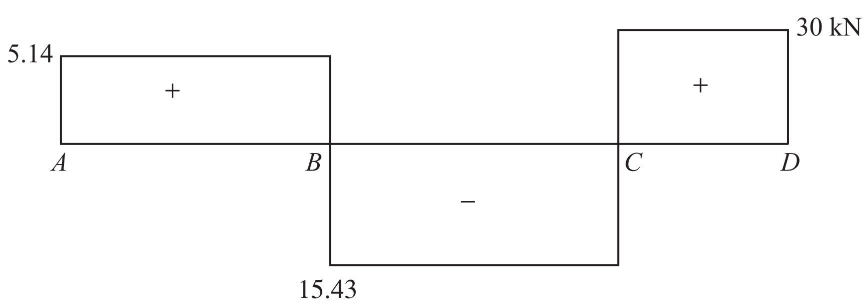


FIG. 11.13d Shear force diagram

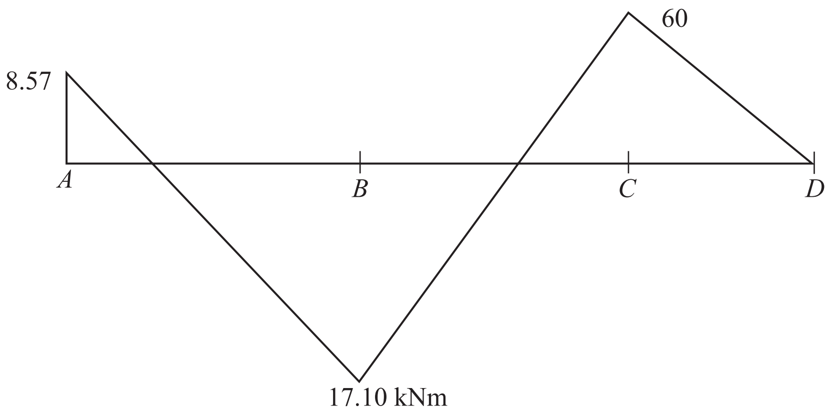


FIG. 11.13e Bending moment diagram

EXAMPLE 11.14: Analyse the continuous beam by the theorem of three moments. Draw neat sketches of *SFD* and *BMD*. Clearly indicate all the salient values.

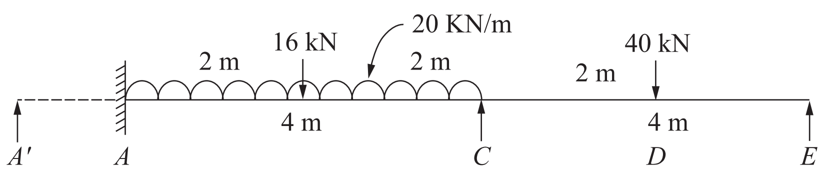


FIG. 11.14a

SOLUTION

The simple beam moments are

$$M_B = \frac{wl^2}{8} + \frac{wl}{4} = 20 \times \frac{4^2}{8} + 16 \times \frac{4}{4} = 56 \text{ kNm}$$

$$M_D = \frac{wl}{4} = 40 \times \frac{4}{4} = 40 \text{ kNm}$$

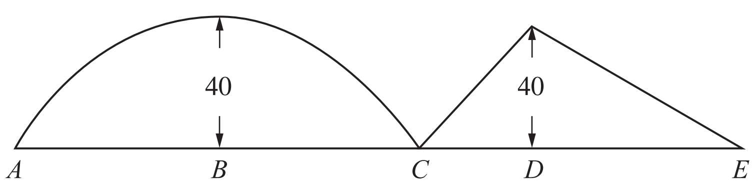


FIG. 11.14b Simple beam moment diagram

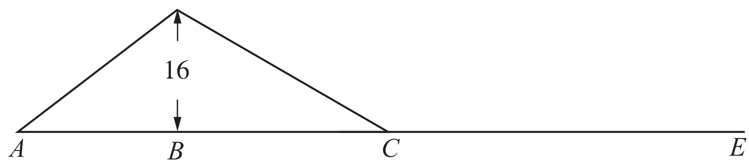


FIG. 11.14c Simple beam moment diagram

Properties of simple beam *BMD*

$$A_1 = \frac{2}{3} \times 4 \times 40 + \frac{1}{2} \times 4 \times 16 = 138.67$$

$$l_1 = 4 \text{ m}$$

$$\bar{x}_1 = 2 \text{ m}$$

$$A_2 = \frac{1}{2} \times 4 \times 40 = 80 \text{ kNm}^2$$

$$\bar{x}_2 = 2 \text{ m}$$

$$l_2 = 4 \text{ m}$$

Applying three moment theorem for span $A'AC$

$$\cancel{M'_A} l_1 + 2M_A\left(\cancel{l_1} + l_2\right) + M_C l_2 = -6\left(\cancel{\frac{A_1\bar{x}_1}{l_1}} + \frac{A_2\bar{x}_2}{l_2}\right)$$

$$2M_A(4) + 4M_C = -6 \times 138.67 \times \frac{2}{4}$$

$$8M_A + 4M_C = -416.01 \qquad \text{(i)}$$

Applying theorem of three moments for the spans ACE

$$M_A(4) + 2M_C(4+4) + \cancel{M_E(4)} = -6\left(\frac{138.67 \times 2}{4} + \frac{80 \times 2}{4}\right)$$

$$4M_A + 16M_C = -6(109.335) = -656.01 \qquad \text{(ii)}$$

Solving Equations (i) and (ii)

$$\boxed{\begin{aligned} M_A &= -36 \text{ kNm} \\ M_C &= -32 \text{ kNm} \end{aligned}}$$

Free body diagram

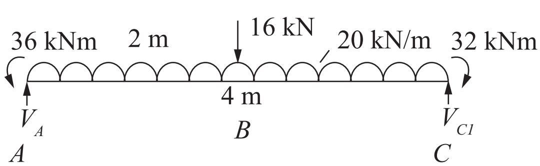


FIG. 11.14d

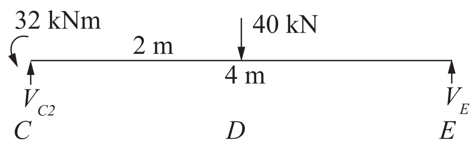


FIG. 11.14e

Static equilibrium of AC

$\sum V = 0;$

$V_A + V_{C1} = 16 + 4(20) = 96$ (i)

$\sum M_A = 0$

$-36 + 32 + 4V_A - 16(2) - 20 \times \frac{4^2}{2} = 0$ (ii)

$V_A = 49$ kN
$V_{C1} = 47$ kN

Static equilibrium of CE

$\sum V = 0;$

$V_{C2} + V_E = 40$ (iii)

$\sum M_E = 0;$

$-32 + 4V_{C2} - 40(2) = 0$ (iv)

$\therefore \quad V_{C2} = 28$ kN
$V_E = 12$ kN

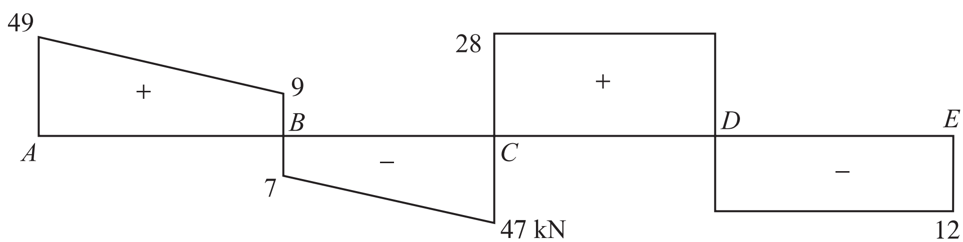


FIG. 11.14f Shear force Diagrams

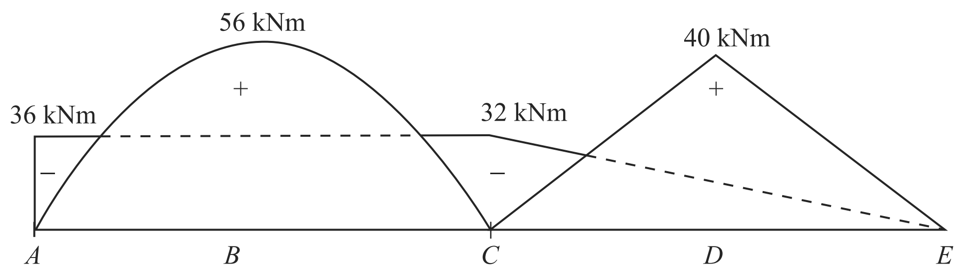


FIG. 11.14g Bending moment diagram

EXAMPLE 11.15: Sketch the BMD for the continuous beam shown in figure.

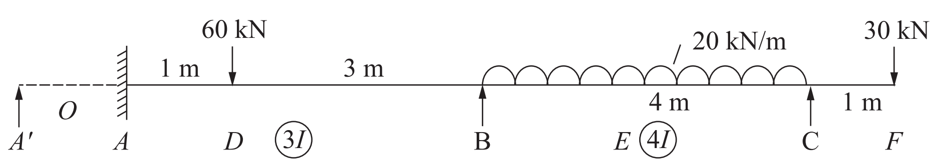


FIG. 11.15a

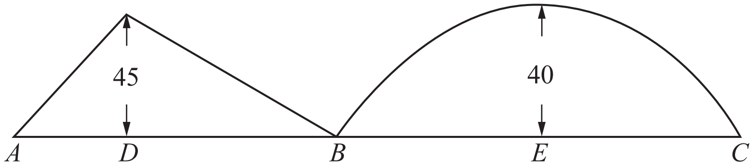


FIG. 11.15b Simple beam moment diagram

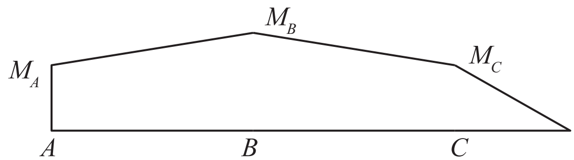


FIG. 11.15c Pure moment diagram

SOLUTION

<u>Properties of the simple beam *BMD*</u>

$A_1 = \frac{1}{2} \times 4 \times 45 = 90\ \text{kNm}^2$	$A_2 = \frac{2}{3} \times 4 \times 40 = 106.7\ \text{kNm}^2$
$\bar{x}_1 = \frac{4+1}{3} = 1.67\ \text{m}$	$\bar{x}_2 = 2\ \text{m}$
$l_1 = 4\ \text{m}$	$l_2 = 4\ \text{m}$

Since A is fixed assume an imaginery span of $A'A$ of zero length with no loading. Assume A' as simply supported. Apply three moment equation for the span $A'AB$,

$$M'_A(0) + 2M_A\left(\frac{0+4}{\cancel{3I}}\right) + M_B\left(\frac{4}{\cancel{3I}}\right) = -6\left(\frac{0 + 90 \times 2.33}{\cancel{4} \times 3I}\right)$$

$$8M_A + 4M_B = -315 \qquad \text{(i)}$$

Applying three moment theorem for the spans AB and BC;

$$M_A\left(\frac{4}{3I}\right) + 2M_B\left(\frac{4}{3I} + \frac{4}{4I}\right) + M_C\left(\frac{4}{4I}\right) = -6\left(\frac{90 \times 1.67}{4 \times 3I} + \frac{106.7 \times 2}{4 \times 4I}\right).$$

$$1.33M_A + 2M_B(1.33 + 1.0) - 30 = -6\ (12.525 + 13.338)$$

$$1.33M_A + 4.66M_B = 30 - (25.863)6$$

$$1.33M_A + 4.66M_B = -125.18 \qquad \text{(ii)}$$

Solving (i) and (ii);

$$\boxed{\begin{aligned} M_A &= -30.3\ \text{kNm.} \\ M_B &= -18.1\ \text{kNm.} \end{aligned}}$$

Free body diagrams of span *AB* and *BC*

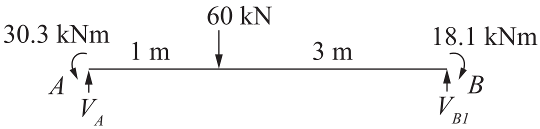


FIG 11.15d

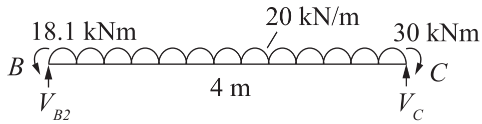


FIG 11.15e

Static equilibrium of AB

$\sum V = 0;$

$V_A + V_B = 60$ (i)

$\sum M_B = 0;$

$-30.3 + 18.1 + 4V_A - 60(3) = 0$

$V_A = 48.05$ kN (ii)

$V_{B2} = 11.95$ kN

Static equilibrium of BC

$\sum V = 0;$

$V_{B2} + V_C = 80$ (iii)

$\sum M_B = 0$

$$-18.10 + 30 + 4V_{B2} - 20 \times \frac{4^2}{2} = 0$$

$V_{B2} = 37.02$ kN.

$V_C = 42.98$ kN.

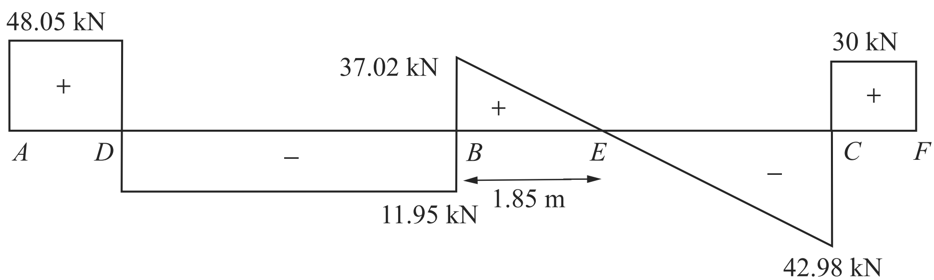


FIG. 11.15f Shear force diagram

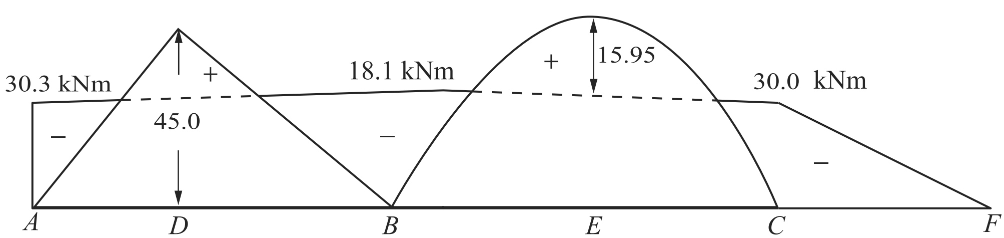


FIG. 11.15g Bending moment diagram

EXAMPLE 11.16: Analyse the continuous beam by three moment theorem. E is constant. Draw the bending moment diagram.

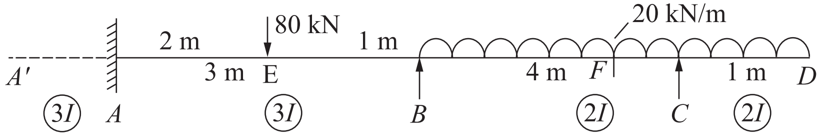


FIG. 11.16a

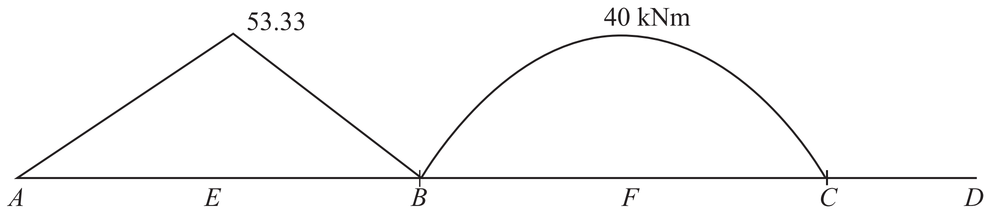


FIG. 11.16b Simple beam BMD

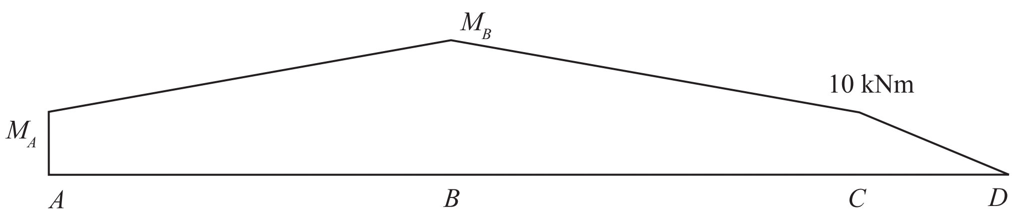


FIG. 11.16c Pure moment diagram

SOLUTION

As the end A is fixed assume an imaginery span A' of zero length with no load and A' is simply supported;

Apply three moment theorem for spans $A'AB$

$$M'_A\left(\cancel{\frac{0}{3I}}\right) + 2M_A\left(\cancel{\frac{0}{3I}} + \frac{3}{3I}\right) + M_B\left(\frac{3}{3I}\right) = -6\left(0 + \frac{80\times 1.33}{3\times 3I}\right)$$

$$2M_A + M_B = -70.93 \qquad \text{(i)}$$

Applying three moment theorem for spans ABC

$$M_A\left(\frac{3}{3I}\right) + 2M_B\left(\frac{3}{3I} + \frac{4}{2I}\right) - 10\left(\frac{4}{2I}\right) = -6\left(\frac{80\times 1.67}{3\times 3I} + \frac{106.67\times 2}{2I\times 4}\right)$$

$$M_A + 6M_B - 20 = -6(14.84 + 26.67)$$

$$M_A + 6M_B = -249.06 \qquad \text{(ii)}$$

Solving (i) and (ii)

$$\boxed{\begin{aligned} M_A &= -16.05 \text{ kNm.} \\ M_B &= -38.84 \text{ kNm.} \end{aligned}}$$

Free Body diagrams of span *AB* and *BC*

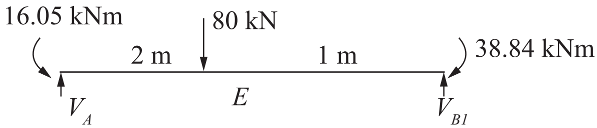


FIG. 11.16d

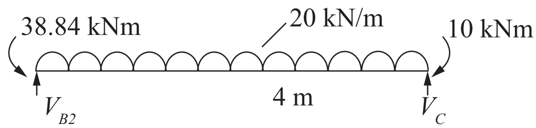


FIG. 11.16e

$$\sum V = 0;$$

$$V_A + V_{B1} = 80 \qquad \text{(i)}$$

$$\sum M_B = 0$$

$$-16.05 + 38.84 - 80(1) + 3V_A = 0 \qquad \text{(ii)}$$

$$\boxed{\begin{aligned} V_A &= +19.07 \text{ kN} \\ V_{B1} &= +60.93 \text{ kN} \end{aligned}}$$

$$\begin{aligned} M_E &= -17.86 + 20.88(2) \\ &= 23.9 \text{ kNm} \end{aligned}$$

$$\sum V = 0;$$

$$V_{B2} + V_c = 80 \qquad \text{(iii)}$$

$$\sum M_C = 0$$

$$10 - 38.84 + 4V_{B2} - \frac{20 \times 4^2}{2} = 0 \qquad \text{(iv)}$$

$$\boxed{\begin{aligned} & V_{B2} = 47.21 \text{ kN} \\ \therefore \quad & V_C = 32.79 \text{ kN} \end{aligned}}$$

The location of zero shear in zone *BC* is obtained from

$$\begin{aligned} 47.21 - 20x &= 0 \\ x &= 2.36 \text{ m} \end{aligned}$$

$$\begin{aligned} \therefore \quad \text{Max +ve BM} &= -38.84 + 47.21 \times 2.36 - 20 \times \frac{2.36^2}{2} \\ &= 16.88 \text{ kNm} \end{aligned}$$

At the midspan of *BC*;

$$M_F = -38.84 + 47.21 \times 2 - 20 \times \frac{2^2}{2} = 15.58 \text{ kNm}$$

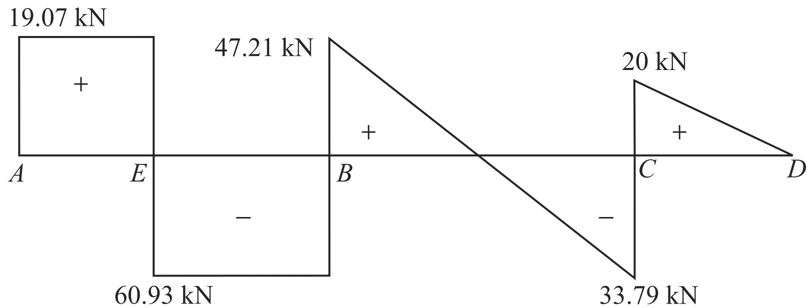


FIG. 11.16f Shear force diagram

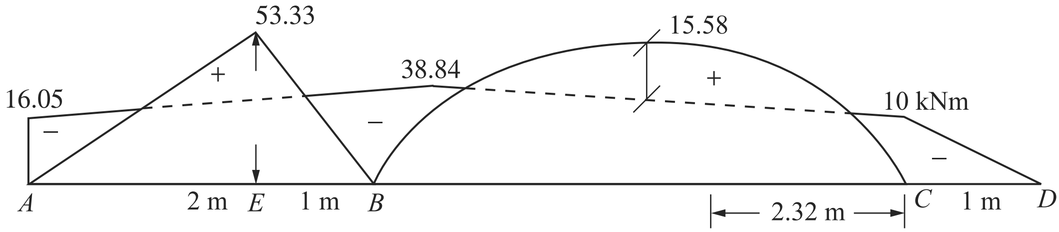


FIG. 11.16g Bending moment diagram

EXAMPLE 11.17: A continuous beam *ABC* is fixed at *A* and *C*. It is continuous over a simple support *B*. Span *AB* is 5 m while *BC* span is 6 m. It is subjected to a concentrated load of 60 kN at 3 m from *A* and the span *BC* is subjected to uniformly distributed load of 10 kN/m. The ratio of flexural rigidity of span *BC* to *BA* is 1.5. Sketch the shear force and bending moment diagram. Use Clapeyron's theorem of three moments.

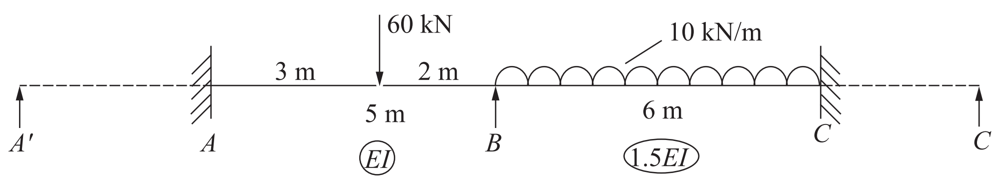


FIG. 11.17a

SOLUTION

The simple beam moments are

$$M_D = \frac{Wab}{l} = \frac{60 \times 3 \times 2}{5} = 72 \text{ kNm}$$

$$M_E = \frac{wl^2}{8} = \frac{10 \times 6^2}{8} = 45 \text{ kNm}$$

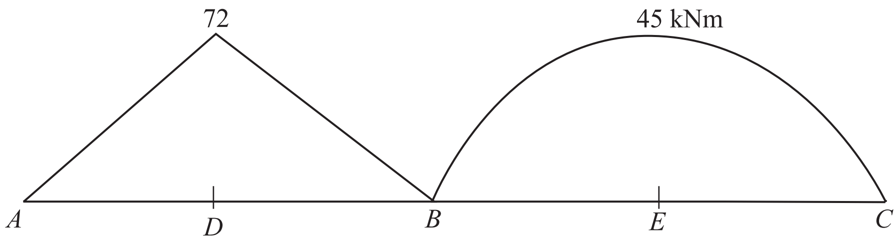


FIG. 11.17b Simple beam BMD

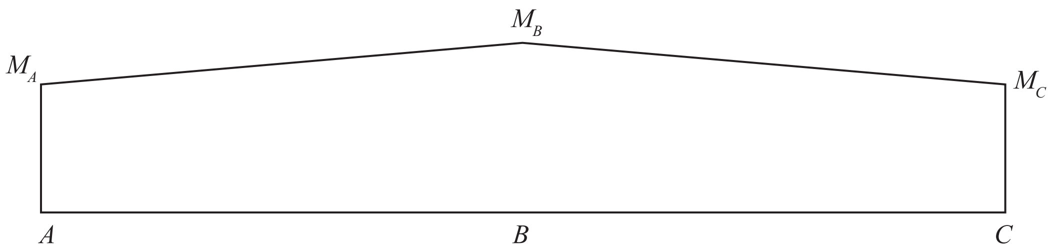


FIG. 11.17c Pure moment diagram

Since A is fixed imagine a span of zero length $A'A$ with no load and A' is simply supported.

Apply three moment theorem for the spans $A'AB$

Properties of the simple beam *BMD*

$A_1 = 0$	$A_2 = \frac{1}{2} \times 5 \times 72 = 180$
$\bar{x}_1 = 0$	$\bar{x}_2 = \frac{5+2}{3} = 2.33$
$l_1 = 0$	$l_2 = 5.0$

$$M'_A\left(\frac{l_1}{I_1}\right) + 2M_A\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right) + M_B\left(\frac{l_2}{I_2}\right) = -6\left(\frac{A_1\bar{x}_1}{l_1} + \frac{A_2\bar{x}_2}{l_2}\right)$$

$$2M_A\left(\frac{5}{I}\right) + M_B\left(\frac{5}{I}\right) = -6\left(\frac{180 \times 2.33}{5 \times I}\right)$$

$$10M_A + 5M_B = -503.28 \tag{i}$$

Apply three moment theorem for the spans ABC

Properties of the simple beam *BMD*

$A_1 = 180\ \text{kNm}^2$	$A_2 = \frac{2}{3} \times 6 \times 45 = 180\ \text{kNm}^2$
$\bar{x}_1 = \frac{5+3}{3} = 2.67\ \text{m}$	$\bar{x}_2 = 3\ \text{m}$
$l_1 = 5\ \text{m}$	$l_2 = 6\ \text{m}$

$$M_A\left(\frac{5}{I}\right)+2M_B\left(\frac{5}{I}+\frac{6}{1.5I}\right)+M_C\left(\frac{6}{1.5I}\right)=-6\left(\frac{180\times 2.67}{5}+\frac{180\times 3}{6\times 1.5}\right)$$

$$5M_A+18M_B+4M_C=-6(96.12+60)$$

$$5M_A+18M_B+4M_C=-936.72 \quad \text{(ii)}$$

Applying three moment theorem BCC′

As the end C is fixed imagine a span CC' of zero length and C' is simply supported

$$M_B\left(\frac{6}{1.5I}\right)+2M_C\left(\frac{6}{1.5I}+0\right)+M_C'\left(\frac{0}{1.5I}\right)=-6\left(\frac{180\times 3}{6\times 1.5}+0\right)$$

$$4M_B+8M_C=-360 \quad \text{(iii)}$$

Solving equations (i), (ii) and (iii)

$$M_A = 31.62 \text{ kNm.}$$
$$M_B = 37.4 \text{ kNm.}$$
$$M_C = 26.29 \text{ kNm.}$$

Shear force and bending moment values for the spans *AB* and *BC* respectively.

Span AB

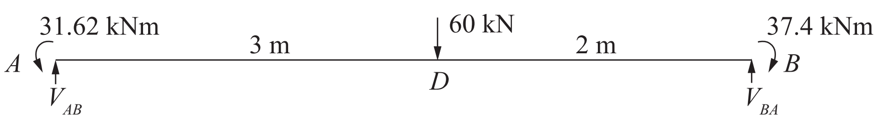


FIG. 11.17d

$$\sum V=0; \quad V_{AB}+V_{BA}=60 \quad \text{(i)}$$

$$\sum M_B=0; \quad 5V_{AB}-60(2)-31.62+37.4=0 \quad \text{(ii)}$$

$$V_{AB}=22.84 \text{ kN}$$

$$V_{BA}=37.16 \text{ kN}$$

Span BC

$$M_D=22.84(3)-31.62=36.9 \text{ kNm}$$

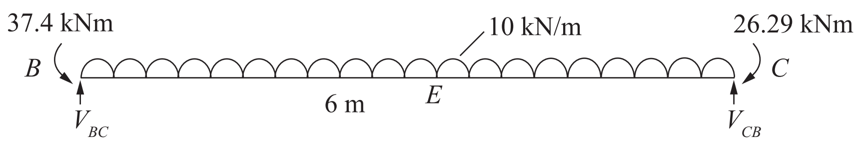


FIG. 11.17e

$$\sum V = 0; \quad V_{BC} + V_{CB} = 60 \tag{iii}$$

$$\sum M_C = 0; \quad -37.4 + 26.29 + 6V_{BC} - 10 \times \frac{6^2}{2} = 0$$

$$\boxed{\begin{aligned} V_{BC} &= 31.85 \text{ kN} \\ V_{CB} &= 28.15 \text{ kN} \end{aligned}}$$

$$M_E = 31.85(3) - 37.4 - 10 \times \frac{3^2}{2} = 13.15 \text{ kNm}$$

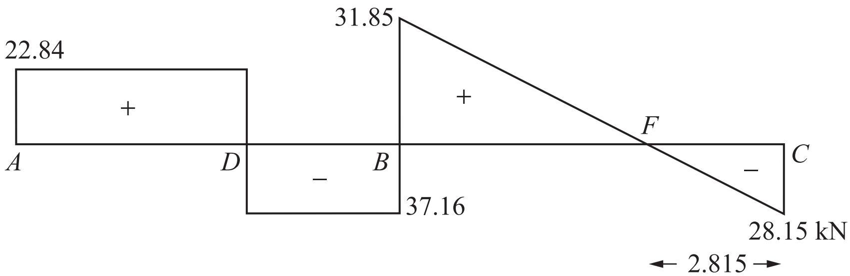


FIG. 11.17f Shear force diagram

The location of zero shear in span *CB* is obtained by equating the shear force equation to zero as

$$(SF)_{xx} = 28.15 - 10x = 0$$

$$x = 2.815 \text{ m}$$

$$M_F = 28.15(2.815) - 10(2.815)^2/2 - 26.29$$

$$= 13.2 \text{ kNm}$$

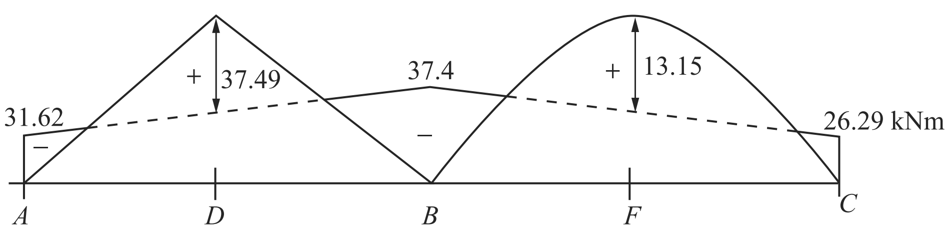


FIG. 11.17g Bending moment diagram

EXAMPLE 11.18: A continuous beam *ABCD* is of uniform section as shown in figure. *EI* is constant. Draw the *SFD* and *BMD*

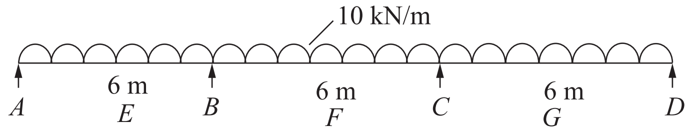


FIG. 11.18a

SOLUTION

The simple beam moments are

$$M_E = M_F = M_G = \frac{10 \times 6^2}{8} = 45 \text{ kNm}$$

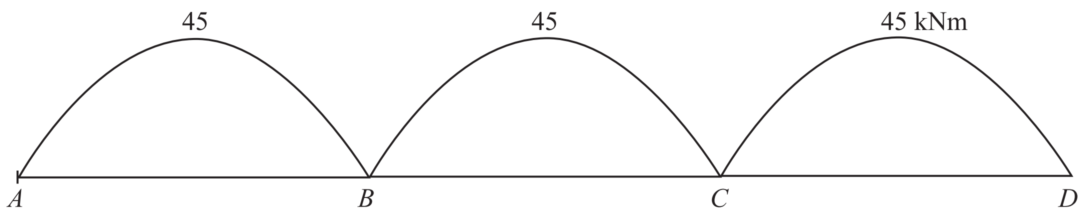


FIG. 11.18b Simple beam BMD

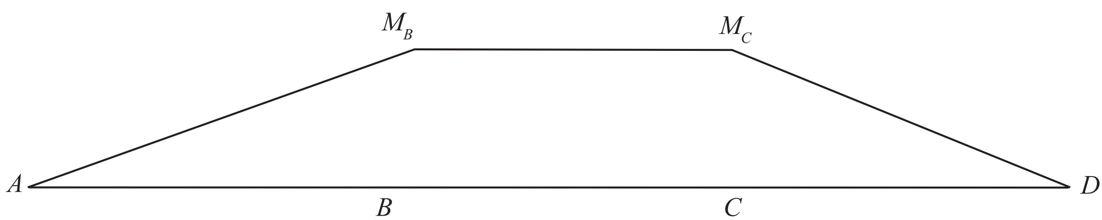


FIG. 11.18c Pure moment diagram

Considering spans ABC

$$A_1 = \frac{2}{3} \times 6 \times 45 = 180 \text{ KNm}^2$$

$$\bar{x}_1 = 3 \text{ m}$$

$$\cancel{6M_A} + 2M_B(6+6) + 6M_C = -6\left(\frac{180 \times 3}{6} + \frac{180 \times 3}{6}\right)$$

$$24M_B + 6M_C = -6(90+90) = -1080 \qquad (1)$$

Considering span BCD

$$6M_B + 2M_C(6+6) + \cancel{6M_D} = -6\left(\frac{180\times 3}{6} + \frac{180\times 3}{6}\right)$$

$$6M_B + 24M_C = -1080 \tag{2}$$

Solving Equations (1) and (2)

$$\boxed{\begin{aligned} M_B &= -36\ \text{kNm} \\ M_C &= -36\ \text{kNm} \end{aligned}}$$

Shear force and bending moment values in the spans *ABC, BCD*

Consider span AB

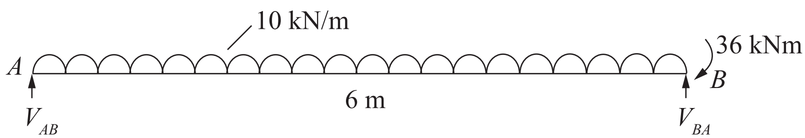


FIG. 11.18d

$$\sum V = 0;\quad V_{AB} + V_{BA} = 6(10) = 60 \tag{i}$$

$$\sum M_B = 0;\quad 6V_{AB} + 36 - 10\times\frac{6^2}{2} = 0 \tag{ii}$$

$$\therefore\ \boxed{\begin{aligned} V_{AB} &= 24\ \text{kN} \\ V_{BA} &= 36\ \text{kN} \end{aligned}}$$

Consider span BC

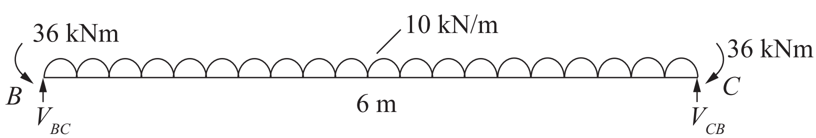


FIG. 11.18e

$$\sum V = 0;\quad V_{BC} + V_{CB} = 60 \tag{iii}$$

$$\sum M_C = 0;\quad 6V_{BC} - 36 + 36 - 10\times\frac{6^2}{2} = 0 \tag{iv}$$

$$V_{BC} = 30\ \text{kN},\ V_{CB} = 30\ \text{kN}$$

Span CD

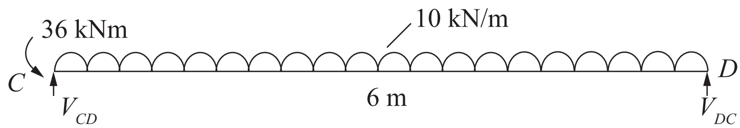


FIG. 11.18f

$$\sum V = 0; \quad V_{CD} + V_{DC} = 10 \times 6 = 60 \tag{v}$$

$$\sum M_D = 0; \quad 6V_{CD} - 36 - 10 \times \frac{6^2}{2} = 0$$

$$\boxed{\begin{aligned} V_{CD} &= 36 \text{ kN} \\ V_{DC} &= 24 \text{ kN} \end{aligned}}$$

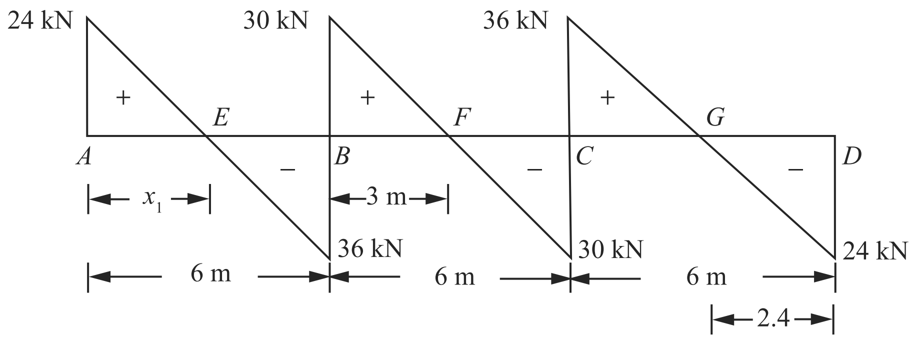


FIG. 11.18g Shear force diagram

The location of zero shear is calculated as

$$24 - 10x_1 = 0$$

$$x_1 = 2.4 \text{ m}$$

$$M_E = 24(2.4) - 10(2.4)^2/2 = 28.8 \text{ kNm}$$

$$M_F = 30(3) - 36 - 10 \times 3^2/2 = 9.0 \text{ kNm}$$

$$M_G = 24\ (2.4) - 10(2.4)^2/2 = 28.8 \text{ kNm}$$

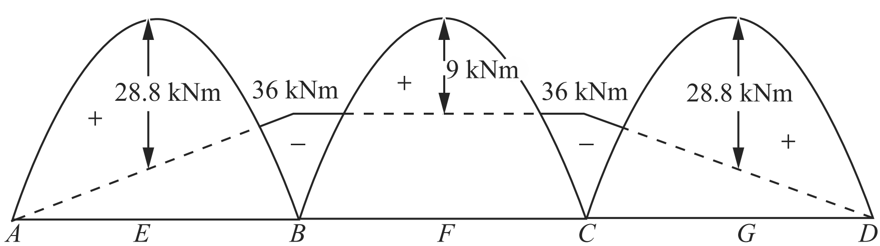


FIG. 11.18h Bending moment diagram

EXAMPLE 11.19: Analyse the continuous beam by three moment theorem. Also draw SFD and BMD.

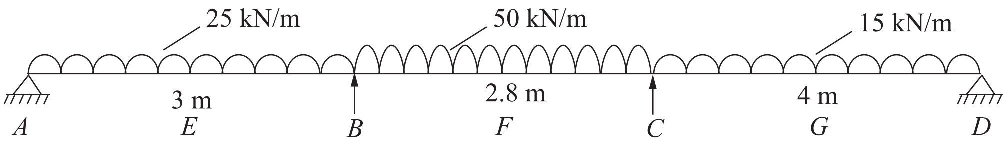


FIG. 11.19a

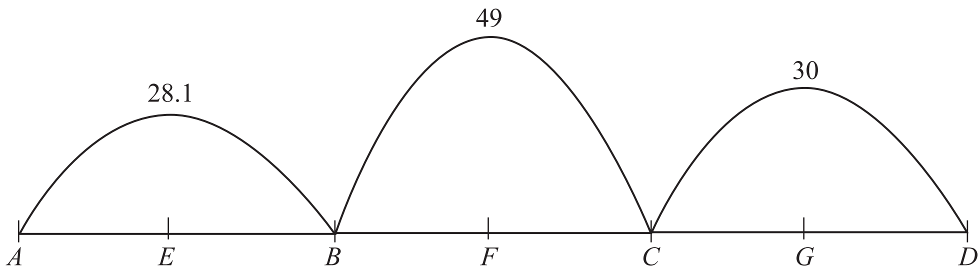


FIG. 11.19b Simply supported BMD

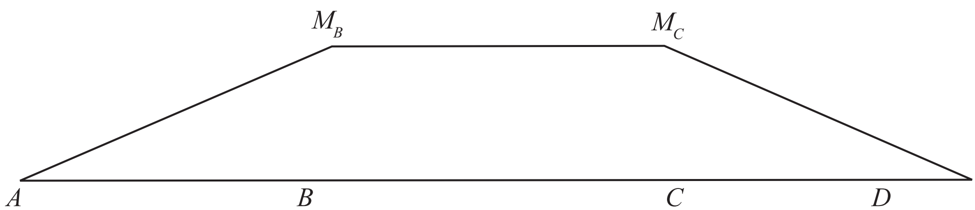


FIG. 11.19c Pure moment diagram

SOLUTION

<u>Properties of the simple beam *BMD*</u>

$A_1 = \dfrac{2}{3} \times 3 \times 28.1 = 56.2\ \text{kNm}^2$	$A_2 = \dfrac{2}{3} \times 2.8 \times 49 = 91.47\ \text{kNm}^2$	$A_3 = 80\ \text{kNm}^2$
$\bar{x}_1 = 1.5\ \text{m}$	$\bar{x}_2 = 1.4\ \text{m}$	$\bar{x}_3 = 2\ \text{m}$
$l_1 = 3\ \text{m}$	$l_2 = 2.8\ \text{m}$	$l_3 = 4\ \text{m}$

Applying three moment theorem for spans ABC

$$\cancel{M_A}(3) + 2M_B(3 + 2.8) + 2.8M_C = -6\left(\frac{56.2 \times 1.5}{3} + \frac{91.47 \times 1.4}{2.8}\right)$$

$$11.6M_B + 2.8M_C = -6(28.1 + 45.74)$$

$$11.6M_B + 2.8M_C = -443 \qquad \text{(i)}$$

Applying three moment theorem for spans BCD

$$2.8M_B + 2M_C(2.8+4) + 4M_D = -6\left(\frac{91.47\times 1.4}{2.8} + \frac{80\times 2}{4}\right)$$

$$2.8M_B + 13.6M_C = -6\ (45.74+40)$$

$$2.8M_B + 13.6M_C = -514.44 \qquad \text{(ii)}$$

Solving (i) and (ii)

$$\boxed{M_B = -30.58\text{ kNm},\ M_C = -31.53\text{ kNm}}$$

Free body diagrams of AB, BC and CD

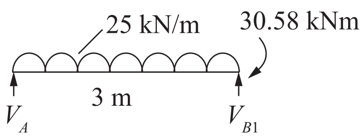


FIG. 11.19d

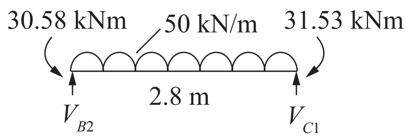


FIG. 11.19e

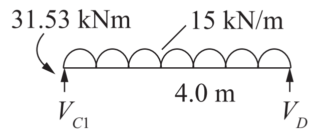


FIG. 11.19f

Static equilibrium of spans AB, BC and CD

$$\sum V = 0;$$

$$V_A + V_{B1} = 75 \qquad \text{(i)}$$

$$\sum M_B = 0;$$

$$3V_A + 30.58 - \frac{25}{2}\times 3^2 = 0 \qquad \text{(ii)}$$

$$V_A = 27.3\text{ kN}$$

$$V_{B1} = 47.7\text{ kN}$$

$$\sum V = 0;$$

$$V_{B_2} + V_{C_1} = 140 \qquad \text{(iii)}$$

$$\sum M_C = 0;$$

$$31.53 - 30.58 + 2.8V_{B2} - 50\times\frac{2.8^2}{2} = 0 \qquad \text{(iv)}$$

$$V_{B2} = 69.66\text{ kN}$$

$$V_C = 70.34\text{ kN}$$

$$\sum V = 0;$$

$$V_{C_2} + V_D = 60 \qquad \text{(v)}$$

$$\sum M_D = 0;$$

$$-31.53 + 4V_{C_2} - 15\times\frac{4^2}{2} = 0 \qquad \text{(vi)}$$

$$V_{C_2} = 151.53/4 = 37.88\text{ kN}$$

$$V_D = 22.12\text{ kN}$$

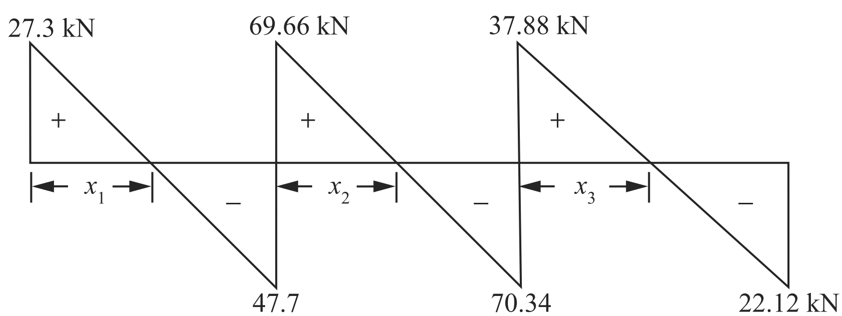


FIG. 11.19g Shear force diagram

The locations of shear forces in zones AB, BC and CD are

$27.3 - 25x_1 = 0$	$69.66 - 50x_2 = 0$	$37.88 - 15x_3 = 0$
$x_1 = 1.09$ m	$x_2 = 1.39$ m	$x_3 = 2.52$ m

$$M_1 = 27.3(1.09) - 25 \times 1.09^2/2 = 14.9 \text{ kNm}$$

$$M_2 = -30.58 + 69.66(1.39) - 50 \times 1.39^2/2 = 17.94 \text{ kNm}$$

$$M_3 = -31.53 + 37.88(2.52) - 15 \times 2.52^2/2 = 16.3 \text{ kNm}$$

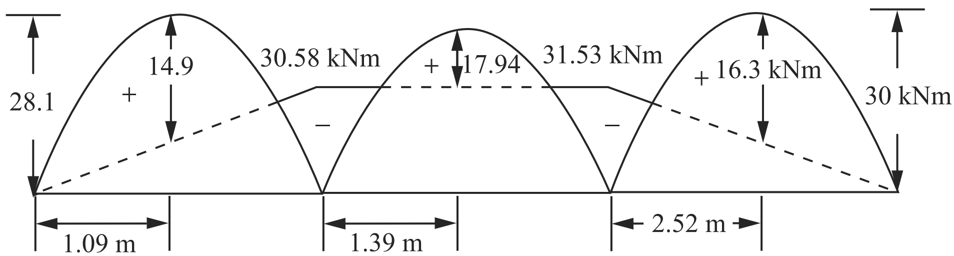


FIG. 11.19h Bending moment diagram

EXAMPLE 11.20: Analyse the continuous beam by theorem of three moments and draw SFD and BMD. EI is constant.

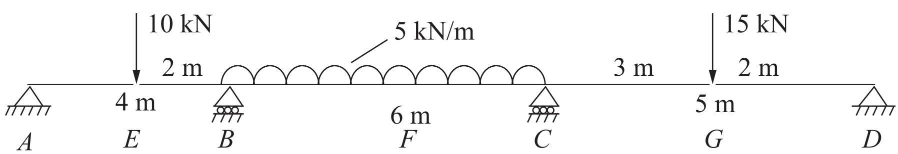


FIG. 11.20a

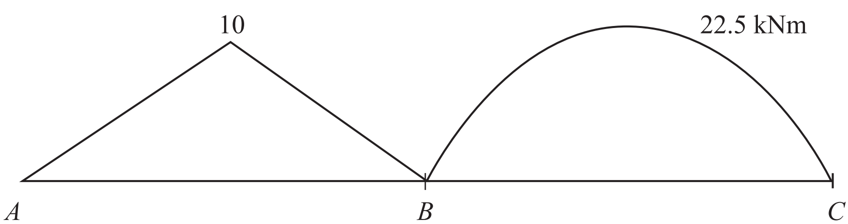


FIG. 11.20b Simple beam BMD for span *ABC*

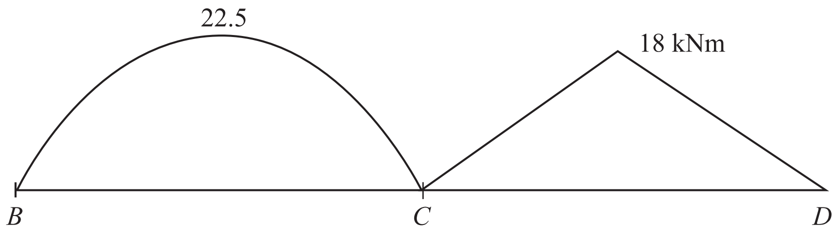


FIG. 11.20c Simple beam BMD for span *BCD*

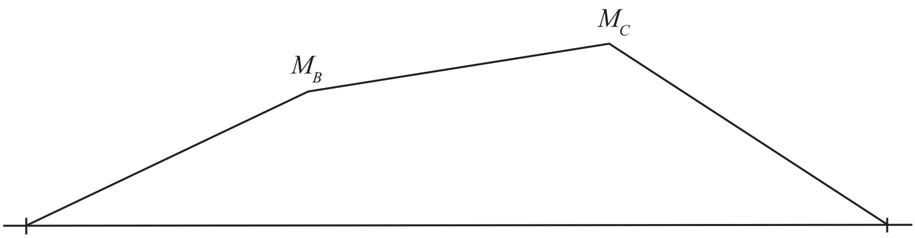


FIG. 11.20d Pure moment diagram

SOLUTION

Referring to Fig. 11.20 b

Properties of simple beam *BMD*

$A_1 = \frac{1}{2} \times 10 \times 4 = 20\ \text{kNm}^2$	$A_2 = \frac{2}{3} \times 6 \times 22.5 = 90\ \text{kNm}^2$
$\bar{x}_1 = 2\ \text{m}$	$\bar{x}_2 = 3\ \text{m}$
$l_1 = 4\ \text{m}$	$l_2 = 6\ \text{m}$

Applying three moment theorem for spans ABC,

$$\cancel{4M_A} + 2M_B(4+6) + 6M_C = -6\left(\frac{20 \times 2}{4} + \frac{90 \times 3}{6}\right)$$

$$20M_B + 6M_C = -6(10+45)$$

$$20M_B + 6M_C = -330 \qquad \text{(i)}$$

Referring to Fig. 11.20 c

Properties of simple beam *BMD*

$A_1 = \frac{2}{3} \times 6 \times 22.5 = 90\ \text{kNm}^2$	$A_2 = \frac{1}{2} \times 5 \times 18 = 45\ \text{kNm}^2$
$\bar{x}_1 = 3\ \text{m}$	$\bar{x}_2 = \frac{5+2}{3} = 2.33\ \text{m}$
$l_1 = 6\ \text{m}$	$l_2 = 5\ \text{m}$

Applying three moment theorem for spans BCD
Considering span BCD

$$6M_B + 2M_C(6+5) + 5M_D = -6\left(\frac{90\times 3}{6} + \frac{45\times 2.33}{5}\right)$$

$$6M_B + 22M_C = -6(45 + 20.97)$$

$$6M_B + 22M_C = -395.82 \qquad \text{(ii)}$$

Solving (i) and (ii)

$$M_B = 12.09 \text{ kNm.}$$

$$M_C = 14.69 \text{ kNm.}$$

Shear force and bending moment values for spans *AB, BC* and *CD*.

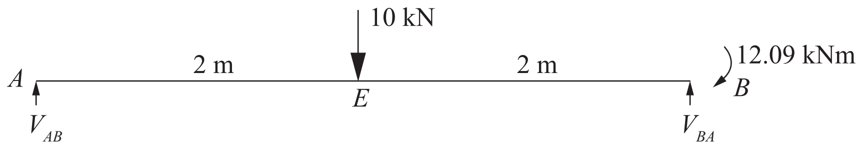


FIG. 11.20e

$$\sum V = 0; \quad V_{AB} + V_{BA} = 10 \qquad \text{(i)}$$

$$\sum M_B = 0; \quad 4V_{AB} + 12.09 - 10(2) = 0 \qquad \text{(ii)}$$

Solving (i) and (ii)

$$\boxed{\begin{aligned} V_{AB} &= 1.98 \text{ kN} \\ V_{BA} &= 8.02 \text{ kN} \end{aligned}}$$

$$M_E = 1.98(2) = 3.96 \text{ kNm}$$

Span BC

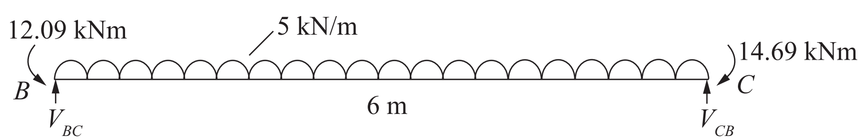


FIG. 11.20f

$$\sum V = 0; \quad V_{BC} + V_{CB} = 6(5) = 30 \text{ kN} \qquad \text{(iii)}$$

$$\sum M_B = 0; \quad 6V_{BC} + 14.69 - 12.09 - \frac{5\times 6^2}{2} = 0$$

$$\boxed{\begin{aligned} V_{BC} &= 14.56 \text{ kN} \\ V_{CB} &= 15.44 \text{ kN} \end{aligned}}$$

The location of shear force is zero is found out as

$$14.56 - 5x = 0$$
$$x = 2.91 \text{ m}$$

$$\text{Hence Max +ve BM} = 14.56(2.91) - 12.09 - 5 \times \frac{2.91^2}{2} = 9.11 \text{ kNm}$$

Span CD

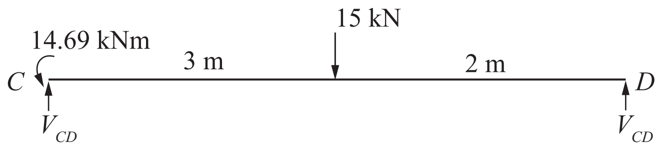


FIG. 11.20g

$$\sum V = 0; \quad V_{CD} + V_{DC} = 15 \qquad \text{(iv)}$$
$$\sum M_D = 0; \quad 5V_{CD} - 14.69 - 15(2) = 0$$

$$\boxed{\begin{aligned} V_{CD} &= 8.94 \text{ kN} \\ V_{DC} &= 6.06 \text{ kN} \end{aligned}}$$

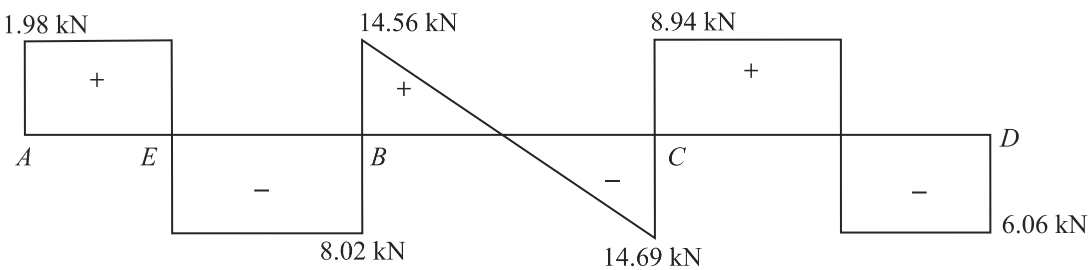


FIG. 11.20h Shear force diagram

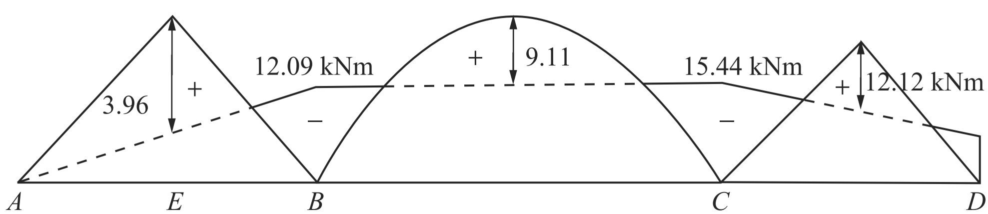


FIG. 11.20i Bending moment diagram

EXAMPLE 11.21: A continuous beam *ABCD* is simply supported at *A* and *D*. It is continuous over supports *B* and *C*. $AB = BC = CD = 4$ m. EI is constant. It is subjected to uniformly distributed load of 8 kN/m over the span *BC*. Draw the shear force diagram and bending moment diagram.

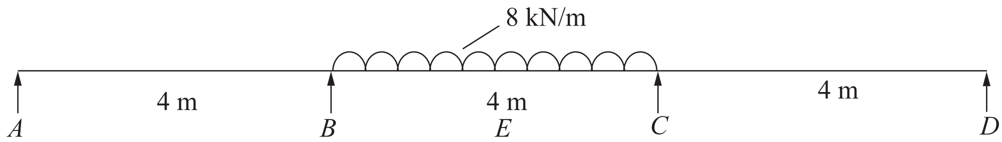


FIG. 11.21a

SOLUTION

The simple beam moment

$$M_E = \frac{8 \times 4^2}{8} = 16 \text{ kNm}$$

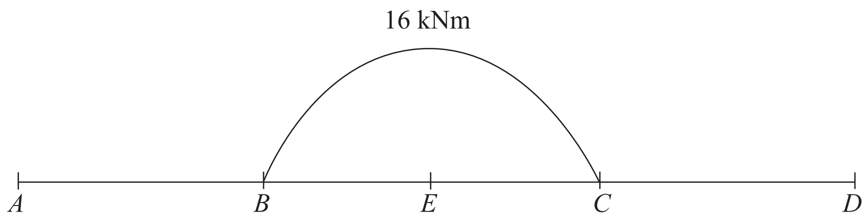


FIG. 11.21b Simple beam bending moment diagram

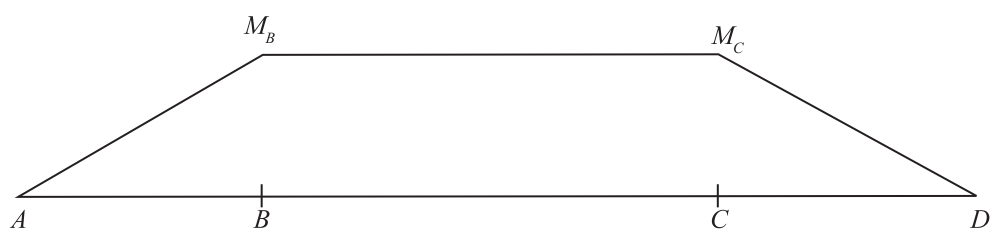


FIG. 11.21c Pure moment diagram

Consider span ABC

Applying three moment theorem;

$$4M_A + 2M_B(4+4) + 4M_C = -6\left(0 + \frac{2}{3} \times \frac{4 \times 16 \times 2}{4}\right)$$

$$16M_B + 4M_C = -128 \quad \text{(i)}$$

Consider span BCD

$$4M_B + 2M_C(4+4) + \cancel{4M_D} = -6\left(\frac{2}{3} \times \frac{4 \times 16 \times 2}{4} + 0\right)$$

$$4M_B + 16M_C = -128 \quad \text{(ii)}$$

Solving (i) and (ii)

$$\boxed{\begin{aligned} M_B &= -6.4\ \text{kNm} \\ M_C &= -6.4\ \text{kNm} \end{aligned}}$$

Shear force and bending moment values of spans *AB*, *BC* and *CD*

span AB

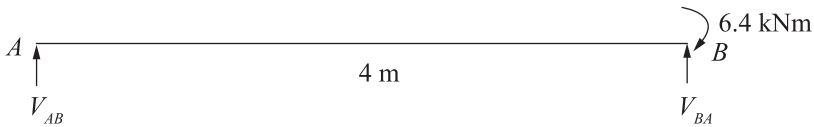


FIG. 11.21d

$$\sum V = 0; \quad V_{AB} + V_{BA} = 0 \tag{i}$$

$$\sum M_B = 0; \quad 4V_{AB} + 6.4 = 0 \tag{ii}$$

$$\boxed{\begin{aligned} V_{AB} &= -1.6\ \text{kN} \\ V_{BA} &= +1.6\ \text{kN} \end{aligned}}$$

span BC

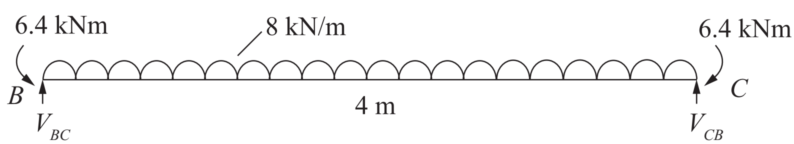


FIG. 11.21e

$$\sum V = 0; \quad V_{BC} + V_{CB} = 8(4) = 32 \tag{iii}$$

$$\sum M_C = 0; \quad -6.4 + 6.4 + 4V_{BC} - \frac{8 \times 4^2}{2} = 0 \tag{iv}$$

$$\boxed{\begin{aligned} V_{BC} &= 16\ \text{kN} \\ V_{CB} &= 16\ \text{kN} \end{aligned}}$$

span CD

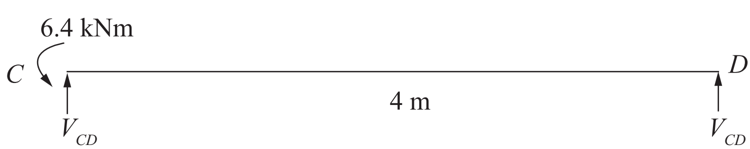


FIG. 11.21f

$$\sum V = 0; \quad V_{CD} + V_{DC} = 0 \tag{v}$$

$$\sum M_D = 0; \quad -6.4 + 4V_{CD} = 0 \tag{vi}$$

$$\boxed{\begin{aligned} V_{CD} &= 1.6\text{ kN} \\ V_{DC} &= -1.6\text{ kN} \end{aligned}}$$

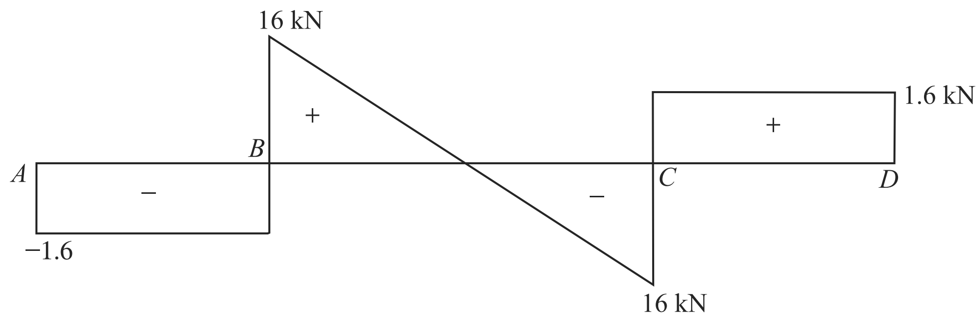


FIG. 11.21g Shear force diagram

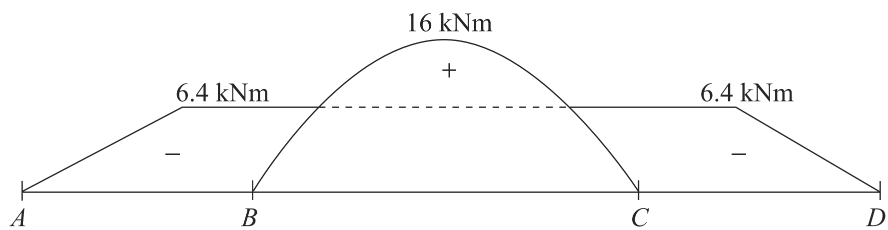


FIG. 11.21h Bending moment diagram

EXAMPLE 11.22: Analyse the beam shown in figure by SFD and BMD. EI is constant.

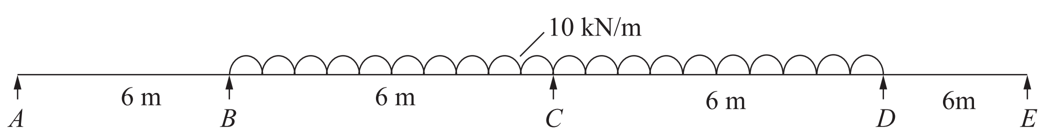


FIG. 11.22a

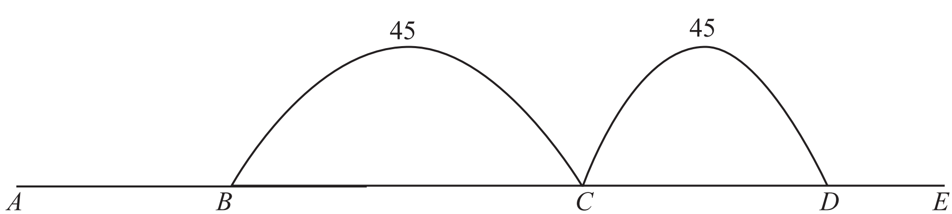


FIG. 11.22b Simple beam BMD

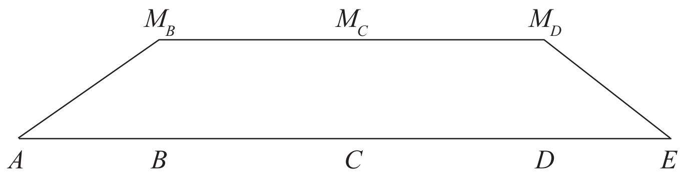


FIG. 11.22c Pure moment diagram

Properties of simple beam *BMD*

$$A_1 = \frac{2}{3} \times 6 \times 45 = 180 \qquad A_2 = \frac{2}{3} \times 6 \times 45 = 180 \text{ KNm}^2$$

$$\bar{x}_1 = 3 \text{ m} \qquad \bar{x}_2 = 3 \text{ m}$$

$$l_1 = 6 \text{ m} \qquad l_2 = 6 \text{ m}$$

Applying 3 moment theorem for the spans ABC

$$6M_A + 2M_B(6+6) + 6M_C = -6\left(0 + \frac{180 \times 3}{6}\right)$$

$$24M_B + 6M_C = -540 \qquad \text{(i)}$$

Applying 3 moment theorem for the spans BCD

$$6M_B + 2M_C(6+6) + 6M_D = -6\left(\frac{180 \times 3}{6} + \frac{180 \times 3}{6}\right)$$

$$6M_B + 24M_C + 6M_D = -6(180) = -1080$$

$$M_B + 4M_C + M_D = -180 \qquad \text{(ii)}$$

Applying 3 moment theorem for spans CDE

$$6M_C + 2M_D(6+6) + 6M_E = -6\left(\frac{180 \times 3}{6} + 0\right)$$

$$6M_C + 24M_D = -540 \qquad \text{(iii)}$$

solving equations (i), (ii) and (iii)

$$\boxed{\begin{aligned} M_B &= -12.86 \text{ kNm.} \\ M_C &= -38.57 \text{ kNm.} \\ M_D &= -12.86 \text{ kNm.} \end{aligned}}$$

Free body diagram of *AB, BC, CD* and *DE*

span AB

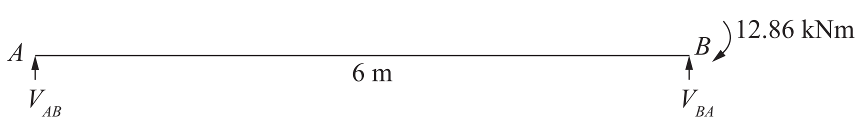


FIG. 11.22d

$$\sum V = 0; \quad V_{AB} + V_{BA} = 0 \tag{i}$$
$$\sum M_B = 0; \quad 6V_{AB} + 12.86 = 0 \tag{ii}$$

$$\boxed{\begin{aligned} V_{AB} &= -2.14\text{ kN} \\ V_{BA} &= +2.14\text{ kN.} \end{aligned}}$$

span BC

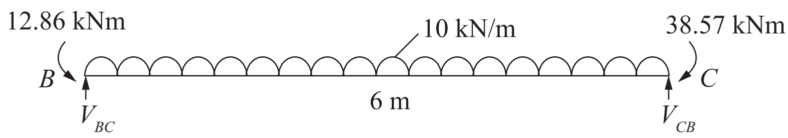


FIG. 11.22e

$$\sum V = 0; \quad V_{BC} + V_{CB} = 60 \tag{iii}$$
$$\sum M_C = 0; \quad -12.86 + 38.57 + 6V_{BC} - \frac{10 \times 6^2}{2} = 0 \tag{iv}$$

$$\boxed{\begin{aligned} V_{BC} &= 25.72\text{ kN} \\ V_{CB} &= 34.28\text{ kN} \end{aligned}}$$

span CD

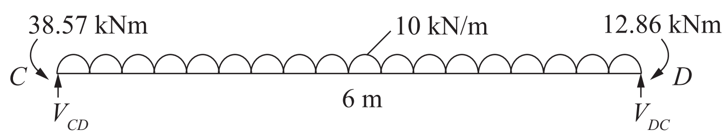


FIG. 11.22f

$$\sum V = 0; \quad V_{CD} + V_{DC} = 6(10) = 60\text{ kN} \tag{v}$$
$$\sum M_D = 0; \quad 12.86 - 38.57 + 6V_{CD} - 10 \times \frac{6^2}{2} = 0 \tag{vi}$$

$$\boxed{\begin{aligned} V_{CD} &= 34.28\text{ kN} \\ V_{DC} &= 25.72\text{ kN} \end{aligned}}$$

span DE

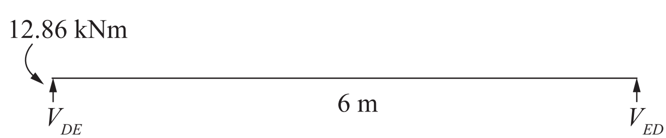


FIG. 11.22g

Solution

$$M_B = -20(1) = -20 \text{ kNm}$$

The simple beam moments are

$$M_F = \frac{20 \times 4^2}{8} = 40 \text{ kNm}$$

$$M_G = \frac{60 \times 4}{4} = 60 \text{ kNm}$$

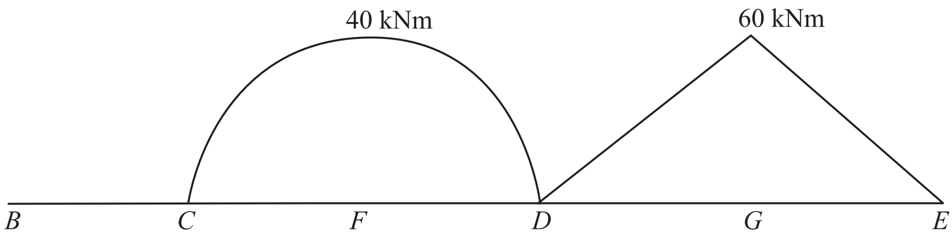


FIG. 11.23b (b) Simple beam BMD

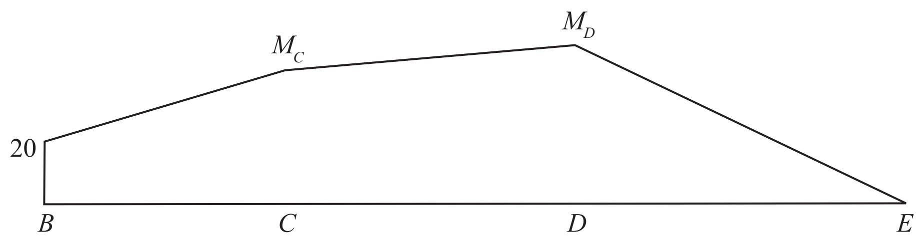


FIG. 11.23c (c) Pure moment diagram

Apply 3 moment theorem for the spans BCD

$$-20(3) + 2M_C(3+4) + M_D(4) = -6\left(0 + \frac{2}{3} \times 4 \times 40 \times \frac{2}{4}\right)$$

$$-60 + 14M_C + 4M_D = -320$$

$$14M_C + 4M_D = -260 \qquad \text{(i)}$$

Apply 3 moment theorem for the spans CDE

$$M_C(4) + 2M_D(4+4) + \cancel{4M_E} = -6\left(\frac{2}{3} \times 4 \times 40 \times \frac{2}{4} + \frac{1}{2} \times 4 \times 60 \times \frac{2}{4}\right)$$

$$4M_C + 16M_D = -6(53.33 + 60) = -680 \qquad \text{(ii)}$$

Solving (i) and (ii)

$$\boxed{\begin{aligned} M_C &= -6.92 \text{ kNm} \\ M_D &= -40.77 \text{ kNm} \end{aligned}}$$

$$+6EI\left(\frac{\delta_A}{L_1}+\frac{\delta_C}{L_2}\right)$$

$$A_1\bar{x}_1 = 960$$

$$A_2\bar{x}_2 = \frac{2}{3}\times 6\times 180\times 3 = 2160$$

Substituting, $M_A\times 4+2M_B(4+6) = -6\left[\frac{960}{4}+\frac{2160}{6}\right]+6EI\left(\frac{240}{EI\times 4}+\frac{120}{EI\times 6}\right)$

$$4M_A+20M_B = -3120 \quad\rightarrow\quad M_A+5M_B = -780 \qquad (2)$$

Solving (1) and (2), $\left.\begin{array}{l} M_A = -163.33\text{ kNm} \\ M_B = -123.33\text{ kNm}\end{array}\right\}$ hogging BM.

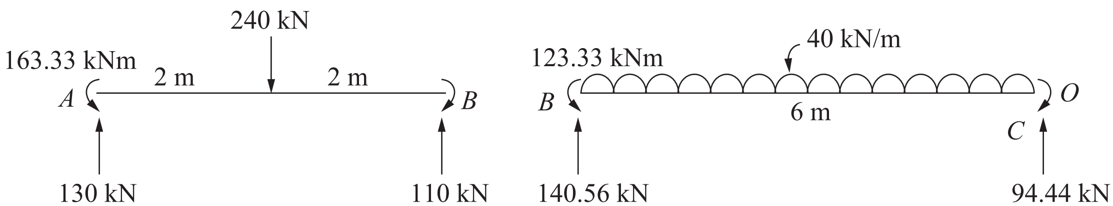


FIG. 11.25d Free body diagram of spans *AB* and *BC*

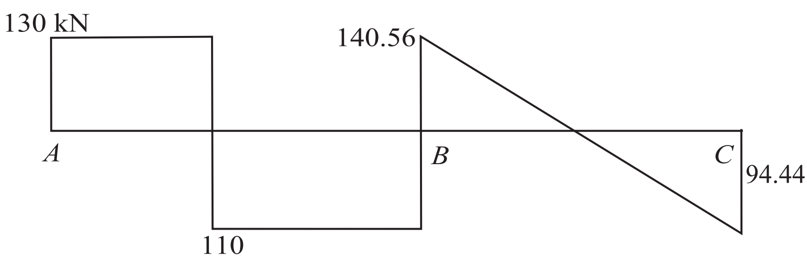


FIG. 11.25e Shear force diagram

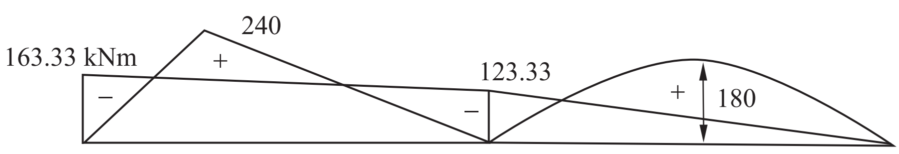
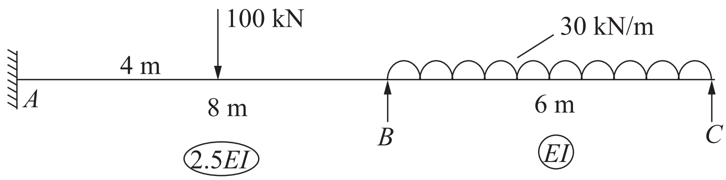


FIG. 11.25f Bending moment diagram

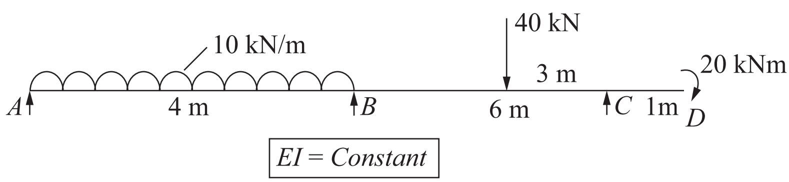


Ans:

$$R_A = 45\text{kN}, \quad R_B = 165.5\text{kN}, \quad R_C = 69.5\text{kN}$$

(11.3) A continuous beam of uniform section *ABCD* is supported and loaded as shown in figure. If the support *B* sinks by 10 mm, determine the resultants and moments at the supports.

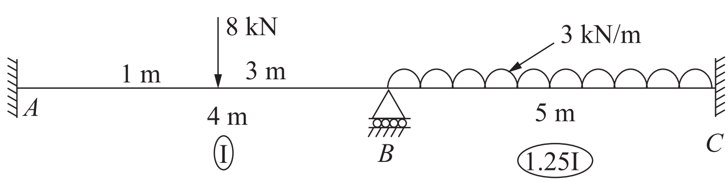
Assume $E = 2(10)^5$ N/mm^2; $I = 6(10)^7$ mm^4



Ans:

$$V_{AB} = +16.5\text{ kN}, \quad V_{BA} = +23.5, \quad V_{BC} = +19, \quad V_{CB} = +21.0$$
$$M_B = -14\text{ kNm}$$

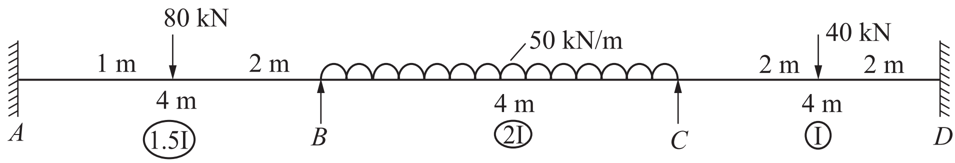
(11.4) Determine the reactions at A, B and C of the continuous beam shown in figure.



Ans:

$$V_{AB} = 6.75\text{ kN}, \quad V_{BA} = 1.25, \quad V_{BC} = 6.31; \quad V_{CB} = 8.69$$
$$M_A = -3.31\text{ kNm}, \quad M_B = -3.87, \quad M_{CB} = +7.44$$

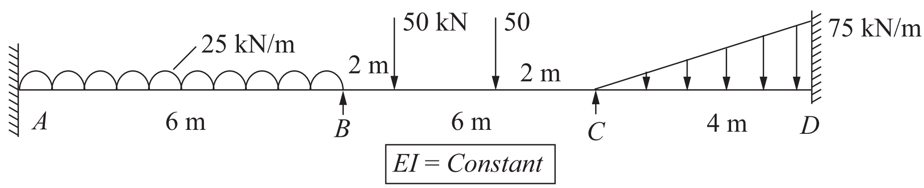
(11.5) Analyse the continuous beam shown in Figure and determine the reactions



Ans:

$V_{AB} = 41.68$ kN, $V_{BA} = 38.32$, $V_{BC} = 102.88$, $V_{CB} = 97.12$
$V_{CD} = 28.03$, $V_{DC} = 11.97$, $M_A = -17.98$ kNm,
$M_B = -52.93$, $M_C = -41.42$, $M_D = -9.29$ kNm

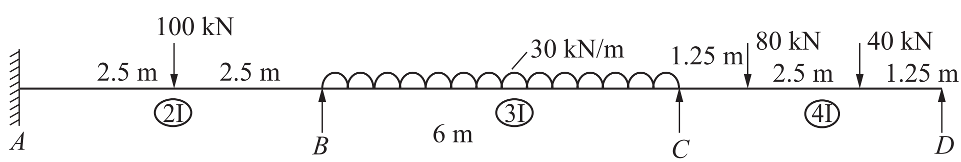
(11.6) Analyse three span continuous beam by three moment theorems. Draw the BMD and shear force diagram. Determine the end moments and reactions *EI* is constant.



Ans:

(i) $R_A = 75.39$ kN, $R_B = 127.59$ kN, $R_C = 97.85$ kN, $R_D = 99.17$ kN
$M_A = -75.78$ kNm, $M_B = -73.44$ kNm, $M_C = -55.55$ kNm, $M_d = -55.2$ kNm.

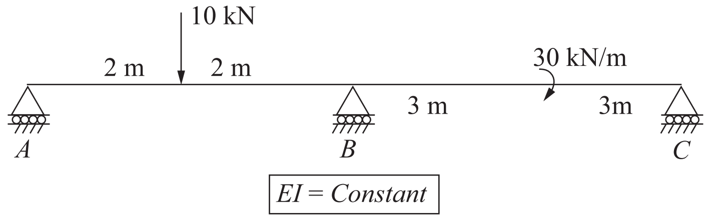
(11.7) Analyse and draw BMD and SFD for the beam shown in Figure. The values of second moment area of each span are indicated along the members. Modulus of elasticity is constant.



Ans:

$M_A = -56.02$ kNm, $M_B = -75.47$ kNm, $M_C = -94.3$ kNm, $M_D = 0$

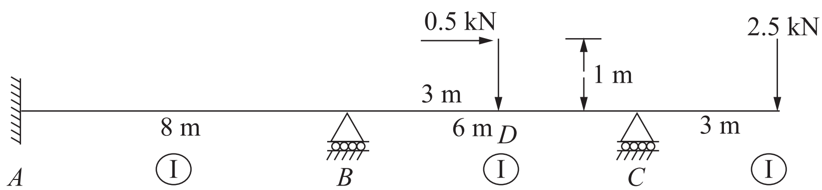
(11.11) Determine the reactions and the support moment at B. Using Clapeyron's three moment theorem.



Ans:

$$V_A = 4.81\text{ kN},\quad V_B = 0.31,\quad V_C = 4.88\text{ kN},\ M_B = -0.72\text{ kNm}$$

(11.12) Analyse the continuous beam by three moment theorem, determine the support moments. No loads on span AB.



Ans:

$$M_A = -1.09\text{ kNm},\quad M_B = -2.188\text{ kNm},\quad M_C = -7.5\text{ kNm}$$