Analysis of Statically Determinate Trusses

THEORY OF STRUCTURES

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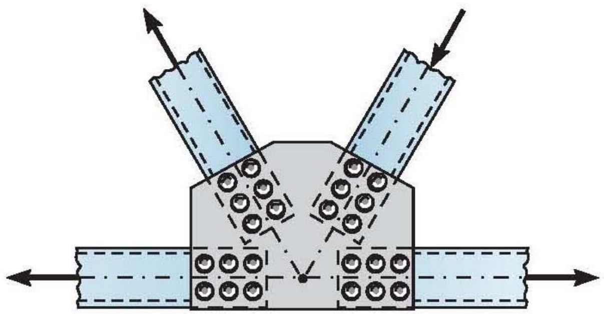
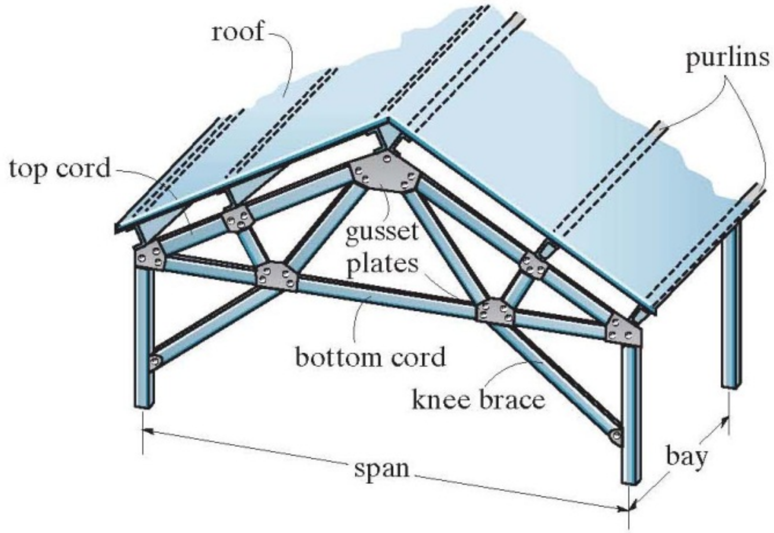
Common Types of Trusses

- A truss is one of the major types of engineering structures which provides a practical and economical solution for many engineering constructions, especially in the design of bridges and buildings that demand large spans.
- A truss is a structure composed of slender members joined together at their end points
- The joint connections are usually formed by bolting or welding the ends of the members to a common plate called gusset
- Planar trusses lie in a single plane & is often used to support roof or bridges



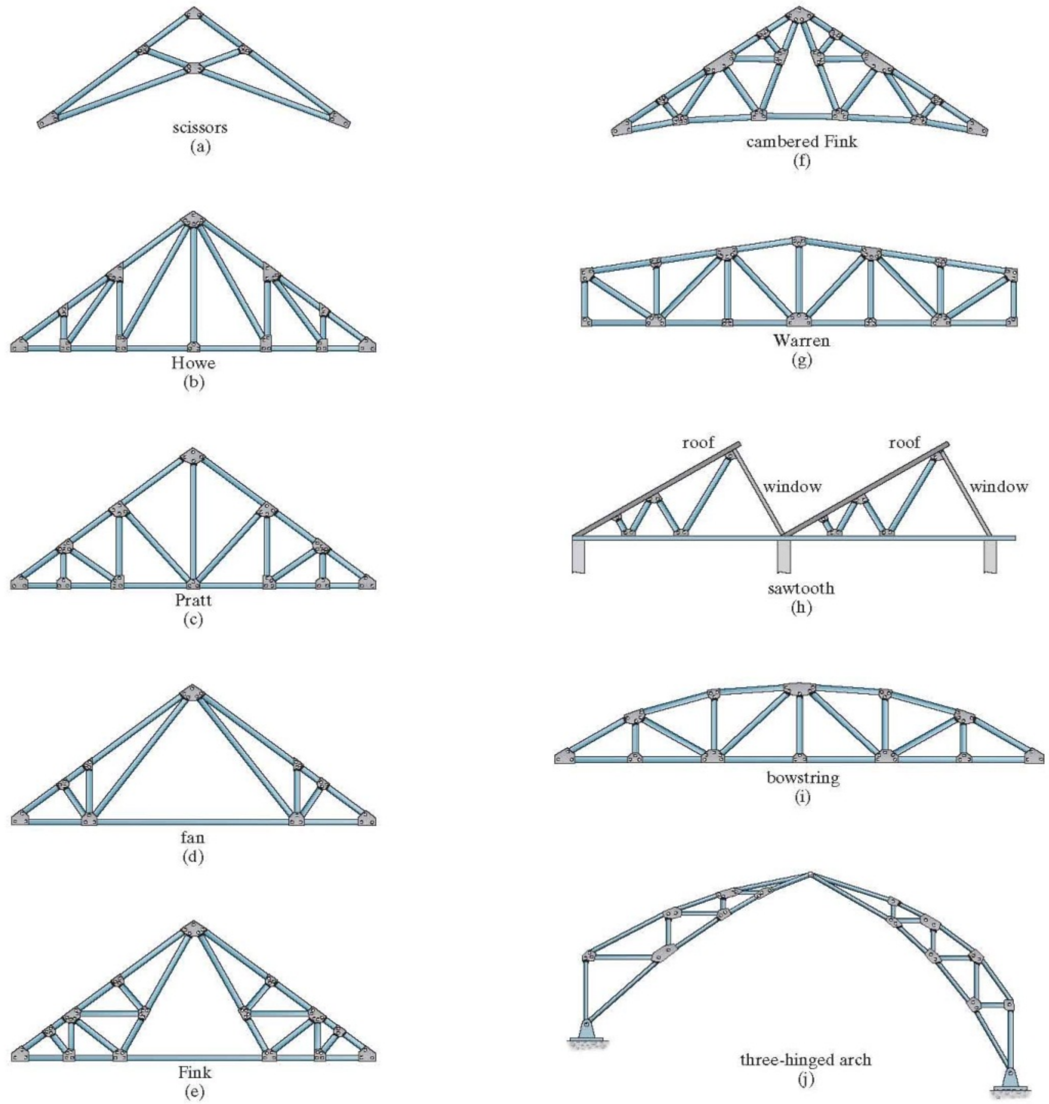
Common Types of Trusses

- Roof Trusses
  - They are often used as part of an industrial building frame
  - Roof load is transmitted to the truss at the joints by means of a series of purlins
  - To keep the frame rigid & thereby capable of resisting horizontal wind forces, knee braces are sometimes used at the supporting column



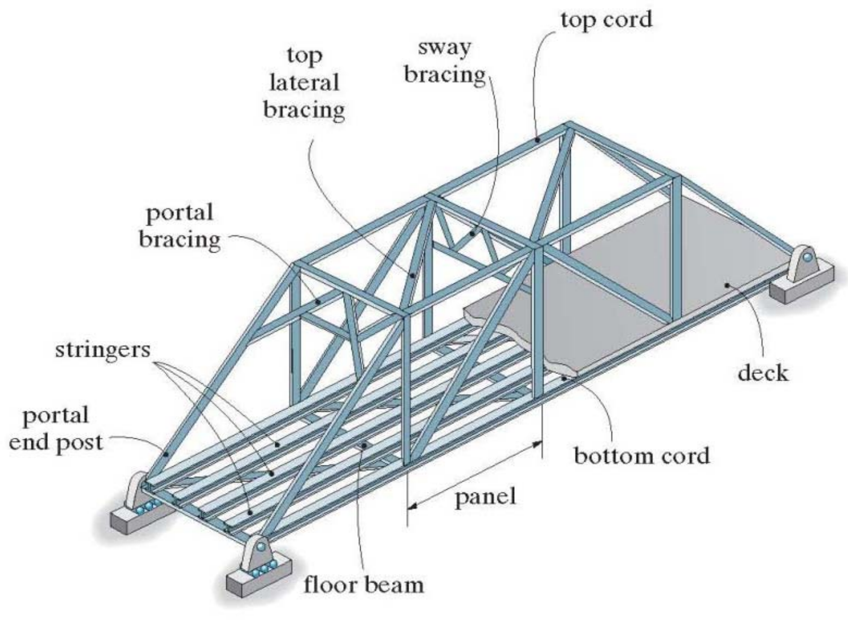
Common Types of Trusses

- Roof Trusses



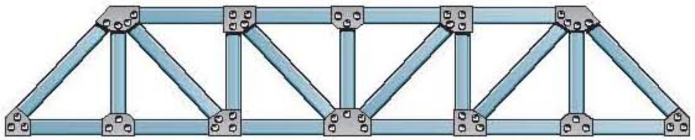
Common Types of Trusses

- Bridge Trusses
  - The main structural elements of a typical bridge truss are shown in figure. Here it is seen that a load on the deck is first transmitted to stringers, then to floor beams, and finally to the joints of the two supporting side trusses.
  - The top and bottom cords of these side trusses are connected by top and bottom lateral bracing, which serves to resist the lateral forces caused by wind and the sidesway caused by moving vehicles on the bridge.
  - Additional stability is provided by the portal and sway bracing. As in the case of many long-span trusses, a roller is provided at one end of a bridge truss to allow for thermal expansion.

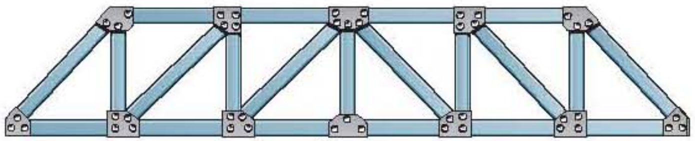


Common Types of Trusses

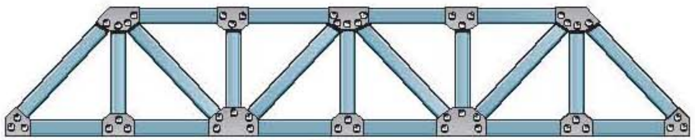
- Bridge Trusses
  - In particular, the **Pratt**, **Howe**, and **Warren** trusses are normally used for spans up to 61 m in length. The most common form is the Warren truss with verticals.
  - For larger spans, a truss with a polygonal upper cord, such as the **Parker** truss, is used for some savings in material.
  - The Warren truss with verticals can also be fabricated in this manner for spans up to 91 m.



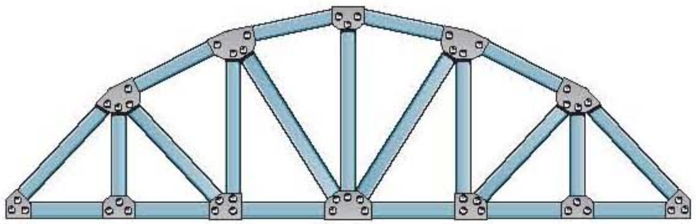
Pratt
(a)



Howe
(b)



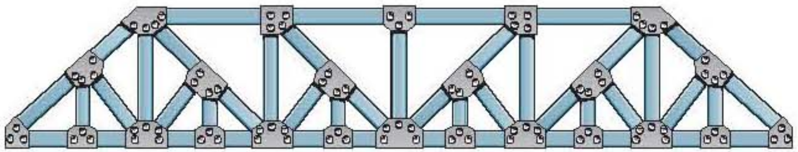
Warren (with verticals)
(c)



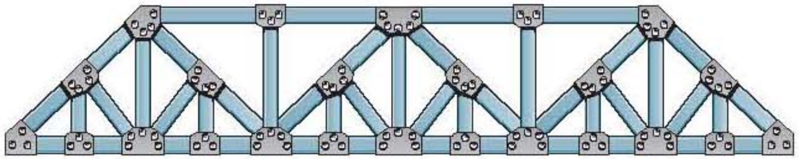
Parker
(d)

Common Types of Trusses

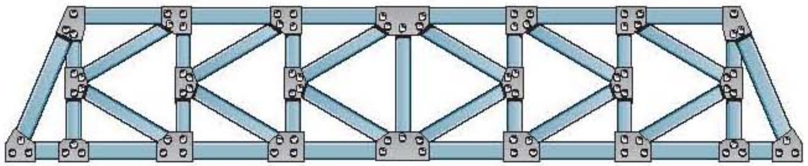
- Bridge Trusses
  - The greatest economy of material is obtained if the diagonals have a slope between 45° and 60° with the horizontal. If this rule is maintained, then for spans greater than 91 m, the depth of the truss must increase and consequently the panels will get longer.
  - This results in a heavy deck system and, to keep the weight of the deck within tolerable limits, subdivided trusses have been developed. Typical examples include the **Baltimore** and **subdivided Warren** trusses.
  - The **K-truss** shown can also be used in place of a subdivided truss, since it accomplishes the same purpose.



Baltimore
(e)



subdivided Warren
(f)



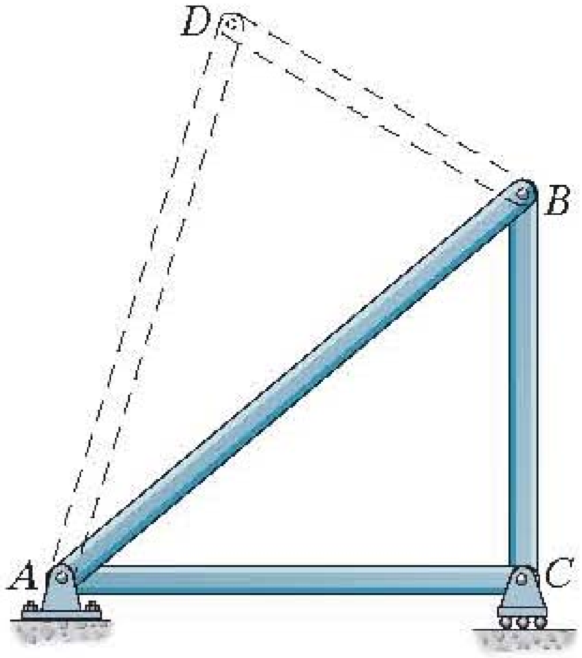
K-truss
(g)

Common Types of Trusses

- Assumptions for Design
  - The members are joined together by smooth pins
  - All loadings are applied at the joints
- Due to the 2 assumptions, each truss member acts as an axial force member

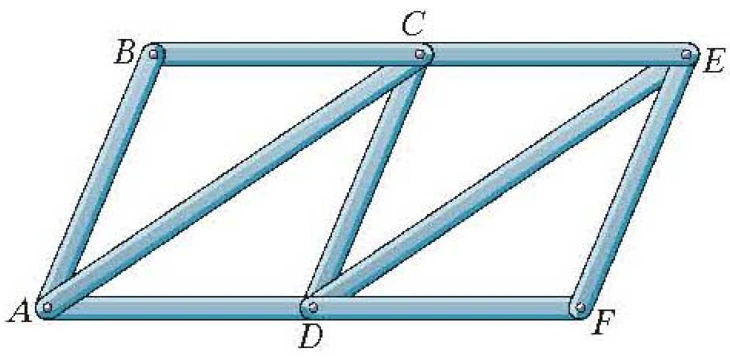
Classification of Coplanar Trusses

- Simple , Compound or Complex Truss
- Simple Truss
  - To prevent collapse, the framework of a truss must be rigid
  - The simplest framework that is rigid or stable is a triangle

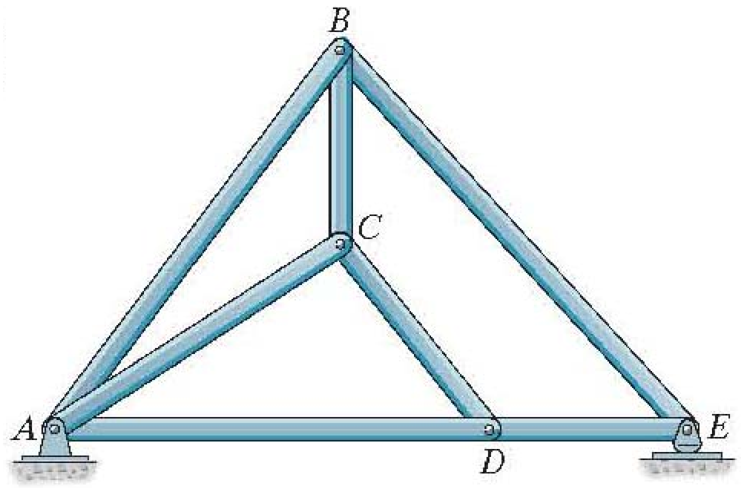


Classification of Coplanar Trusses

- Simple Truss
  - The basic “stable” triangle element is ABC
  - The remainder of the joints D, E & F are established in alphabetical sequence
  - Simple trusses do not have to consist entirely of triangles



simple truss



simple truss

Classification of Coplanar Trusses

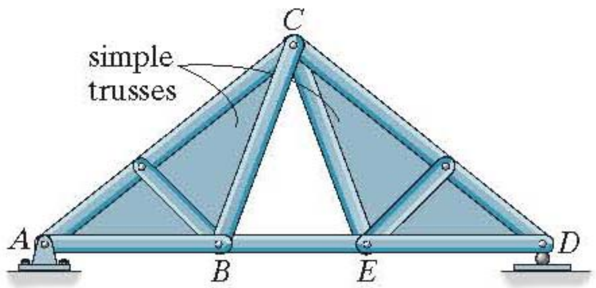
- Compound Truss
  - It is formed by connecting 2 or more simple truss together
  - Often, this type of truss is used to support loads acting over a larger span as it is cheaper to construct a lighter compound truss than a heavier simple truss

Classification of Coplanar Trusses

- Compound Truss
  - Type 1
    - The trusses may be connected by a common joint & bar
  - Type 2
    - The trusses may be joined by 3 bars
  - Type 3
    - The trusses may be joined where bars of a large simple truss, called the main truss, have been substituted by simple truss, called secondary trusses

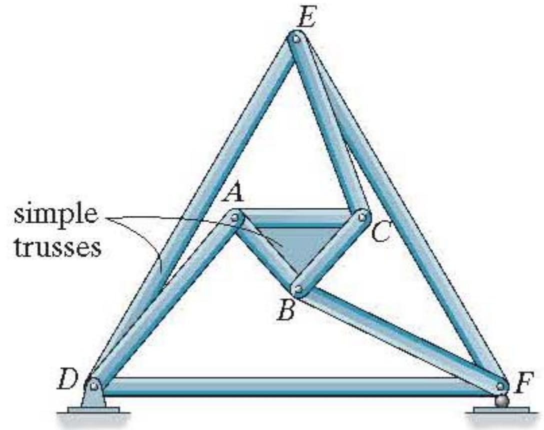
Classification of Coplanar Trusses

- Compound Truss



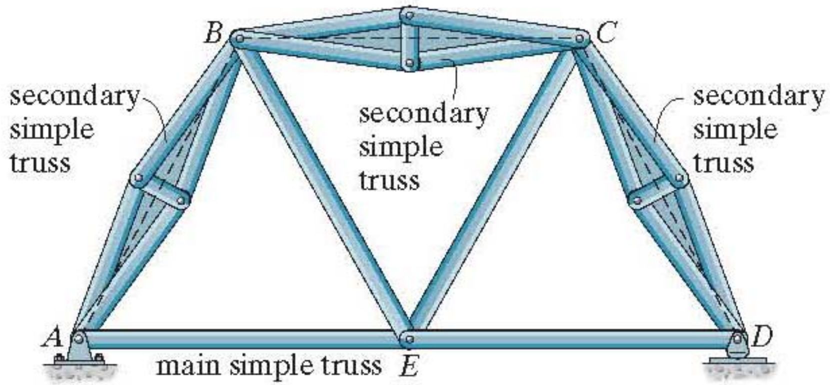
compound truss

(a)



compound truss

(b)

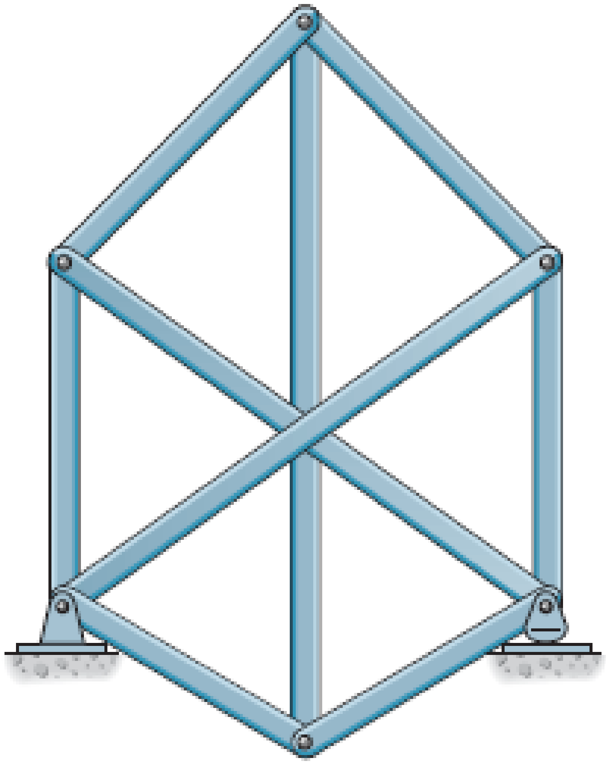


compound truss

(c)

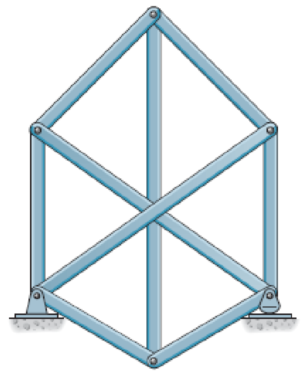
Classification of Coplanar Trusses

- Complex Truss
  - A complex truss is one that cannot be classified as being either simple or compound



Classification of Coplanar Trusses

- Determinacy
  - The total number of unknowns includes the forces in b number of bars of the truss and the total number of external support reactions r.
  - Since the truss members are all straight axial force members lying in the same plane, the force system acting at each joint is coplanar and concurrent.
  - Consequently, rotational or moment equilibrium is automatically satisfied at the joint (or pin).



Classification of Coplanar Trusses

- Determinacy
  - Therefore only

$$\sum F_x = 0 \text{ and } \sum F_y = 0$$

  - By comparing the total unknowns with the total number of available equilibrium equations, we have:

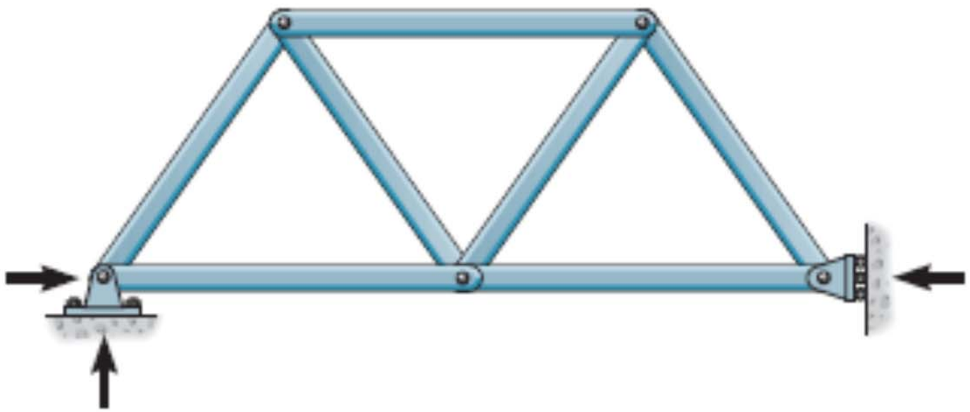
$$\begin{aligned} b + r &= 2j \quad \text{statically determinate} \\ b + r &> 2j \quad \text{statically indeterminate} \end{aligned}$$

Classification of Coplanar Trusses

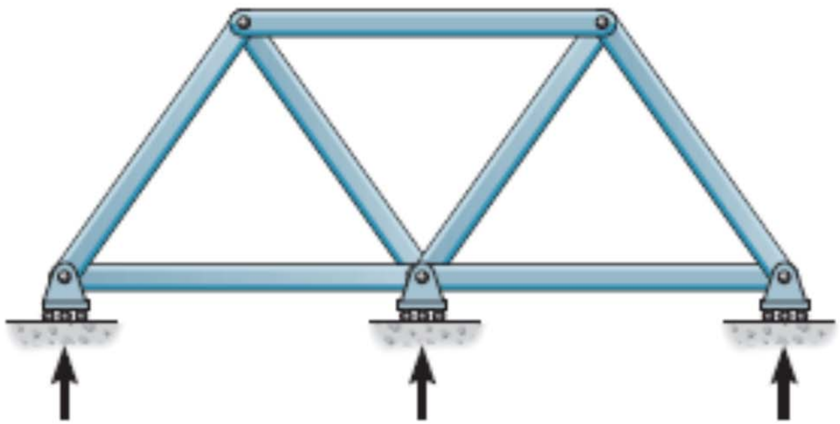
- Stability
  - If $b + r < 2j$ => collapse
  - A truss can be unstable if it is statically determinate or statically indeterminate
  - Stability will have to be determined either through inspection or by force analysis

Classification of Coplanar Trusses

- Stability
  - External Stability
    - A structure is externally unstable if all of its reactions are concurrent or parallel
    - The trusses are externally unstable since the support reactions have lines of action that are either concurrent or parallel



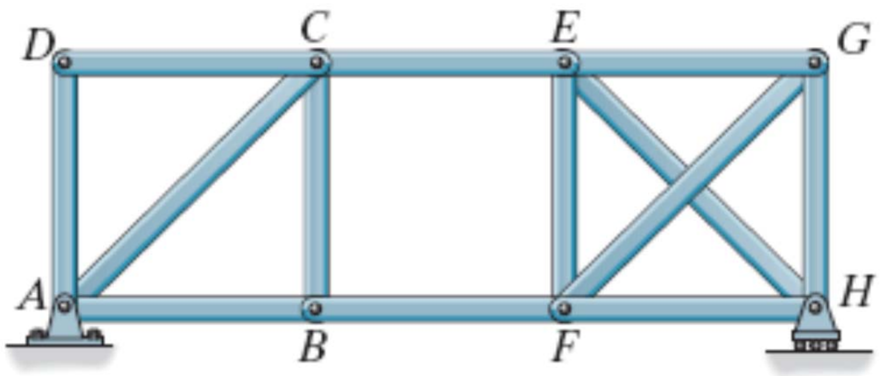
unstable concurrent reactions



unstable parallel reactions

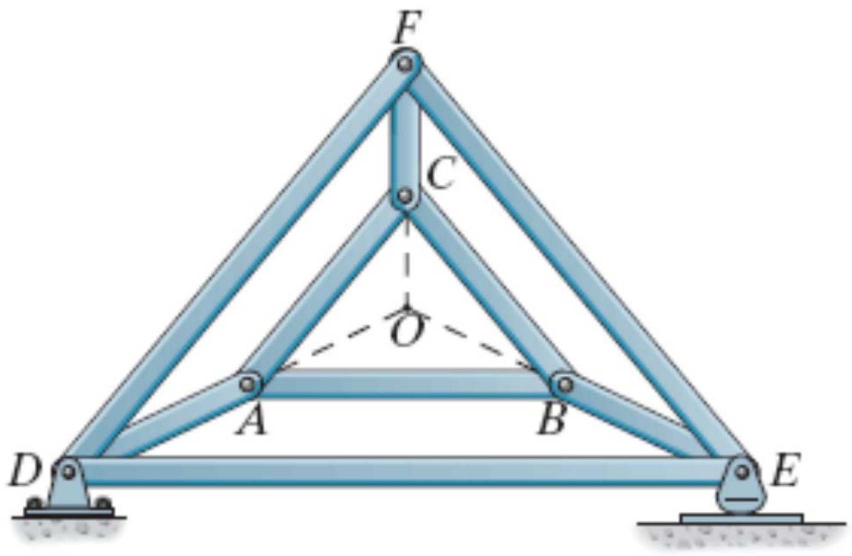
Classification of Coplanar Trusses

- Internal Stability
  - The internal stability can be checked by careful inspection of the arrangement of its members
  - If it can be determined that each joint is held fixed so that it cannot move in a "rigid body" sense with respect to the other joints, then the truss will be stable
  - A simple truss will always be internally stable
  - If a truss is constructed so that it does not hold its joints in a fixed position, it will be unstable or have a "critical form"



Classification of Coplanar Trusses

- Internal Stability
  - To determine the internal stability of a compound truss, it is necessary to identify the way in which the simple truss are connected together
  - The truss shown is unstable since the inner simple truss ABC is connected to DEF using 3 bars which are concurrent at point O

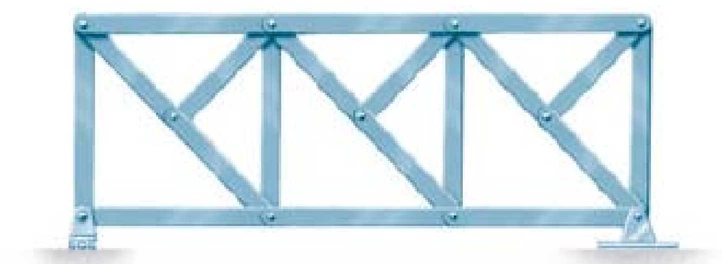


Classification of Coplanar Trusses

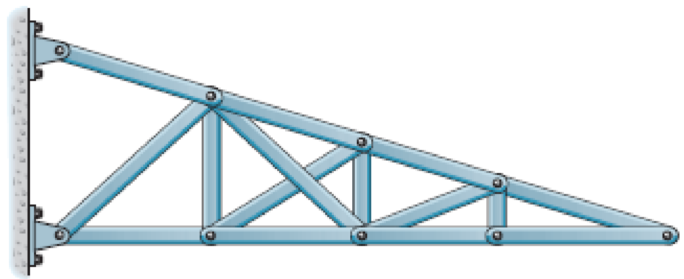
- Internal Stability
  - Thus an external load can be applied at A, B or C & cause the truss to rotate slightly
  - For complex truss, it may not be possible to tell by inspection if it is stable
  - The instability of any form of truss may also be noticed by using a computer to solve the 2j simultaneous equations for the joints of the truss
  - If inconsistent results are obtained, the truss is unstable or have a critical form

Example 3.1

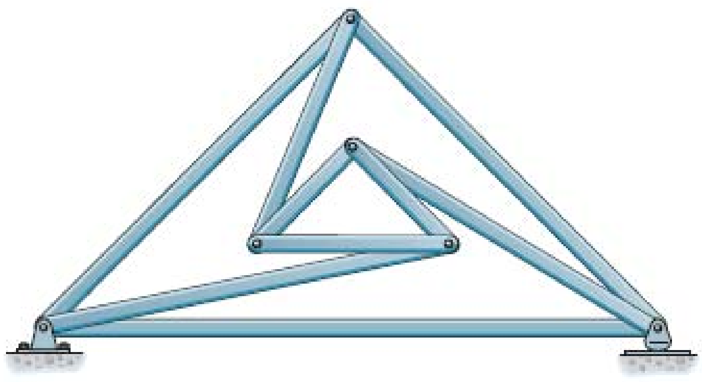
Classify each of the trusses as stable, unstable, statically determinate or statically indeterminate. The trusses are subjected to arbitrary external loadings that are assumed to be known & can act anywhere on the trusses.



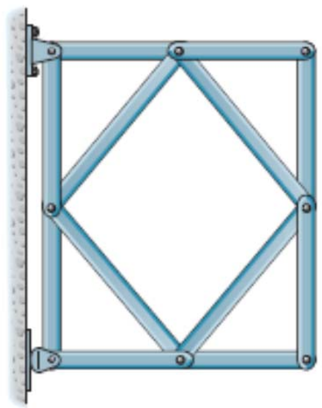
(a)



(b)



(c)



(d)

Solution

For (a),

Externally stable

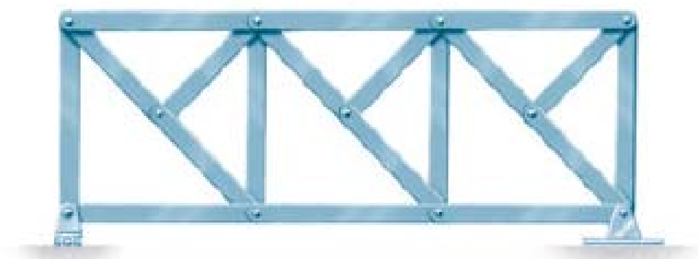
Reactions are not concurrent or parallel

$b = 19, r = 3, j = 11$

$b + r = 2j = 22$

Truss is statically determinate

By inspection, the truss is internally stable



(a)

Solution

For (b),

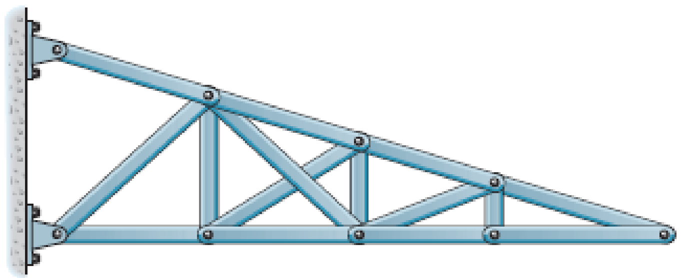
Externally stable

$b = 15, r = 4, j = 9$

$b + r = 19 > 2j$

Truss is statically indeterminate

By inspection, the truss is internally stable



(b)

Solution

For (c),

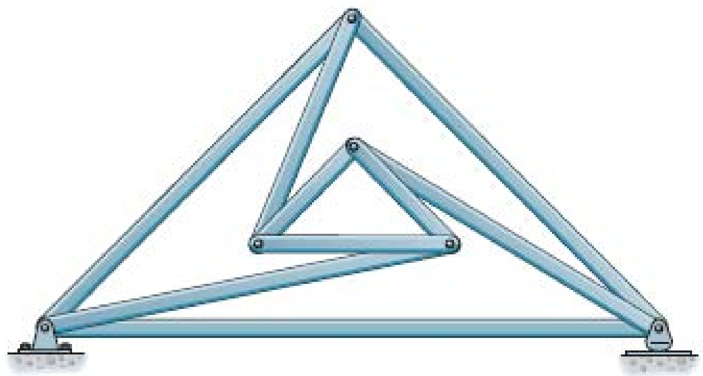
Externally stable

$b = 9, r = 3, j = 6$

$b + r = 12 = 2j$

Truss is statically determinate

By inspection, the truss is internally stable



(c)

Solution

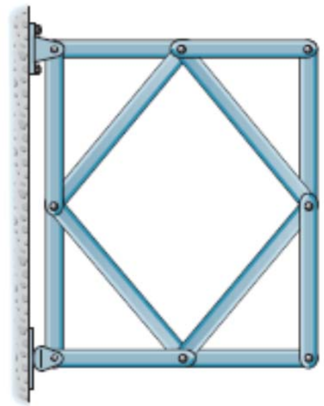
For (d),

Externally stable

$b = 12$, $r = 3$, $j = 8$

$b + r = 15 < 2j$

The truss is internally unstable



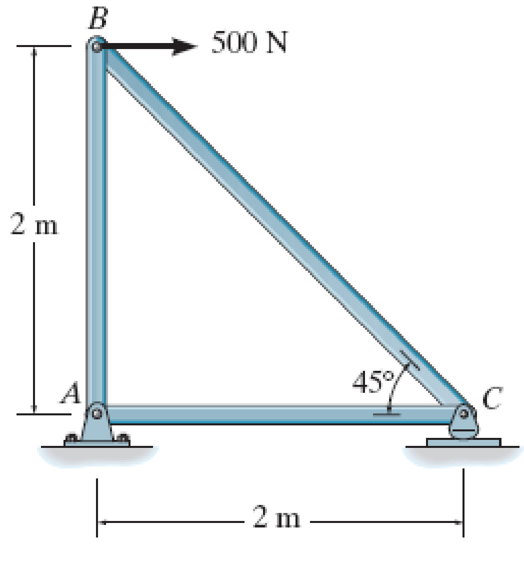
(d)

Determination of the member forces

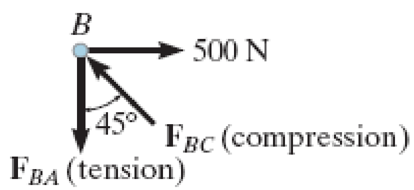
- The Method of Joints
- The Method of Sections (Ritter Method)
- The Graphical Method (Cremona Method)

The Method of Joints

- Satisfying the equilibrium equations for the forces exerted on the pin at each joint of the truss
- Applications of equations yields 2 algebraic equations that can be solved for the 2 unknowns



(a)



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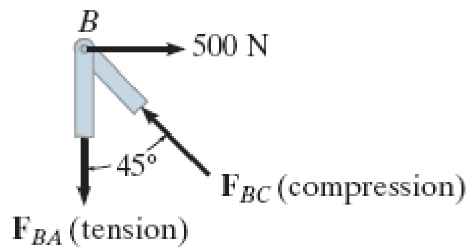


(c)

The Method of Joints

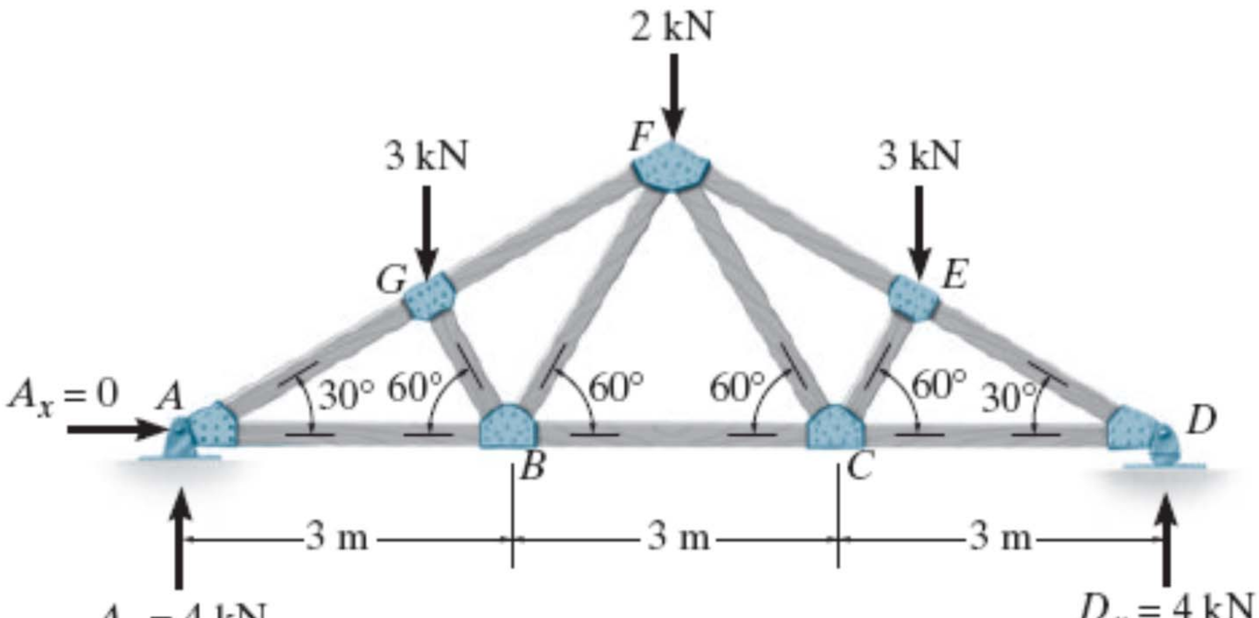
- Always assume the unknown member forces acting on the joint's free body diagram to be in tension
- Numerical solution of the equilibrium eqns will yield positive scalars for members in tension & negative for those in compression
- The correct sense of direction of an unknown member force can in many cases be determined by inspection

The Method of Joints

- A positive answer indicates that the sense is correct, whereas a negative answer indicates that the sense shown on the free-body diagram must be reversed

Example 3.2

Determine the force in each member of the roof truss as shown. State whether the members are in tension or compression. The reactions at the supports are



(a)

Solution

Only the forces in half the members have to be determined as the truss is symmetric with respect to both loading & geometry,

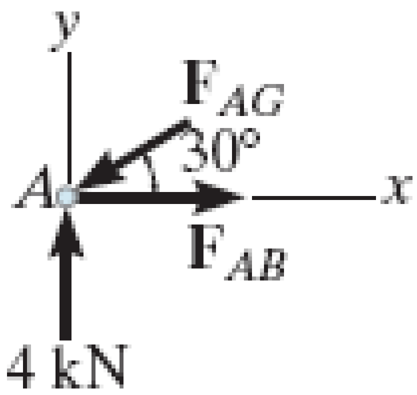
Joint A,

$$+\uparrow \sum F_y = 0; \quad 4 - F_{AG}\sin 30^0 = 0$$

$$F_{AG} = 8kN(C)$$

$$\underset{\rightarrow}{+} \sum F_x = 0; \quad F_{AB} - 8\cos 30^0 = 0$$

$$F_{AB} = 6.93kN(T)$$



(b)

Solution

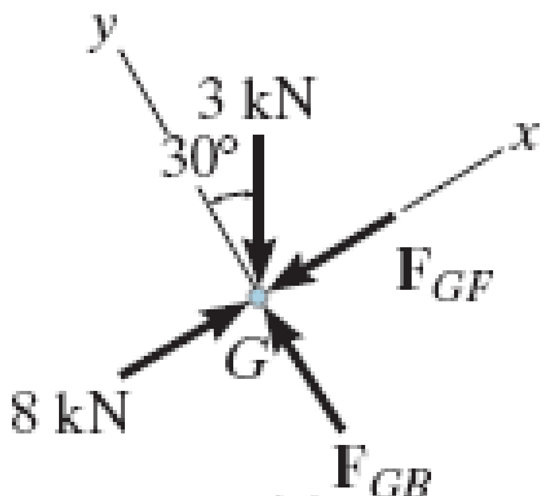
Joint G,

$$+\uparrow \sum F_y = 0; \quad F_{GB} - 3\cos 30^0 = 0$$

$$F_{GB} = 2.60 kN (C)$$

$$\underrightarrow{+} \sum F_x = 0; \quad 8 - 3\sin 30^0 - F_{GF} = 0$$

$$F_{GF} = 6.50 kN (C)$$



(c)

Solution

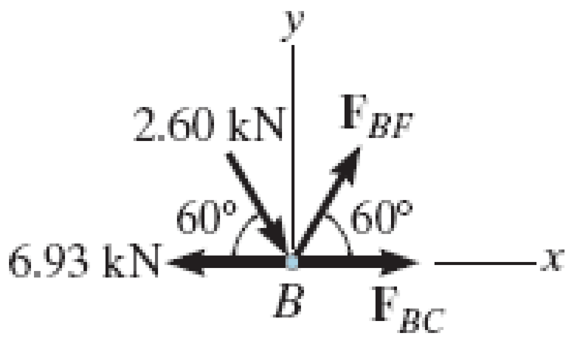
Joint B,

$$+\uparrow \sum F_y = 0;\ \ F_{BF}\sin 60^0 - 2.60\sin 60^0 = 0$$

$$F_{BF} = 2.60kN(T)$$

$$\underset{\rightarrow}{+}\sum F_x = 0;\quad F_{BC} + 2.60\cos 60^0 + 2.60\cos 60^0 - 6.93 = 0$$

$$F_{BC} = 4.33kN(T)$$



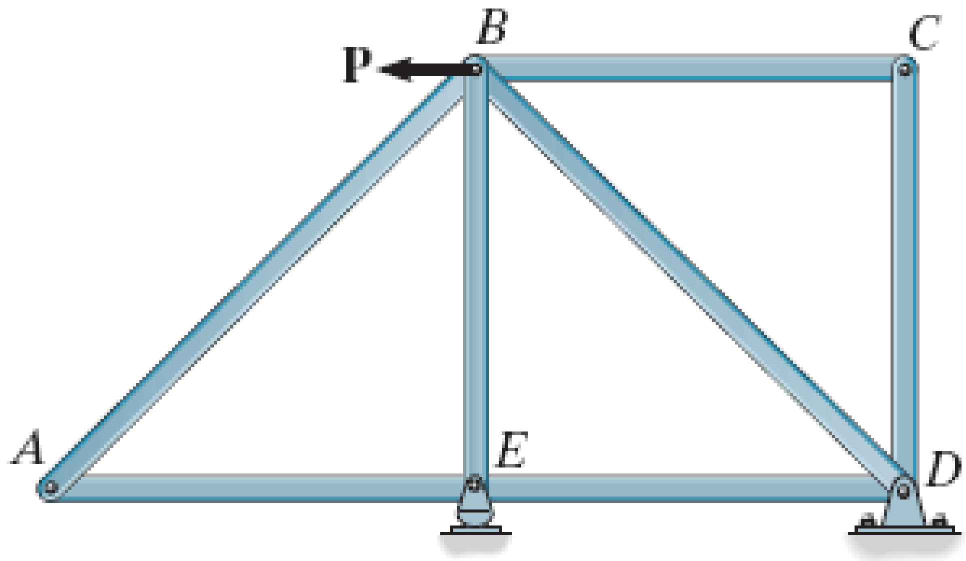
(d)

Zero-Force Members

- Truss analysis using method of joints is greatly simplified if one is able to first determine those members that support no loading
- These zero-force members may be necessary for the stability of the truss during construction & to provide support if the applied loading is changed
- The zero-force members of a truss can generally be determined by inspection of the joints & they occur in 2 cases.

Zero-Force Members

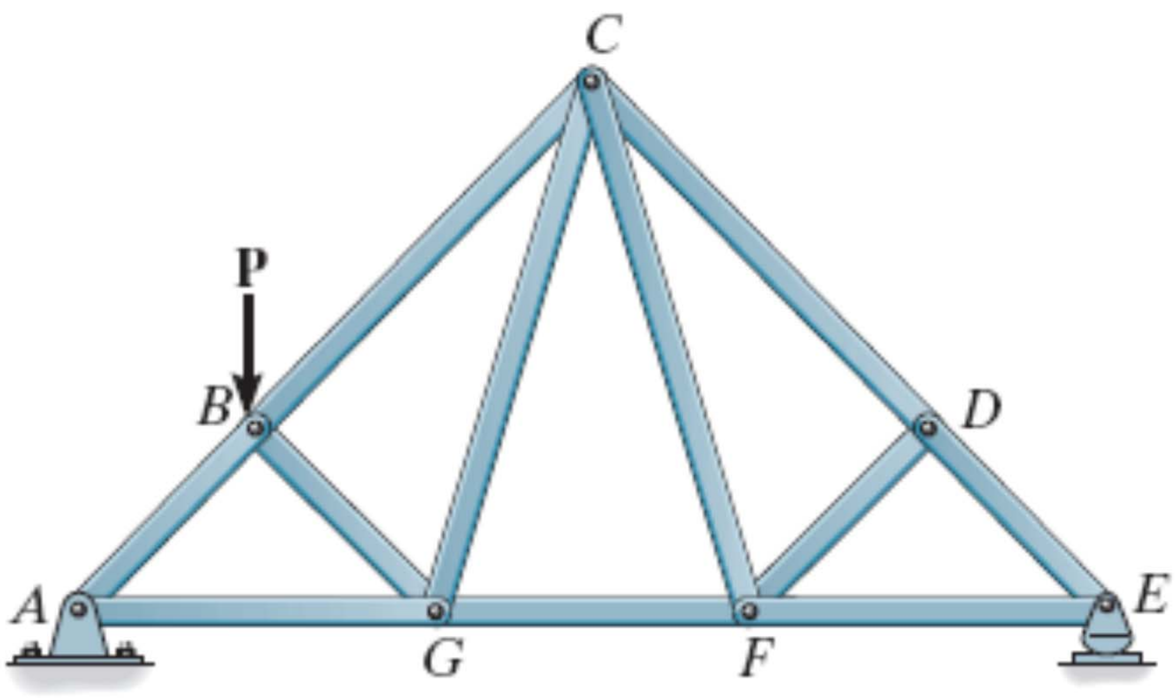
- Case 1
  - The 2 members at joint C are connected together at a right angle & there is no external load on the joint
  - The free-body diagram of joint C indicates that the force in each member must be zero in order to maintain equilibrium



(a)

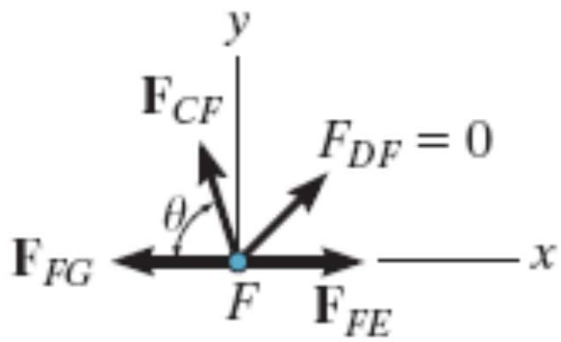
Zero-Force Members

- Case 2
  - Zero-force members also occur at joints having a geometry as joint D



Zero-Force Members

- Case 2
  - No external load acts on the joint, so a force summation in the y-direction which is perpendicular to the 2 collinear members requires that $F_{DF} = 0$
  - Using this result, FC is also a zero-force member, as indicated by the force analysis of joint F



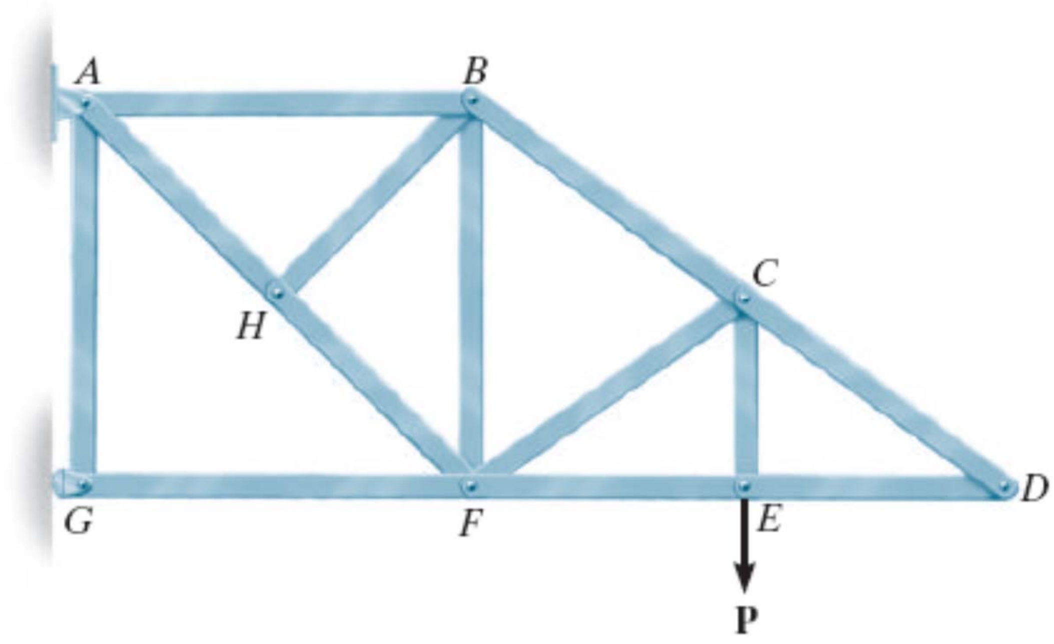
(c)

$$+\uparrow \Sigma F_y = 0;\ F_{CF} \sin\theta + 0 = 0$$

$$F_{CF} = 0 \text{ (since } \sin\theta \neq 0)$$

Example 3.4

Using the method of joints, indicate all the members of the truss that have zero force.



Solution

We have,

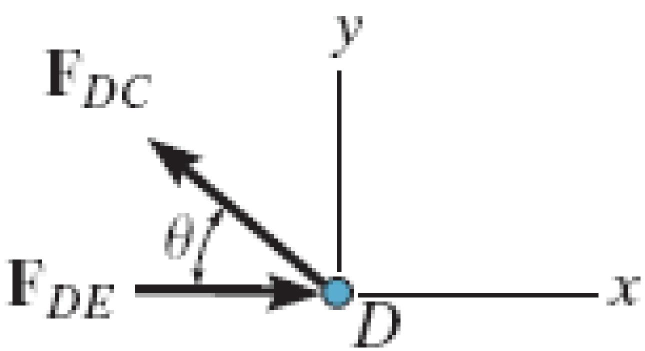
Joint D,

$$+\uparrow \sum F_y = 0; \quad F_{DC} \sin\theta = 0$$

$$F_{DC} = 0$$

$$\underset{\rightarrow}{+} \sum F_x = 0; \quad F_{DE} + 0 = 0$$

$$F_{DE} = 0$$

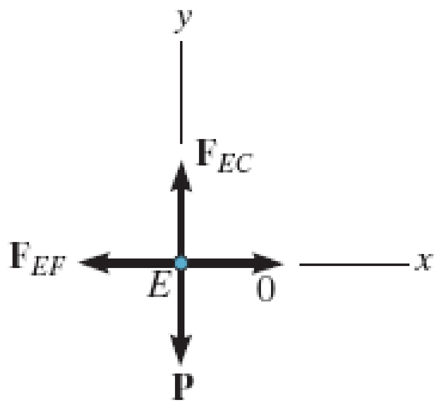


(b)

Solution

Joint E,

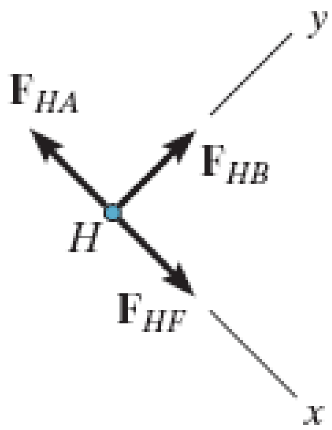
$$\overset{+}{\rightarrow}\sum F_x = 0; \quad F_{EF} = 0$$



(c)

Joint H,

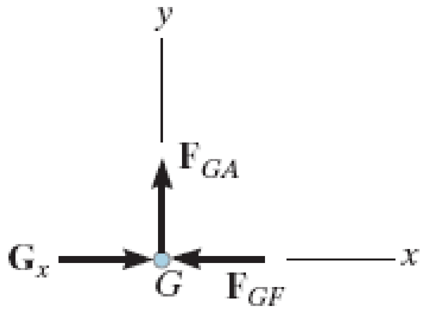
$$+\uparrow\sum F_y = 0; \quad F_{HB} = 0$$



(d)

Joint G,

$$+\uparrow\sum F_y = 0; \quad F_{GA} = 0$$



(e)

The Method of Sections (Ritter Method)

- If the forces in only a few members of a truss are to be found, the method of sections generally provide the most direct means of obtaining these forces
- The method is created the German scientist August Ritter (1826 - 1908).

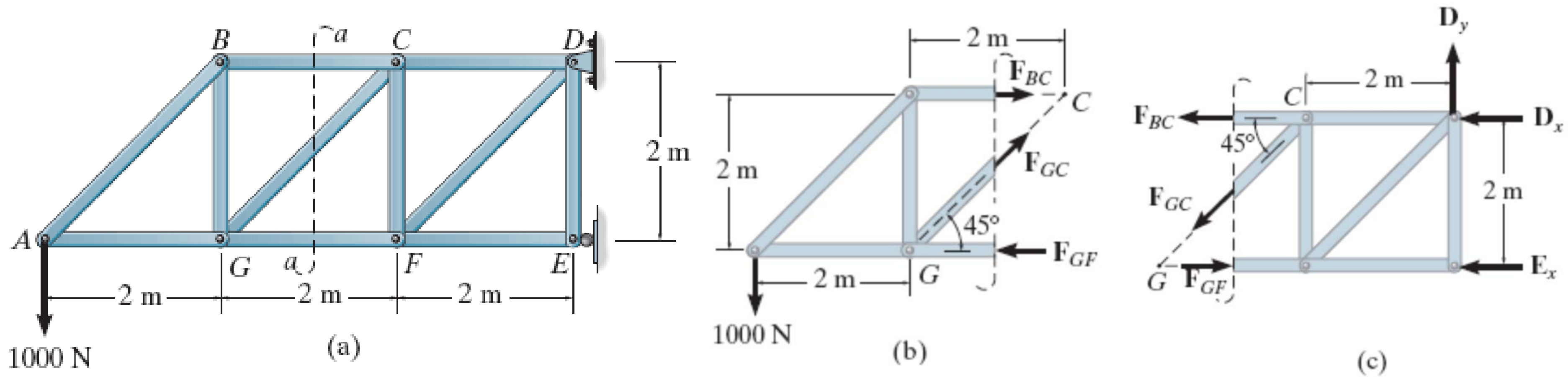
- This method consists of passing an imaginary section through the truss, thus cutting it into 2 parts
- Provided the entire truss is in equilibrium, each of the 2 parts must also be in equilibrium

The Method of Sections (Ritter Method)

- The 3 eqns of equilibrium may be applied to either one of these 2 parts to determine the member forces at the "cut section"
- A decision must be made as to how to "cut" the truss
- In general, the section should pass through not more than 3 members in which the forces are unknown

The Method of Sections (Ritter Method)

- If the force in GC is to be determined, section a-a will be appropriate
- Also, the member forces acting on one part of the truss are equal but opposite
- The 3 unknown member forces, F_{BC}, F_{GC} & F_{GF} can be obtained by applying the 3 equilibrium equations

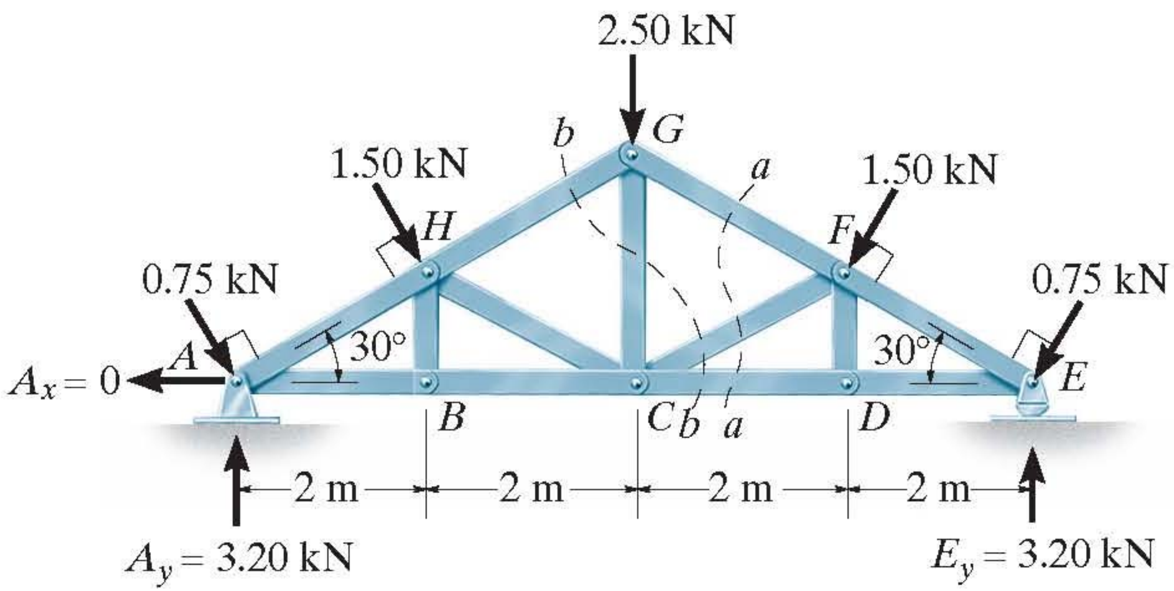


The Method of Sections

- When applying the equilibrium equations, consider ways of writing the equations to yield a direct solution for each of the unknown, rather than to solve simultaneous equations

Example 3.5

Determine the force in members CF and GC of the roof truss. State whether the members are in tension or compression. The reactions at the supports have been calculated.



Solution

The free-body diagram of member CF can be obtained by considering the section a-a,

A direct solution for F_{CF} can be obtained by applying $\sum M_E = 0$

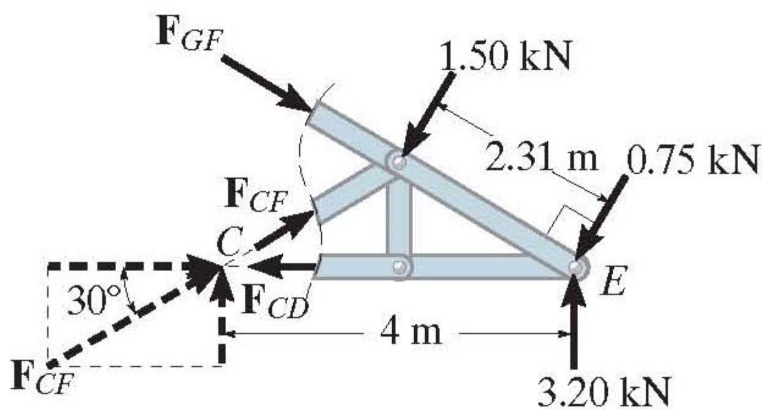
Applying Principal of transmissibility,

F_{CF} is slide to point C for simplicity.

With anti-clockwise moments as +ve, $\sum M_E = 0$

$$-F_{CF}\sin 30^o(4) + 1.50(2.31) = 0$$

$$F_{CF} = 1.73kN(C)$$



Solution

The free-body diagram of member GC can be obtained by considering the section b-b,

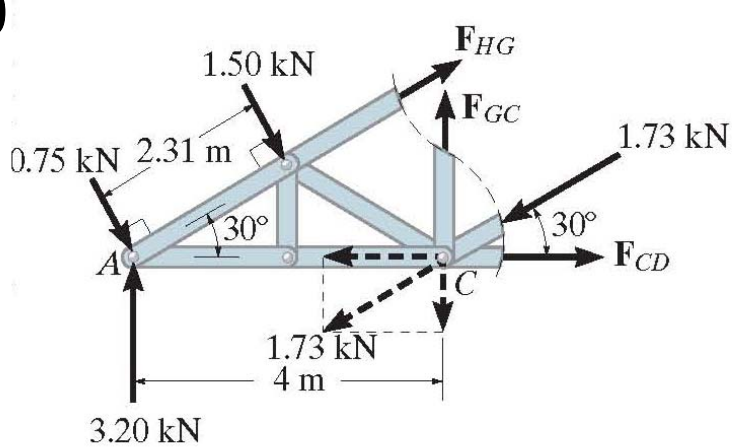
Moments will be summed about point A in order to eliminate the unknowns F_{HG} and F_{CD}. Sliding to F_{CF} point C, we have:

F_{CF} is slide to point C for simplicity.

With anti-clockwise moments as +ve, $\sum M_A = 0$

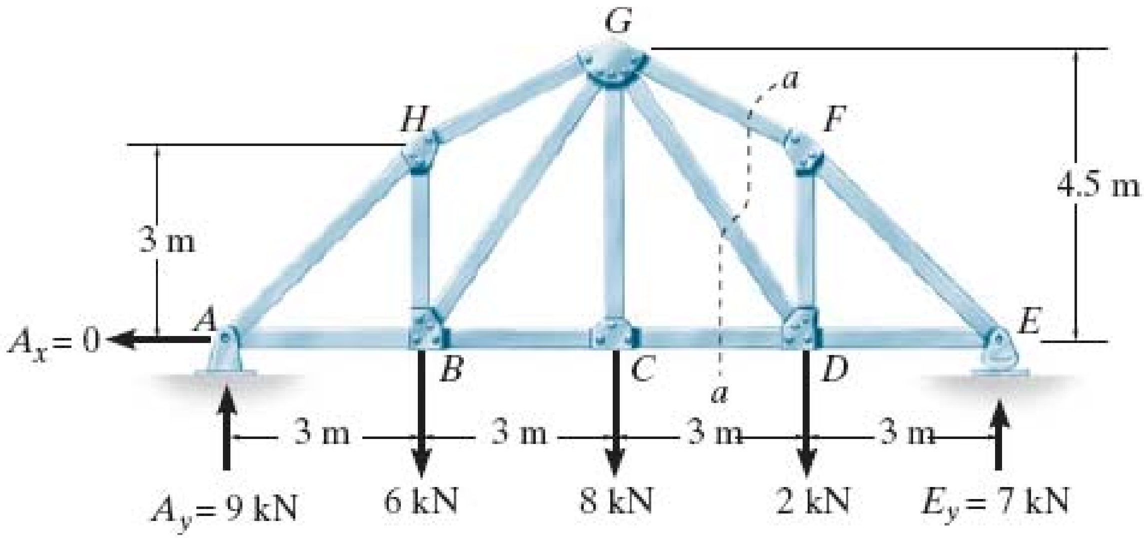
$$-1.50(2.31) + F_{GC}(4) - 1.73\sin 30^o(4) = 0$$

$$F_{GC} = 1.73kN(T)$$



Example 3.6

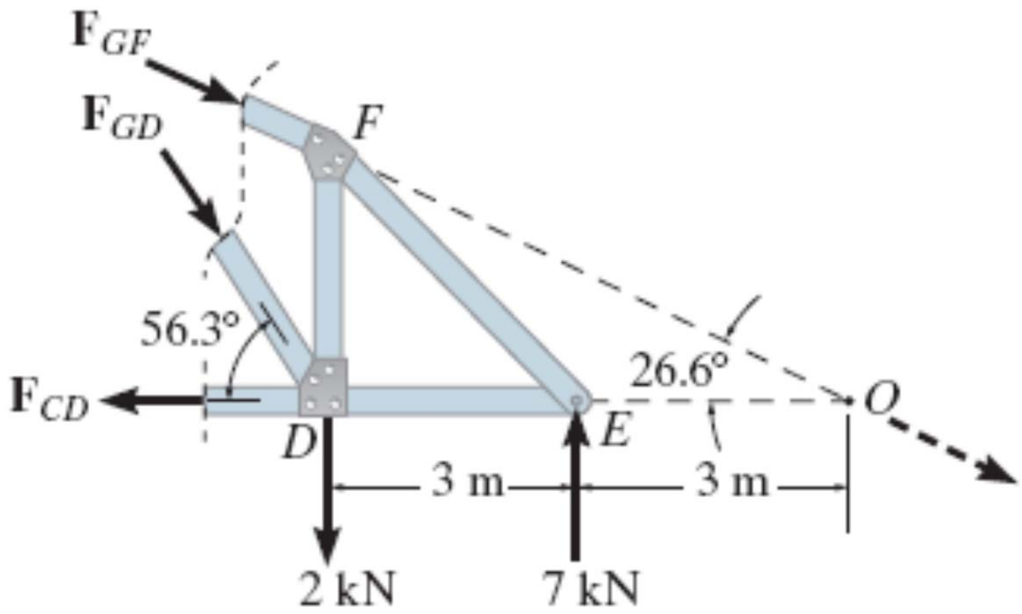
Determine the force in member GF and GD of the truss. State whether the members are in tension or compression. The reactions at the supports have been calculated.



Solution

The distance EO can be determined by proportional triangles or realizing that member GF drops vertically 4.5 – 3 = 1.5m in 3m.

Hence, to drop 4.5m from G the distance from C to O must be 9m



Solution

The angles F_{GD} and F_{GF} make with the horizontal are

$\tan^{-1}(4.5/3) = 56.3°$

$\tan^{-1}(4.5/9) = 26.6°$

The force in GF can be determined directly by applying

$\sum M_D = 0$

F_{GF} is slide to point O.

With anti-clockwise moments as +ve, $\sum M_D = 0$

$-F_{GF} \sin 26.6^o (6) + 7(3) = 0$

$F_{GF} = 7.83kN(C)$

Solution

The force in GD can be determined directly by applying

$\sum M_O = 0$

F_{GD} is slide to point D.

With anti-clockwise moments as +ve, $\sum M_O = 0$

$$-7(3) + 2(6) + F_{GD} \sin 56.3^o (6) = 0$$

$$F_{GD} = 1.80 kN (C)$$

The Graphical Method (Cremona Method)

- This method deals mainly with the graphical representation of equilibrium for each joint. The basic advantage that makes the method attractive, is its ability to unify all the force polygons, resulting from graphical equilibrium of each joint, into one only force polygon, known as *Cremona's diagram*. The method was created by the Italian mathematician Luigi Cremona.

The Graphical Method (Cremona Method)

- Although graphical, this method leads to a quick determination of the member forces and is useful specifically in the cases where the external loads and/or the truss members form random angles.
- Consider the case of graphical analyzing the equilibrium of a point, acted upon 3 forces, one of which is completely known while the other 2 are known in direction only (for example, a lamp hanged by two wires). The procedure:
  - Draw the vector of the completely known force, in the proper direction, scale, magnitude and sense.
  - From **one end** of the vector, draw a line parallel to the direction of **one** of the 2 forces, while from the **other end** draw a second line parallel to the **other** direction.

The Graphical Method (Cremona Method)

- The vector and the point of section of the two lines define a triangle.
- Now, following the path of the vector by laying out the 2 unknown forces tip to tail, thus closing the force triangle, we find both the **magnitudes** and the **senses** of the other 2 forces.

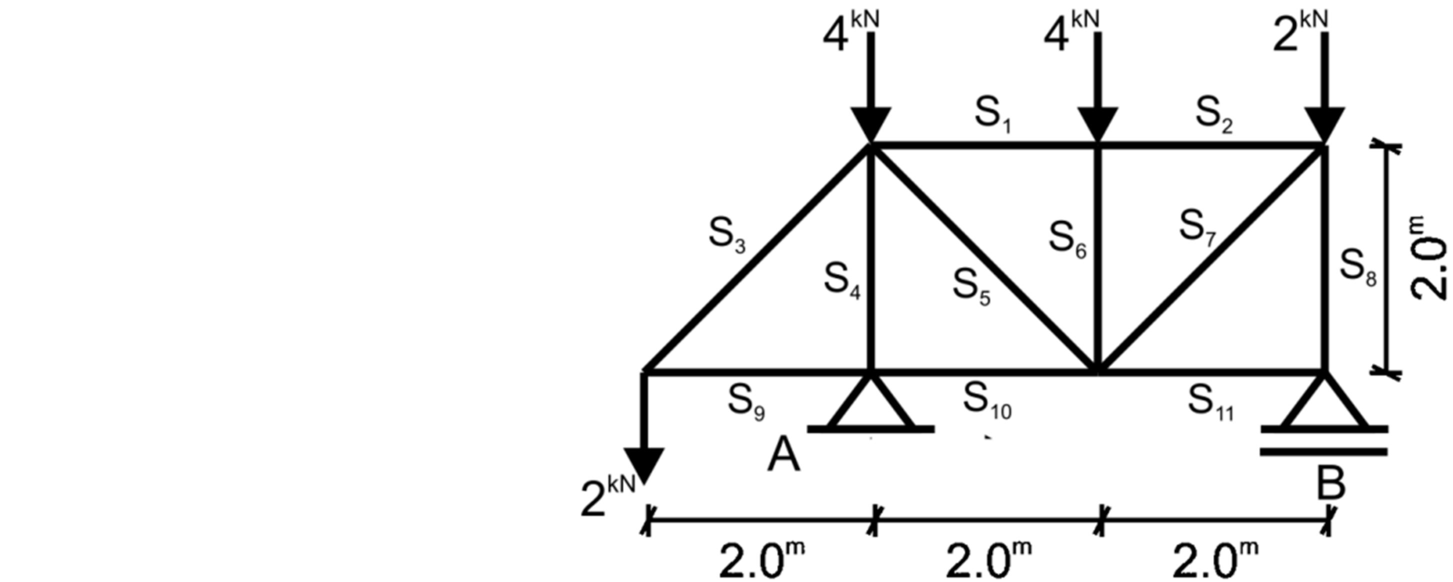
- Of course the completely known force can be considered as the resultant of other known forces, through a force polygon. From this procedure we realize that the basic characteristic which appears to be common in the method of joints and Cremona’s diagram lies in the main strategic.

The Graphical Method (Cremona Method)

- For analyzing the equilibrium of a joint, in the first method available were **2 equations** only, whereas in the second, the **two ends** of the known-force-vector only.
- Keeping in mind this similarity for the new method, we can also start and continue with the equilibrium of a joint, where at least one known load exists, while not more than two unknown forces are present.
- Compared to the analytical method of joints, the graphical method of Cremona's diagram is less precise. However, the 'loss of precision' is unimportant and theoretical. Nevertheless, the speed and the elegance of the method are the main characteristics that make it popular and attractive by many designers.

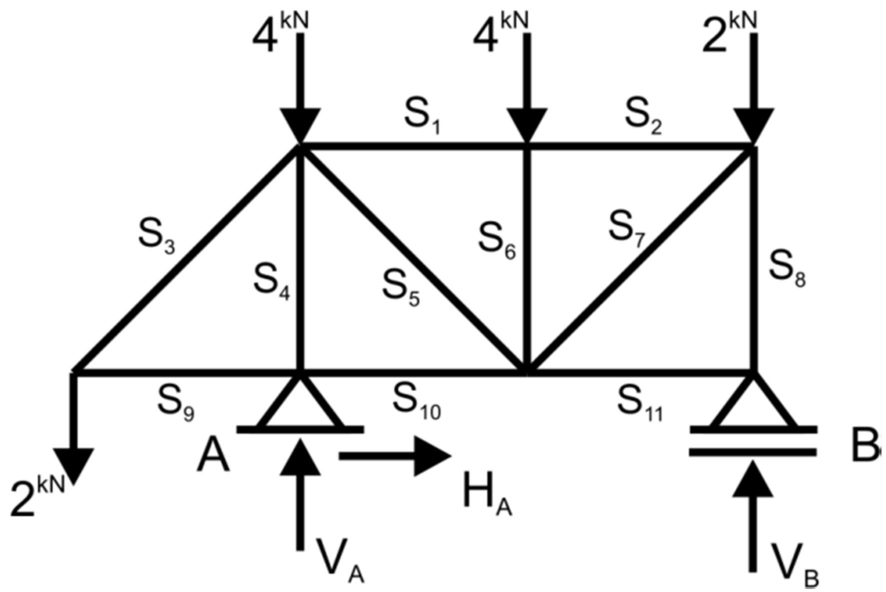
Example 3.7

Determine graphically the force in each of the eleven members of the following truss by the method of Cremona's diagram.



Solution

We first calculate the support reactions:



$\rightarrow \qquad \Sigma X = 0 \quad \mathbf{H_A = 0}$

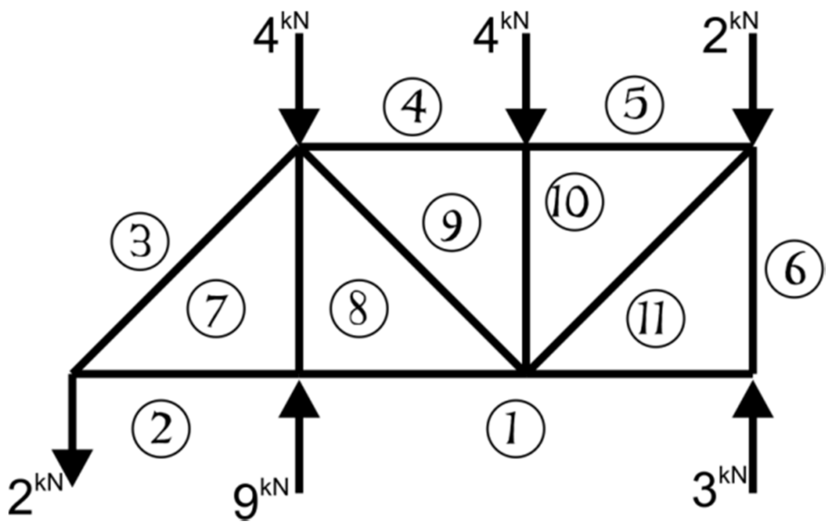
$\circlearrowleft \qquad \Sigma M_B = 0 \quad V_A = (\,4^{kN} \uparrow 4^{m} + 4^{kN} \uparrow 2^{m} + 2^{kN} \uparrow 6^{m}\,) \,/\, 4^{m} = 9^{kN} \qquad \mathbf{V_A = 9^{kN}}$

$\uparrow \qquad \Sigma Y = 0 \quad V_B = 2^{kN} + 4^{kN} + 4^{kN} + 2^{kN} \rightarrow 9^{kN} = 3^{kN} \qquad \mathbf{V_B = 3^{kN}}$

Solution

After drawing the the free body diagram we follow the next steps:

1) Covering the whole area of the free body diagram, we name, say with numbers in circles 1, 2, 3… both the triangles formed by the members and by the external loads so that each member or load separates two areas.

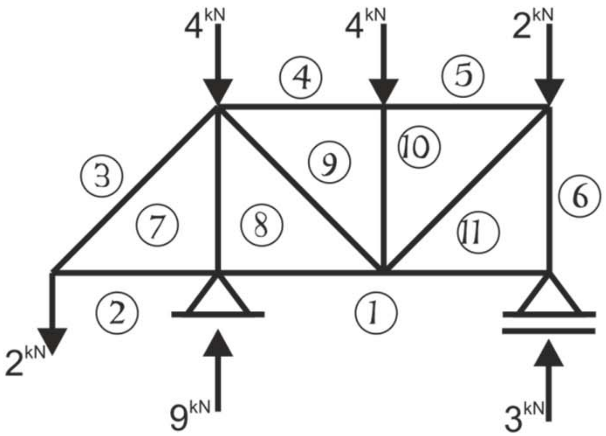


Solution

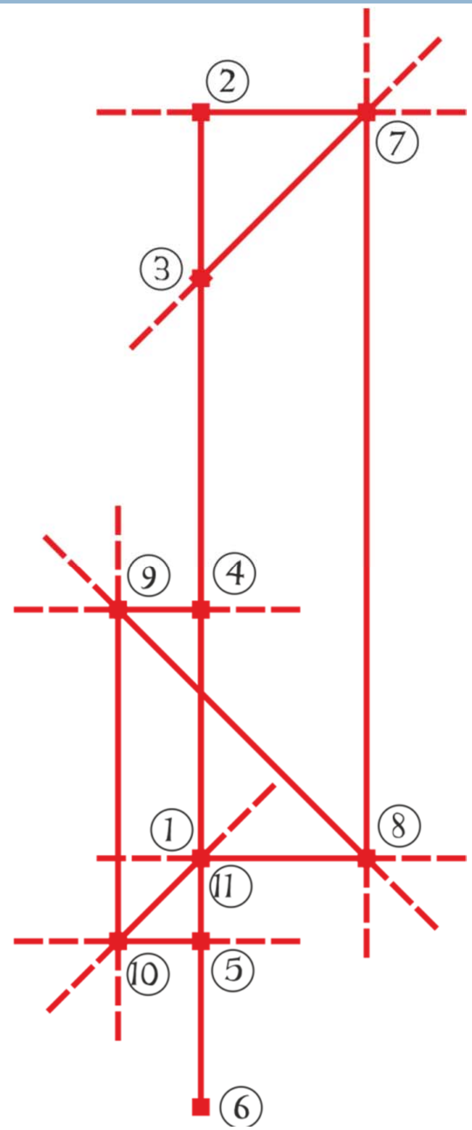
2) Using an appropriate scale (for example 1kN = 1cm) we next draw the force polygon of the external loadings. Each region is represented by a point at the force polygon. The intersection of the parallel lines drawn from one region (point) to the adjacent region gives the point corresponding to the adjacent region.

3) Next, we define a clockwise sequence of forces around a joint. This means that if we start drawing the force triangle for equilibrium of joint B, from, say, the calculated reaction force of 3 kN, the next force considered will be that of member S_{11} and not of S_8.

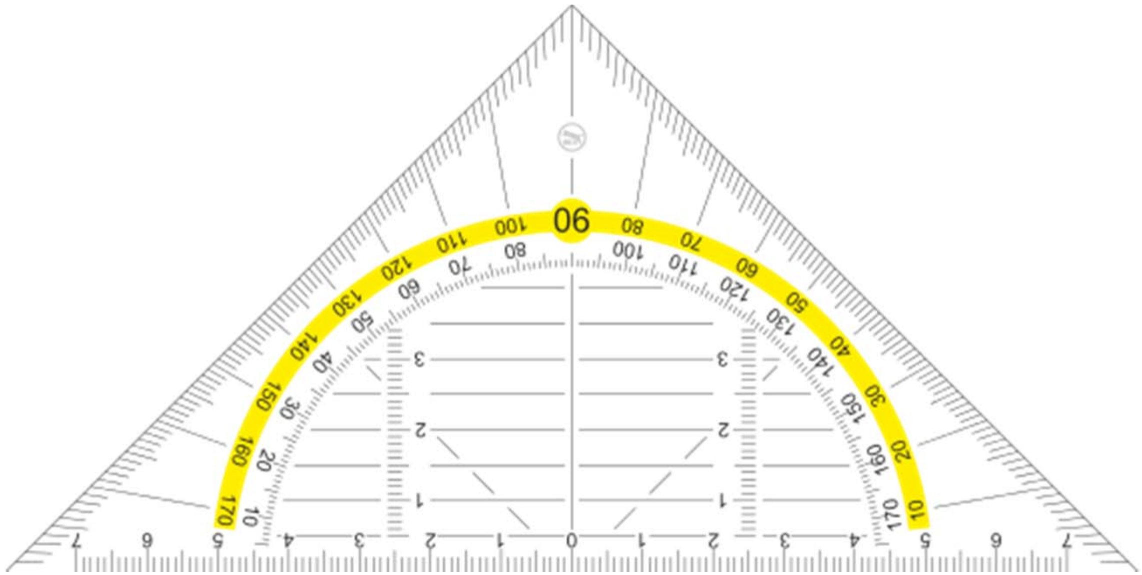
Solution



	Tension	Compression
	[kN]	
S_1		-1.00
S_2		-1.00
S_3	2.83	
S_4		-9.00
S_5	4.24	
S_6		-4.00
S_7	1.41	
S_8		-3.00
S_9		-2.00
S_{10}		-2.00
S_{11}	0.00	



The Graphical Method
(Cremona Method)



A set square with integrated protractor

DEFLECTIONS

THEORY OF STRUCTURES

Asst. Prof. Dr. Cenk Üstündağ

Deflection diagrams & the elastic curve

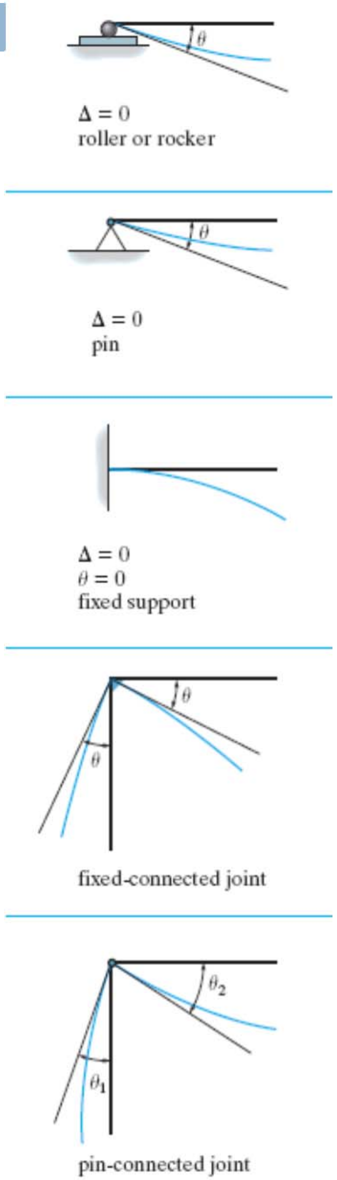
- Deflections of structures can come from loads, temperature, fabrication errors or settlement
- In designs, deflections must be limited in order to prevent cracking of attached brittle materials
- A structure must not vibrate or deflect severely for the comfort of occupants
- Deflections at specified points must be determined if one is to analyze statically indeterminate structures

Deflection diagrams & the elastic curve

- In this topic, only linear elastic material response is considered
- This means a structure subjected to load will return to its original undeformed position after the load is removed
- It is useful to sketch the shape of the structure when it is loaded in order to visualize the computed results & to partially check the results

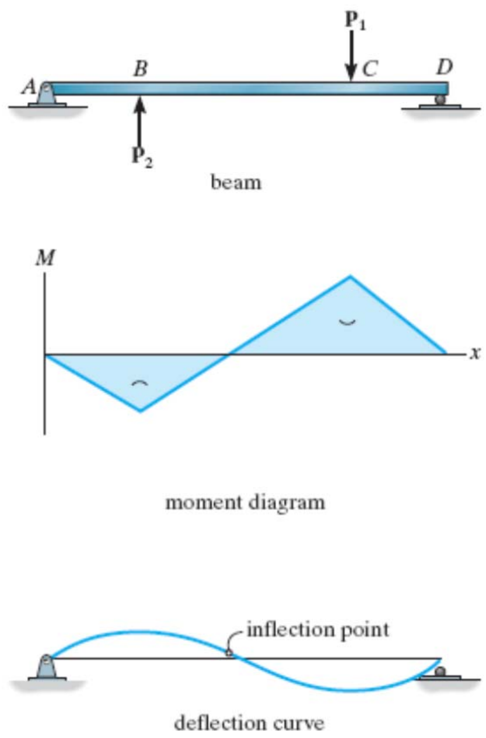
Deflection diagrams & the elastic curve

- This deflection diagram rep the elastic curve for the points at the centroids of the cross-sectional areas along each of the members
- If the elastic curve seems difficult to establish, it is suggested that the moment diagram be drawn first
- From there, the curve can be constructed



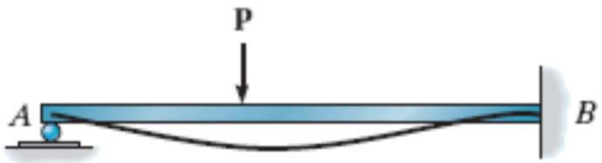
Deflection diagrams & the elastic curve

- Due to pin-and-roller support, the disp at A & D must be zero
- Within the region of –ve moment, the elastic curve is concave downward
- Within the region of +ve moment, the elastic curve is concave upward
- There must be an inflection point where the curve changes from concave down to concave up

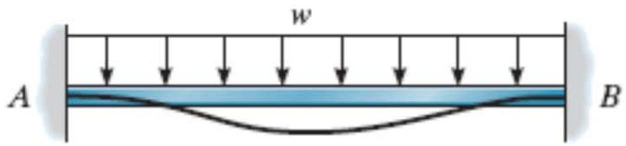


Example 8.1

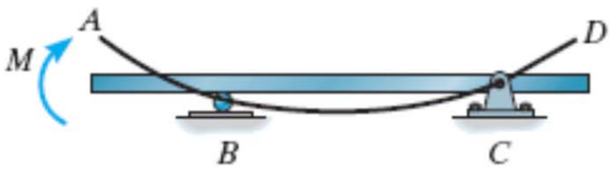
Draw the deflected shape of each of the beams.



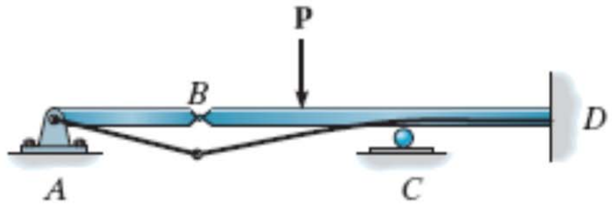
(a)



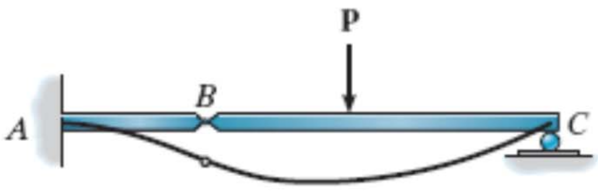
(b)



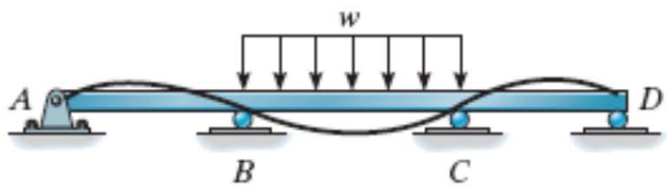
(c)



(d)



(e)



(f)

Solution

In (a), the roller at A allows free rotation with no deflection while the fixed wall at B prevents both rotation & deflection. The deflected shape is shown by the bold line.

In (b), no rotation or deflection occur at A & B

In (c), the couple moment will rotate end A. This will cause deflections at both ends of the beam since no deflection is possible at B & C. Notice that segment CD remains un-deformed since no internal load acts within.

Solution

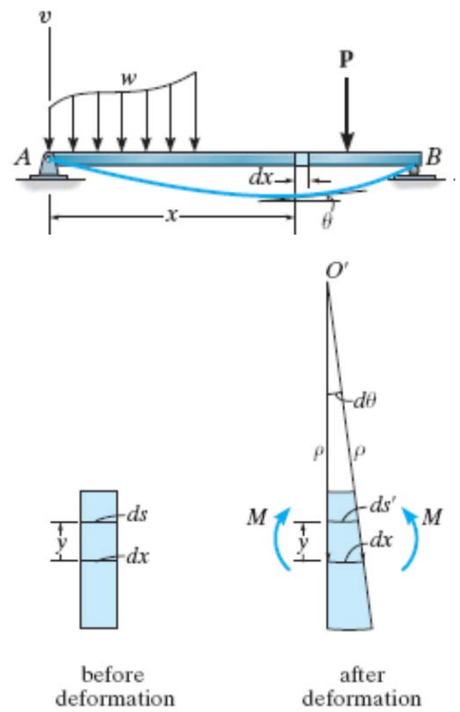
In (d), the pin at B allows rotation, so the slope of the deflection curve will suddenly change at this point while the beam is constrained by its support.

In (e), the compound beam deflects as shown. The slope changes abruptly on each side of B.

In (f), span BC will deflect concave upwards due to load. Since the beam is continuous, the end spans will deflect concave downwards.

Elastic Beam Theory

- To derive the DE, we look at an initially straight beam that is elastically deformed by loads applied perpendicular to beam's x-axis & lying in x-v plane of symmetry
- Due to loading, the beam deforms under shear & bending
- If beam L >> d, greatest deformation will be caused by bending
- When M deforms, the angle between the cross sections becomes dθ



Elastic Beam Theory

- The arc dx that rep a portion of the elastic curve intersects the neutral axis
- The radius of curvature for this arc is defined as the distance, ρ, which is measured from ctr of curvature O' to dx
- Any arc on the element other than dx is subjected to normal strain
- The strain in arc ds located at position y from the neutral axis is

$$\boxed{\varepsilon = (ds' - ds) / ds}$$

Elastic Beam Theory

$$ds = dx = \rho d\theta \;\text{ and }\; ds' = (\rho - y)d\theta$$

$$\varepsilon = \frac{(\rho - y)d\theta - \rho d\theta}{\rho d\theta} \Rightarrow \frac{1}{\rho} = -\frac{\varepsilon}{y}$$

- If the material is homogeneous & behaves in a linear manner, then Hooke's law applies

$$\varepsilon = \sigma / E$$

- The flexure formula also applies

$$\sigma = -My / I$$

Elastic Beam Theory

- Combining these eqns, we have:

$$\frac{1}{\rho} = \frac{M}{EI}$$

ρ = the radius of curvature at a specific point on the elastic curve

M = internal moment in the beam at the point where ρ is to be determined

E = the material's modulus of elasticity

I = the beam's moment of inertia computed about the neutral axis

Elastic Beam Theory

$$EI = \text{flexural rigidity}; \quad dx = \rho d\theta$$

$$d\theta = \frac{M}{EI} dx$$

$$\text{v} - \text{axis as} + \text{ve} \uparrow, \quad \frac{1}{\rho} = \frac{d^2 v / dx^2}{[1 + (dv/dx)^2]^{3/2}}$$

$$\text{Therefore,} \quad \frac{M}{EI} = \frac{d^2 v / dx^2}{[1 + (dv/dx)^2]^{3/2}}$$

Elastic Beam theory

- This eqn rep a non-linear second DE
- V=f(x) gives the exact shape of the elastic curve
- The slope of the elastic curve for most structures is very small
- Using small deflection theory, we assume dv/dx ~ 0

$$\boxed{\frac{d^2v}{dx^2} = \frac{M}{EI}}$$

Elastic Beam Theory

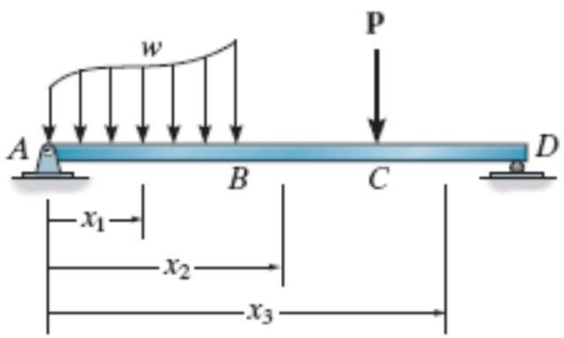
- By assuming dv/dv ~ 0 $\Rightarrow$ ds, it will approximately equal to dx

$$ds = \sqrt{dx^2 + dv^2} = \sqrt{1 + (dv/dx)^2}\,dx \approx dx$$

- This implies that points on the elastic curve will only be displaced vertically & not horizontally

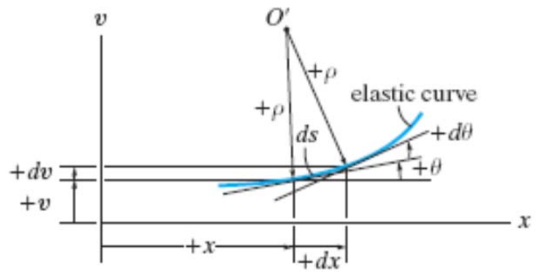
The double integration method

- $M = f(x)$, successive integration of eqn 8.4 will yield the beam's slope
  - $\theta \approx \tan\theta = dv/dx = \int M/EI\,dx$
- Eqn of elastic curve
  - $V = f(x) = \int M/EI\,dx$
- The internal moment in regions AB, BC & CD must be written in terms of x_1, x_2 and x_3



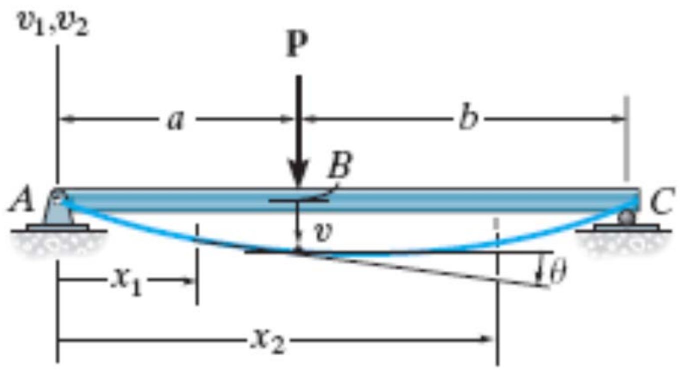
The double integration method

- Once these functions are integrated & the constants determined, the functions will give the slope & deflection for each region of the beam
- It is important to use the proper sign for M as established by the sign convention used in derivation
- +ve v is upward, hence, the +ve slope angle, θ will be measured counterclockwise from the x-axis



The double integration method

- The constants of integration are determined by evaluating the functions for slope or displacement at a particular point on the beam where the value of the function is known
- These values are called boundary conditions
- Here x_1 and x_2 coordinates are valid within the regions AB & BC

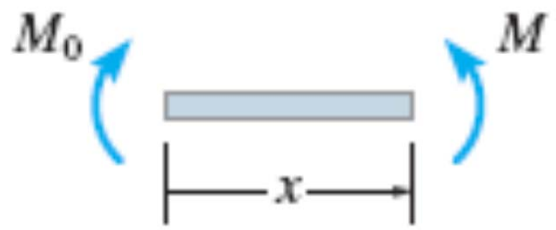


The double integration method

- Once the functions for the slope & deflections are obtained, they must give the same values for slope & deflection at point B
- This is so as for the beam to be physically continuous

Example 8.1

The cantilevered beam is subjected to a couple moment M_o at its end. Determine the eqn of the elastic curve. EI is constant.



Solution

By inspection, the internal moment can be represented throughout the beam using a single x coordinate. From the free-body diagram, with M acting in +ve direction, we have: $M = M_o$

Integrating twice yields:

$$EI\frac{d^2v}{dx^2} = M_o$$

$$EI\frac{dv}{dx} = M_o x + C_1$$

$$EI\,v = \frac{M_o x^2}{2} + C_1 x + C_2$$

Solution

Using boundary conditions, dv/dx = 0 at x = 0 & v = 0 at x = 0 then $C_1 = C_2 = 0$.

Substituting these values into earlier eqns, we get:

$$\text{with } \theta = dv/dx$$

$$\theta = \frac{M_o x}{EI}; \quad v = \frac{M_o x^2}{2EI}$$

Max slope & disp occur at A (x = L) for which

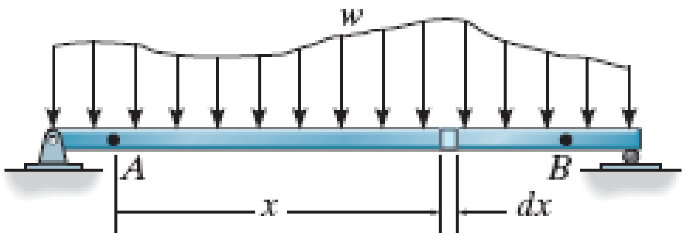
$$\theta_A = \frac{M_o L}{EI}; \quad v_A = \frac{M_o L^2}{2EI}$$

The +ve result for θ_A indicates counterclockwise rotation & the +ve result for v_A indicates that it is upwards.

Moment-Area Theorems

- If we draw the moment diagram for the beam & then divide it by the flexural rigidity, EI, the “M/EI diagram” is the results

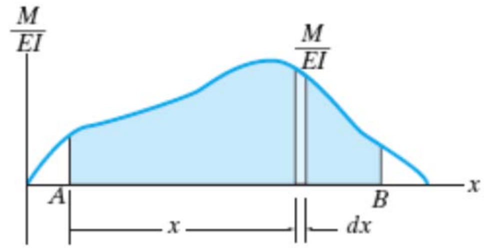
$$d\theta = \left(\frac{M}{EI} \right) dx$$



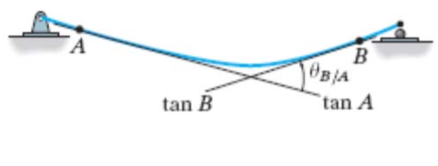
Moment-Area Theorems

- dθ on either side of the element dx = the lighter shade area under the M/EI diagram
- Integrating from point A on the elastic curve to point B, we have

$$\theta_{B/A} = \int_A^B \frac{M}{EI}\,dx$$



- This eqn forms the basis for the first moment-area theorem



elastic curve

Moment-Area Theorems

- Theorem 1
  - The change in slope between any 2 points on the elastic curve equals the area of the M/EI diagram between the 2 points
- The second moment-area theorem is based on the relative derivation of tangents to the elastic curve
- Shown in Fig 8.12(c) is a greatly exaggerated view of the vertical deviation dt of the tangents on each side of the differential element, dx

Moment-Area theorems

- Since slope of elastic curve & its deflection are assumed to be very small, it is satisfactory to approximate the length of each tangent line by x & the arc ds' by dt
- Using $\theta s = r \Rightarrow dt = x d\theta$
- Using eqn 8.2, $d\theta = (M/EI)\, dx$
- The vertical deviation of the tangent at A with respect to the tangent at B can be found by integration

$$\boxed{t_{A/B} = \int_A^B x \frac{M}{EI}\, dx}$$

Moment-Area Theorems

- Centroid of an area

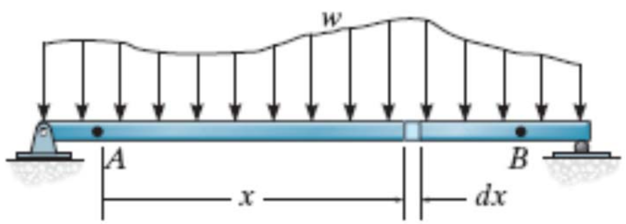
$$\bar{x}\int dA = \int x\,dA$$

$$t_{A/B} = \bar{x}\int_A^B \frac{M}{EI}\,dx$$

$\bar{x} =$ distance from the vertical axis through A to the centroid of the area between A & B.

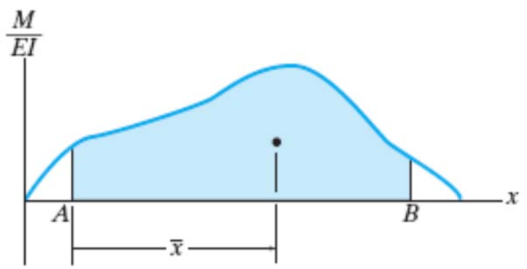
Moment-Area Theorems

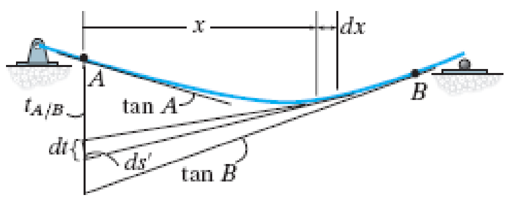
- Theorem 1
  - The vertical deviation of the tangent at a point (A) on the elastic curve with respect to the tangent extended from another point (B) equals the "moment" of the area under the M/EI diagram between the 2 points (A & B)
  - This moment is computed about point A where the derivation is to be determined



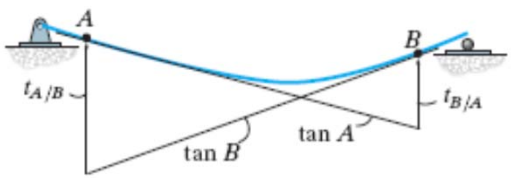
Moment-Area Theorems

- Provided the moment of a +ve M/EI area from A to B is computed, it indicates that the tangent at point A is above the tangent to the curve extended from point B
- -ve areas indicate that the tangent at A is below the tangent extended from B





elastic curve



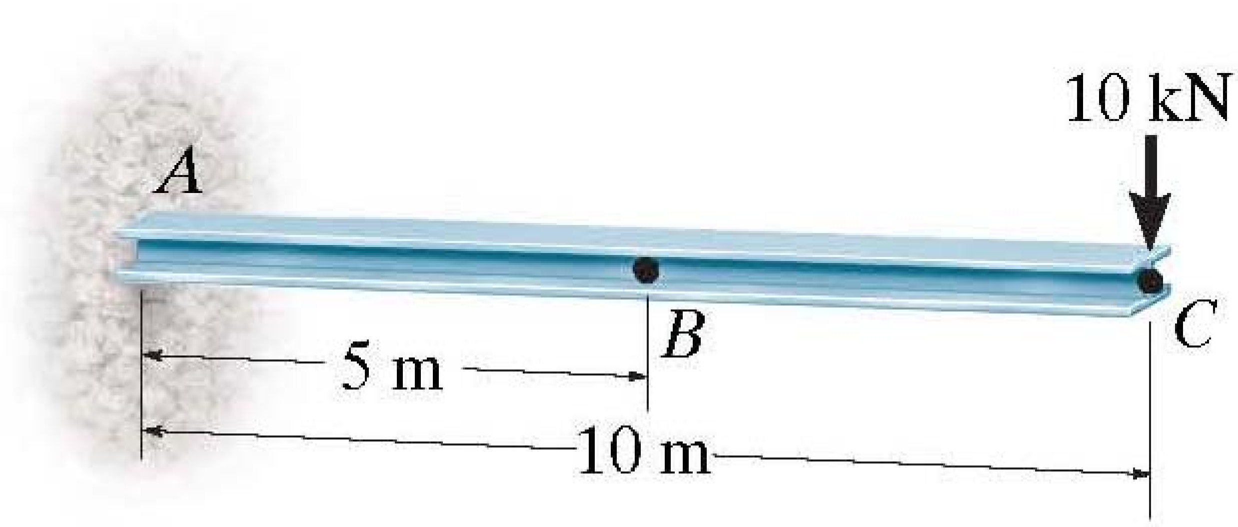
elastic curve

Moment-Area Theorems

- It is important to realise that the moment-area theorems can only be used to determine the angles & deviations between 2 tangents on the beam's elastic curve
- In general, they do not give a direct solution for the slope or disp. at a point

Example 8.5

Determine the slope at points B & C of the beam. Take E = 200GPa, I = 360(10^6)mm^4



Solution

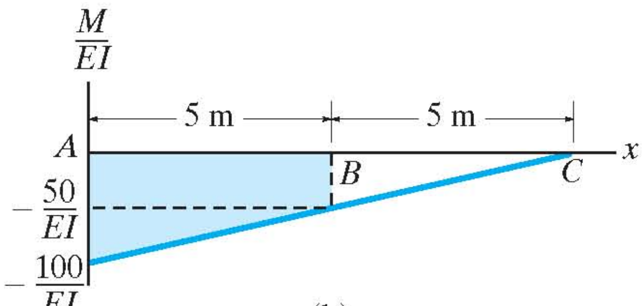
It is easier to solve the problem in terms of EI & substitute the numerical data as a last step.

The 10kN load causes the beam to deflect.

Here the tangent at A is always horizontal.

The tangents at B & C are also indicated.

By construction, the angle between tan A and tan B ($\theta_{B/A}$) is equivalent to θ_B.



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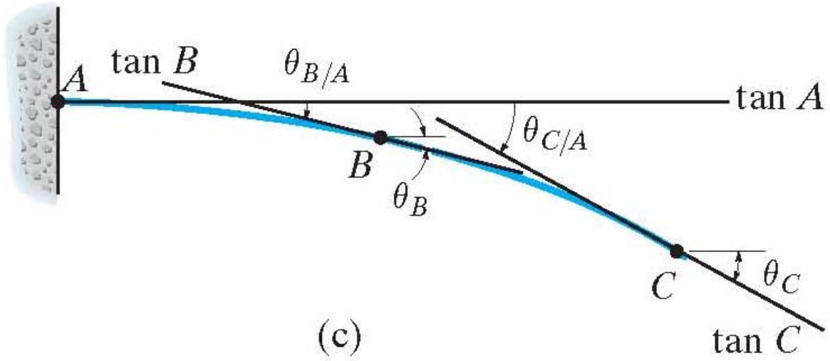


(c)

Solution

$$\theta_B = \theta_{B/A}; \quad \theta_C = \theta_{C/A}$$

Applying Theorem 1, is equal to the area under the M/EI diagram between points A & B.

$$\theta_B = \theta_{B/A} = -\left(\frac{50kNm}{EI}\right)(5m) - \frac{1}{2}\left(\frac{100kNm}{EI} - \frac{50kNm}{EI}\right)(5m)$$

$$= -\frac{375kNm^2}{EI}$$

Solution

Substituting numerical data for E & I

$$-\frac{375kNm^2}{[200(10^6)kN/m^2][360(10^6)(10^{-12})m^4]} = -0.00521rad$$

The –ve sign indicates that the angle is measured clockwise from A.

In a similar manner, the area under the M/EI diagram between points A & C equals ($\theta_{C/A}$).

Solution

$$\theta_C = \theta_{C/A} = \frac{1}{2}\left(-\frac{100kNm}{EI}\right)(10m) = -\frac{500kNm^2}{EI}$$

Substituting numerical values of EI, we have :

$$\frac{-500kNm^2}{[200(10^6)kN/m^2][360(10^6)(10^{-12})m^4]} = -0.00694rad$$

Conjugate-Beam Method

- The basis for the method comes from similarity equations
- To show this similarity, we can write these eqn as shown

$\dfrac{dV}{dx} = w$	$\dfrac{d^2M}{dx_2} = w$
$\dfrac{d\theta}{dx} = \dfrac{M}{EI}$	$\dfrac{d^2v}{dx_2} = \dfrac{M}{EI}$

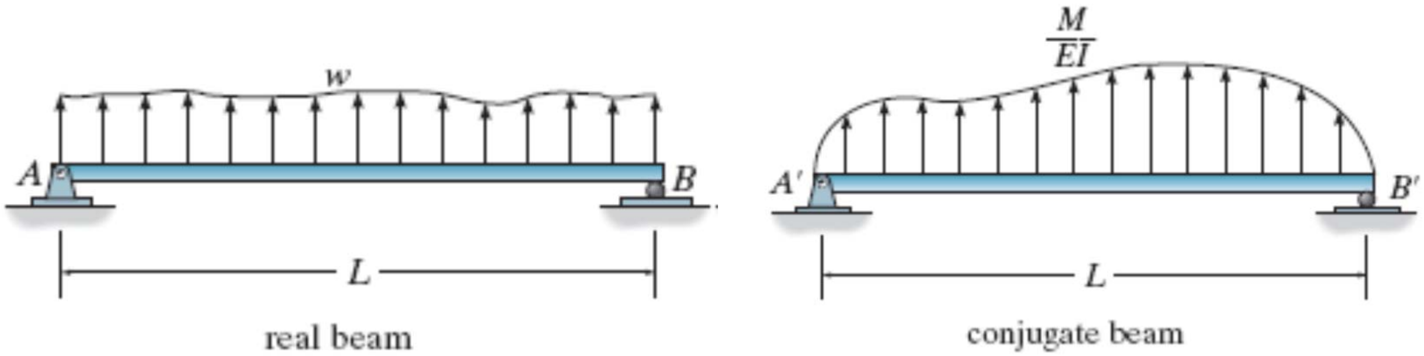
Conjugate-Beam Method

- Or integrating,

$V = \int w dx$	$M = \int \left[w dx \right] dx$
$\theta = \int \left(\frac{M}{EI} \right) dx$	$v = \int \left[\left(\frac{M}{EI} \right) dx \right] dx$

Conjugate-Beam Method

- Here the shear V compares with the slope θ , the moment M compares with the disp v & the external load w compares with the M/EI diagram
- To make use of this comparison we will now consider a beam having the same length as the real beam but referred to as the "conjugate beam",



Conjugate-Beam Method

- The conjugate beam is loaded with the M/EI diagram derived from the load w on the real beam
- From the above comparisons, we can state 2 theorems related to the conjugate beam
- Theorem 1
  - The slope at a point in the real beam is numerically equal to the shear at the corresponding point in the conjugate beam

Conjugate-Beam Method

- Theorem 2
  - The disp. of a point in the real beam is numerically equal to the moment at the corresponding point in the conjugate beam
- When drawing the conjugate beam, it is important that the shear & moment developed at the supports of the conjugate beam account for the corresponding slope & disp of the real beam at its supports

Conjugate-Beam Method

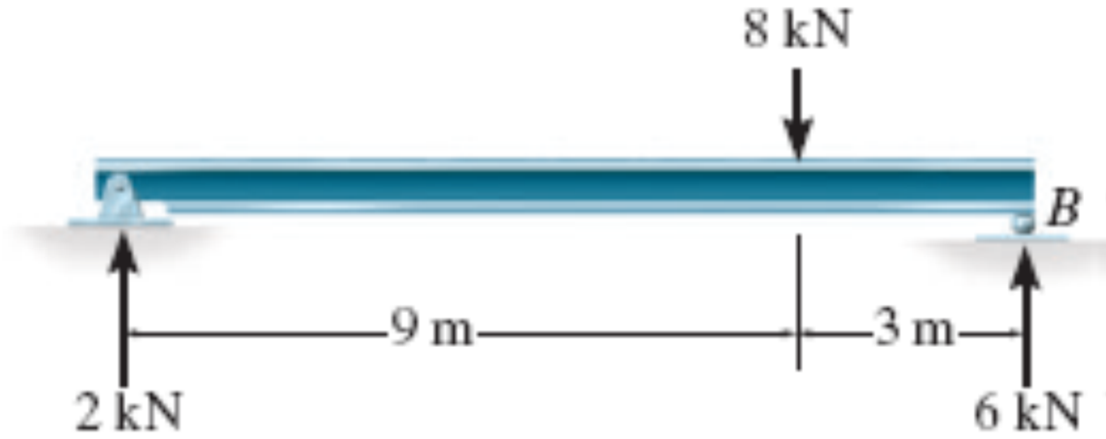
- Consequently from Theorem 1 & 2, the conjugate beam must be supported by a pin or roller since this support has zero moment but has a shear or end reaction
- When the real beam is fixed supported, both beam has a free end since at this end there is zero shear & moment

Conjugate-Beam Method

	Real Beam		Conjugate Beam	
1)	θ $\Delta = 0$	pin	V $M = 0$	pin
2)	θ $\Delta = 0$	roller	V $M = 0$	roller
3)	$\theta = 0$ $\Delta = 0$	fixed	$V = 0$ $M = 0$	free
4)	θ Δ	free	V M	fixed
5)	θ $\Delta = 0$	internal pin	V $M = 0$	hinge
6)	θ $\Delta = 0$	internal roller	V $M = 0$	hinge
7)	θ Δ	hinge	V M	internal roller

Example 8.5

Determine the max deflection of the steel beam. The reactions have been computed. Take E = 200GPa, I = $60(10^6)\text{mm}^4$



real beam

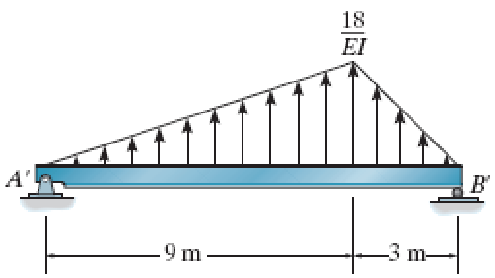
Solution

The conjugate beam loaded with the M/EI diagram is shown. Since M/EI diagram is +ve, the distributed load acts upward.

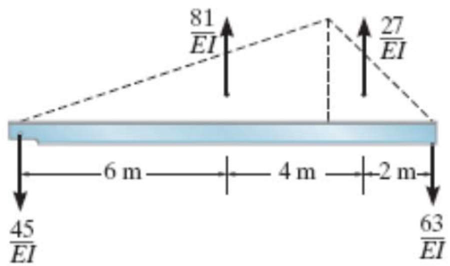
The external reactions on the conjugate beam are determined first and are indicated on the free-body diagram.

Max deflection of the real beam occurs at the point where the slope of the beam is zero.

Assuming this point acts within the region $0 \le x \le 9$m from A' we can isolate the section.



conjugate beam



external reactions

Solution

Note that the peak of the distributed loading was determined from proportional triangles,

$$w/x = (18/EI)/9$$
$$V' = 0$$

$$+\uparrow \sum F_y = 0$$
$$-\frac{45}{EI} + \frac{1}{2}\left(\frac{2x}{EI}\right)x = 0$$
$$x = 6.71m \quad (0 \le x \le 9m) \;\; OK$$

Solution

Using this value for x, the max deflection in the real beam corresponds to the moment M'.

Hence,

$$\text{With anticlockwise moments as} + \text{ve,} \sum M = 0$$

$$\frac{45}{EI}(6.71) - \left[\frac{1}{2}\left(\frac{2(6.71)}{EI}\right)6.71\right]\frac{1}{3}(6.71) + M' = 0$$

Solution

The –ve sign indicates the deflection is downward .

$$\Delta_{\max} = M' = -\frac{201.2kNm^3}{EI}$$

$$= \frac{-201.2kNm^3}{[200(10^6)kN/m^2][60(10^6)mm^4(1m^4/(10^3)^4mm^4)]}$$

$$= -0.0168m = -16.8mm$$

DEFLECTIONS USING ENERGY METHODS

THEORY OF STRUCTURES

Asst. Prof. Dr. Cenk Üstündağ

External Work & Strain Energy

- For more complicated loadings or for structures such as trusses & frames, it is suggested that energy methods be used for the computations
- Most energy methods are based on the conservation of energy principal
- Work done by all external forces acting on a structure, U_e is transformed into internal work or strain energy U_i
  - $U_e = U_i$

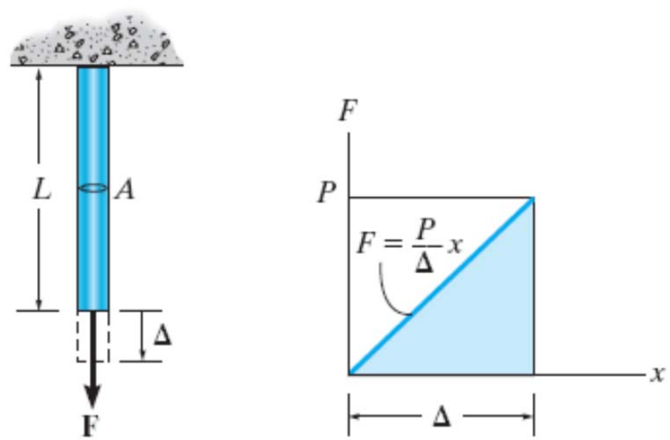
External Work & Strain Energy

- If the material's elastic limit is not exceeded, the elastic strain energy will return the structure to its undeformed state when the loads are removed
- When a force F undergoes a disp dx in the same direction as the force, the work done is
  - $d\,U_e = F\,dx$
- If the total disp is x, the work becomes:

$$\boxed{U_e = \int_0^x F dx}$$

External Work & Strain Energy

- Consider the effect caused by an axial force applied to the end of a bar
- F is gradually increased from 0 to some limiting value F = P
- The final elongation of the bar becomes Δ
- If the material has a linear elastic response, then $F = (P/\Delta)x$

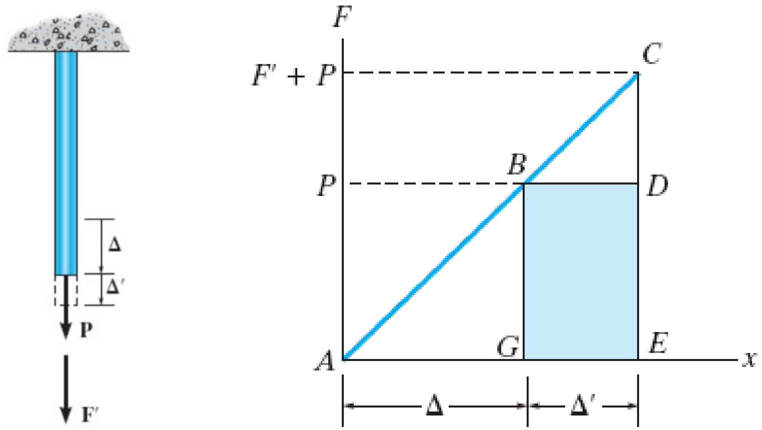


External Work & Strain Energy

- By integrating we have:

$$U_e = \frac{1}{2} P\Delta$$

- Suppose P is already applied to the bar & that another force F' is now applied, so that the bar deflects further by an amount Δ



External Work & Strain Energy

- The work done by P when the bar undergoes the further deflection is then
  - $d\ U_e' = P\Delta'$
- Here the work represents the shaded rectangular area
- In this case, P does not change its magnitude since Δ' is caused only by F'
  - Work = force x disp

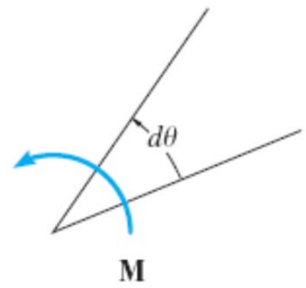
External Work & Strain Energy

- When a force P is applied to the bar, followed by an application of the force F', the total work done by both forces is rep by the triangular area ACE
- The triangular area ABG rep the work of P that is caused by disp Δ
- The triangular area BCD rep the work of F' since this force causes a dsip Δ'
- Lastly the shaded rectangular area BDEG rep the additional work done by P

External Work & Strain Energy

- The work of a moment = magnitude of the moment (M) x the angle ($d\theta$) through which it rotates
  - $dU_e = M\, d\theta$
- If the total angle of rotation is θ rad, the work becomes

$$\boxed{Ue = \int_0^{\theta} M d\theta}$$



External Work & Strain Energy

- If the moment is applied gradually to a structure having a linear elastic response from 0 to M, then the work done is

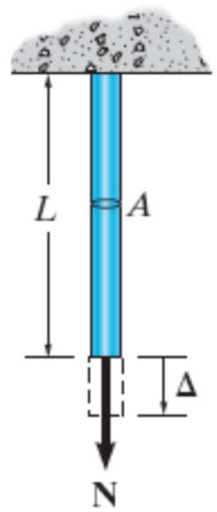
$$\boxed{Ue = \frac{1}{2} M\theta}$$

- However, if the moment is already applied to the structure & other loadings further distort the structure an amount θ', then M rotates θ' & the work done is

$$\boxed{Ue' = M\theta'}$$

External Work & Strain Energy

- When an axial force N is applied gradually to the bar, it will strain the material such that the external work done by N will be converted into strain energy
- Provided the material is linearly elastic, Hooke's Law is valid
  - $\sigma = E\varepsilon$
- If the bar has a constant x-sectional area A and length L



External Work & Strain Energy

- The normal stress is $\sigma = N/A$
- The final strain is $\varepsilon = \Delta/L$
- Consequently, $N/A = E(\Delta/L)$
- Final deflection:

$$\boxed{\Delta = \frac{NL}{AE}}$$

- Substituting with $P = N$,

$$\boxed{U_i = \frac{N^2 L}{2AE}}$$

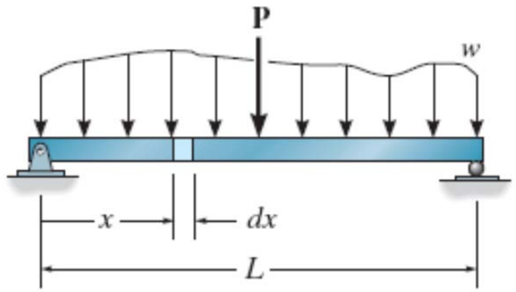
External Work & Strain Energy

- Consider the beam, P & w are gradually apply
- These loads create an internal moment M in the beam at a section located a distance x from the left support
- Consequently, the strain energy or work stored in the element can be determined since the internal moment is gradually developed

External Work & Strain Energy

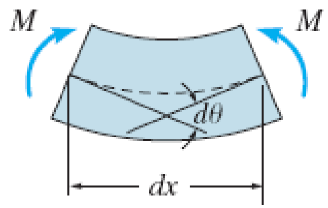
- Hence,

$$dU_i = \frac{M^2 dx}{2EI}$$



- The strain energy for the beam is determined by integrating this result over the beam's length

$$U_i = \int_0^L \frac{M^2 dx}{2EI}$$



Principle of Work & Energy

- Consider finding the disp at a point where the force P is applied to the cantilever beam
- The external work:

$$\boxed{U_e = \frac{1}{2} P \Delta}$$

- To obtain the resulting strain energy, we must first determine the internal moment as a function of position x in the beam

Principle of Work & Energy

- In this case, M = - Px so that:

$$U_i = \int_0^L \frac{M^2 dx}{2EI} = \int_0^L \frac{(-Px)^2 dx}{2EI} = \frac{1}{6}\frac{P^2L^3}{EI}$$

- Equating the ext work to int strain energy & solving for the unknown disp, we have:

$$U_e = U_i$$

$$\frac{1}{2}P\Delta = \frac{1}{6}\frac{P^2L^3}{EI}$$

$$\Delta = \frac{PL^3}{3EI}$$

Principle of Work & Energy

- Limitations
  - It will be noted that only one load may be applied to the structure
  - Only the disp under the force can be obtained

Principle of Work & Energy

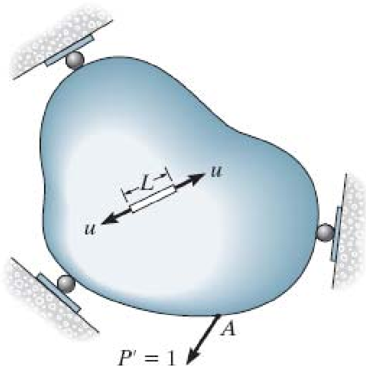
- If we take a deformable structure of any shape or size & apply a series of external loads P to it, it will cause internal loads u at points throughout the structure
- As a consequence of these loadings, external disp Δ will occur at the P loads & internal disp δ will occur at each point of internal loads u
- In general, these disp do not have to be elastic, & they may not be related to the loads

Principle of Virtual Work

- In general, the principle states that:

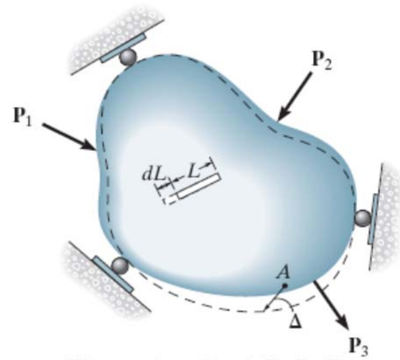
$$\sum P\Delta = \sum u\delta$$

Work of Ext loads	Work of Int loads



Apply virtual load P′ first.

- Consider the structure (or body) to be of arbitrary shape
- Suppose it is necessary to determine the disp Δ of point A on the body caused by the "real loads" P_1, P_2 and P_3



Then apply real loads P_1, P_2, P_3.

(b)

Principle of Virtual Work

- It is to be understood that these loads cause no movement of the supports
- They can strain the material beyond the elastic limit
- Since no external load acts on the body at A and in the direction of Δ, the disp Δ, the disp can be determined by first placing on the body a “virtual” load such that this force P’ acts in the same direction as Δ

Principle of Virtual Work

- We will choose P' to have a unit magnitude, P' =1
- Once the virtual loadings are applied, then the body is subjected to the real loads P_1, P_2 and P_3,
- Point A will be displaced an amount Δ causing the element to deform an amount dL

Principle of Virtual Work

- As a result, the external virtual force P' & internal load u "ride along" by Δ and dL & therefore, perform external virtual work of 1. Δ on the body and internal virtual work of u.dL on the element

$$\boxed{1.\Delta = \sum u.dL}$$

- By choosing P' = 1, it can be seen from the solution for Δ follows directly since Δ = ΣudL
- A virtual couple moment M' having a unit magnitude is applied at this point

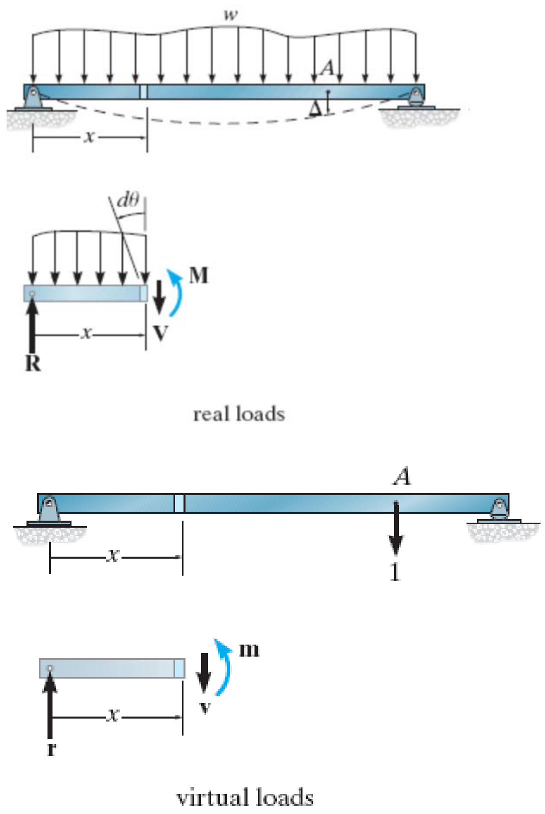
Principle of Virtual Work

- This couple moment causes a virtual load u_θ in one of the elements of the body
- Assuming that the real loads deform the element an amount dL, the rotation θ can be found from the virtual – work eqn

$$\boxed{1.\theta = \sum u_\theta . dL}$$

Method of virtual work: Beams & Frames

- To compute Δ a virtual unit load acting in the direction of Δ is placed on the beam at A
- The internal virtual moment m is determined by the method of sections at an arbitrary location x from the left support
- When point A is displaced Δ, the element dx deforms or rotates $d\theta = (M/EI)dx$



Method of virtual work: Beams & Frames

$$1.\Delta = \int_0^L \frac{mM}{EI} dx$$

$=$ external virtual unit load acting on the beam or frame in the direction of Δ
$=$ internal virtual moment in the beam or frame, expressed as a function of x & caused by the ext virtual unit load
$=$ ext disp of the point caused by real loads acting on the beam or frame
$=$ int moment in the beam or frame, expressed as a function of x & caused by the real loads
$=$ modulus of elasticity of the material
$=$ moment of inertia of cross-sectional area, computed about the neutral axis

Method of virtual work: Beams & Frames

- If the tangent rotation or slope angle θ at a point on the beam's elastic curve is to be determined, a unit couple moment is applied at the point
- The corresponding int moment m_θ have to be determined

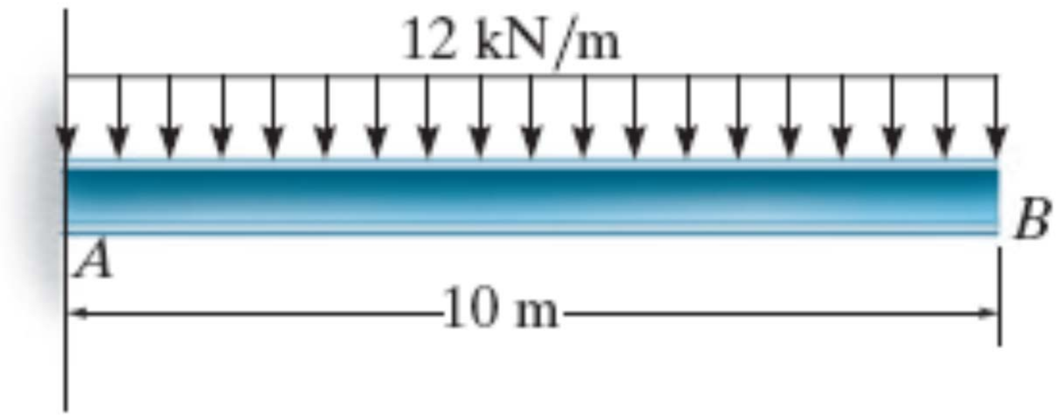
$$\boxed{1.\theta = \int_0^L \frac{m_\theta M}{EI} dx}$$

Method of virtual work: Beams & Frames

- If concentrated forces or couple moments act on the beam or the distributed load is discontinuous, separate x coordinates will have to chosen within regions that have no discontinuity of loading

Example 9.4

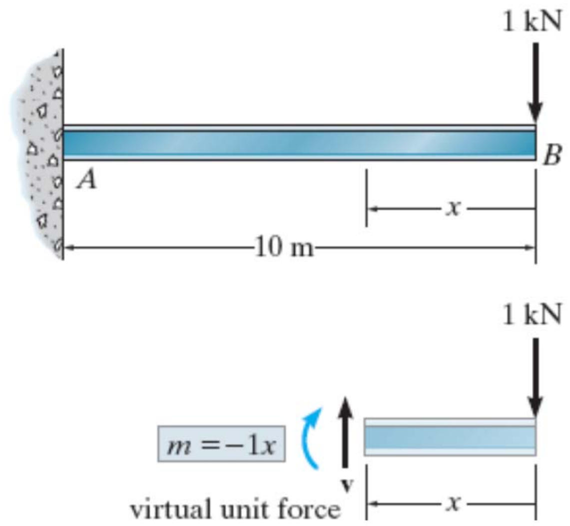
Determine the disp of point B of the steel beam. Take E = 200GPa and I = $500(10^6)$ mm^4.



Solution

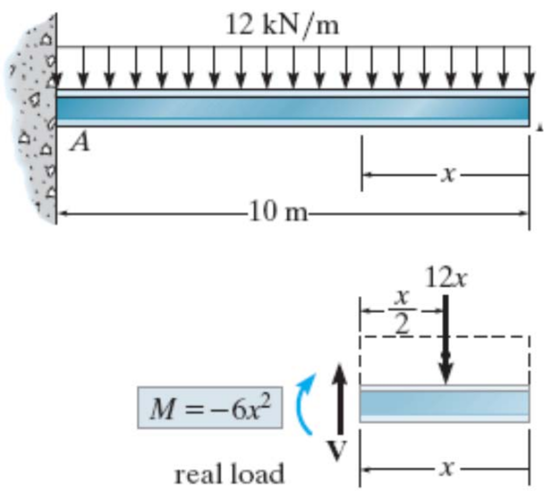
Virtual moment m

The vertical disp of point B is obtained by placing a virtual unit load of 1kN at B. Using method of sections, the internal moment m is formulated.



Real moment M

Using the same x coordinate, M is formulated.



Solution

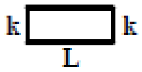
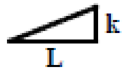
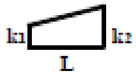
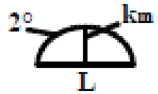
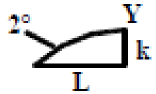
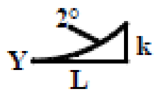
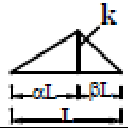
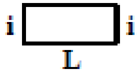
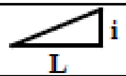
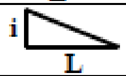
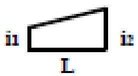
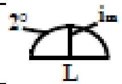
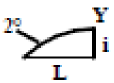
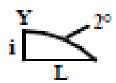
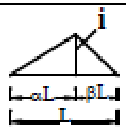
Virtual work eqn

$$1kN.\Delta_B = \int_0^L \frac{mM}{EI}dx = \int_0^{10} \frac{(-1x)(-6x^2)}{EI}dx$$

$$1kN.\Delta_B = \frac{15(10^3)kN^2m^3}{EI}$$

$$\Delta_B = 0.150m = 150mm$$

Method of virtual work: $\int M_i M_k$ Table

$\int_0^L M_i M_k \, ds$							
M_k \ M_i							
	$L i k$	$\frac{1}{2} L i k$	$\frac{1}{2} L i (k_1 + k_2)$	$\frac{2}{3} L i k_m$	$\frac{2}{3} L i k$	$\frac{1}{3} L i k$	$\frac{1}{2} L i k$
	$\frac{1}{2} L i k$	$\frac{1}{3} L i k$	$\frac{1}{6} L i (k_1 + 2k_2)$	$\frac{1}{3} L i k_m$	$\frac{5}{12} L i k$	$\frac{1}{4} L i k$	$\frac{1}{6} L (1+\alpha) i k$
	$\frac{1}{2} L i k$	$\frac{1}{6} L i k$	$\frac{1}{6} L i (2k_1 + k_2)$	$\frac{1}{3} L i k_m$	$\frac{1}{4} L i k$	$\frac{1}{12} L i k$	$\frac{1}{6} L (1+\beta) i k$
	$\frac{1}{2} L (i_1 + i_2) k$	$\frac{1}{6} L (i_1 + 2i_2) k$	$\frac{1}{6} L (2i_1k_1 + i_1k_2 + i_2k_1 + 2i_2k_2)$	$\frac{1}{3} L (i_1 + i_2) k_m$	$\frac{1}{12} L (3i_1 + 5i_2) k$	$\frac{1}{12} L (i_1 + 3i_2) k$	$\frac{1}{6} L k [(1+\beta) i_1 + (1+\alpha) i_2]$
	$\frac{2}{3} L i_m k$	$\frac{1}{3} L i_m k$	$\frac{1}{3} L i_m (k_1 + k_2)$	$\frac{8}{15} L i_m k_m$	$\frac{7}{15} L i_m k$	$\frac{1}{5} L i_m k$	$\frac{1}{3} L (1+\alpha\beta) i_m k$
	$\frac{2}{3} L i k$	$\frac{5}{12} L i k$	$\frac{1}{12} L i (3k_1 + 5k_2)$	$\frac{7}{15} L i k_m$	$\frac{8}{15} L i k$	$\frac{3}{10} L i k$	$\frac{1}{12} L (5-\beta-\beta^2) i k$
	$\frac{2}{3} L i k$	$\frac{1}{4} L i k$	$\frac{1}{12} L i (5k_1 + 3k_2)$	$\frac{7}{15} L i k_m$	$\frac{11}{30} L i k$	$\frac{2}{15} L i k$	$\frac{1}{12} L (5-\alpha-\alpha^2) i k$
	$\frac{1}{3} L i k$	$\frac{1}{4} L i k$	$\frac{1}{12} L i (k_1 + 3k_2)$	$\frac{1}{5} L i k_m$	$\frac{3}{10} L i k$	$\frac{1}{5} L i k$	$\frac{1}{12} L (1+\alpha+\alpha^2) i k$
	$\frac{1}{3} L i k$	$\frac{1}{12} L i k$	$\frac{1}{12} L i (3k_1 + k_2)$	$\frac{1}{5} L i k_m$	$\frac{2}{15} L i k$	$\frac{1}{30} L i k$	$\frac{1}{12} L (1+\beta+\beta^2) i k$
	$\frac{1}{2} L i k$	$\frac{1}{6} L (1+\alpha) i k$	$\frac{1}{6} L i [(1+\beta) k_1 + (1+\alpha) k_2]$	$\frac{1}{3} L (1+\alpha\beta) i k_m$	$\frac{1}{12} L (5-\beta-\beta^2) i k$	$\frac{1}{12} L (1+\alpha+\alpha^2) i k$	$\frac{1}{3} L i k$

Ends with the letter Y represent horizontal tangents

Analysis of Statically Indeterminate Structures by the Force Method

THEORY OF STRUCTURES

Asst. Prof. Dr. Cenk Üstündağ

Statically Indeterminate Structures

- Advantages & Disadvantages
  - For a given loading, the max stress and deflection of an indeterminate structure are generally smaller than those of its statically determinate counterpart
  - Statically indeterminate structure has a tendency to redistribute its load to its redundant supports in cases of faulty designs or overloading

Statically Indeterminate Structures

- Advantages & Disadvantages
  - Although statically indeterminate structure can support loading with thinner members & with increased stability compared to their statically determinate counterpart, the cost savings in material must be compared with the added cost to fabricate the structure since often it becomes more costly to construct the supports & joints of an indeterminate structure
  - Because statically indeterminate structures have redundant support reactions, one has to be very careful to prevent differential displacement of the supports, since this effect will introduce internal stress in the structure.

Statically Indeterminate Structures

- Method of Analysis

When analyzing any indeterminate structure, it is necessary to satisfy *equilibrium*, *compatibility*, and *force-displacement* requirements for the structure.

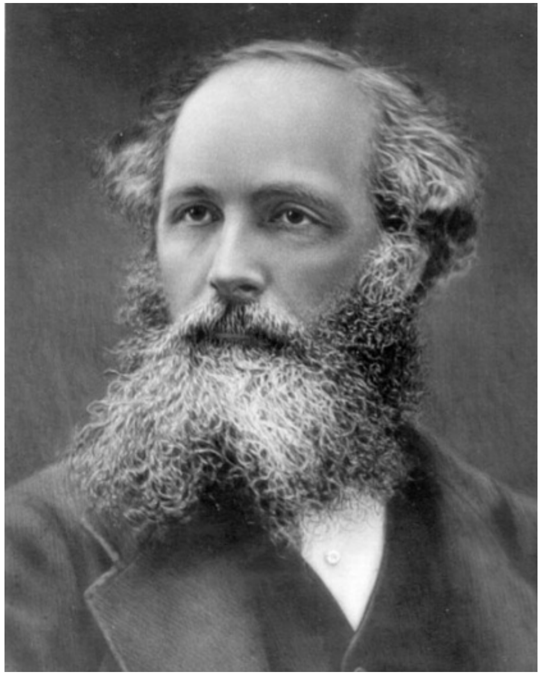
Equilibrium is satisfied when the reactive forces hold the structure at rest, and *compatibility* is satisfied when the various segments of the structure fit together without intentional breaks or overlaps. The *force-displacement* requirements depend upon the way the material responds; (*Here we assume linear elastic response*).

Statically Indeterminate Structures

- Force Method

  The force method was originally developed by James Clerk Maxwell (*a Scottish physicist and mathematician*) in 1864 and later refined by Otto Mohr and Heinrich Müller-Breslau (*German civil engineers*).

- Displacement Method



Statically Indeterminate Structures

- Force Method:

  The force method consists of writing equations that satisfy the *compatibility* and *force-displacement* requirements for the structure in order to determine the redundant forces. Once these forces have been determined, the remaining reactive forces on the structure are determined by satisfying the equilibrium requirements.

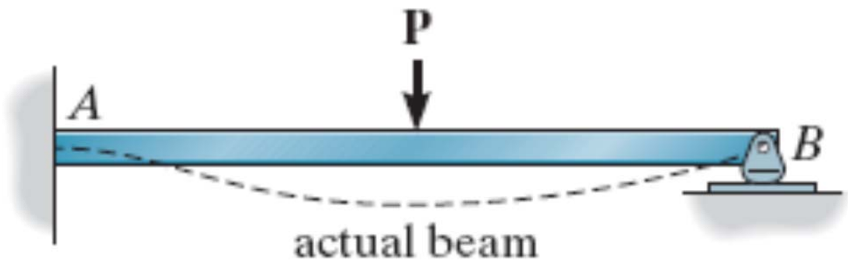
- Displacement Method:

  The displacement method of analysis is based on first writing *force-displacement* relations for the members and then satisfying the *equilibrium requirements* for the structure. In this case the unknowns in the equations are displacements. Once the displacements are obtained, the forces are determined from the compatibility and force-displacement equations.

Statically Indeterminate Structures

	Unknowns	Equations Used for Solution	Coefficents of the Unknowns
Force Method	Forces	Compatibility and Force Displacement	Flexibility Coefficents
Displacement Method	Displacements	Equilibirum and Force Displacement	Stiffness Coefficients

Force Method of Analysis: General Procedure

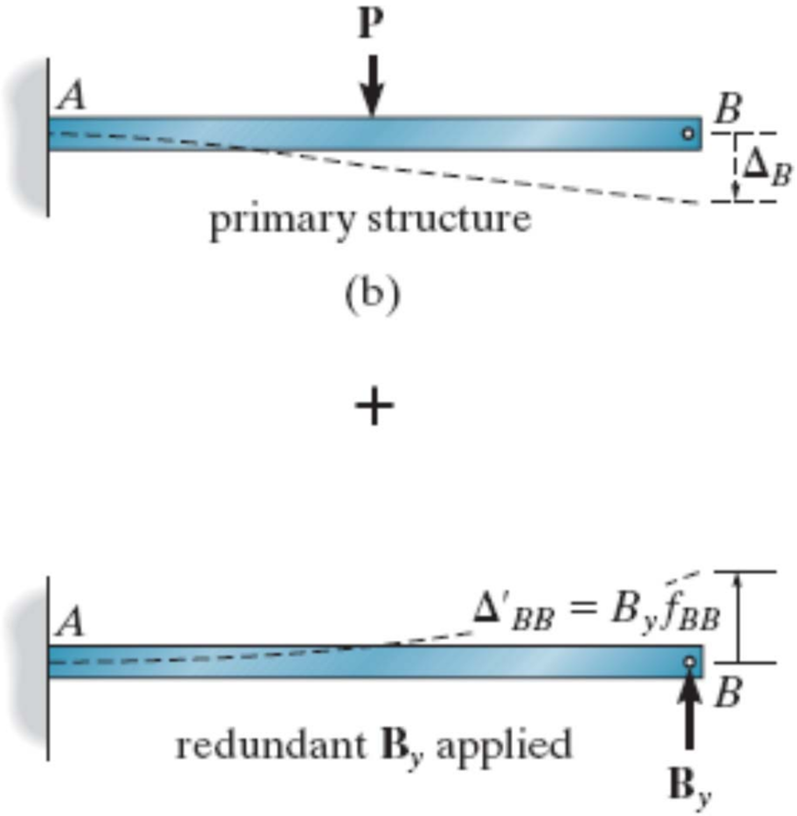


(a)

- From free-body diagram, there would be 4 unknown support reactions
- 3 equilibrium equations
- Beam is indeterminate to first degree
- Use principle of superposition & consider the compatibility of displacement at one of the supports
- Choose one of the support reactions as redundant & temporarily removing its effect on the beam

Force Method of Analysis: General Procedure

- This will allow the beam to be statically determinate & stable
- Here, we will remove the rocker at B
- As a result, the load P will cause B to be displaced downward
- By superposition, the unknown reaction at B causes the beam at B to be displaced upward

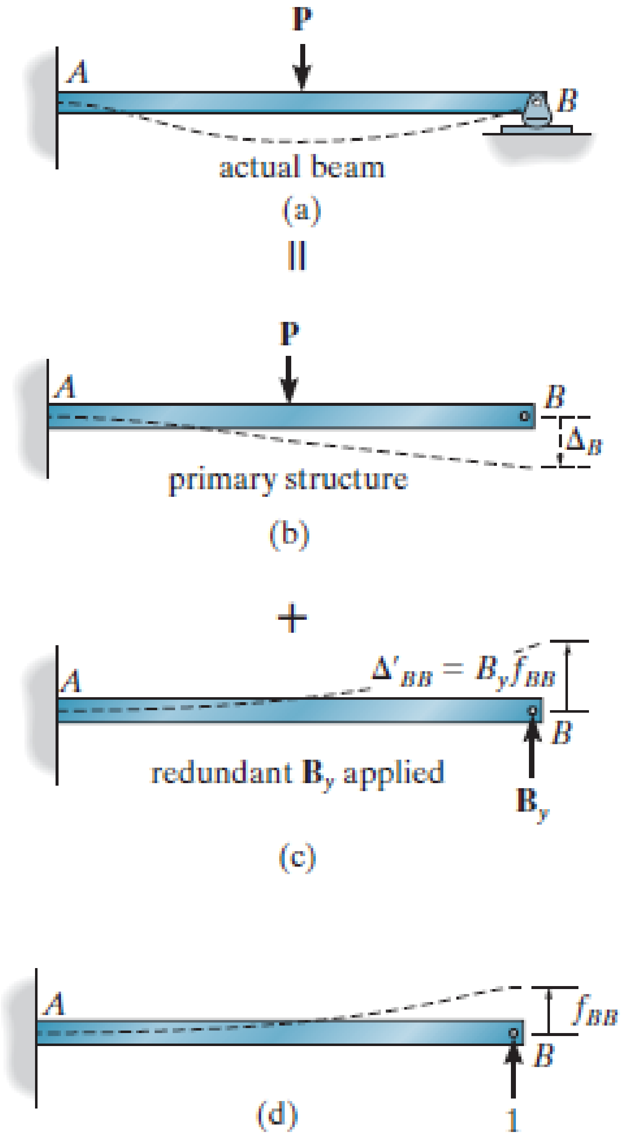


Force Method of Analysis: General Procedure

- Assuming positive displacements act upward, then we can write the necessary compatibility equation at the rocker as

$$0 = -\Delta_B + \Delta'_{BB}$$

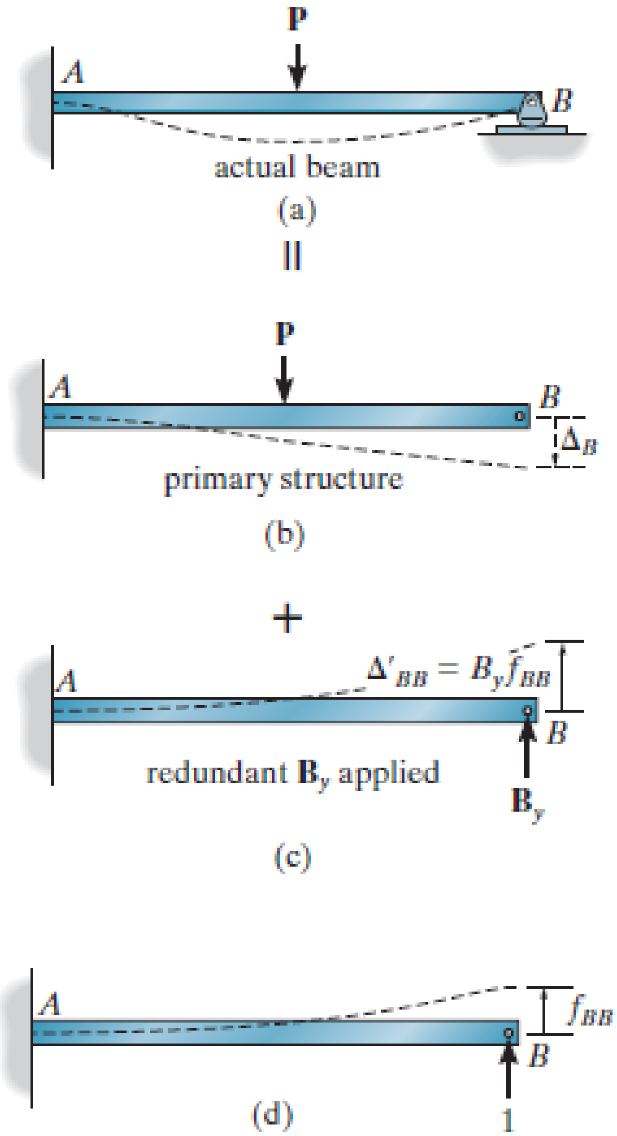
- Here the first letter in this double-subscript notation refers to the point (B) where the deflection is specified, and the second letter refers to the point (B) where the unknown reaction acts.



Force Method of Analysis: General Procedure

- Let us denote the displacement at B caused by a unit load acting in the direction of B_y as the *linear flexibility coefficient* f_{BB}.
- Since the material behaves in a linear-elastic manner, a force of B_y acting at B, instead of the unit load, will cause a proportionate increase in f_{BB}.

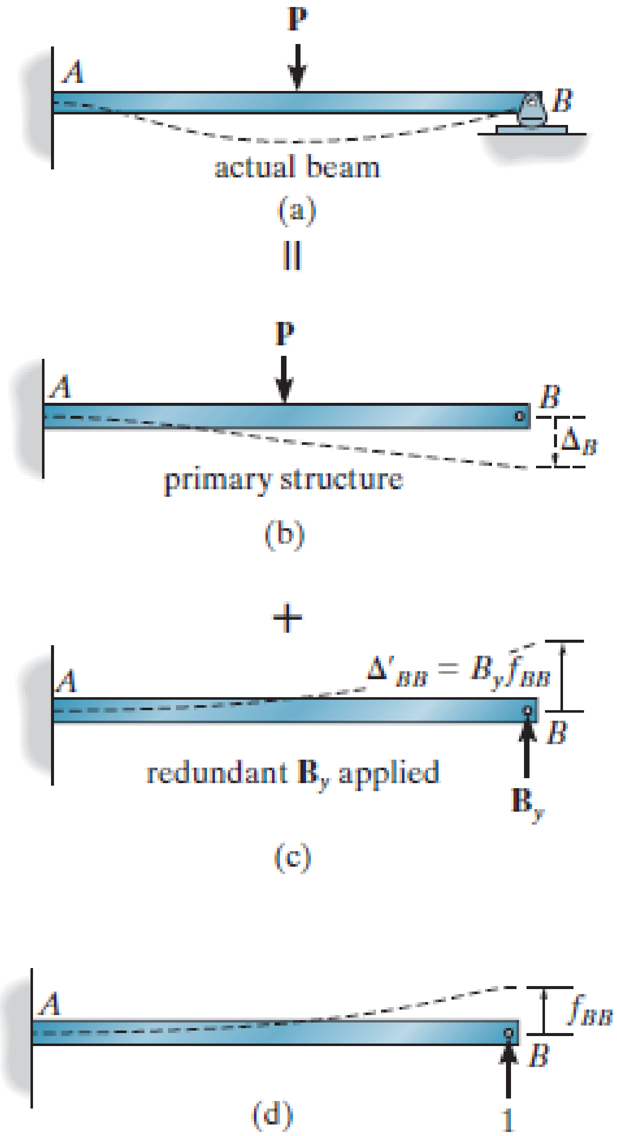
$$\Delta'_{BB} = B_y f_{BB}$$



Force Method of Analysis: General Procedure

- The linear flexibility coefficient f_{BB} is a measure of the deflection per unit force, and so its units are m/N.
- The compatibility equation above can therefore be written in terms of the unknown B_y as

$$0 = -\Delta_B + B_y f_{BB}$$

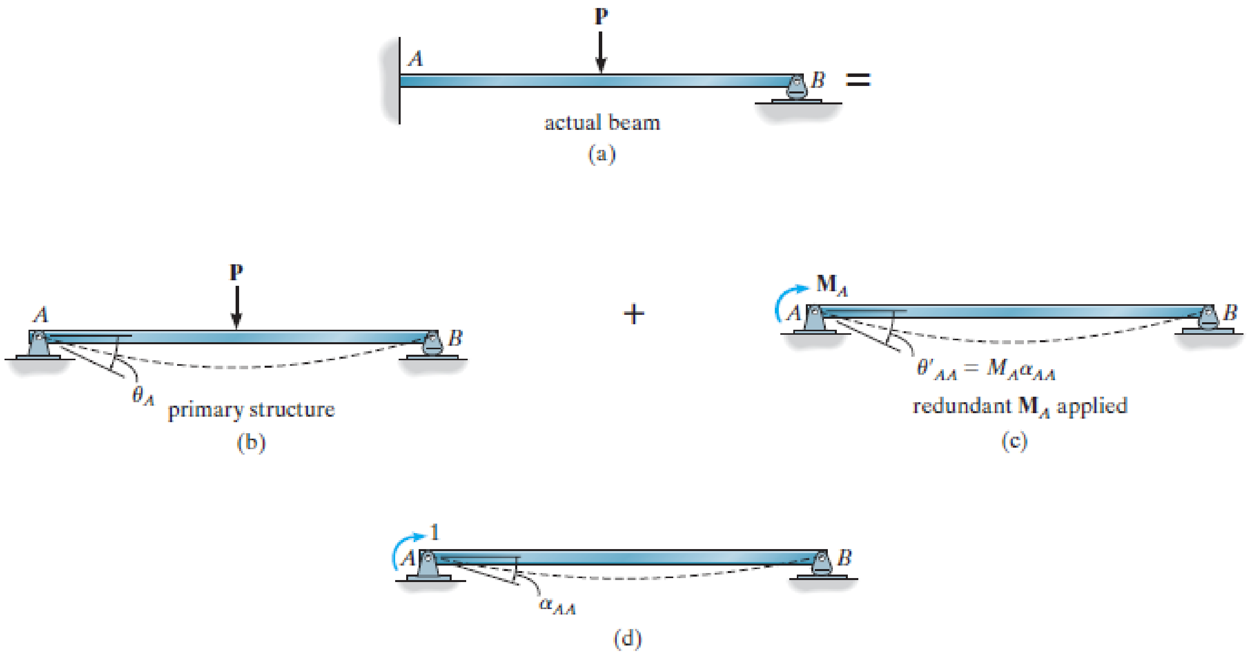


Force Method of Analysis: General Procedure

- Using the method of virtual work the appropriate load-displacement relations for the deflection Δ_B and the flexibility coefficient f_{BB}, can be obtained and the solution for B_y can be determined.
- Once this is accomplished, the three reactions at the wall A can then be found from the equations of equilibrium.
- The choice of redundant is arbitrary

Force Method of Analysis: General Procedure

- The moment at A can be determined directly by removing the capacity of the beam to support moment at A, replacing fixed support by pin support
- The rotation at A caused by P is θ_A
- The rotation at A caused by the redundant M_A at A is θ'_{AA}



Force Method of Analysis: General Procedure

- If we denote an *angular flexibility coefficient* α_{AA} as the angular displacement at A caused by a unit couple moment applied to A, then

$$\theta'_{AA} = M_A \alpha_{AA}$$

- Thus, the angular flexibility coefficient measures the angular displacement per unit couple moment, and therefore it has units of rad/N. The compatibility equation for rotation at A therefore requires

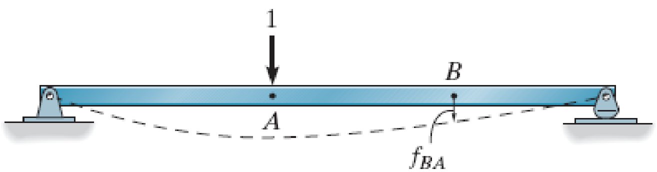
$$0 = \theta_A + M_A \alpha_{AA}$$

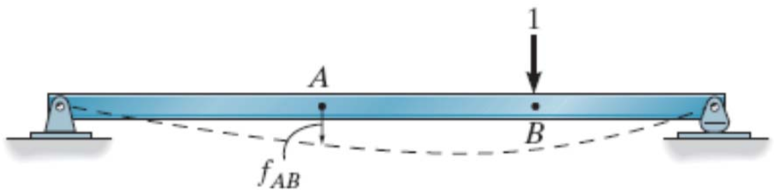
Maxwell's Theorem of Reciprocal Displacements: Betti's Law

- The displacement of a point B on a structure due to a unit load acting at point A is equal to the displacement of point A when the load is acting at point B

$$\boxed{f_{BA} = f_{AB}}$$

- Proof of this theorem is easily demonstrated using the principle of virtual work



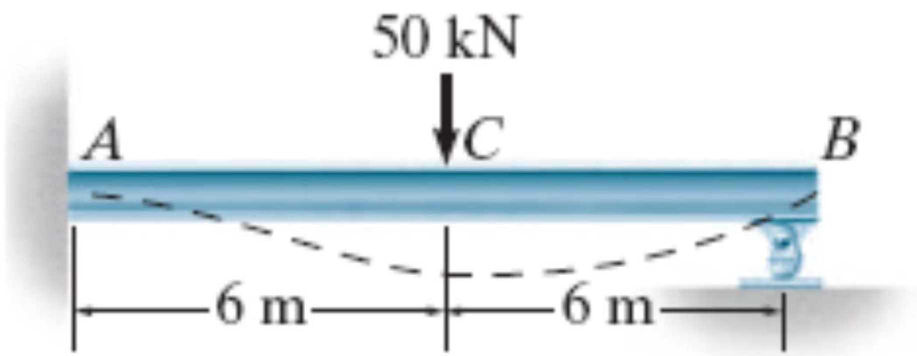


Maxwell's Theorem of Reciprocal Displacements: Betti's Law

- The theorem also applies for reciprocal rotations
- The rotation at point B on a structure due to a unit couple moment acting at point A is equal to the rotation at point A when the unit couple is acting at point B

Example 10.1

Determine the reaction at the roller support B of the beam. EI is constant.

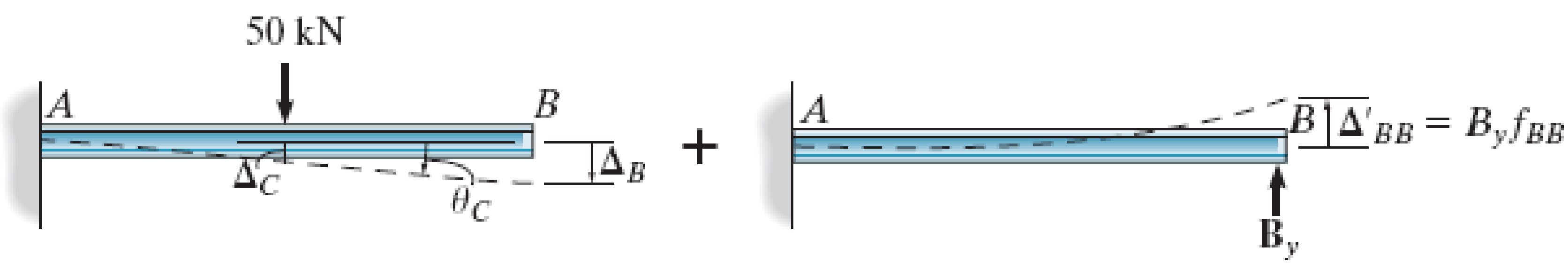


actual beam

Solution

Principle of superposition

By inspection, the beam is statically indeterminate to the first degree. The redundant will be taken as B_y. We assume B_y acts upward on the beam.



primary structure

redundant $\mathbf{B}_y$ applied

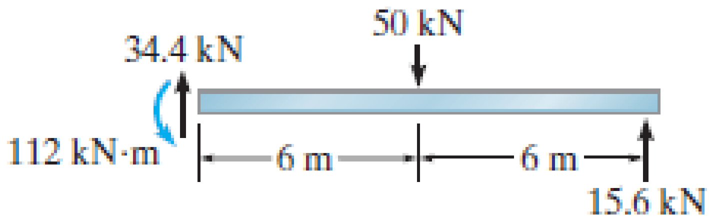
Solution

Compatibility equation

$$(+\uparrow)\ \ 0 = -\Delta_B + B_y f_{BB} \quad \text{eqn}(1)$$

Δ_B and f_{BB} are easily obtained using standard table.

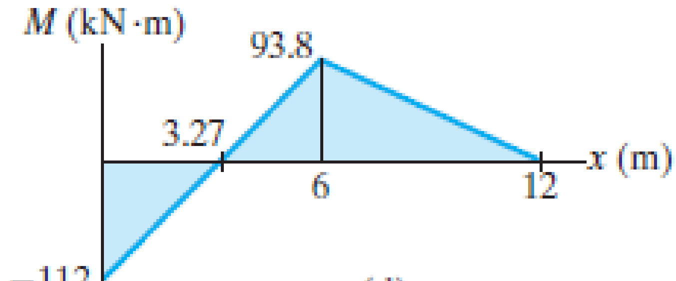
$$\Delta_B = \frac{9000kNm^3}{EI}; \quad f_{BB} = \frac{576m^3}{EI}$$



(c)

Sub into eqn (1) :

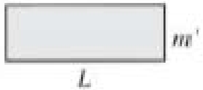
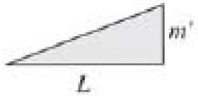
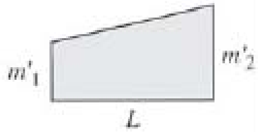
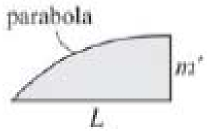
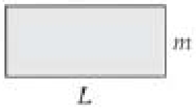
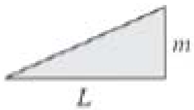
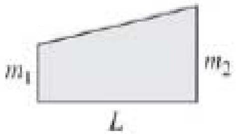
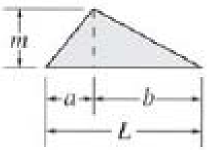
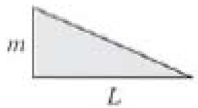
$$0 = -\frac{9000}{EI} + B_y \frac{576}{EI} \Rightarrow B_y = 15.6kN$$



(d)

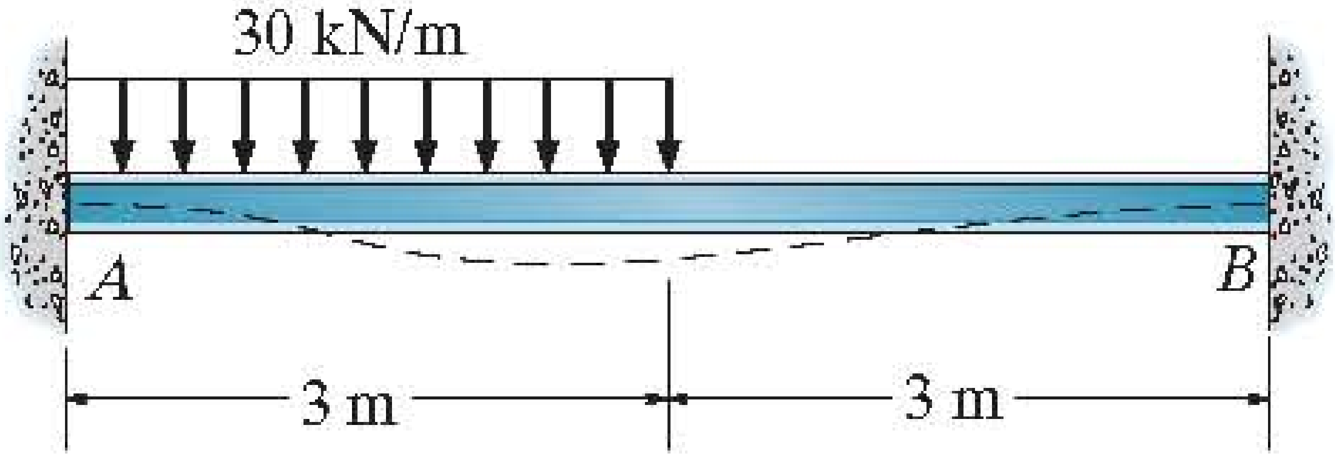
Solution

Table for Evaluating $\int_0^L m\, m'\, dx$

$\int_0^L m\, m'\, dx$				
	$mm'L$	$\frac{1}{2}mm'L$	$\frac{1}{2}m(m'_1 + m'_2)L$	$\frac{2}{3}mm'L$
	$\frac{1}{2}mm'L$	$\frac{1}{3}mm'L$	$\frac{1}{6}m(m'_1 + 2m'_2)L$	$\frac{5}{12}mm'L$
	$\frac{1}{2}m'(m_1 + m_2)L$	$\frac{1}{6}m'(m_1 + 2m_2)L$	$\frac{1}{6}[m'_1(2m_1 + m_2) + m'_2(m_1 + 2m_2)]L$	$\frac{1}{12}[m'(3m_1 + 5m_2)]L$
	$\frac{1}{2}mm'L$	$\frac{1}{6}mm'(L + a)$	$\frac{1}{6}m[m'_1(L + b) + m'_2(L + a)]$	$\frac{1}{12}mm'\left(3 + \frac{3a}{L} - \frac{a^2}{L^2}\right)L$
	$\frac{1}{2}mm'L$	$\frac{1}{6}mm'L$	$\frac{1}{6}m(2m'_1 + m'_2)L$	$\frac{1}{4}mm'L$

Example 10.4

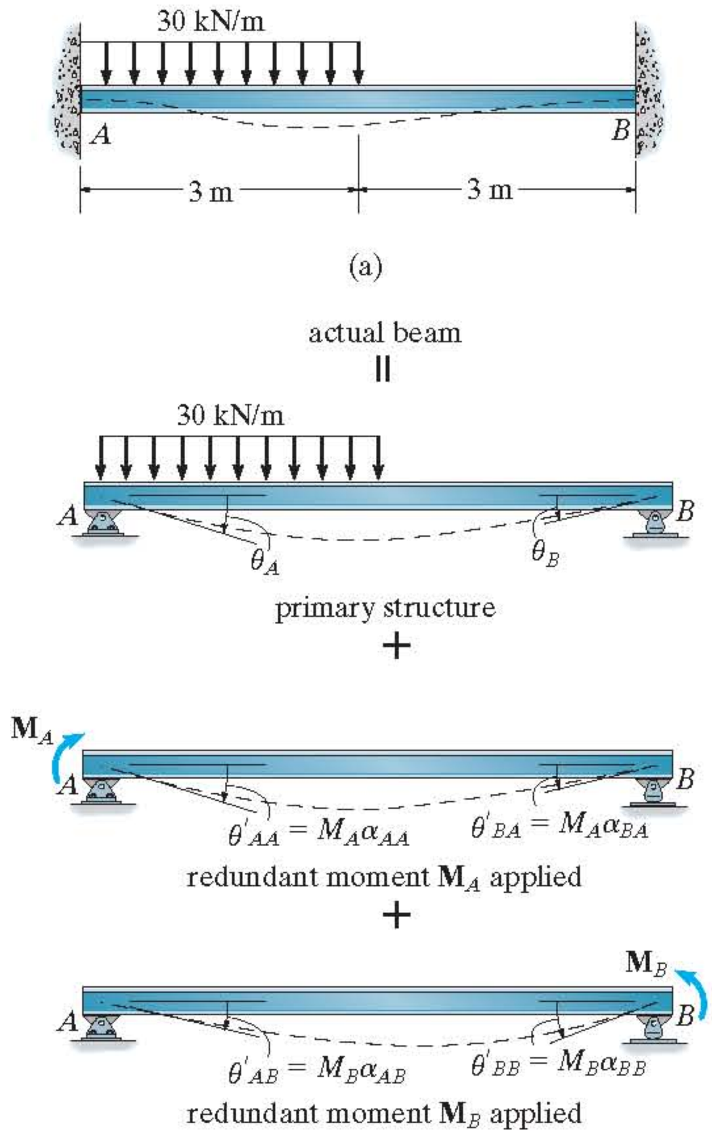
Draw the shear and moment diagrams for the beam. *EI* is constant. Neglect the effects of axial load.



Solution

Principle of Superposition

Since axial load is neglected, the beam is indeterminate to the second degree. The 2 end moments at A & B will be considered as the redundant. The beam's capacity to resist these moments is removed by placing a pin at A and a rocker at B.



Solution

Compatibility eqn

Reference to points A & B requires

$$0 = \theta_A + M_A \alpha_{AA} + M_B \alpha_{AB} \quad \text{eqn (1)}$$

$$0 = \theta_B + M_A \alpha_{BA} + M_B \alpha_{BB} \quad \text{eqn (2)}$$

The required slopes and angular flexibility coefficients can be determined using standard tables.

$$\theta_A = \frac{151.9}{EI}; \quad \theta_A = \frac{118.1}{EI}$$

$$\alpha_{AA} = \frac{2}{EI}; \quad \alpha_{BB} = \frac{2}{EI}; \quad \alpha_{AB} = \alpha_{BA} = \frac{1}{EI}$$

Solution

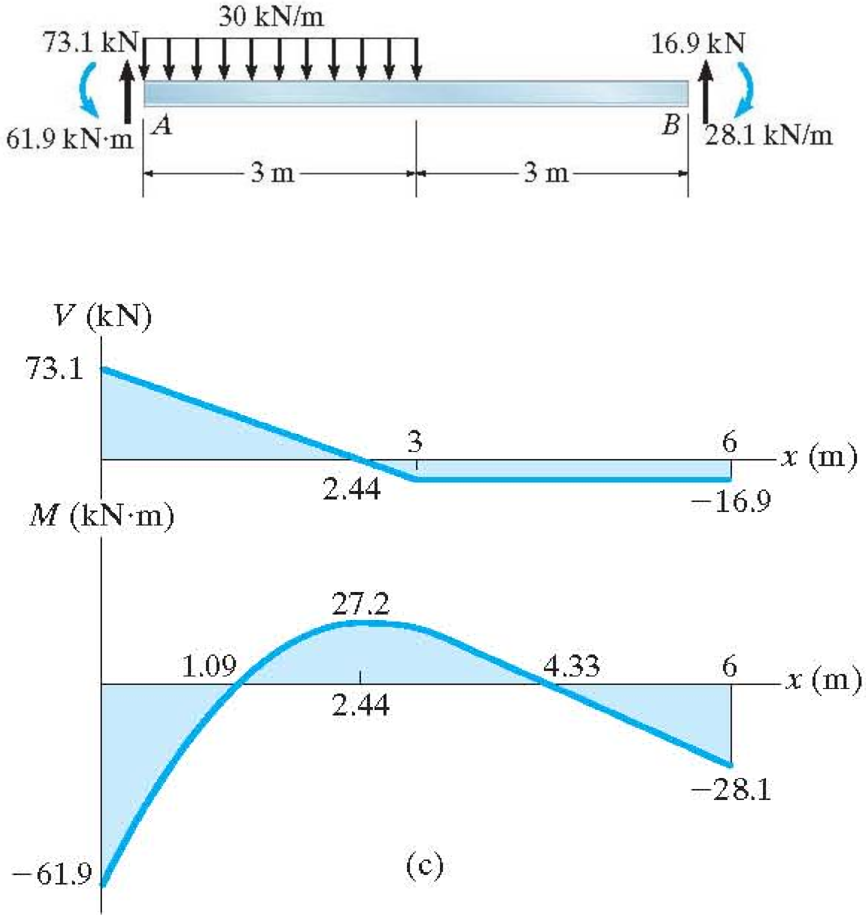
Compatibility eqn

Sub into eqn (1) and (2)gives :

$$0 = \frac{151.9}{EI} + M_A\left(\frac{2}{EI}\right) + M_B\left(\frac{1}{EI}\right)$$

$$0 = \frac{118.1}{EI} + M_A\left(\frac{1}{EI}\right) + M_B\left(\frac{2}{EI}\right)$$

$$M_A = -61.9kNm; \quad M_B = -28.1kNm,$$



(c)

Displacement Method of Analysis: Moment Distribution

THEORY OF STRUCTURES

Asst. Prof. Dr. Cenk Üstündağ

General Principles & Definition

- Displacement method requires satisfying equilibrium equations for the structures
- The unknowns displacement are written in terms of the loads by using the load-displacement relations
- These equations are solved for the displacement
- Once the displacement are obtained, the unknown loads are determined from the compatibility equations using the load displacement relations

General Principles & Definition

The method of analyzing beams and frames using moment distribution was developed by Hardy Cross, in 1930.

At the time this method was first published it attracted immediate attention, and it has been recognized as one of the most notable advances in structural analysis during the twentieth century.

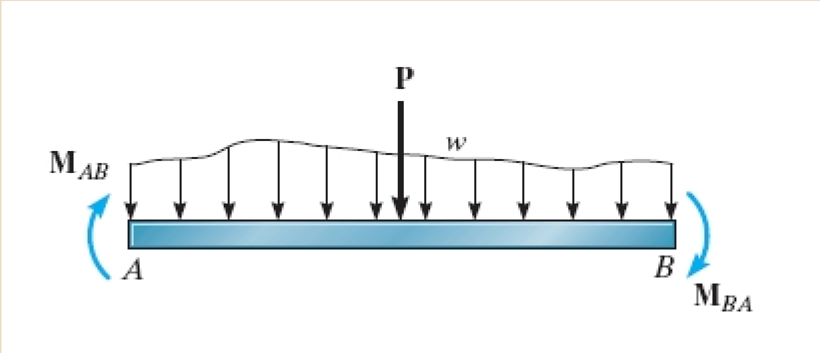
General Principles & Definition

- Moment distribution is a method of successive approximations that may be carried out to any desired degree of accuracy
- The method begins by assuming each joint of a structure is fixed
- By unlocking and locking each joint in succession, the internal moments at the joints are "distributed" & balanced until the joints have rotated to their final or nearly final positions

General Principles & Definition

- Sign Convention

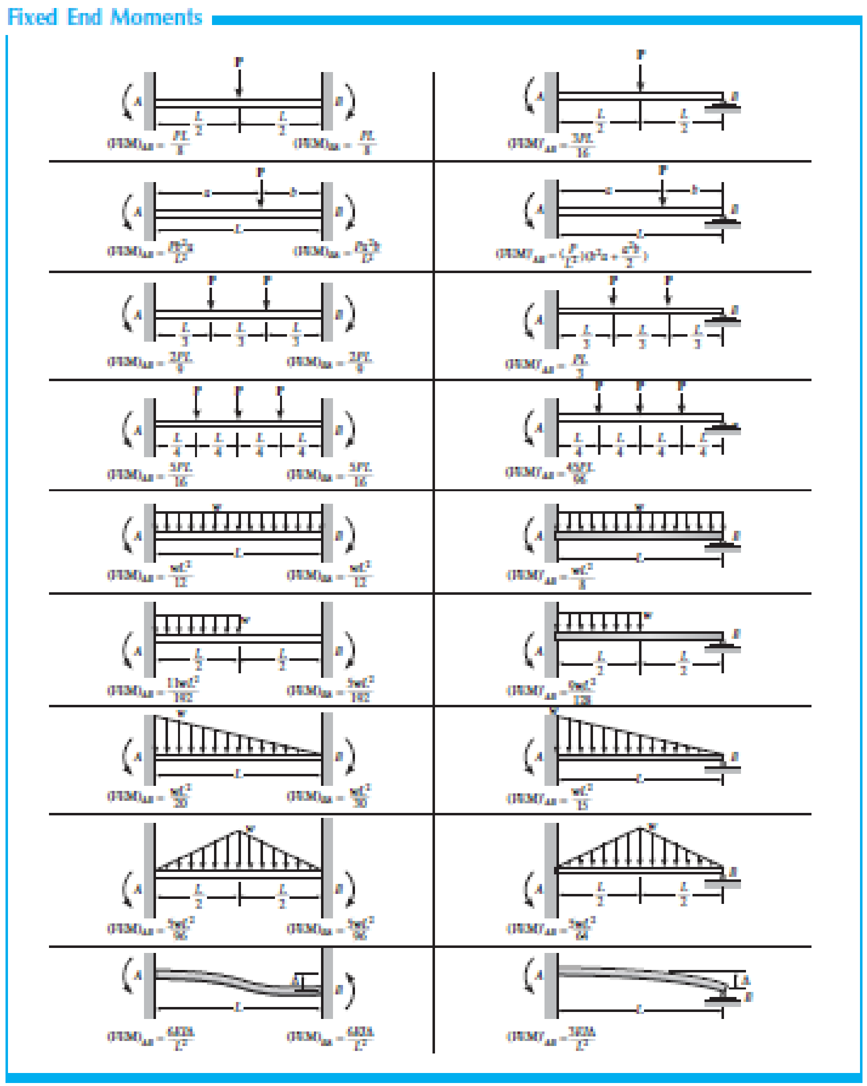
  We will establish the same sign convention as that established for the slope-deflection equations: *Clockwise moments* that act *on the member* are considered *positive,* whereas *counterclockwise moments* are *negative.*



General Principles & Definition

- Fixed-End Moments (FEMs)

  The moments at the "walls" or fixed joints of a loaded member are called *fixed-end moments*. These moments can be determined from the table on the right side depending upon the type of loading on the member.



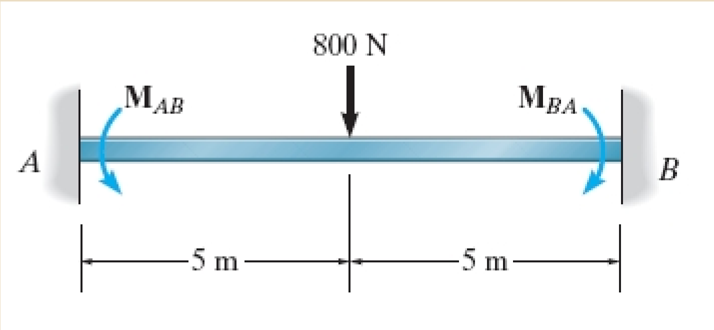
General Principles & Definition

For example, the beam loaded as shown in figure has fixed-end moments of

FEM = PL/8 = 800(10)/8 = 1000 Nm.

Noting the action of these moments *on the beam* and applying our sign convention, it is seen that

M_{AB} = -1000Nm and M_{BA}=1000Nm



General Principles & Definition

- Member stiffness factor

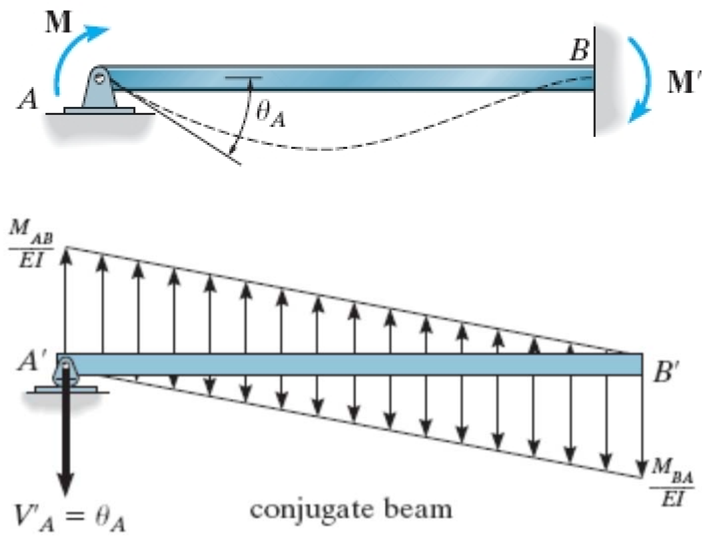
Consider the beam in the figure, which is pinned at one end and fixed at the other. Application of the moment **M** causes the end *A* to rotate through an angle θ_A. Using the conjugate-beam method M can be related to θ_A as follows:

$$M = \left(\frac{4EI}{L}\right)\theta_A$$

The term in parentheses

$$K = \left(\frac{4EI}{L}\right)$$

is referred to as the *stiffness factor* at *A* and can be defined as the amount of moment *M* required to rotate the end *A* of the beam θ_A=1 rad.



(b)

$$\sum M_{A'} = 0$$

$$\left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{2L}{3} = 0$$

$$\sum M_{B'} = 0$$

$$\left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{2L}{3} + \theta_A L = 0$$

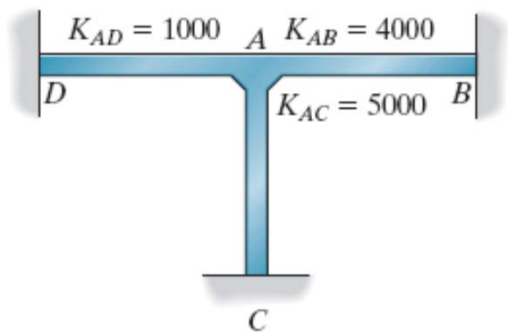
General Principles & Definition

- Joint stiffness factor

  If several members are fixed connected to joint and each of their far ends is fixed, then by the principle of superposition, the *total stiffness factor* at the joint is the sum of the member stiffness factors at the joint.

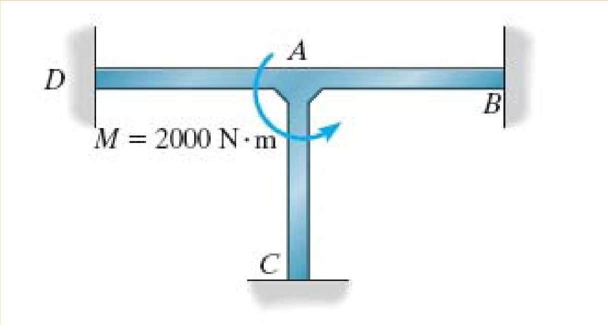
  - The total stiffness factor of joint A is

$$K_T = \sum K = 4000 + 5000 + 1000 = 10000$$



General Principles & Definition

- Distribution Factor (DF)
  - If a moment M is applied to a fixed connected joint, the connecting members will each supply a portion of the resisting moment necessary to satisfy moment equilibrium at the joint. That fraction of the total resisting moment supplied by the member is called the distribution factor (DF).



$$DF_i = \frac{M_i}{M} = \frac{K_i \theta}{\theta \sum K_i}$$

$$DF = \frac{K}{\sum K}$$

General Principles & Definition

- Member relative stiffness factor
  - Quite often a continuous beam or a frame will be made from the same material
  - E will therefore be constant
  - In the case, the common factor *4E* will cancel from the numerator and denominator when the distribution factor for a joint is determined.

$$K_R = \frac{I}{L}$$

General Principles & Definition

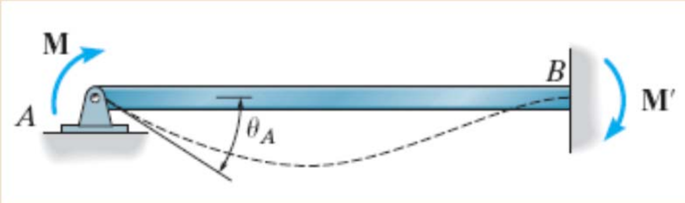
- Carry-over (CO) factor

$$M_{AB} = \left(\frac{4EI}{L}\right)\theta_{\mathrm{A}}; \qquad M_{BA} = \left(\frac{2EI}{L}\right)\theta_{\mathrm{A}}$$

  - Solving for θ and equating these equations,

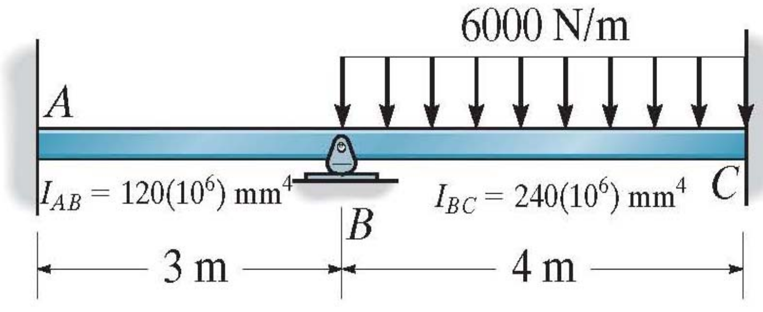
$$M_{BA} = 0.5M_{AB}$$

  - The moment M at the pin induces a moment of M' = 0.5M at the wall
  - In the case of a beam with the far end fixed, the CO factor is +0.5



General Principles & Definition

- Carry-over (CO) factor
  - The plus sign indicates both moments act in the same direction
- Consider the beam



$$K_{BA} = \frac{4E(120)(10^6)}{3} = 4E(40)(10^6)mm^4/m$$

$$K_{BC} = \frac{4E(240)(10^6)}{4} = 4E(60)(10^6)mm^4/m$$

$$DF_{BA} = \frac{4E(40)}{4E(40)+4E(60)} = 0.4$$

$$DF_{BC} = \frac{4E(60)}{4E(40)+4E(60)} = 0.6$$

General Principles & Definition

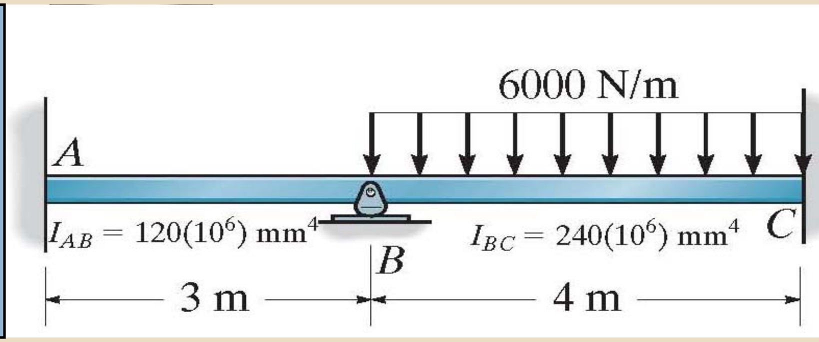
$$DF_{AB} = \frac{4E(40)}{\infty + 4E(40)} = 0$$

$$DF_{CB} = \frac{4E(60)}{\infty + 4E(60)} = 0$$

- Note that the above results could also have been obtained if the relative stiffness factor is used

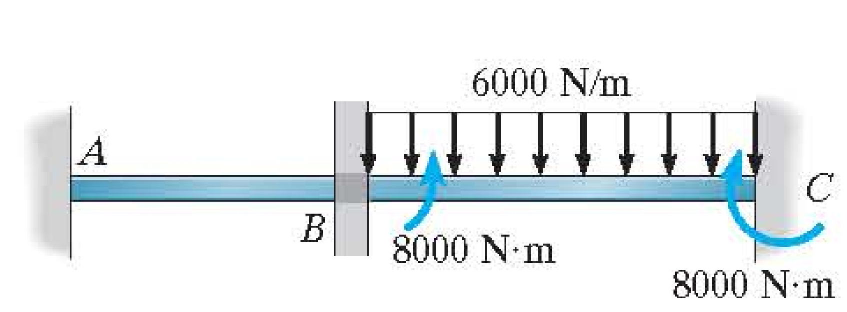
$$(FEM)_{BC} = -\frac{wL^2}{12} = -8000kNm$$

$$(FEM)_{CB} = \frac{wL^2}{12} = 8000kNm$$

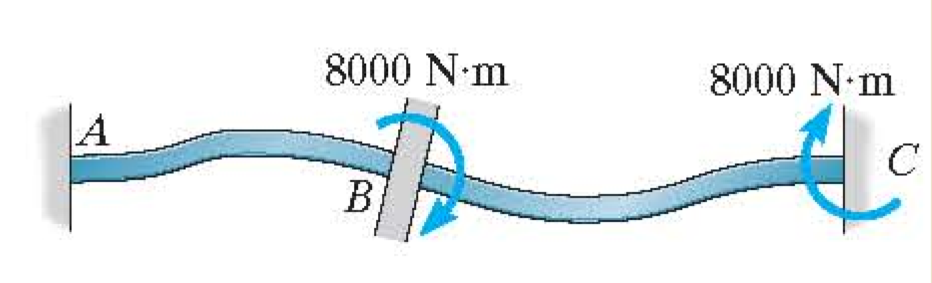


General Principles & Definition

- We begin by assuming joint B is fixed or locked
- The fixed end moment at B then holds span BC in this fixed or locked position
- To correct this, we will apply an equal but opposite moment of 8000Nm to the joint and allow the joint to rotate freely



joint B held fixed



correction moment applied to joint B

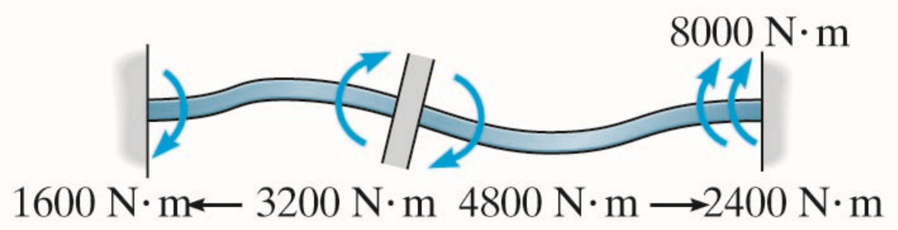
General Principles & Definition

- As a result, portions of this moment are distributed in spans BC and BA in accordance with the DFs of these spans at the joint
- Moment in BA is 0.4(8000) = 3200Nm
- Moment in BC is 0.6(8000) = 4800Nm
- These moment must be carried over since moments are developed at the far ends of the span

General Principles & Definition

- Using the carry-over factor of +0.5, the results are shown
- The steps are usually presented in tabular form
- CO indicates a line where moments are distributed then carried over
- In this particular case only one cycle of moment distribution is necessary
- The wall supports at A and C “absorb” the moments and no further joints have to be balanced to satisfy joint equilibrium

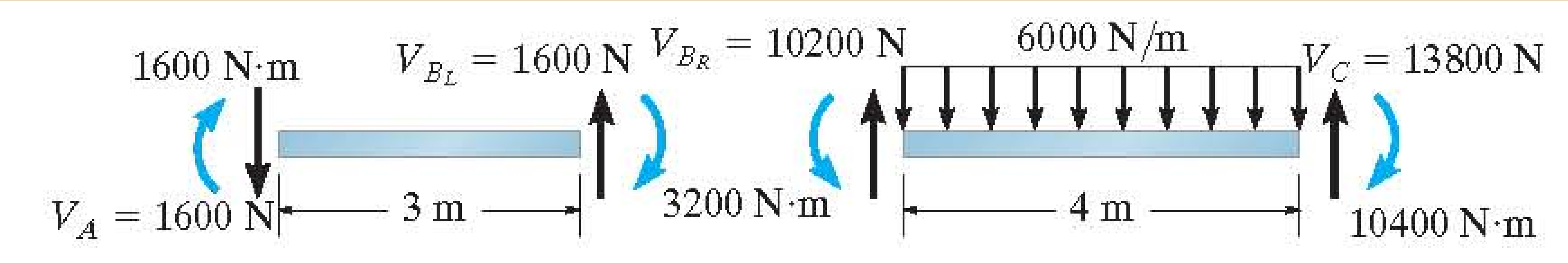
Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.4	0.6	0
FEM			−8000	8000
Dist,CO	1600 ←	3200	4800 →	2400
ΣM	1600	3200	−3200	10 400



moment at B distributed

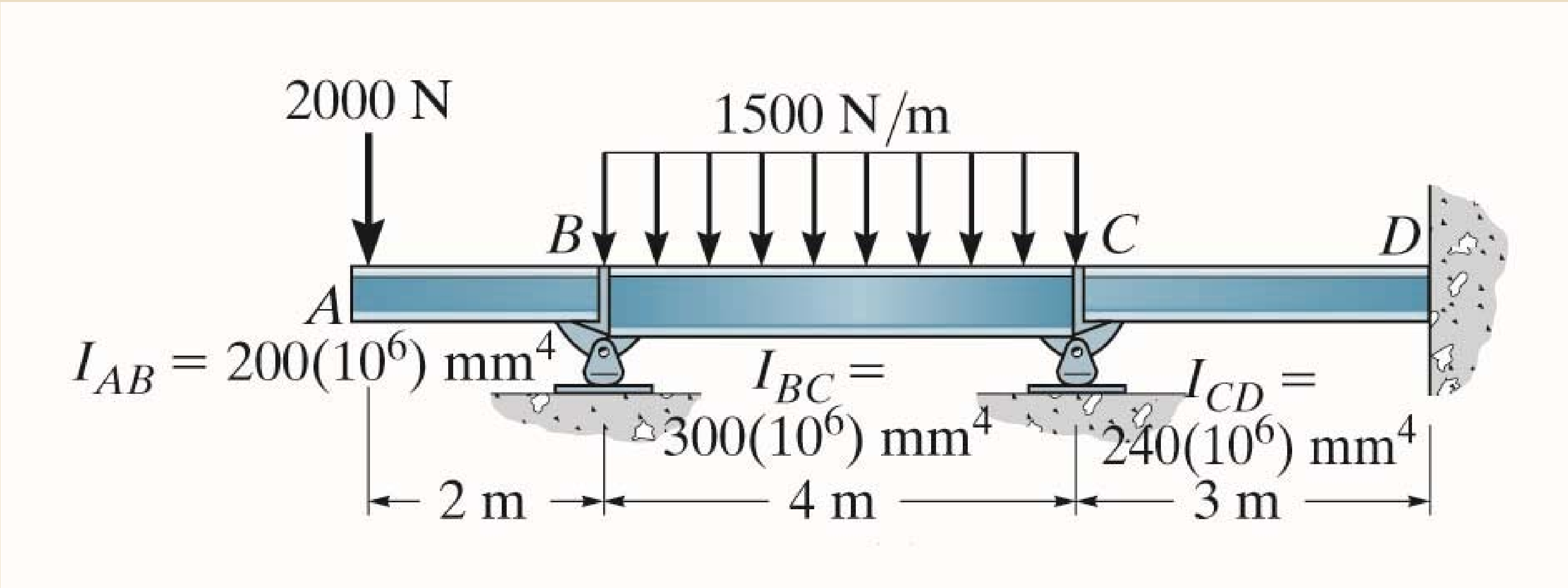
General Principles & Definition

Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.4	0.6	0
FEM			−8000	8000
Dist,CO	1600 ←	3200	4800 →	2400
ΣM	1600	3200	−3200	10 400



Example

- Determine the internal moment at each support of the beam. The moment of inertia of each span is indicated.



Solution

A moment does not get distributed in the overhanging span AB

So the distribution factor $(DF)_{BA}$ =0

Span BC is based on 4EI/L since the pin rocker is not at the far end of the beam

$$K_{BC} = \frac{4E(300)(10^6)}{4} = 300(10^6)E$$

$$K_{CD} = \frac{4E(240)(10^6)}{3} = 320(10^6)E$$

Solution

$$DF_{BC} = 1 - (DF)_{BA} = 1 - 0 = 1$$

$$DF_{CB} = \frac{300E}{300E + 320E} = 0.484$$

$$DF_{CD} = 0.516; \quad DF_{DC} = 0$$

Due to overhang, $(FEM)_{BA} = 2000N(2m) = 4000Nm$

$$(FEM)_{BC} = -\frac{wL^2}{12} = -2000Nm$$

$$(FEM)_{CB} = \frac{wL^2}{12} = 2000Nm$$

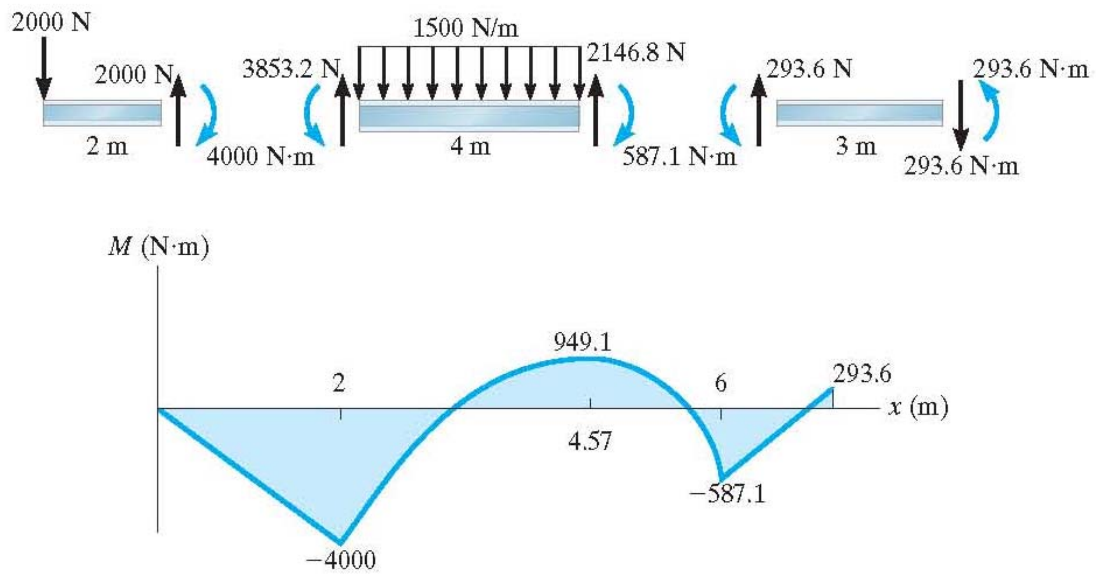
Solution

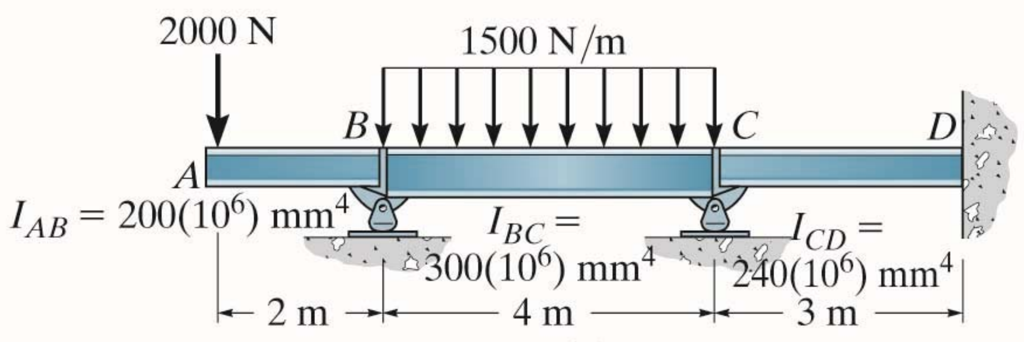
- The overhanging span requires the internal moment to the left of B to be +4000Nm.
- Balancing at joint B requires an internal moment of –4000Nm to the right of B.
- -2000Nm is added to BC in order to satisfy this condition.
- The distribution & CO operations proceed in the usual manner.

Since the internal moments are known, the moment diagram for the beam can be constructed.

Solution

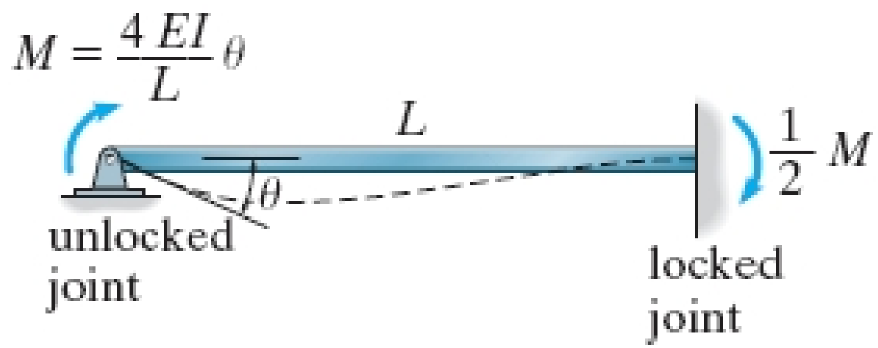
Joint	B		C		D
Member		BC	CB	CD	DC
DF	0	1	0.484	0.516	0
FEM	4000	−2000	2000		
Dist.		−2000	−968	−1032	
CO		−484	−1000		−516
Dist.		484	484	516	
CO		242	242		258
Dist.		−242	−117.1	−124.9	
CO		−58.6	−121		−62.4
Dist.		58.6	58.6	62.4	
CO		29.3	29.3		31.2
Dist.		−29.3	−14.2	−15.1	
CO		−7.1	−14.6		−7.6
Dist.		7.1	7.1	7.6	
CO		3.5	3.5		3.8
Dist.		−3.5	−1.7	−1.8	
CO		−0.8	−1.8		−0.9
Dist.		0.8	0.9	0.9	
CO		0.4	0.4		0.4
Dist.		−0.4	−0.2	−0.2	
CO		−0.1	−0.2		−0.1
Dist.		0.1	0.1	0.1	
ΣM	4000	−4000	587.1	−587.1	−293.6





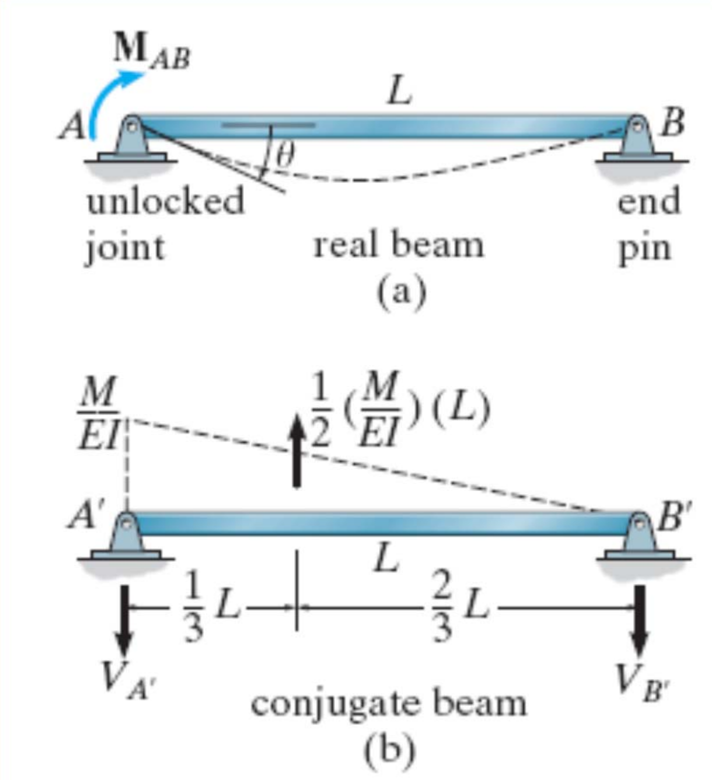
Stiffness-Factor Modifications

- The previous e.g. of moment distribution, we have considered each beam span to be constrained by a fixed support at its far end when distributing & carrying over the moments. For this reason we have computed the stiffness factors, distribution factors, and the carry-over factors based on the case shown in figure below.
- In some cases, it is possible to modify the stiffness factor of a particular beam span & thereby simplify the process of moment distribution



Stiffness-Factor Modifications

- Member pin supported at far end
  - As shown the applied moment M rotates end A by an amount θ
  - To determine θ, the shear in the conjugate beam at A' must be determined



$$\sum M_{B'} = 0 \quad V'_A(L) - \frac{1}{2}\left(\frac{M}{EI}\right)L\left(\frac{2}{3}L\right) = 0$$

$$V'_A = \theta = \frac{3L}{EI} \Rightarrow M = \frac{3EI}{L}\theta$$

Stiffness-Factor Modifications

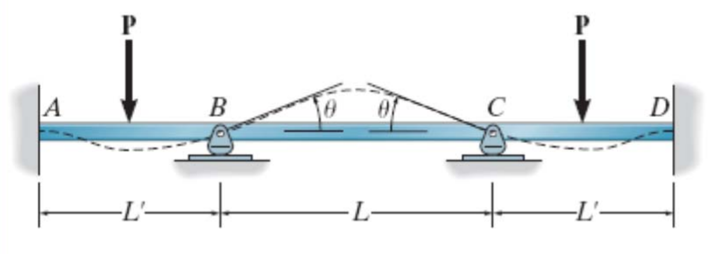
- Member pin supported at far end (cont'd)
  - The stiffness factor in the beam is

$$K = \frac{3EI}{L}$$

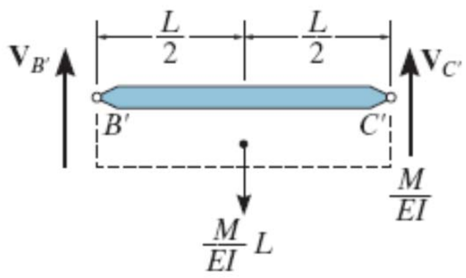
  - The CO factor is zero, since the pin at B does not support a moment
  - By comparison, if the far end was fixed supported, the stiffness factor would have to be modified by ¾ to model the case of having the far end pin supported. If this modification is considered, the moment distribution process is simplified since the end pin does not have to be unlocked–locked successively when distributing the moments

Stiffness-Factor Modifications

- Symmetric beam & loading
  - The bending-moment diagram for the beam will also be symmetric
  - To develop the appropriate stiffness-factor modification consider the beam
  - Due to symmetry, the internal moment at B & C are equal
  - Assuming this value to be M, the conjugate beam for span BC is shown



real beam



conjugate beam

Stiffness-Factor Modifications

- Symmetric beam & loading (cont'd)

$$\sum M_{C'} = 0 \quad -V'_B(L) + \left(\frac{M}{EI}\right)L\left(\frac{L}{2}\right) = 0$$

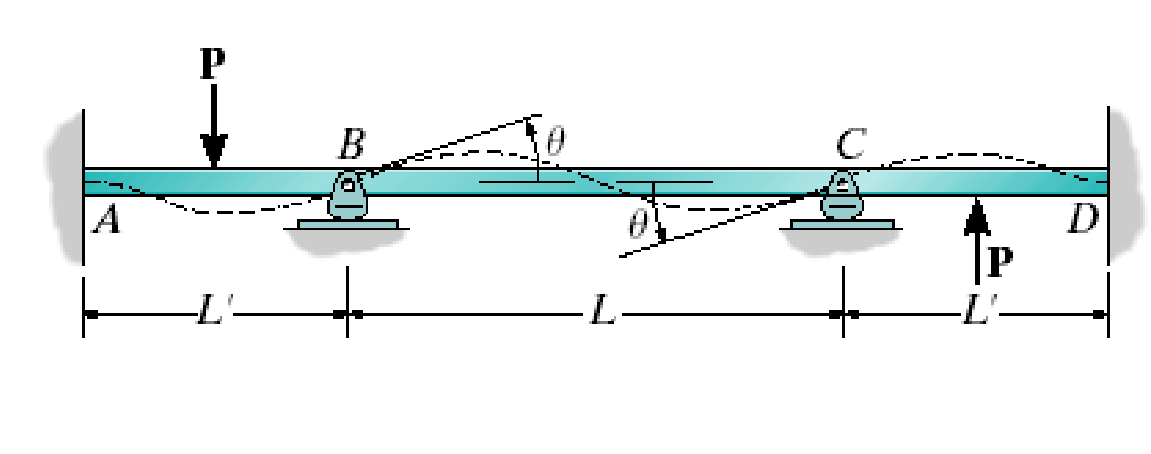
$$V'_B = \theta = \frac{ML}{2EI} \Rightarrow M = \frac{2EI}{L}\theta$$

$$K = \frac{2EI}{L}\theta$$

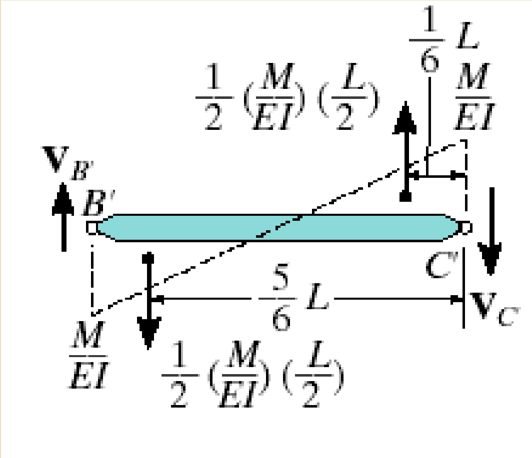
  - Moments for only half the beam can be distributed provided the stiffness factor for the center span is computed

Stiffness-Factor Modifications

- Symmetric beam with asymmetric loading
  - Consider the beam as shown
  - The conjugate beam for its center span BC is shown
  - Due to its asymmetric loading, the internal moment at B is equal but opposite to that at C



real beam



conjugate beam

Stiffness-Factor Modifications

- Symmetric beam with asymmetric loading
  - Assuming this value to be M, the slope θ at each end is determined as follows:

$$\sum M_{C'} = 0$$

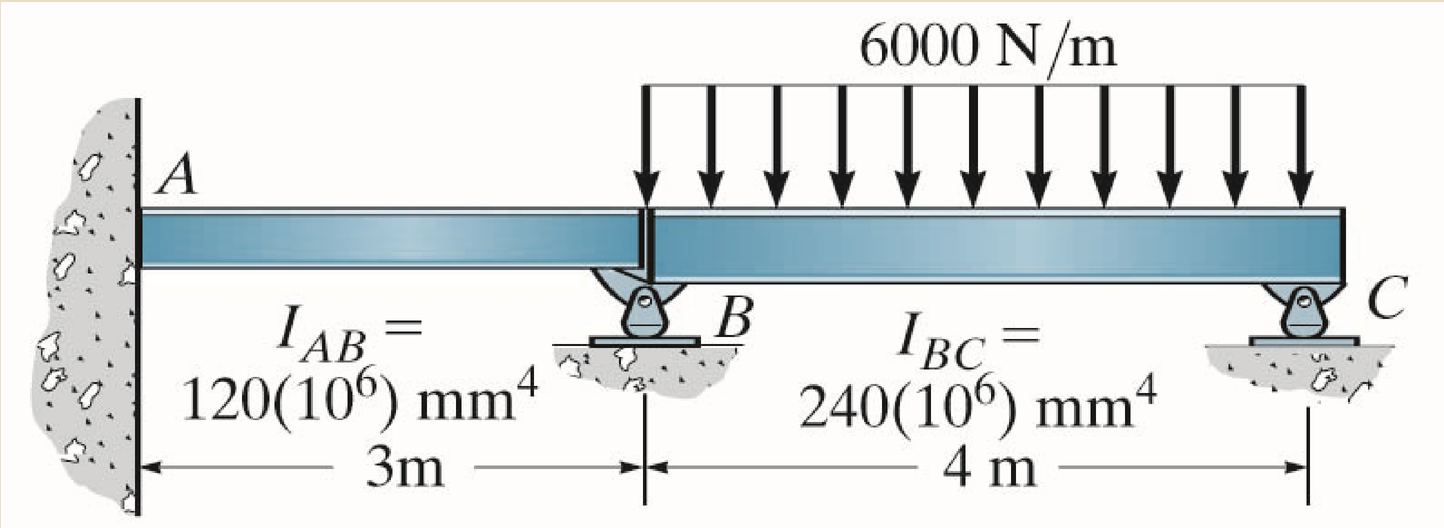
$$-V'_B(L) + \left(\frac{1}{2}\right)\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{5L}{6}\right) - \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{6}\right) = 0$$

$$V'_B = \theta = \frac{ML}{6EI} \Rightarrow M = \frac{6EI}{L}\theta$$

$$K = \frac{6EI}{L}$$

Example

Determine the internal moments at the supports of the beam shown below. The moment of inertia of the two spans is shown in the figure.



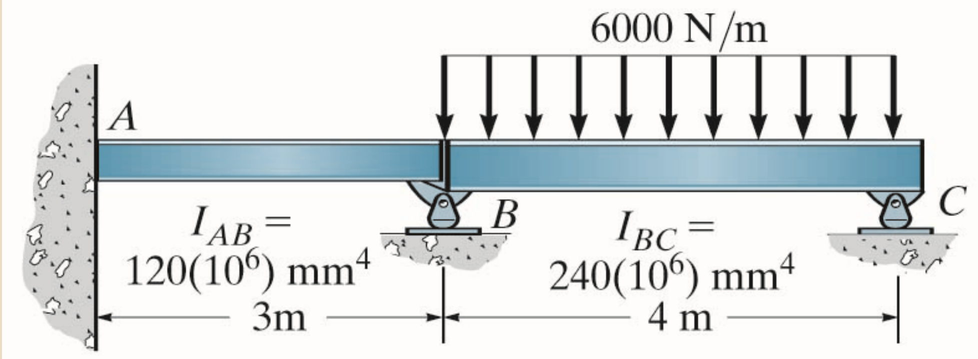
Solution

- The beam is roller supported at its far end C.
- The stiffness of span BC will be computed on the basis of K = 3EI/L
- We have:

$$K_{AB} = \frac{4EI}{L} = \frac{4E(120)(10^6)}{3} = 160(10^6)E$$

$$K_{BC} = \frac{3EI}{L} = \frac{3E(240)(10^6)}{4} = 180(10^6)E$$

Solution



$$DF_{AB} = \frac{160E}{\infty + 160E} = 0$$

$$DF_{BA} = \frac{160E}{160E + 180E} = 0.4706$$

$$DF_{BC} = \frac{180E}{160E + 180E} = 0.5294$$

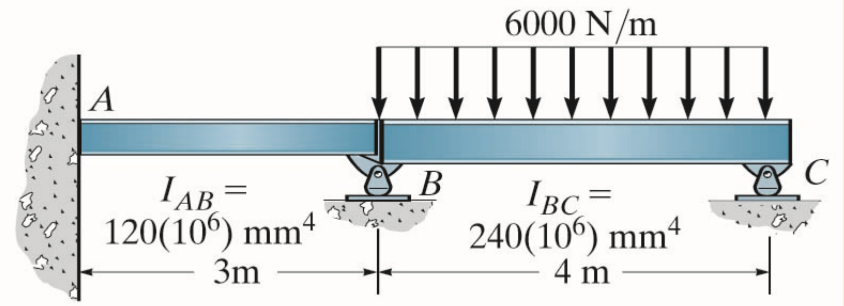
$$DF_{CB} = \frac{180E}{180E} = 1$$

$$(FEM)_{BC} = -\frac{wL^2}{8} = \frac{-6000(4)^2}{8} = -12000\,Nm$$

Solution

The forgoing data are entered into table as shown. The moment distribution is carried out.

By comparison, the method considerably simplifies the distribution.



Joint	A	B		C
Member	AB	BA	BC	CB
DF	0	0.4706	0.5294	1
FEM Dist.		 5647.2	−12 000 6352.8	
CO	2823.6			
ΣM	2823.6	5647.2	−5647.2	0