Module 5 : Force Method - Introduction and applications

Lecture 4 : Three Moment Equation

Objectives

In this course you will learn the following

- Derivation of three moment equation for analysis of continous beams.
- Demonstration of three moment equation using numerical examples.

5.5 Three Moment Equation

The continuous beams are very common in the structural design and it is necessary to develop simplified force method known as *three moment equation* for their analysis. This equation is a relationship that exists between the moments at three points in continuous beam. The points are considered as three supports of the indeterminate beams. Consider three points on the beam marked as 1, 2 and 3 as shown in Figure 5.25(a). Let the bending moment at these points is M_1, M_2 and M_3 and the corresponding vertical displacement of these points are Δ_1, Δ_2 and Δ_3, respectively. Let L_1 and L_2 be the distance between points 1 – 2 and 2 – 3, respectively.

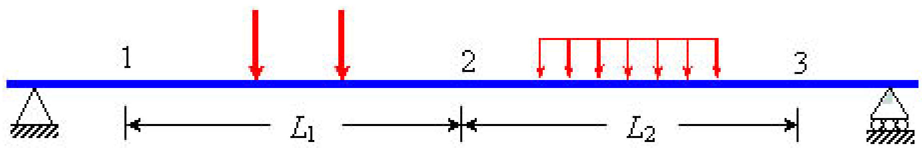

Figure 5.25(a)

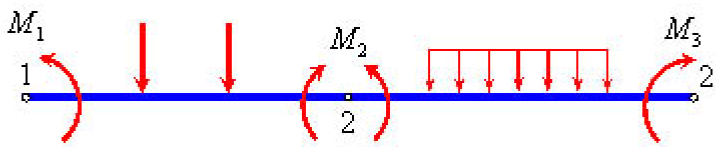

Figure 5.25(b)

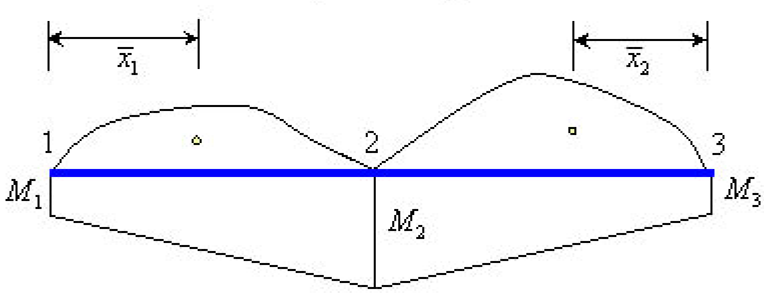

Figure 5.25(c)

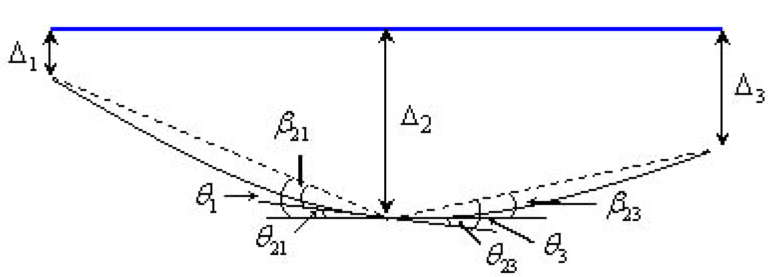

Figure 5.25(d)

The continuity of deflected shape of the beam at point 2 gives

$$\theta_{21} = \theta_{23} \tag{5.4}$$

From the Figure 5.25(d)

$$\theta_{21} = \theta_1 - \beta_{21} \text{ and } \theta_{23} = \theta_3 - \beta_{23} \tag{5.5}$$

where

$$\theta_1 = \frac{\Delta_1 - \Delta_2}{L_1} \quad \text{and} \quad \theta_3 = \frac{\Delta_3 - \Delta_2}{L_2} \tag{5.6}$$

Using the bending moment diagrams shown in Figure 5.25(c) and the second moment area theorem,

$$\theta_{21} = \frac{1}{L_1} \times \frac{1}{EI_1}\left(\frac{M_1 L_1^2}{6} + \frac{M_2 L_1^2}{3} + A_1 \bar{x}_1\right) \tag{5.7}$$

$$\theta_{23} = \frac{1}{L_2} \times \frac{1}{EI_2}\left(\frac{M_3 L_1^2}{6} + \frac{M_2 L_1^2}{3} + A_2 \bar{x}_2\right) \tag{5.8}$$

where A_1 and A_2 are the areas of the bending moment diagram of span 1-2 and 2-3, respectively considering the applied loading acting as simply supported beams.

Substituting from Eqs. (5.7) and Eqs. (5.8) in Eqs. (5.4) and Eqs. (5.5).

$$M_1\left(\frac{L_1}{I_1}\right) + 2M_2\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) + M_3\left(\frac{L_2}{I_2}\right) = -\frac{6A_1\bar{x}_1}{I_1 L_1} - \frac{6A_2\bar{x}_2}{I_2 L_2} + 6E\left[\frac{(\Delta_2 - \Delta_1)}{L_1} + \frac{(\Delta_2 - \Delta_3)}{L_2}\right] \tag{5.9}$$

The above is known as ***three moment equation*** .

Sign Conventions

The M_1, M_2 and M_3 are positive for sagging moment and negative for hogging moment. Similarly, areas A_1, A_2 and A_3 are positive if it is sagging moment and negative for hogging moment. The displacements Δ_1, Δ_2 and Δ_3 are positive if measured downward from the reference axis.

Example 5.22 Analyze the continuous beam shown in Figure 5.26(a) by the three moment equation. Draw the shear force and bending moment diagram.

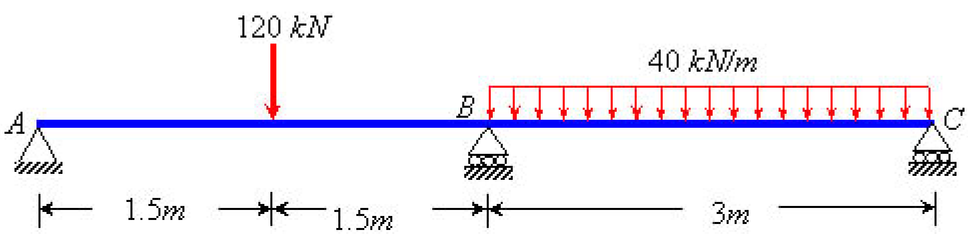

Figure 5.26(a)

Solution: The simply supported bending moment diagram on *AB* and *AC* are shown in Fig 5.26 (b). Since supports *A* and *C* are simply supported

$$M_A = M_C = 0$$

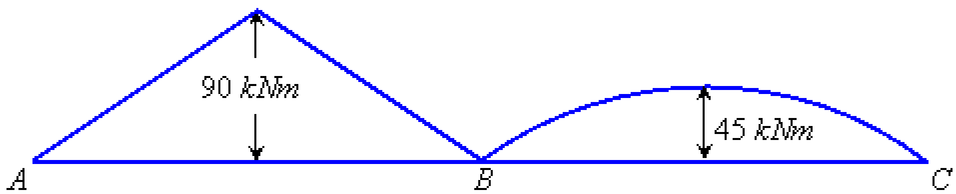

Figure 5.26(b)

Applying the three moment equation to span *AB* and *BC* ($\Delta_1 = \Delta_2 = \Delta_3 = 0$)

$$M_A\left(\frac{3}{I}\right) + 2M_B\left(\frac{3}{I} + \frac{3}{I}\right) + M_C\left(\frac{3}{I}\right) = -\frac{6\times 1/2\times 90\times 3\times 1.5}{3\times I} - \frac{6\times 2/3\times 45\times 3\times 1.5}{3\times I}$$

or M_B = -56.25 kN.m

The reactions at support *A* , *B* and *C* are given as

$$V_A = \frac{120\times 1.5 - 56.25}{3} = 41.25 \text{ kN}$$

$$V_C = \frac{40\times 3\times 1.5 - 56.25}{3} = 41.25 \text{ kN}$$

$$V_B = 120 + 40 \times 3 - 41.25 - 41.25 = 157.5 \text{ kN}$$

The bending moment and shear force diagram are shown in Figures 5.26(c) and (d), respectively

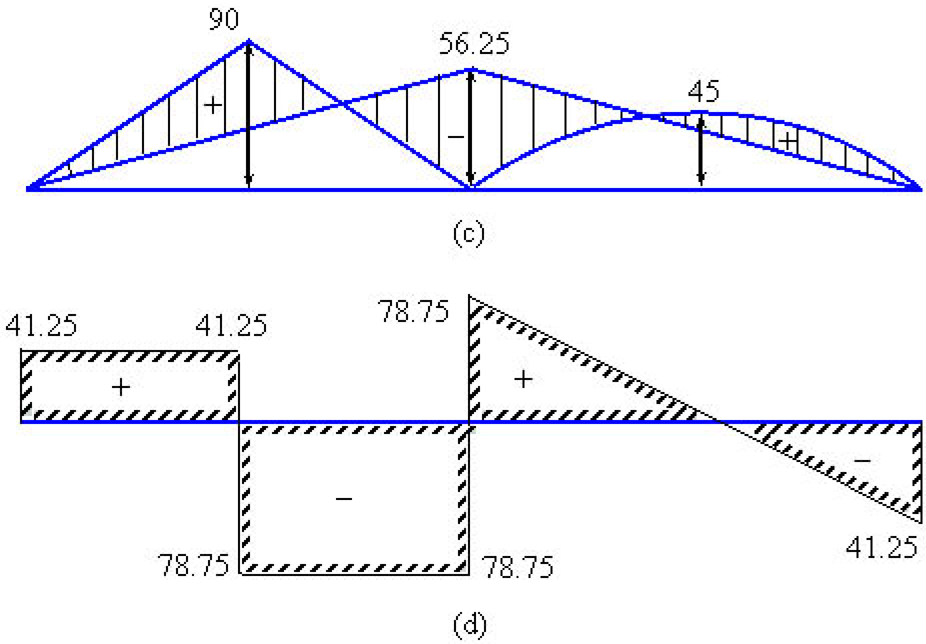

Figure 5.26(c)-(d)

Example 5.23 Analyze the continuous beam shown in Figure 5.27(a) by the three moment equation. Draw the shear force and bending moment diagram.

Solution: The effect of a fixed support is reproduced by adding an imaginary span A_0A as shown in Figure 5.27 (b). The moment of inertia, I_0 of the imaginary span is infinity so that it will never deform and the compatibility condition at the end *A* , that slope should be is zero, is satisfied.

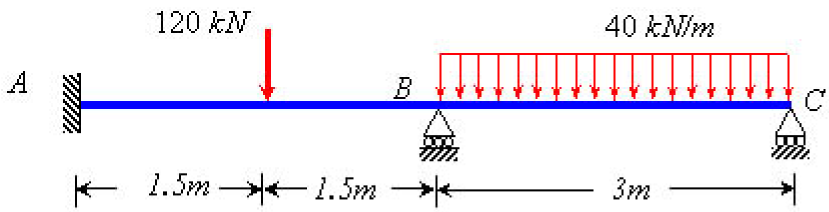

Figure 5.27(a)

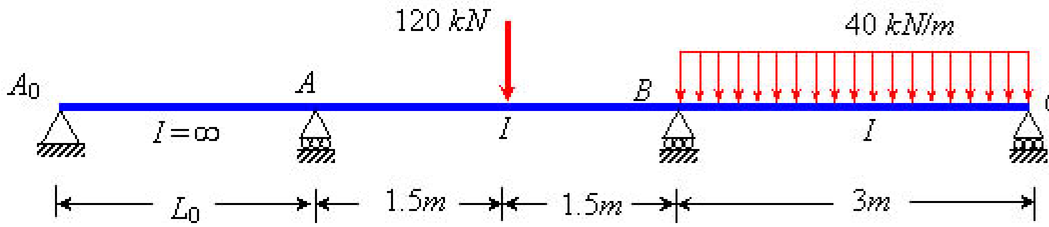

Figure 5.27(b)

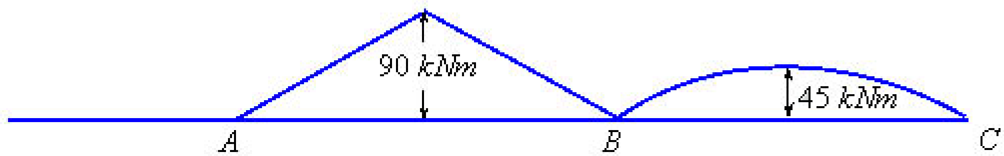

Figure 5.27(c) Simply supported moment diagram

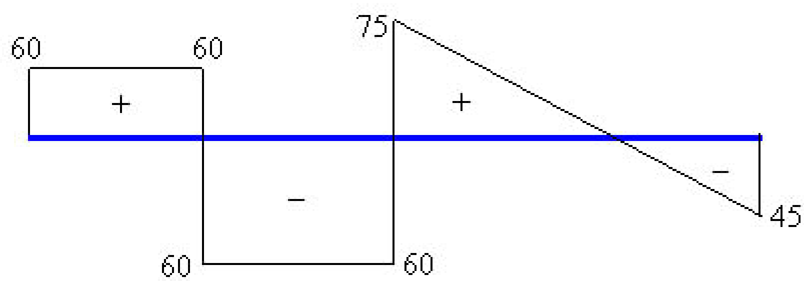

Figure 5.27(d) Shear force diagram (kN)

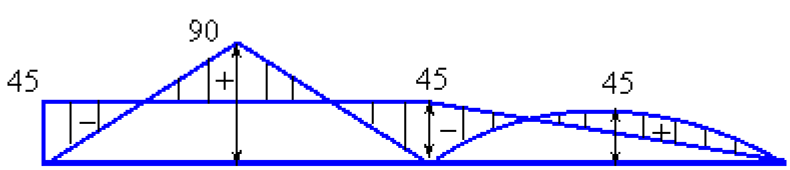

Figure 5.27(e) Bending moment diagram (kNm)

Applying three moment equation to the span A_0A and *AB* :

$$M_{A0}\left(\frac{L_0}{\infty}\right)+2M_A\left(\frac{L_0}{\infty}+\frac{3}{I}\right)+M_B\left(\frac{3}{I}\right)=-\frac{6\times 1/2\times 90\times 3\times 1.5}{3\times I}$$

or $$2M_A + M_B = -135 \qquad \text{(i)}$$

Span *AB* and *BC* :

$$M_A\left(\frac{3}{I}\right)+2M_B\left(\frac{3}{I}+\frac{3}{I}\right)+M_C\left(\frac{3}{I}\right)=-\frac{6\times 1/2\times 90\times 3\times 1.5}{3\times I}-\frac{6\times 2/3\times 45\times 3\times 1.5}{3\times I}$$

or $M_A + 4M_B = -225$ (ii)

Solving Eqs. (i) and (ii), $M_A = -45$ kNm and $M_B = -45$ kNm

The shear force and bending moment diagram are shown in Figures 5.27(d) and (e), respectively.

Example 5.24 Analyze the continuous beam shown in Figure 5.28(a) by the three moment equation. Draw the shear force and bending moment diagram.

Solution: The simply supported moment diagram on *AB* , *BC* and *CD* are shown in Figure 5.28(b). Since the support *A* is simply supported, $M_A = 0$ The moment at *D* is $M_D = -20\times 2 = -40\,\text{kNm}$.

Applying three moment equation to the span *AB* and *BC* :

$$M_A\left[\frac{4}{I}\right]+2M_B\left[\frac{4}{I}+\frac{6}{3I}\right]+M_C\left[\frac{6}{3I}\right]=-\frac{6\times 1/2\times 80\times 4\times 2}{4\times I}-\frac{6\times 2/3\times 108\times 6\times 3}{6\times 3I}$$

or $6M_B + M_C = -456$ (i)

Span *BC* and *CD* : ($M_D = -20\,\text{kNm}$)

$$M_B\left[\frac{6}{3I}\right]+2M_C\left[\frac{6}{3I}+\frac{6}{2I}\right]+M_D\left[\frac{6}{2I}\right]=-\frac{6\times 2/3\times 108\times 6\times 3}{6\times 3I}-\frac{6\times 1/2\times 160\times 6\times (6+4)/3}{6\times 2I}$$

or $M_B + 5M_C = -556$ (ii)

Solving Eqs. (i) and (ii) will give $M_B = -59.448\,\text{kNm}$ and $M_C = -99310\,\text{kNm}$.

The bending moment and shear force diagram are shown in Figures 5.28(d) and (c), respectively.

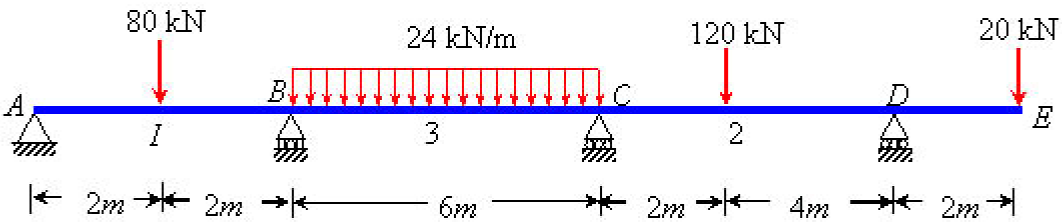

Figure 5.28(a) Given Beam

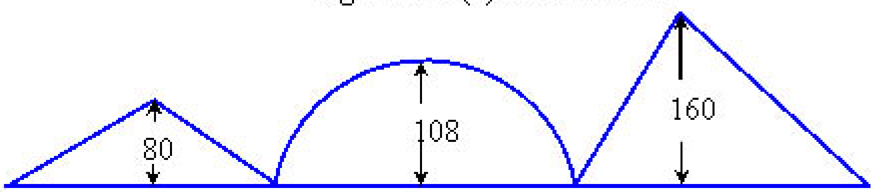

Figure 5.28(b) Simply supported Bending moment diagram (kNm)

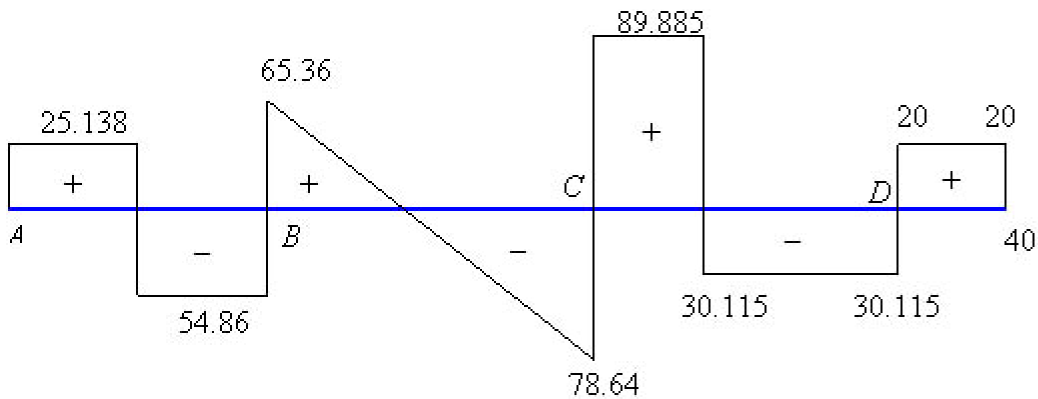

Figure 5.28(c) Shear force diagram (kN)

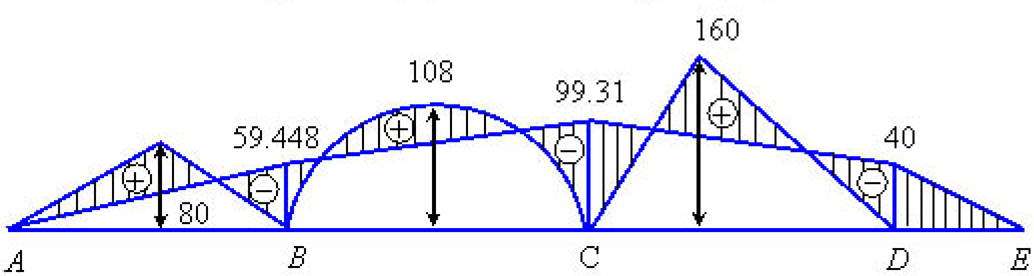

Figure 5.28(d) Bending moment diagram (kNm)

Example 5.25 Analyze the continuous beam show in Fig. 5.29(a) by the three moment equation method if support *B* sinks by an amount of 10 mm. Draw the shear force and bending moment diagram. Take flexural rigidity $EI = 48000\,kNm^2$.

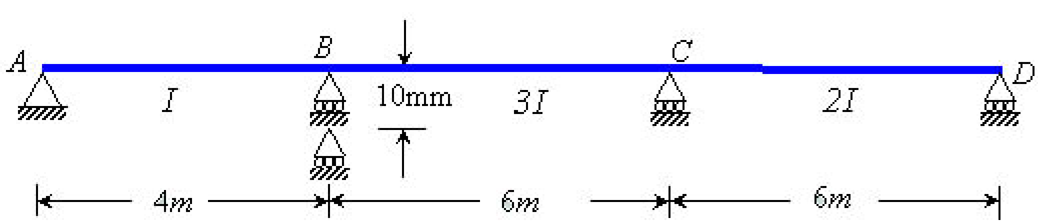

Figure 5.29(a) Given Beam

Solution: Since support *A* and *D* are simply supported, $M_A = M_D = 0$.

Applying the three moment equation for span *AB* and *BC* : ($M_A = 0$)

$$M_A\left[\frac{4}{I}\right] + 2M_B\left[\frac{4}{I} + \frac{6}{3I}\right] + M_C\left[\frac{6}{3I}\right] = \frac{6 \times E \times 10 \times 10^{-3}}{4} + \frac{6E(10 \times 10^{-3})}{6}$$

or $6M_B + M_C = 600$ (i)

Span *BC* and *CD* :

$$M_B\left[\frac{6}{3I}\right] + 2M_C\left[\frac{6}{3I} + \frac{6}{2I}\right] + M_D\left[\frac{6}{2I}\right] = -\frac{6 \times E \times 10 \times 10^{-3}}{6}$$

or $M_B + 5M_C = -240$ (ii)

Solving Eqs. (i) and (ii), $M_B = 111.72\,\text{kNm}$ and $M_C = -70.344\,\text{kNm}$.

The bending moment diagram is shown in Figure 5.29(b).

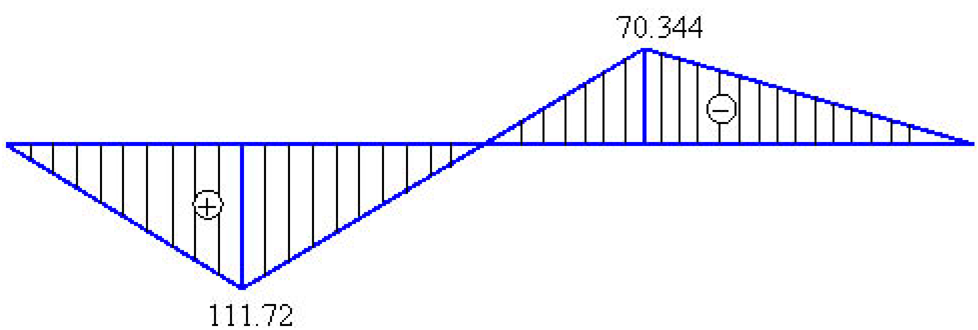

Figure 5.29(b) Bending moment diagram (kNm)

Recap

In this course you have learnt the following

- Derivation of three moment equation for analysis of continous beams.
- Demonstration of three moment equation using numerical examples.