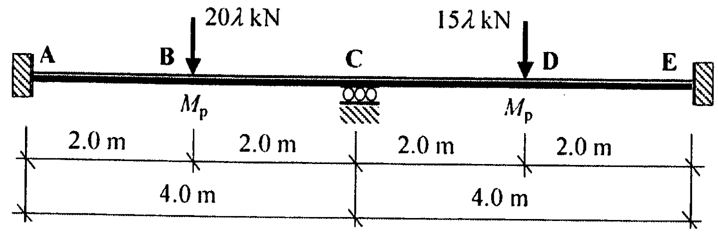
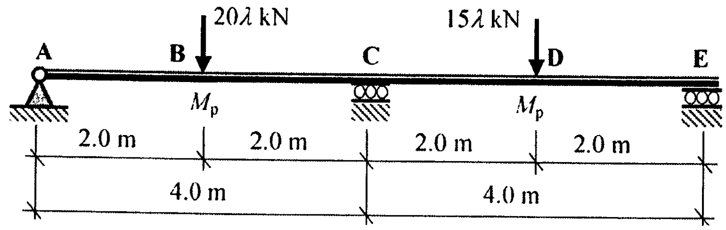
8.4 *Problems: Plastic Analysis – Continuous Beams*

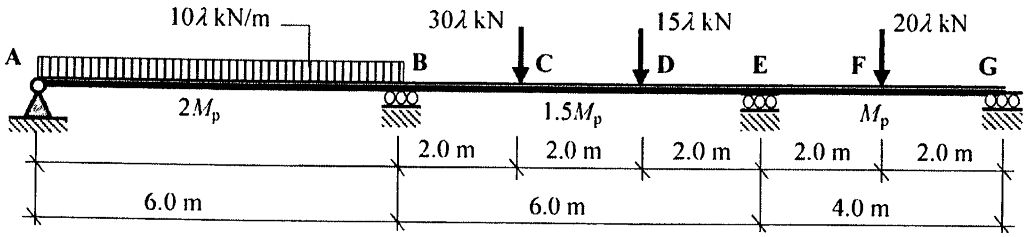
A series of continuous beams are indicated in which the relative M_p values and the applied **collapse** loadings are given in Problems 8.1 to 8.5. Determine the required value of M_p to ensure a minimum load factor $\lambda = 1.7$.



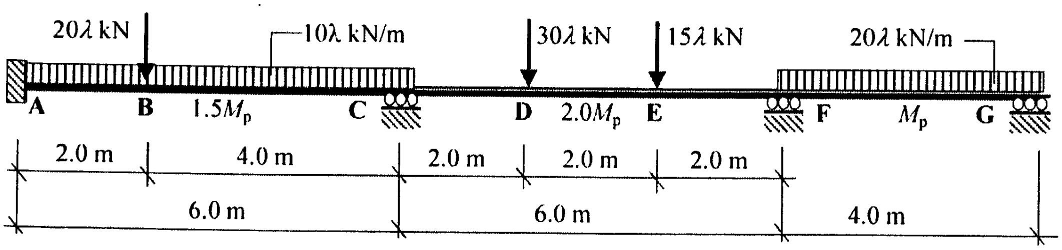
Problem 8.1



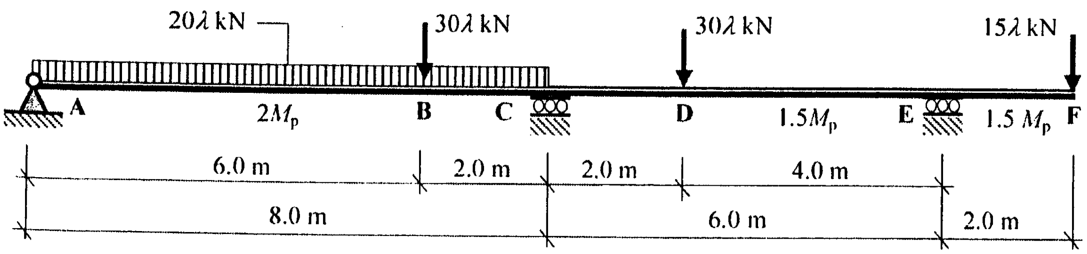
Problem 8.2



Problem 8.3



Problem 8.4

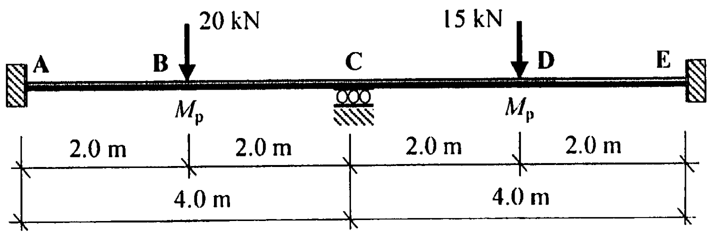


Problem 8.5

8.5 *Solutions: Plastic Analysis – Continuous Beams*

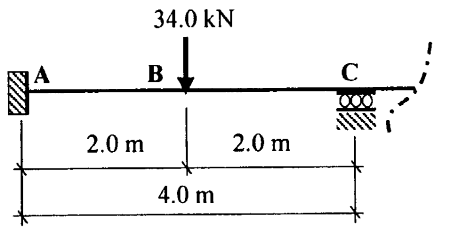
Solution

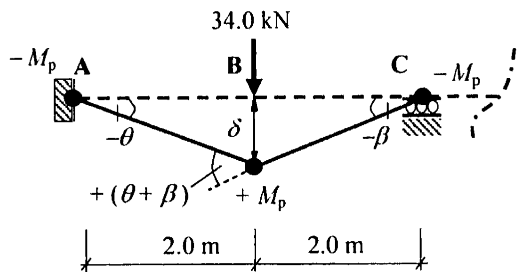
Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.1 – Kinematic Method **Page No. 1**



$\lambda = 1.7$
Factored loads: Beam ABC = $(1.7 \times 20) = 34$ kN, Beam CDE = $(1.7 \times 15) = 25.5$ kN

Kinematic Method:
Span ABC



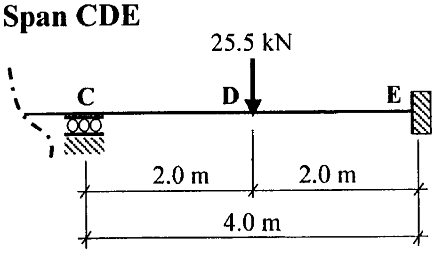


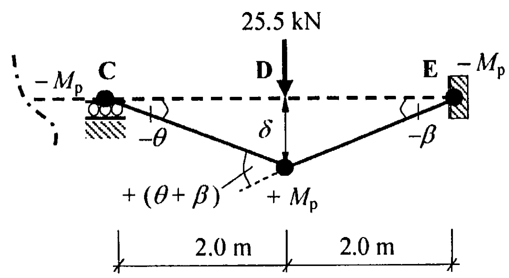
$\delta = 2\beta = 2\theta \quad \therefore \beta = \theta$

Internal Work = External Work
$M_p(\theta) + M_p(\theta+\beta) + M_p(\beta) = (34 \times 2\theta)$
$4M_p\theta = 68\theta$

$\therefore$ **M_p = 17.0 kNm**

Span CDE





$\delta = 2\beta = 2\theta \quad \therefore \beta = \theta$

Internal Work = External Work
$M_p(\theta) + M_p(\theta+\beta) + M_p(\beta) = (25.5 \times 2\theta)$
$4M_p\theta = 51.0\theta$

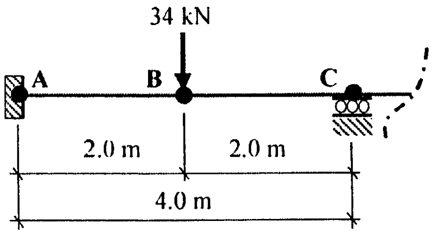
$\therefore$ **M_p = 12.75 kNm**
Critical value of M_p = 17.0 kNm

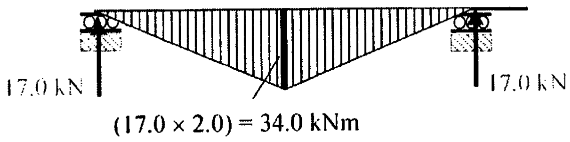
Solution

Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.1 – Static Method **Page No. 2**

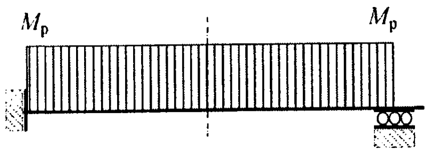
Static Method:

Span ABC

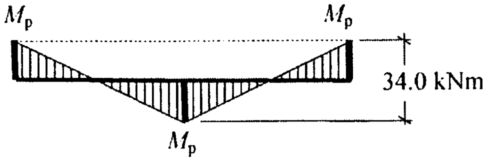




Free Bending Moment Diagram



Fixed Bending Moment Diagram

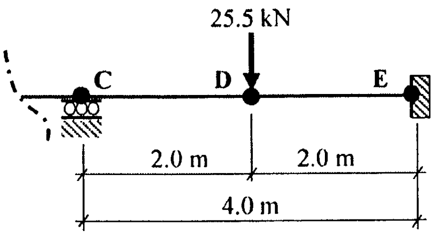


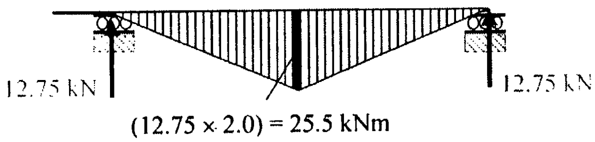
Combined Bending Moment Diagram

$(M_p + M_p) = 2M_p = 34.0$ kNm

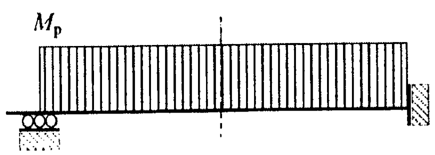
$\therefore$ $\boldsymbol{M_p = 17.0}$ **kNm**

Span CDE

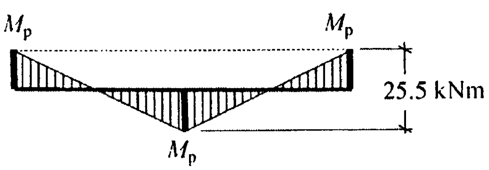




Free Bending Moment Diagram



Fixed Bending Moment Diagram



Combined Bending Moment Diagram

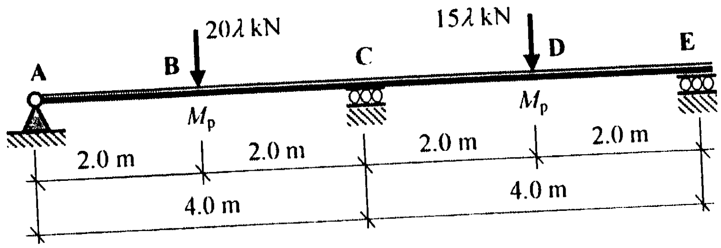
$(M_p + M_p) = 2Mp = 25.5$ kNm

$\therefore$ $\boldsymbol{M_p = 12.75}$ **kNm**

As before the critical value of $\boldsymbol{M_p}$ = 17.0 kNm

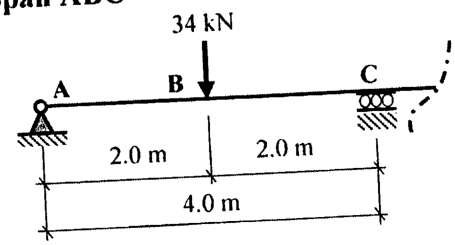
Solution

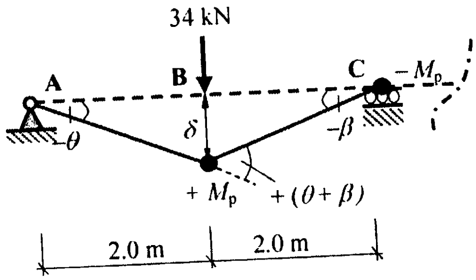
Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.2 – Kinematic Method **Page No. 1**



$\lambda = 1.7$
Factored loads: Beam ABC = (1.7 × 20) = 34 kN, Beam CDE = (1.7 × 15) = 25.5 kN

Kinematic Method:
Span ABC



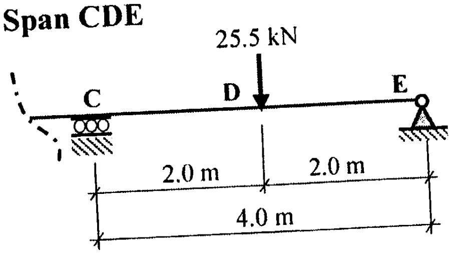


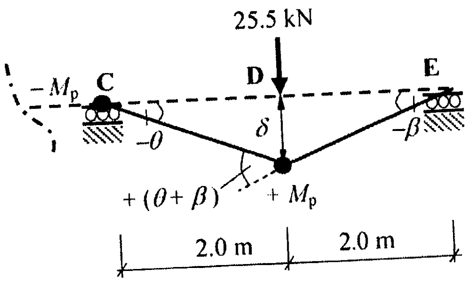
$\delta = 2\beta = 2\theta \quad \therefore \beta = \theta$

Internal Work = External Work
$M_p(\theta + \beta) + M_p(\beta) = (34 \times 2\theta)$
$3M_p\theta = 68\theta$

$\therefore$ **M_p = 22.67 kNm**

Span CDE





$\delta = 2\beta = 2\theta \quad \therefore \beta = \theta$

Internal Work = External Work
$M_p(\theta) + M_p(\theta + \beta) = (25.5 \times 2\theta)$
$3M_p\theta = 51.0\theta$

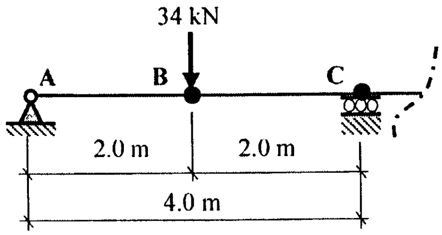
$\therefore$ **M_p = 17.0 kNm**
Critical value of M_p = 22.67 kNm

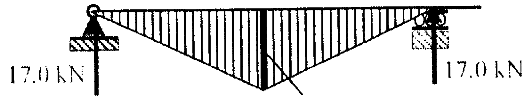
Solution

Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.2 – Static Method **Page No. 2**

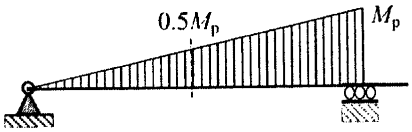
Static Method:

Span ABC

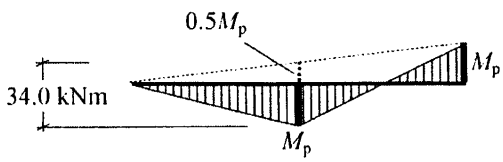




Free Bending Moment Diagram



Fixed Bending Moment Diagram

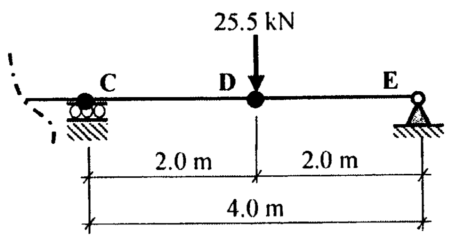


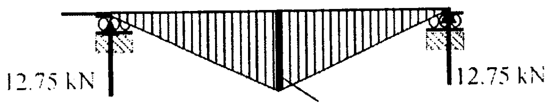
Combined Bending Moment Diagram

$(M_p + 0.5M_p) = 1.5M_p = 34.0$ kNm

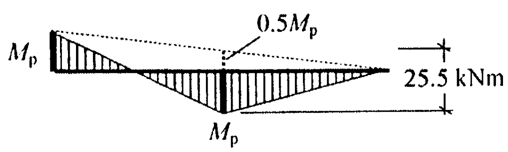
$\therefore$ $\boldsymbol{M_p = 22.67}$ **kNm**

Span CDE

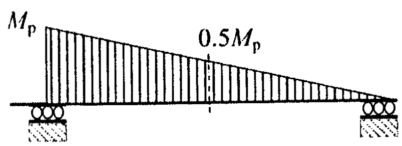




Free Bending Moment Diagram



Combined Bending Moment Diagram



Fixed Bending Moment Diagram

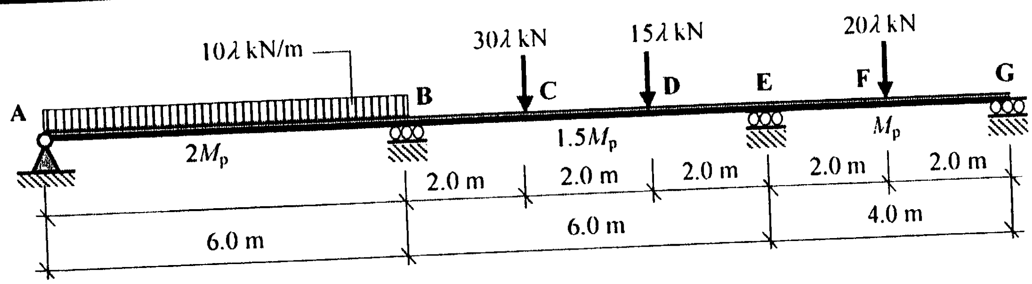
$(M_p + 0.5M_p) = 1.5M_p = 25.5$

$\therefore$ $\boldsymbol{M_p = 17.0}$ **kNm**

As before the critical value of $\boldsymbol{M_p = 22.67}$ kNm

Solution

Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.3 – Kinematic Method **Page No. 1**



$\lambda = 1.7$

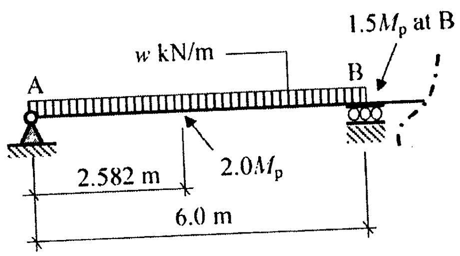
Factored loads $= (1.7 \times 10) = 17.0$ kN/m $\qquad (1.7 \times 15) = 25.5$ kN

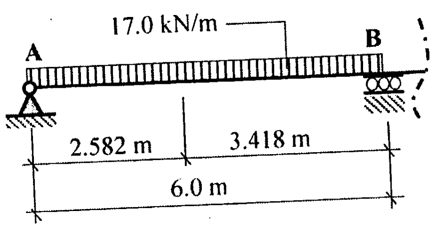
$= (1.7 \times 20) = 34.0$ kN $\qquad (1.7 \times 30) = 51.0$ kN

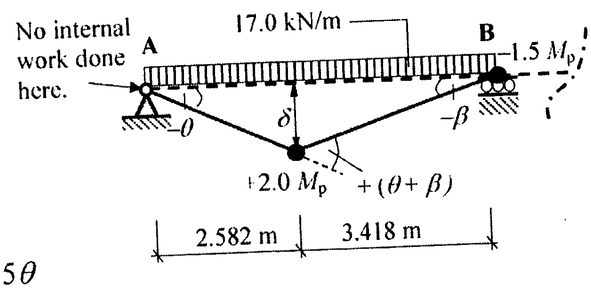
Kinematic Method:

Span AB

Note: Span AB is effectively a propped cantilever and the bending moment diagram is asymmetric. The hinge between A and B does not develop at the mid-span point and should be evaluated in a manner similar to that indicated in Section 8.2.3. The reader should carry-out this calculation to show that the hinge develops at a position equal to 2.582 m from the free support at A as shown below, (see page 3 of this solution).







$\delta = 3.418\beta = 2.582\theta \quad \therefore \ \beta = 0.755\theta$

Internal Work = External Work

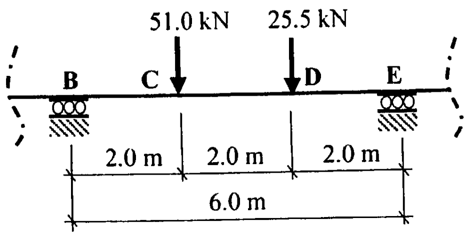
$[2.0M_p\,(\theta + \beta) + (1.5M_p\beta)] = [(17 \times 6.0) \times (0.5 \times \delta)] = (102 \times 0.5 \times 2.582\theta)$

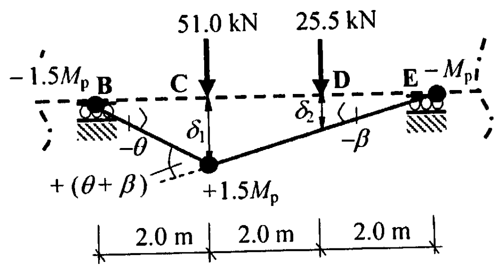
$4.643M_p\theta = 131.682\theta$ $\qquad \therefore \ \boldsymbol{M_p = 28.36}$ **kNm**

Solution

Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.3 – Kinematic Method **Page No. 2**

Span BCDE



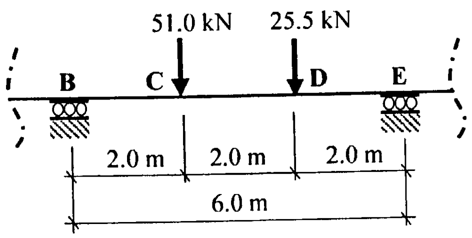


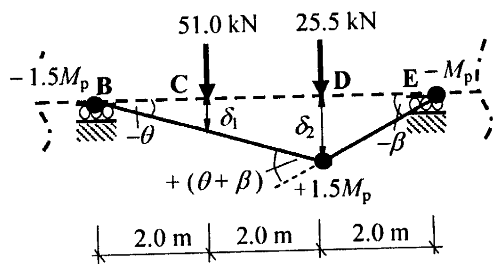
$\delta_1 = 4\beta = 2\theta \quad \therefore\ \beta = 0.5\theta \qquad \delta_2 = 2\beta = \theta$

Internal Work
$1.5M_p\,(\theta) + 1.5M_p\,(\theta+\beta) + M_p\,(\beta) = 4.25M_p\theta$

External Work
$(51.0 \times \delta_1) + (25.5 \times \delta_2) = (51.0 \times 2\theta) + (25.5 \times 2\beta) = 127.5\theta$
$4.25M_p\theta = 127.5\theta$

$\therefore\ \mathbf{M_p = 30.0\ kNm}$





$\delta_1 = 2\theta \qquad \delta_2 = 2\beta = 4\theta \quad \therefore\ \beta = 2\theta$

Internal Work
$1.5M_p\,(\theta) + 1.5M_p\,(\theta+\beta) + M_p\,(\beta) = 8.0M_p\theta$

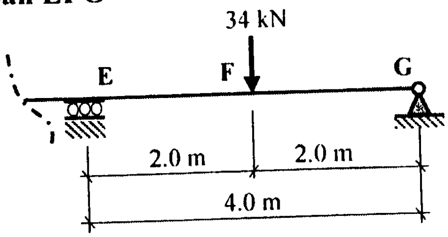
External Work
$(51.0 \times \delta_1) + (25.5 \times \delta_2) = (51.0 \times 2\theta) + (25.5 \times 4\theta) = 204.0\theta$
$8.0M_p\theta = 204\theta$

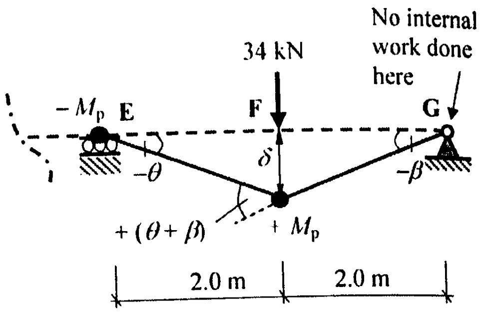
$\therefore\ \mathbf{M_p = 25.5\ kNm}$

Solution

Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.3 – Static Method **Page No. 3**

Span EFG



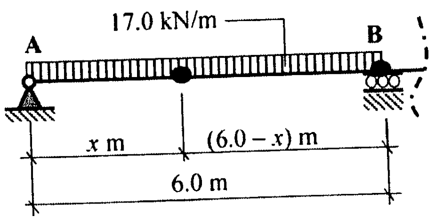


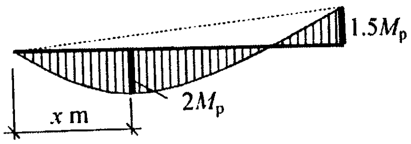
$\delta = 2\beta = 2\theta \quad \therefore \beta = \theta$

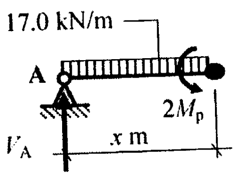
Internal Work = External Work
$M_p(\theta+\beta) + M_p(\beta) = (34 \times 2\theta)$
$3M_p\theta = 68\theta$

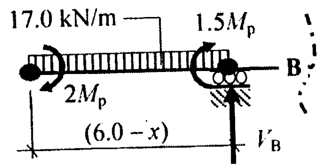
$\therefore$ **M_p = 22.67 kNm**
The critical value of M_p = 30.0 kNm

Static Method:
Span AB









+ve $\Sigma M_A = 0$
$(17.0x^2)/2 - 2M_p = 0$
$8.5x^2 - 2M_p = 0 \quad \therefore M_p = 4.25x^2$

+ve $\Sigma M_B = 0$
$2M_p + 1.5M_p - 17.0(6.0 - x)^2/2 = 0$
$M_p = 2.429(6.0 - x)^2 0$

Equate the M_p values to determine x:
$4.25x^2 = 2.429(36.0 - 12x + x^2) \quad \therefore 1.821x^2 + 29.148x - 87.44 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-29.148 \pm \sqrt{29.148^2 + (4 \times 1.821 \times 87.44)}}{(2 \times 1.821)} = +2.582 \text{ m}$$

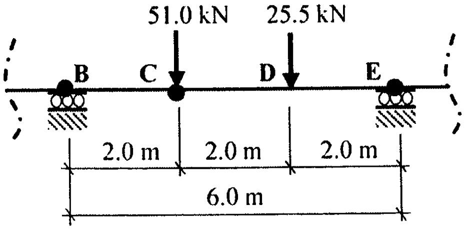
$M_p = 4.25x^2 = (4.25 \times 2.582^2)$

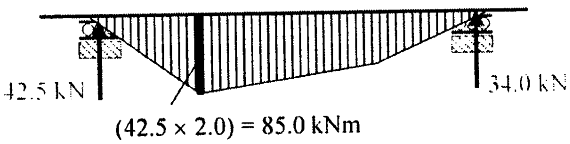
$\therefore$ **M_p = 28.33 kNm**

Solution

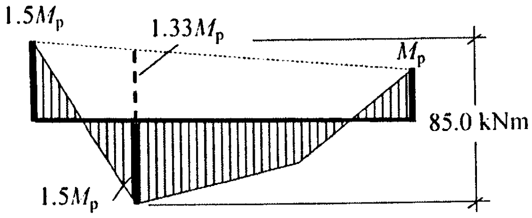
Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.3 – Static Method **Page No. 4**

Span BCDE

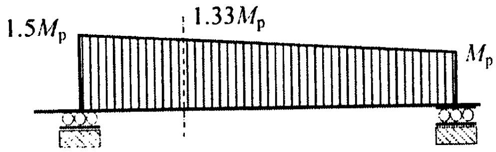




Free Bending Moment Diagram



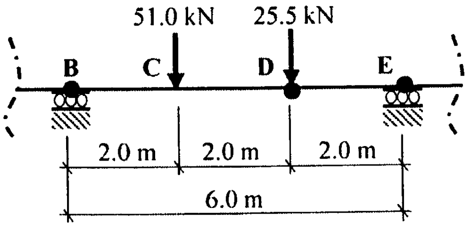
Combined Bending Moment Diagram

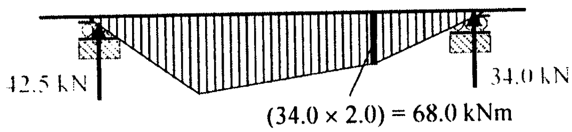


Fixed Bending Moment Diagram

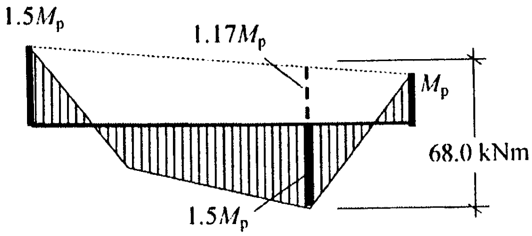
$(1.5M_p + 1.33M_p) = 85.0$ kNm
$2.83M_p = 85.0$ kNm

$\therefore$ **$M_p = 30.0$ kNm**

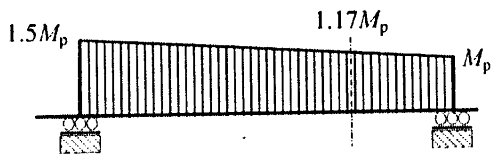




Free Bending Moment Diagram



Combined Bending Moment Diagram



Fixed Bending Moment Diagram

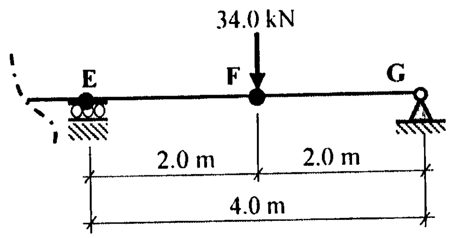
$(1.5M_p + 1.17M_p) = 68.0$ kNm
$2.67M_p = 68.0$ kNm

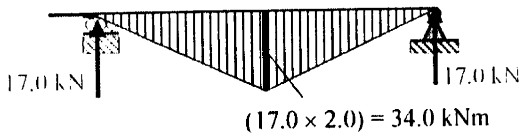
$\therefore$ **$M_p = 25.5$ kNm**

Solution

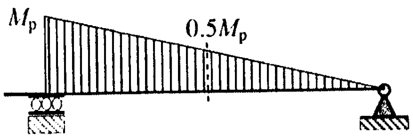
Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.3 – Static Method **Page No. 5**

Span EFG

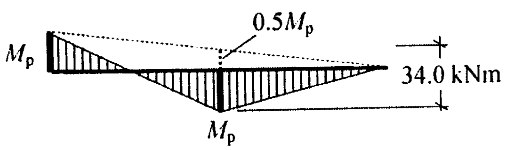




Free Bending Moment Diagram



Fixed Bending Moment Diagram



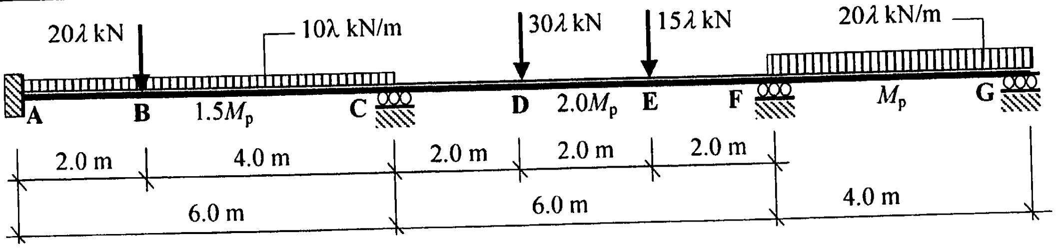
Combined Bending Moment Diagram

$(M_p + 0.5M_p) = 25.5$ $\quad\therefore\ M_p = 22.67$ kNm

As before the critical value of M_p = 30.0 kNm

Solution

Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.4 – Kinematic Method **Page No. 1**



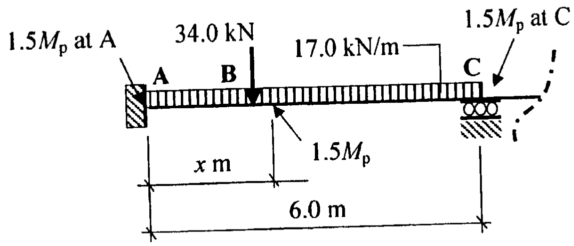
$\lambda = 1.7$

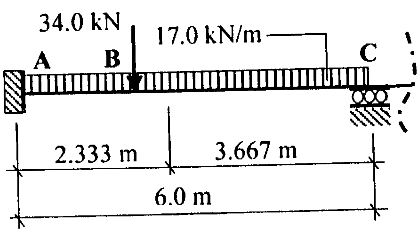
Factored loads: $(1.7 \times 10) = 17.0$ kN $(1.7 \times 20) = 34.0$ kN
$(1.7 \times 15) = 25.5$ kN $(1.7 \times 30) = 51.0$ kN

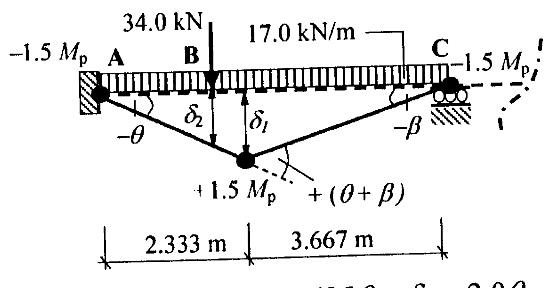
Kinematic Method:

Span ABC

Note: The bending moment diagram on span ABC is asymmetric and in this case the hinge between A and C does not necessarily develop under the point load.
The position should be evaluated in a manner similar to that indicated in Section 8.2.3. The reader should carry-out this calculation to show that the hinge develops at a position equal to 2.333 m from the support at A as shown below, (see page 3 of this solution).







$$\delta_1 = 3.667\beta = 2.333\theta \quad \therefore\ \beta = 0.635\theta \qquad \delta_2 = 2.0\theta$$

Internal Work $= [1.5M_p\,(\theta) + 1.5M_p\,(\theta + \beta) + (1.5M_p\beta)] = 4.91M_p\theta$

External Work $= [(34 \times \delta_2)] + [(17 \times 6.0) \times (0.5 \times \delta_1)]$

$= [(34 \times 2\theta)] + [(102.0) \times (0.5 \times 2.333\theta)] = 186.98\theta$

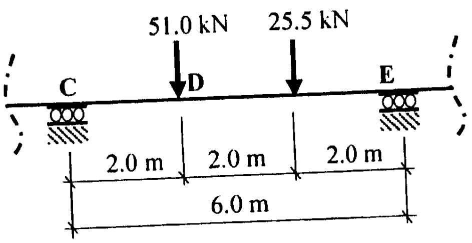
$4.91M_p\theta = 186.98\theta$

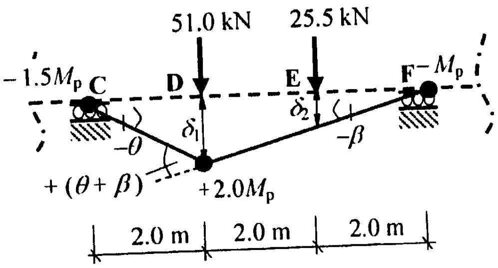
$\therefore$ $\mathbf{M_p = 38.08\ kNm}$

Solution

Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.4 – Kinematic Method **Page No. 2**

Span CDEF



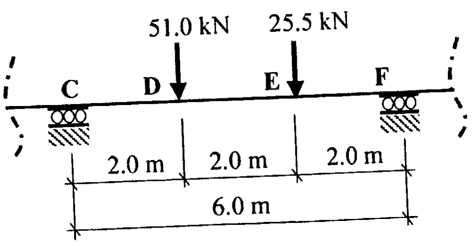


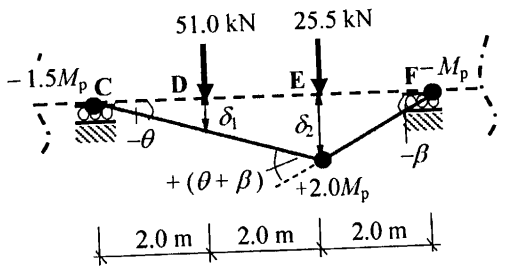
$\delta_1 = 4\beta = 2\theta \quad \therefore \; \beta = 0.5\theta \qquad \delta_2 = 2\beta = \theta$

Internal Work
$1.5M_p\,(\theta) + 2.0M_p\,(\theta + \beta) + M_p\,(\beta) = [1.5M_p\,(\theta) + 2.0M_p\,(1.5\theta) + M_p\,(0.5\theta)]$
$= 5.0M_p\theta$

External Work
$(51.0 \times \delta_1) + (25.5 \times \delta_2) = [(51.0 \times 2\theta) + (25.5 \times 2\beta)] = [(102\theta) + (25.5\theta)]$
$= 127.5\theta$

$5.0M_p\theta = 127.5\theta$ $\therefore$ $\boldsymbol{M_p = 25.5}$ **kNm**





$\delta_1 = 2\theta \qquad \delta_2 = 2\beta = 4\theta \quad \therefore \; \beta = 2\theta$

Internal Work
$1.5M_p\,(\theta) + 2.0M_p\,(\theta + \beta) + M_p\,(\beta) = [1.5M_p\,(\theta) + 2.0M_p\,(3.0\theta) + M_p\,(2.0\theta)]$
$= 9.5M_p\theta$

External Work
$(51.0 \times \delta_1) + (25.5 \times \delta_2) = [(51.0 \times 2\theta) + (25.5 \times 4\theta)] = [(102\theta) + (102\theta)]$
$= 204.0\theta$

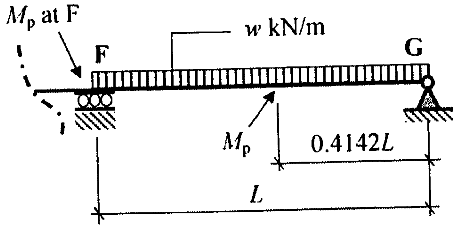
$9.5M_p\theta = 204.0\theta$ $\therefore$ $\boldsymbol{M_p = 21.47}$ **kNm**

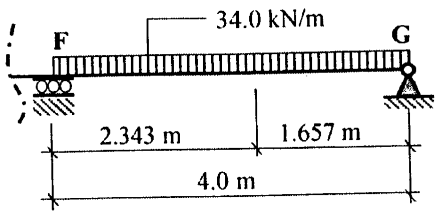
Solution

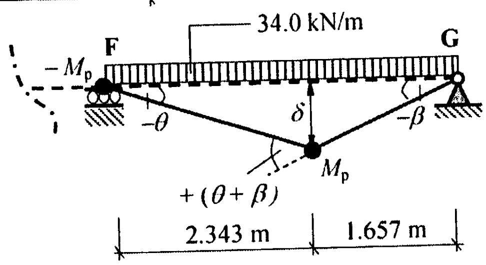
Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.4 – Kinematic Method **Page No. 3**

Span FG

Note: Span FG is effectively a propped cantilever and the bending moment diagram is asymmetric. The hinge between F and G develops at a position $0.4142L$ from the simply supported end as indicated in Section 8.2.3.







$\delta = 1.657\beta = 2.343\theta \quad \therefore\ \beta = 1.414\theta$

Internal Work = External Work

$[M_p(\theta) + M_p(\theta+\beta)] = [(34.0 \times 4.0) \times (0.5 \times \delta)]$

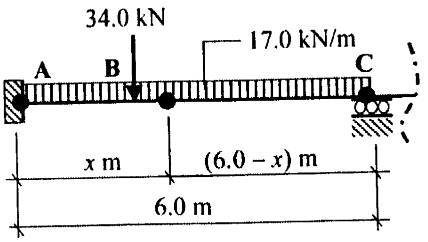
$[M_p(\theta) + M_p(2.414\theta)] = (136 \times 0.5 \times 2.343\theta)$

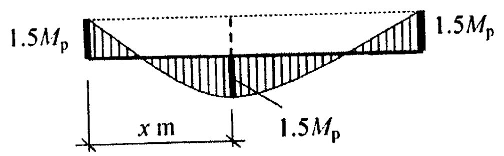
$3.414 M_p\theta = 159.32\theta$

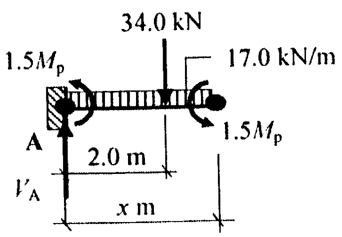
$\therefore$ $\mathbf{M_p = 46.67}$ **kNm**

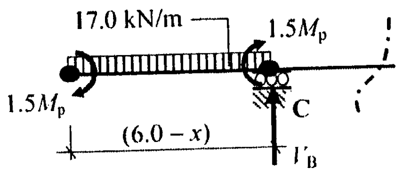
Static Method:

Span ABC









Solution

Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.4 – Static Method **Page No. 4**

+ve $\circlearrowright \Sigma M_A = 0$
$-1.5M_p + (34 \times 2.0) + (17.0x^2)/2 - 1.5M_p = 0$
$68.0 + 8.5x^2 - 3.0M_p = 0$ $\therefore M_p = 22.667 + 2.833x^2$

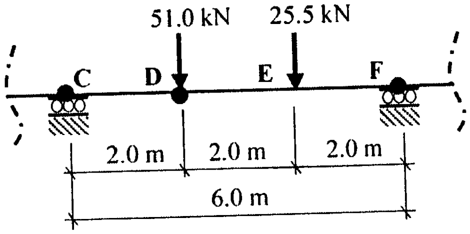
+ve $\circlearrowright \Sigma M_C = 0$
$1.5M_p - 17.0(6.0 - x)^2/2 + 1.5M_p = 0$ $\therefore M_p = 2.833(6.0 - x)^2$

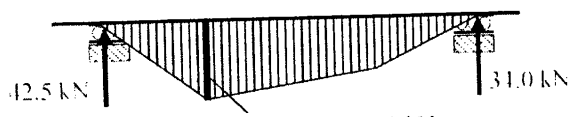
Equate the M_p values to determine x:
$22.667 + 2.833x^2 = 2.833(36.0 - 12x + x^2)$ $\therefore 33.996x - 79.321 = 0$
$x = 2.333$ m

$M_p = 2.833(6.0 - x)^2 = 2.833(6.0 - 2.333)^2$

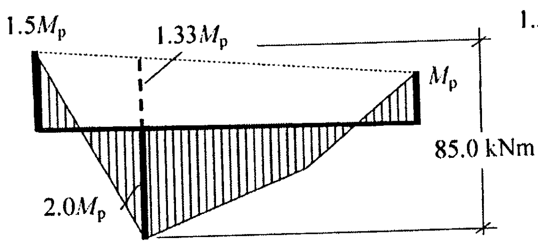
$\therefore$ **M_p = 38.09 kNm**

Span CDEF

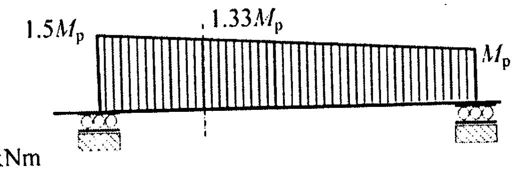




Free Bending Moment Diagram



Combined Bending Moment Diagram



Fixed Bending Moment Diagram

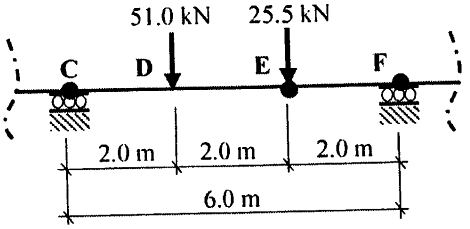
$(2.0M_p + 1.33M_p) = 85.0$ kNm
$3.33M_p = 85.0$ kNm

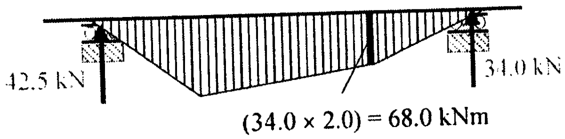
$\therefore$ **M_p = 25.5 kNm**

Solution

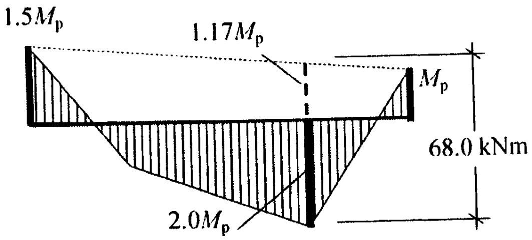
Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.4 – Static Method

Page No. 5

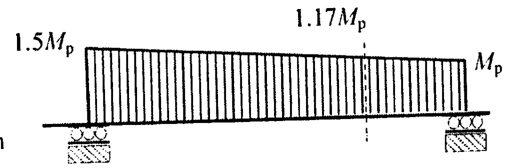




Free Bending Moment Diagram



Combined Bending Moment Diagram

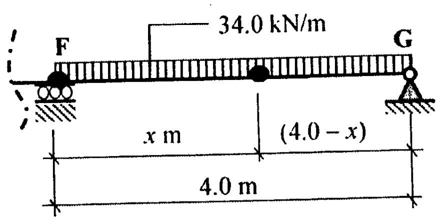


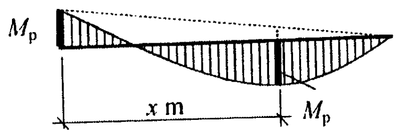
Fixed Bending Moment Diagram

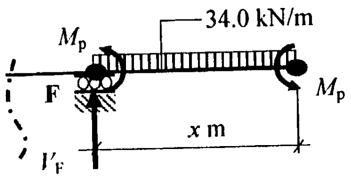
$(2.0M_p + 1.17M_p) = 68.0$ kNm
$3.17M_p = 68.0$ kNm

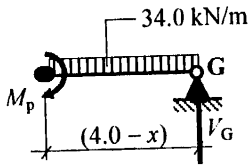
$\therefore$ $\boldsymbol{M_p = 21.47}$ **kNm**

Span FG









+ve $\sum M_F = 0$
$(34.0x^2)/2 - M_p - M_p = 0$
$17.0x^2 - 2.0\ M_p = 0$
$M_p = 8.5x^2$

+ve $\sum M_G = 0$
$M_p - 34.0(4.0 - x)^2/2 = 0$
$M_p = 17.0(4.0 - x)^2$

Solution

Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.4 – Static Method **Page No. 6**

Equate the M_p values to determine x:

$8.5x^2 = 17.0(16.0 - 8x + x^2)$ $\quad\therefore\ 8.5x^2 - 136x + 272 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{136 \pm \sqrt{136^2 - (4 \times 8.5 \times 272)}}{(2 \times 8.5)} = +\,2.343 \text{ m}$$

$M_p = 8.5x^2 = (8.5 \times 2.343)^2$

M_p = 46.67 kNm

As before the critical value of M_p = 46.67 kNm

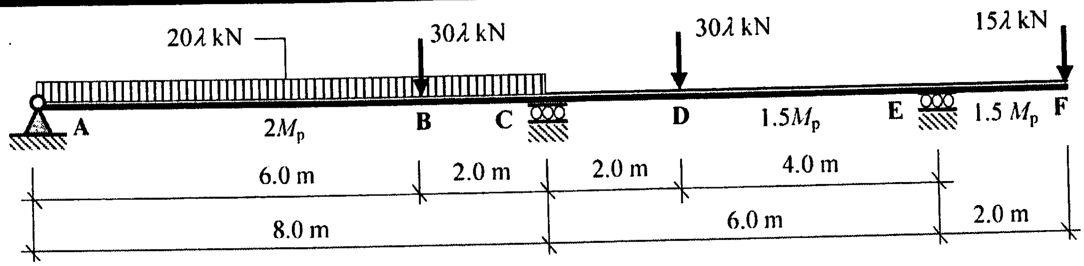
Note: Span FG is the same as the standard propped cantilever in Example 8.3 in which the hinge develops at a point $0.414L$ from the simply supported end and the M_p value equals $0.0858wL^2$, i.e.

Distance of hinge from support F = $[4.0 - 0.414L] = [4.0 - (0.414 \times 4.0)] = 2.344$ m

$\therefore\ M_p = (0.0858 \times 34.0 \times 4.0^2) = 46.67$ kNm

Solution

Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.5 – Kinematic Method **Page No. 1**



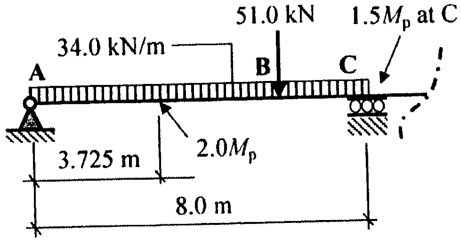
$\lambda = 1.7$

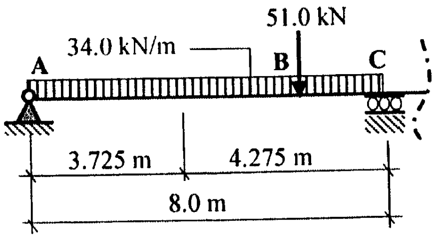
Factored loads: $(1.7 \times 15) = 25.5$ kN $\quad (1.7 \times 20) = 34.0$ kN
$(1.7 \times 30) = 51.0$ kN

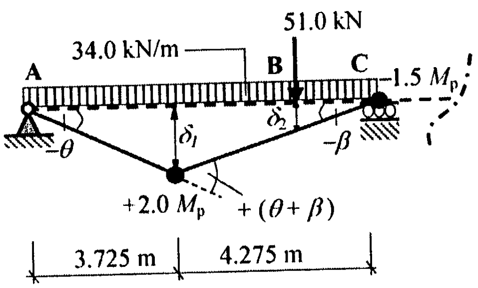
Kinematic Method:

Span ABC

Note: The bending moment diagram on span ABC is asymmetric and in this case the hinge between A and C does not necessarily develop under the point load and its position should be evaluated in a manner similar to that indicated in Section 8.2.3. The reader should carry-out this calculation to show that the hinge develops at a position equal to 3.725 m from the support at A as shown below, (see page 2 of this solution).







$\delta_1 = 4.275\beta = 3.725\theta \;\therefore\; \beta = 0.871\theta; \qquad \delta_2 = 2.0\beta$

Internal Work $= [\,2.0M_p\,(\theta + \beta) + (1.5M_p\beta)] = 5.05M_p\theta$

External Work $= [(51 \times \delta_2)] + [(34 \times 8.0) \times (0.5 \times \delta_1)]$

$= [(51 \times 1.742\theta)] + [(272.0) \times (0.5 \times 3.725\theta)] = 595.44\theta$

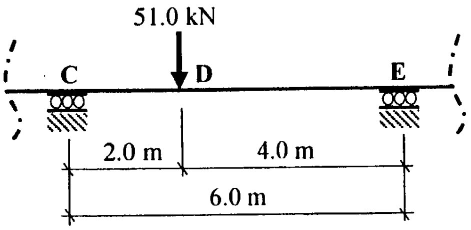
$5.05M_p\theta = 595.44\theta$

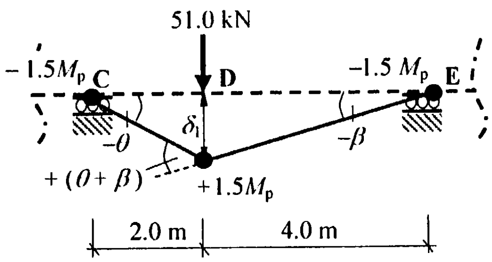
$\therefore \;\; \mathbf{M_p = 117.91\ kNm}$

Solution

Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.5 – Kinematic Method **Page No. 2**

Span CDE





$$\delta_1 = 4\beta = 2\theta \quad \therefore \ \beta = 0.5\theta$$

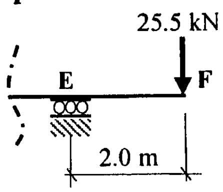
Internal Work = External Work

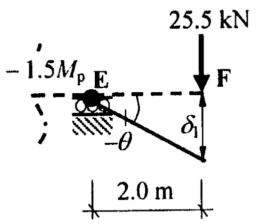
$1.5M_p\,(\theta) + 1.5M_p\,(\theta+\beta) + 1.5M_p\,(\beta) = (51.0 \times \delta_1) = (51.0 \times 2\theta)$

$4.5M_p\theta = 102\theta$

$\therefore$ **M_p = 22.67 kNm**

Span EF





$$\delta_1 = 2\theta$$

Internal Work = External Work

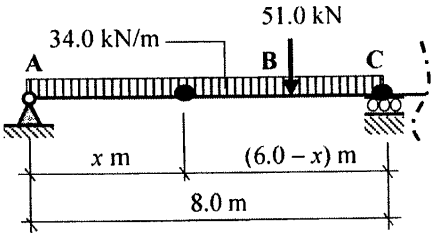
$1.5M_p\,(\theta) = (25.5 \times \delta_1) = (25.5 \times 2\theta) = 51.0\theta$

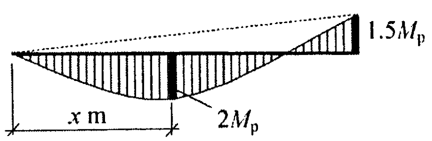
$\therefore$ **M_p = 34.0 kNm**

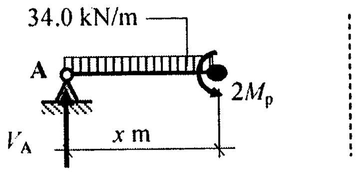
The critical value of M_p = 117.91 kNm

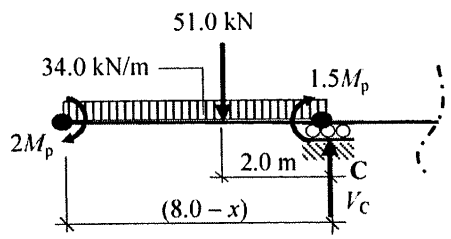
Static Method:

Span ABC









Solution

Topic: Plastic Analysis – Continuous Beams
Problem Number: 8.5 – Static Method **Page No. 3**

+ve $\circlearrowright \Sigma M_A = 0$
$(34.0x^2)/2 - 2M_p = 0$
$17.0x^2 - 2M_p = 0$
$M_p = 8.5x$

+ve $\circlearrowright \Sigma M_C = 0$
$2M_p - 34.0(8.0 - x)^2/2 - (51.0 \times 2.0) + 1.5M_p = 0$
$M_p = 4.857(8.0 - x)^2 - 29.143$

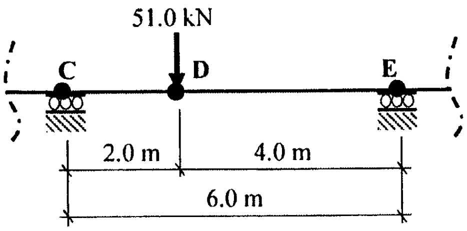
Equate the M_p values to determine x:
$8.5x^2 = 4.857(64.0 - 16x + x^2) - 29.143$ $\quad\therefore\ 3.643x^2 - 77.712x + 339.991 = 0$

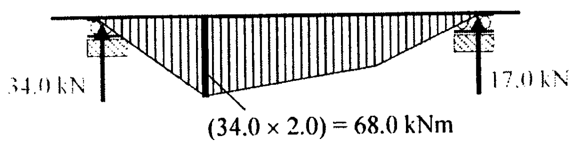
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-77.712 \pm \sqrt{77.712^2 + (4 \times 3.643 \times 339.991)}}{(2 \times 3.643)} = +3.725 \text{ m}$$

$M_p = 8.5x^2 = (8.5 \times 3.725^2)$

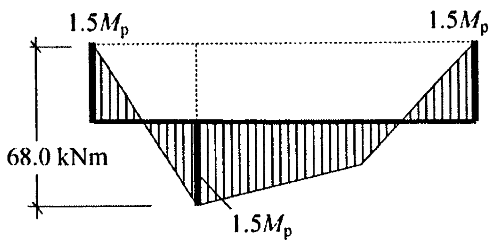
$\therefore$ **M_p = 117.94 kNm**

Span CDE

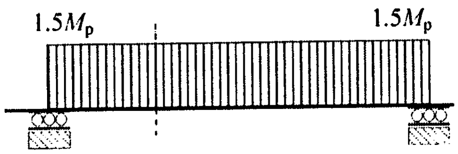




Free Bending Moment Diagram



Combined Bending Moment Diagram

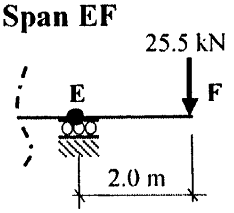


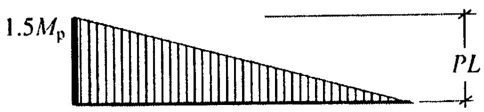
Fixed Bending Moment Diagram

$(1.5M_p + 1.5M_p) = 68.0$ kNm

$\therefore$ **M_p = 22.67 kNm**

Span EF





$1.5M_p = PL = (25.5 \times 2.0) = 51.0$ kNm

$\therefore$ **M_p = 34.0 kNm**
Critical value of M_p = 117.94 kNm