# Plastic Analysis

## *3rd Year*

## *Structural Engineering*

## 2007/8

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# Contents

# 1. Introduction

### *1.1 Background*

Up to now we have concentrated on the elastic analysis of structures. In these analyses we used superposition often, knowing that for a linearly elastic structure it was valid. However, an elastic analysis does not give information about the loads that will actually collapse a structure. An indeterminate structure may sustain loads greater than the load that first causes a yield to occur at any point in the structure. In fact, a structure will stand as long as it is able to find redundancies to yield. It is only when a structure has exhausted all of its redundancies will extra load causes it to fail. Plastic analysis is the method through which the actual failure load of a structure is calculated, and as will be seen, this failure load can be significantly greater than the elastic load capacity.

To summarize this, Prof. Sean de Courcy (UCD) used to say:

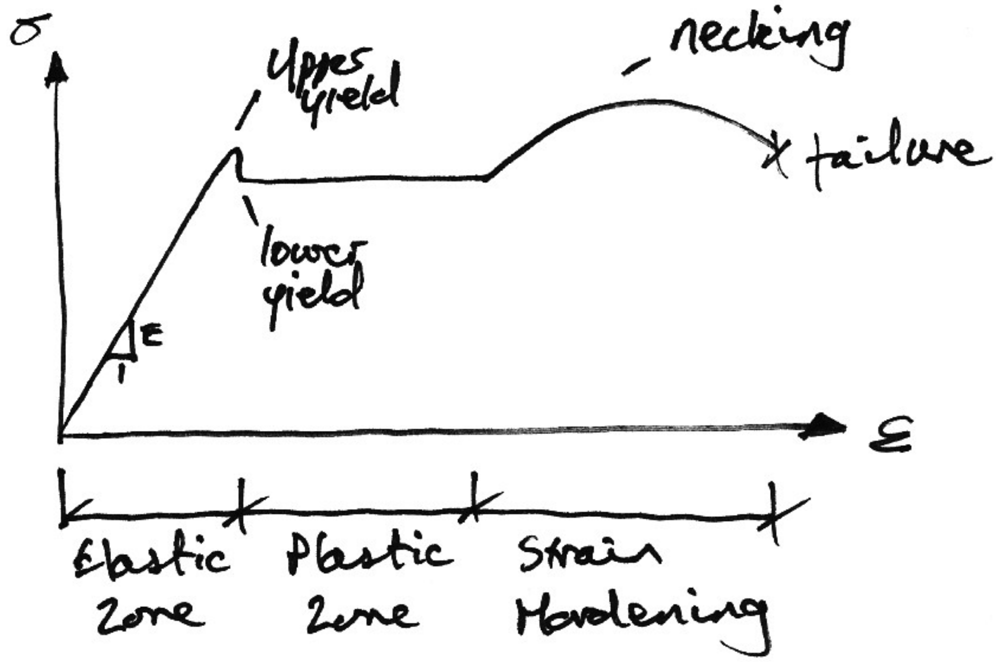
> *"a structure only collapses when it has exhausted all means of standing"*.

Before analysing complete structures, we review material and cross section behaviour beyond the elastic limit.

# 2. Development

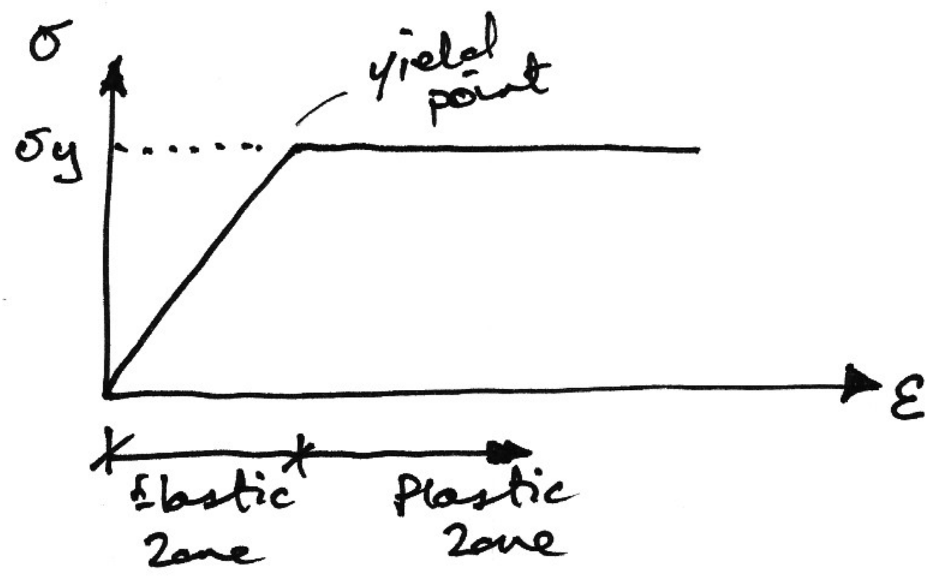
### *2.1 Material Behaviour*

A uniaxial tensile stress on a ductile material such as mild steel typically provides the following graph of stress versus strain:




As can be seen, the material can sustain strains far in excess of the strain at which yield occurs before failure. This property of the material is called its *ductility*.

Though complex models do exist to accurately reflect the above real behaviour of the material, the most common, and simplest, model is the *idealised stress-strain curve*. This is the curve for an ideal elastic-plastic material (which doesn't exist), and the graph is:



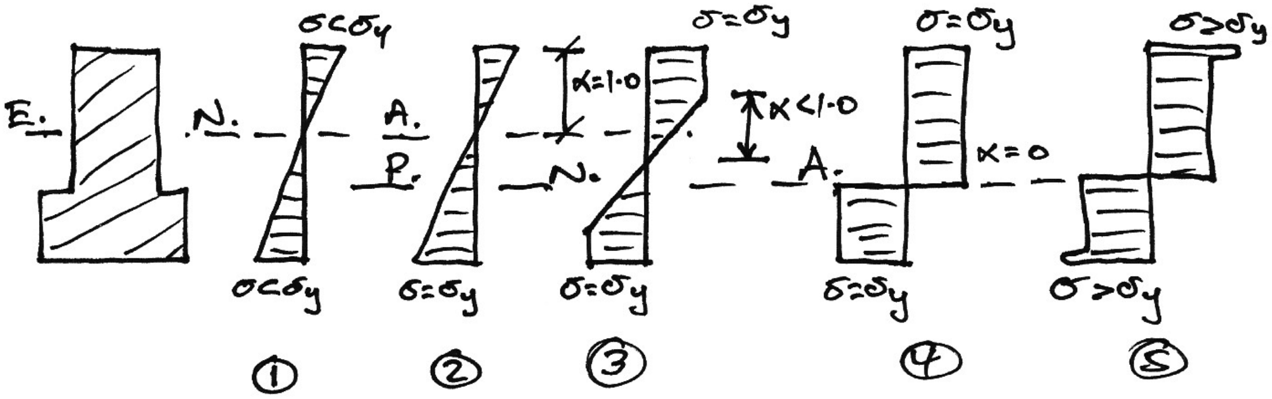

As can be seen, once the yield has been reached it is taken that an indefinite amount of strain can occur. Since so much post-yield strain is modelled, the actual material (or cross section) must also be capable of allowing such strains. That is, it must be sufficiently ductile for the idealised stress-strain curve to be valid.

Next we consider the behaviour of a cross section of an ideal elastic-plastic material subject to bending. In doing so, we seek the relationship between applied moment and the rotation of a cross section.

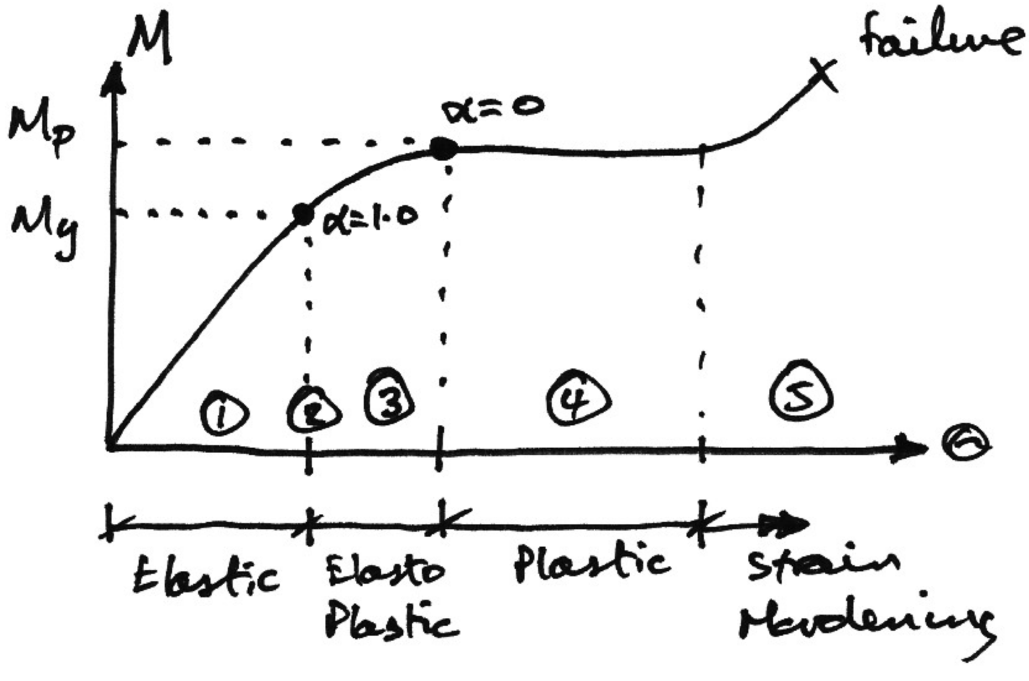
## 2.2 Cross Section Behaviour

### Moment-Rotation Characteristics of General Cross Section

We consider an arbitrary cross-section with a vertical plane of symmetry, which is also the plane of loading. We consider the cross section subject to an increasing bending moment, and assess the stresses at each stage.




**Cross-Section and Stresses**




**Moment-Rotation Curve**

**Stage 1 – Elastic Behaviour**

The applied moment causes stresses over the cross-section that are all less than the yield stress of the material.

**Stage 2 – Yield Moment**

The applied moment is just sufficient that the yield stress of the material is reached at the outermost fibres of the cross-section. All other stresses in the cross section are less than the yield stress. This is limit of applicability of an elastic analysis and of elastic design. Since all fibres are elastic, the ratio of the depth of the elastic to plastic regions, $\alpha = 1.0$.

**Stage 3 – Elasto-Plastic Bending**

The moment applied to the cross section has been increased beyond the yield moment. Since by the idealised stress-strain curve the material cannot sustain a stress greater than yield stress, the fibres at the yield stress have progressed inwards towards the centre of the beam. Thus over the cross section there is an elastic core and a plastic region. The ratio of the depth of the elastic core to the plastic region is $1.0 < \alpha < 0$. Since extra moment is being applied and no stress is bigger than the yield stress, extra rotation of the section occurs: the moment rotation curve losses its linearity and curves, giving more rotation per unit moment (i.e. looses stiffness).

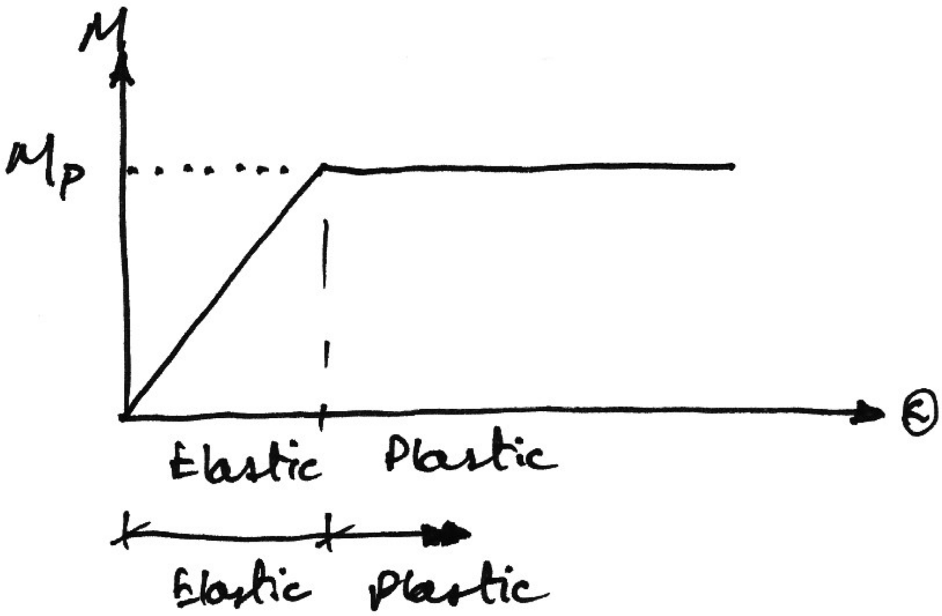
**Stage 4 – Plastic Bending**

The applied moment to the cross section is such that all fibres in the cross section are at yield stress. This is termed the Plastic Moment Capacity of the section. Since there are no fibres at an elastic stress, $\alpha = 0$. An attempt at increasing the moment at this point simply results in more rotation, once the cross-section has sufficient ductility. That is, in steel the cross section classification must be plastic and in concrete the section must be under-reinforced.

**Stage 5 – Strain Hardening**

Due to strain hardening of the material, a small amount of extra moment can be sustained.

The above moment-rotation curve represents the behaviour of a cross section of a regular elastic-plastic material. However, it is usually further simplified as follows:



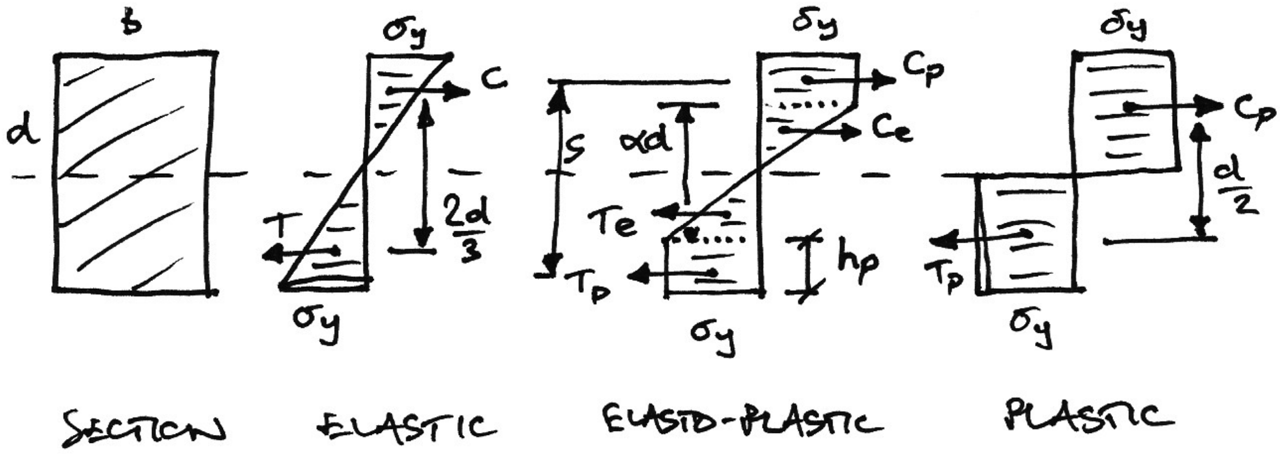

With this idealised moment-rotation curve, the cross section linearly sustains moment up to the plastic moment capacity of the section and then yields in rotation an indeterminate amount. Again, to use this idealisation, the actual section must be capable of sustaining large rotations – that is it must be ductile.

**Plastic Hinge**

Note that once the plastic moment capacity is reached, the section can rotate freely – that is, it behaves like a hinge, except with moment of $M_P$ at the hinge. This is termed a *plastic hinge*, and is the basis for plastic analysis. AT the plastic hinge stresses remain constant, but strains and hence rotations can increase.

## Analysis of Rectangular Cross Section

Since we now know that a cross section can sustain more load than just the yield moment, we are interested in how much more. In other words we want to find the yield moment and plastic moment, and we do so for a rectangular section. Taking the stress diagrams from those of the moment-rotation curve examined previously, we have:




### Elastic Moment

From the diagram:

$$M_Y = C \times \frac{2}{3}d$$

But, the force (or the volume of the stress block) is:

$$C = T = \frac{1}{2}\sigma_Y \frac{d}{2} b$$

Hence:

$$M_Y = \left(\frac{1}{2}\sigma_Y \frac{d}{2} b\right)\left(\frac{2}{3}d\right)$$
$$= \sigma_Y \cdot \frac{bd^2}{6}$$
$$= \sigma_Y \cdot Z$$

The term $bd^2/6$ is thus a property of the cross section called the *elastic section modulus* and it is termed *Z*.

**Elasto-Plastic Moment**

The moment in the section is made up of plastic and elastic components:

$$M_{EP} = M_E^{'} + M_P^{'}$$

The elastic component is the same as previous, but for the reduced depth, $\alpha d$ instead of the overall depth, *d*:

$$M_E^{'} = \left(\frac{1}{2}\sigma_Y \frac{\alpha d}{2}\right)\left(\frac{2\alpha d}{3}\right)$$
$$= \sigma_Y \cdot \alpha^2 \cdot \frac{bd^2}{6}$$

The plastic component is:

$$M_P^{'} = C_P \cdot s$$

The lever arm, *s*, is:

$$s = \alpha d + h_p$$

But

$$h_p = \frac{d - \alpha d}{2} = \frac{d}{2}(1-\alpha)$$

Thus,

$$\begin{aligned} s &= \alpha d + \frac{d}{2} - \frac{\alpha d}{2} \\ &= \frac{d}{2}(1+\alpha) \end{aligned}$$

The force is:

$$\begin{aligned} C_P &= \sigma_Y h_p b \\ &= \sigma_Y b \frac{d}{2}(1-\alpha) \end{aligned}$$

Hence,

$$\begin{aligned} M_P^{'} &= \left[\sigma_Y b \frac{d}{2}(1-\alpha)\right] \cdot \left[\frac{d}{2}(1+\alpha)\right] \\ &= \sigma_Y \frac{bd^2}{4}(1-\alpha^2) \end{aligned}$$

And so the total elasto-plastic moment is:

$$\begin{aligned} M_{EP} &= \sigma_Y \cdot \alpha^2 \cdot \frac{bd^2}{6} + \sigma_Y \frac{bd^2}{4}(1-\alpha^2) \\ &= \sigma_Y \frac{bd^2}{6} \cdot \frac{(3-\alpha^2)}{2} \end{aligned}$$

**Plastic Moment**

From the stress diagram:

$$M_P = C \times \frac{d}{2}$$

And the force is:

$$C = T = \sigma_Y \frac{d}{2} b$$

Hence:

$$\begin{aligned} M_P &= \left( \sigma_Y \frac{bd}{2} \right) \left( \frac{d}{2} \right) \\ &= \sigma_Y \cdot \frac{bd^2}{4} \\ &= \sigma_Y \cdot S \end{aligned}$$

The term $bd^2/4$ is a property of the cross section called the *plastic section modulus*, termed *S*.

## Shape Factor

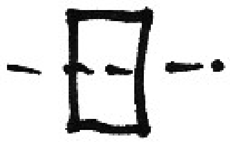
Thus the ratio of elastic to plastic moment capacity is:

$$\frac{M_P}{M_Y} = \frac{\sigma_Y \cdot S}{\sigma_Y \cdot Z} = \frac{S}{Z}$$

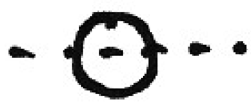
This ration is termed the *shape factor*, *f*, and is a property of a cross section alone. For a rectangular cross-section, we have:

$$f = \frac{S}{Z} = \frac{bd^2/4}{bd^2/6} = 1.5$$

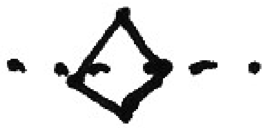
And so a rectangular section can sustain 50% more moment than the yield moment, before a plastic hinge is formed. Therefore the shape factor is a good measure of the efficiency of a cross section in bending. Shape factors for some other cross sections are:



Rectangle: $f = 1.5$, as above;



Circle: $f = 1.698$;

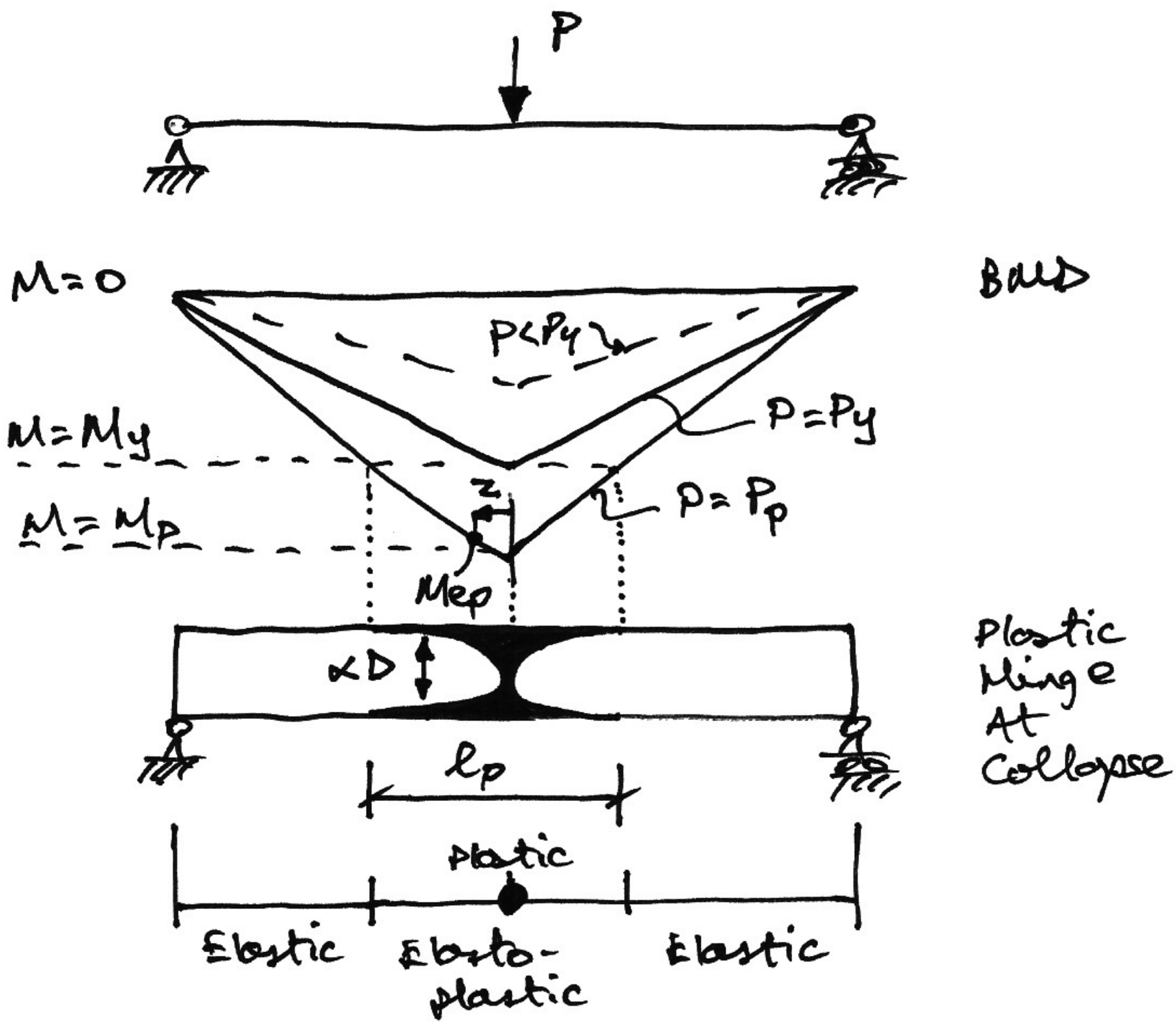


Diamond: $f = 2.0$;

Steel I-beam: $f$ is between 1.12 and 1.15.

### 2.3 Formation of Hinges for Collapse

We investigate the collapse of a simply supported beam under central point load with the information we now have.




The bending moment at the centre of the beam is given by:

$$M_C = \frac{PL}{4}$$

Therefore the load at which yield first occurs is:

$$M_C = M_Y = \frac{P_Y L}{4}$$
$$\therefore P_Y = \frac{4M_Y}{L}$$

Collapse of this beam occurs when the plastic hinge forms at the centre of the beam, since the extra hinge turns the statically determinate beam into a mechanism. The collapse load occurs when the moment at the centre reaches the plastic moment capacity:

$$M_C = M_P = \frac{P_P L}{4}$$
$$\therefore P_P = \frac{4M_P}{L}$$

The ratio collapse to yield load is:

$$\frac{P_P}{P_Y} = \frac{4M_P/L}{4M_Y/L} = \frac{M_P}{M_Y}$$

But since,

$$\frac{M_P}{M_Y} = \frac{S}{Z} = f$$

The ratio is just the shape factor of the section.

We are also interested in the plastic hinge, and the zone of elasto-plastic bending. As can be seen from the diagram, the plastic material zones extend from the centre out to the point where the moment equals the yield moment.

Using similar triangles from the bending moment diagram at collapse, we see that:

$$\frac{M_P}{L} = \frac{M_P - M_Y}{l_p} = \frac{M_P - M_{EP}}{2z}$$

In which $M_{EP}$ is the elasto-plastic moment at a distance $z$ from the plastic hinge, and where $z \le \frac{l_p}{2}$, where $l_p$ is the total length of the plastic region.

Equating the first two equations gives:

$$l_p = \frac{L}{M_P}\left(M_P - M_Y\right) = L\left(1 - \frac{M_Y}{M_P}\right) = L\left(1 - \frac{1}{f}\right)$$

And so for a beam with a rectangular cross section ( $f = 1.5$) the plastic hinge extends for a length:

$$l_p = L\left(1 - \frac{1}{1.5}\right) = \frac{L}{3}$$

Lastly, the shape of the hinge follows from the first and third equation:

$$\frac{M_P}{L} = \frac{M_P - M_{EP}}{2z}$$

$$\frac{z}{L} = \frac{1}{2M_P}\left(M_P - M_{EP}\right)$$

$$\frac{z}{L} = \frac{1}{2}\left(1 - \frac{M_{EP}}{M_P}\right)$$

From our expressions for the elasto-plastic and plastic moments, we have:

$$\begin{aligned}\frac{z}{L} &= \frac{1}{2}\left(1-\frac{\sigma_Y\left(bd^2/6\right)\left(1/2\right)\left(3-\alpha^2\right)}{\sigma_Y\left(bd^2/4\right)}\right)\\ &= \frac{1}{2}\left(1-\frac{2}{3}\cdot\frac{1}{2}\cdot\left(3-\alpha^2\right)\right)\\ \frac{z}{L} &= \frac{\alpha^2}{6}\end{aligned}$$

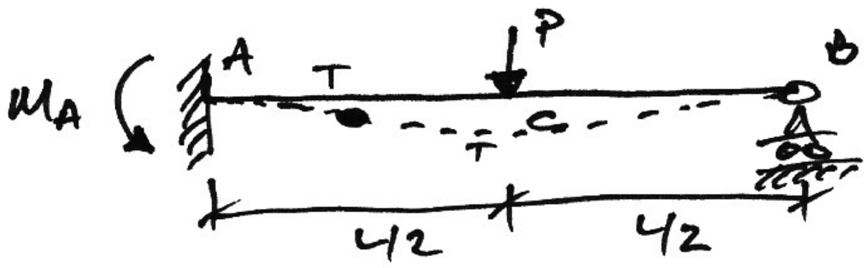
This shows that the plastic region has a parabolic profile, and confirms that the total length of the hinge, $l_p = 2z$, is $L/3$ at the location where $\alpha = 1.0$.

Using a similar form of analysis, we can show that under a UDL the plastic hinge has a linear profile given by $z/L = 2\alpha\sqrt{3}$ and that its length is $L/\sqrt{3}$.

## 2.4 Plastic Hinge Development

### Illustrative Example – Propped Cantilever

We now assess the behaviour of a simple statically indeterminate structure under increasing load. We consider a propped cantilever with mid-span point load:




From previous analyses we know that:

$$M_A = \frac{3PL}{16} \qquad M_C = \frac{5PL}{32}$$

We will take the span to be $L = 1\text{ m}$ and the cross section to have the following capacities:

$$M_Y = 7.5\text{ kNm} \qquad M_P = 9.0\text{ kNm}$$

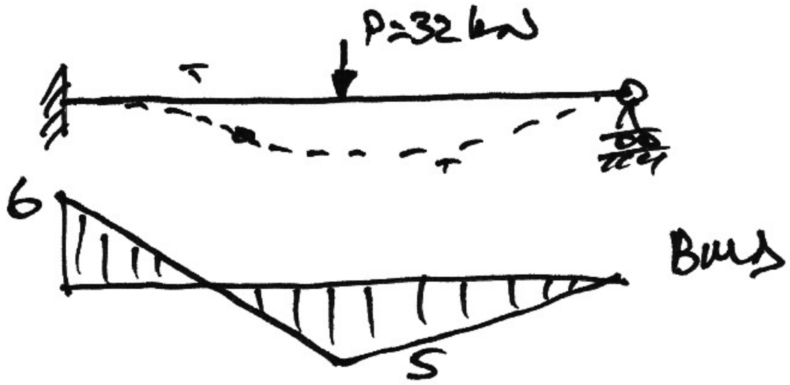
Further, we want this beam to be safe at a working load of 32 kN, so we start there.

**Load of 32 kN**

At this value of load the BMD is as shown, with:

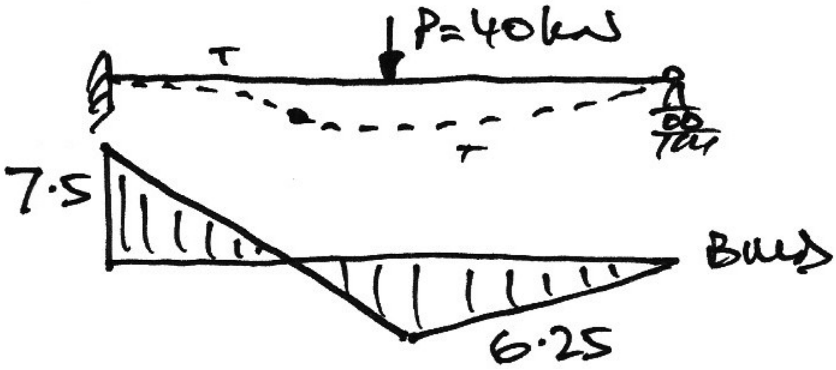
$$M_A = \frac{3(32)(1)}{16} = 6\,\text{kNm} \qquad M_C = \frac{5(32)(1)}{32} = 5\text{ kNm}$$

Since the peak moments are less than the yield moments, we know that yield stress has not been reached at any point in the beam. Also, the maximum moment occurs at *A* and so this point will first reach the yield moment.



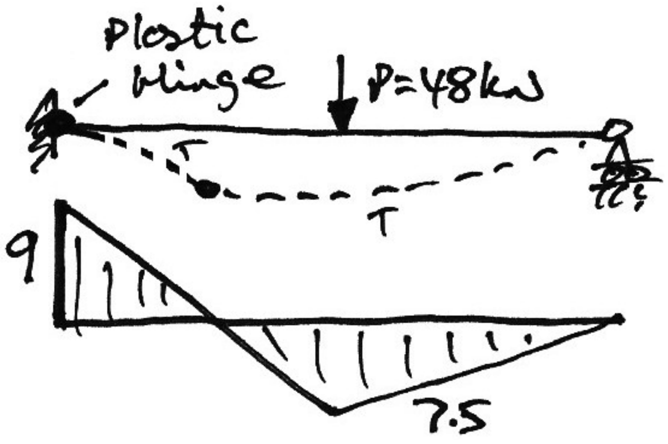

**Load of 40 kN**

At this load the BDM becomes that as shown. The moment at *A* has now reached the yield moment and so the outer fibres at *A* are at yield stress.



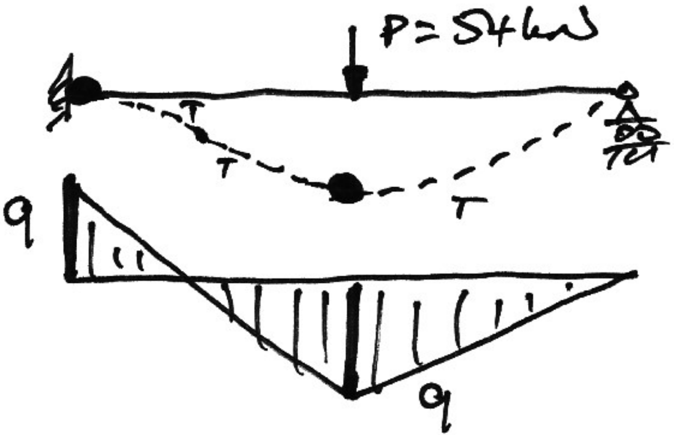

**Load of 48 kN**

The BMD is as shown. The moment at *A* is now 9 kNm – the plastic moment capacity of the section – and so the cross section at *A* has fully yielded. Thus a plastic hinge has formed at *A* and so no extra moment can be taken at *A*, but *A* can rotate freely with constant moment of 9 kNm. Also, the moment at *C* has reached the yield moment. Note that the structure does not collapse since there are not sufficient hinges for it to be a mechanism yet.




**Load of 54 kN**

Since the moment at *A* has already reached the plastic moment of the section, no extra moment can be taken there and $M_A$ must remain 9 kNm whilst allowing rotation to freely occur. Therefore, all of the extra moment caused by the increase in load of $54-48=6$ kN must be taken by the structure as if it were a simply-supported beam. That is, a beam free to rotate at both ends. The extra moment at *C* is thus $6 \cdot 1/4 = 1.5$ kNm bring the total moment at *C* to 9 kNm – the plastic moment capacity of the section. Therefore a plastic hinge forms at *C* and the structure is not capable of sustaining anymore load – becomes a mechanism – and so collapse ensues.



**Discussion**

There are several things to note from this analysis:

1. The actual load carried by the beam is 54 kN, greater than the load at which yield first occurs, 40 kN, the elastic limit. This difference of 35% represents the extra capacity of the structure over the elastic capacity, so to ignore it would be inefficient.
2. At the end of the analysis $M_A = M_C = 9$ kNm and so $M_A / M_C = 1$. Since for an elastic analysis $M_A / M_C = (3PL/16)/(5PL/32) = 1.2$, it is evident that our analysis is not an elastic analysis and so is a plastic analysis.
3. The height of the free bending moment diagram was $PL/4$ throughout, as required by equilibrium – only the height of the reactant bending moment diagram varied.
4. At the point of collapse we had 4 reactions and 2 plastic hinges giving a statical indeterminacy of $R - C - 3 = 4 - 2 - 3 = -1$ which is a mechanism and so collapse occurs.
5. The load can only increase from 48 kN to 54 kN once the cross section at *A* has sufficient ductility to allow it rotate thereby allowing the extra load to be taken at *C*. If there was not sufficient ductility there may have a brittle-type catastrophic failure at *A* resulting in the beam failing by rotating about *B* before the full plastic capacity of the structure is realized. Therefore it is only by having sufficient ductility that a plastic analysis can be used.

Some of these points are general for any plastic analysis and these generalities are known as the *Theorems of Plastic Analysis*. However, before looking at these theorems we need a simpler way of analysing for the collapse of structures: the incremental loading approach works, but is very laborious.

## 2.5 Important Definitions

### Load Factor

The load factor for a possible collapse mode *i*, denoted $\lambda_i$, is of prime importance in plastic analysis:

$$\lambda_i = \frac{\text{Collapse Load for mode } i}{\text{Working Load}}$$

The working load is the load which the structure is expected to carry in the course of its lifetime.

The collapse load factor, $\lambda_C$, is the load factor at which the structure will actually fail. It is therefore the minimum of the load factors for the $n_m$ different possible collapse modes:

$$\lambda_C = \min_{1 \le i \le n_m} \lambda_i$$

In our previous analysis the working load was 32 kN and the collapse load for the single mode was found to be 54 kN. Hence:

$$\lambda_C = \frac{54}{32} = 1.6875$$

### Factor of Safety

This is defined as:

$$\text{FoS} = \frac{\text{First yield load}}{\text{Working Load}}$$

The FoS is an elastic analysis measure of the safety of a design. For our example:

$$\text{FoS} = \frac{40}{32} = 1.25$$

## 2.6 Virtual Work in Plastic Analysis

### Introduction

The easiest way to carry out a plastic analysis is to use virtual work. To do this we allow the presumed shape at collapse to be the compatible displacement set, and the external loading and internal bending moments to be the equilibrium set. We can then equate external and internal virtual work, and solve for the collapse load factor for that supposed mechanism.

Remember:

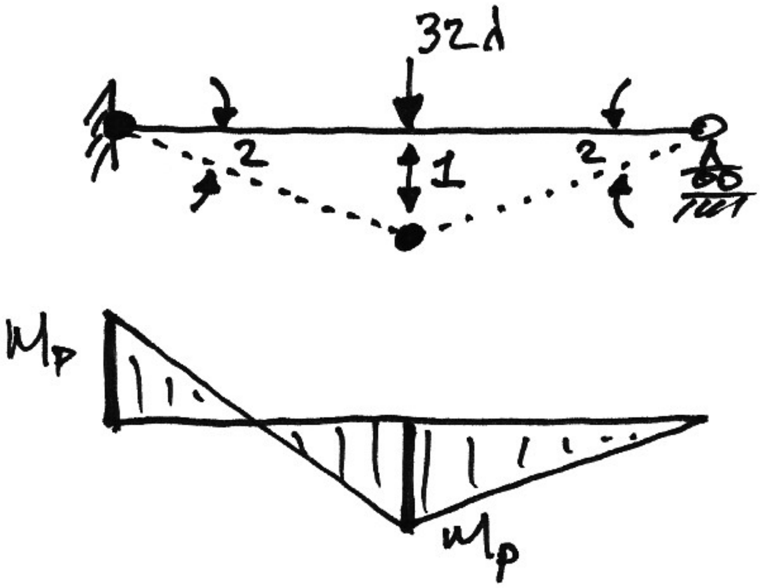
- Equilibrium set: the internal bending moments at collapse;
- Compatible set: the virtual collapsed configuration (see below).

Note that in the actual collapse configuration the members will have elastic deformation in between the plastic hinges. However, since a virtual displacement does not have to be real, only compatible, we will choose to ignore the elastic deformations between plastic hinges, and take the members to be straight between them.

**Illustrative Example Cont'd**

**Actual Collapse Mode**

So for our previous beam, we know that we require two hinges for collapse (one more than its degree of redundancy), and we think that the hinges will occur under the points of peak moment, *A* and *C*. Therefore impose a unit virtual displacement at *C* and relate the corresponding virtual rotations of the hinges using $S = R\theta$, giving:




Notice that the collapse load is the working load times the collapse load factor. So:

$$\delta W_e = \delta W_I$$

$$(32\lambda)(1) = \underbrace{(M_P)(2)}_{\text{At } A} + \underbrace{(M_P)(4)}_{\text{At } C}$$

$$32\lambda = 6M_P$$

$$\lambda = \frac{6(9)}{32} = 1.69$$

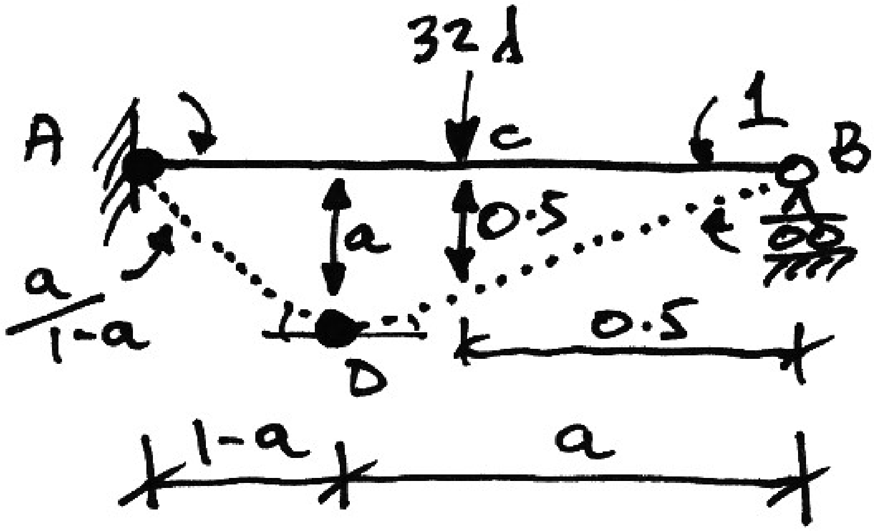
Since $M_P = 9$ kNm and this is as found before.

**Other Collapse Modes**

For the collapse mode looked at previously, it seemed obvious that the plastic hinge in the span should be beneath the load. But why? Using virtual work we can examine any possible collapse mode. So let's consider the following collapse modes and see why the plastic hinge should have been beneath the load.

**Plastic Hinge between *A* and *C*:**

Imposing a unit virtual deflection at *B*, we get the following collapse mode:




And so the virtual work equation becomes:

$$\delta W_e = \delta W_I$$

$$(32\lambda)(0.5) = \underbrace{(M_P)\left(\frac{a}{1-a}\right)}_{\text{At } A} + \underbrace{(M_P)\left(\frac{a}{1-a}+1\right)}_{\text{At } D}$$

$$16\lambda = M_P\left[\frac{2a+(1-a)}{1-a}\right]$$

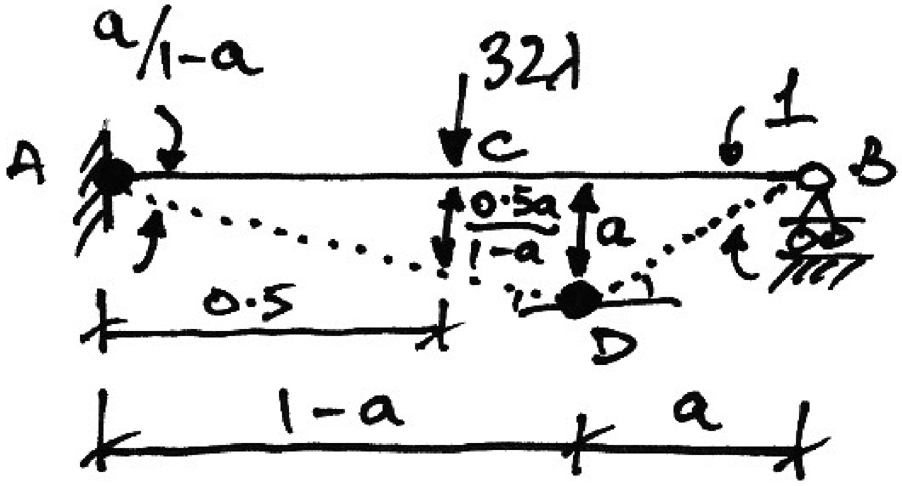
And since $M_P = 9$ kNm:

$$\lambda_{1<a\leq 0.5} = \frac{9}{16}\left[\frac{a+1}{1-a}\right] \qquad \textbf{Eq. (1)}$$

And so we see that the collapse load factor for this mode depends on the position of the plastic hinge in the span.

**Plastic Hinge between *C* and *B*:**

Again imposing a unit virtual deflection at *B* we get:




And so the virtual work equation becomes:

$$\delta W_e = \delta W_I$$

$$(32\lambda)\left(\frac{0.5a}{1-a}\right) = \underbrace{(M_P)\left(\frac{a}{1-a}\right)}_{\text{At } A} + \underbrace{(M_P)\left(\frac{a}{1-a}+1\right)}_{\text{At } D}$$

$$16\lambda\left(\frac{a}{1-a}\right) = M_P\left[\frac{2a+(1-a)}{1-a}\right]$$

$$16\lambda a = M_P(1+a)$$

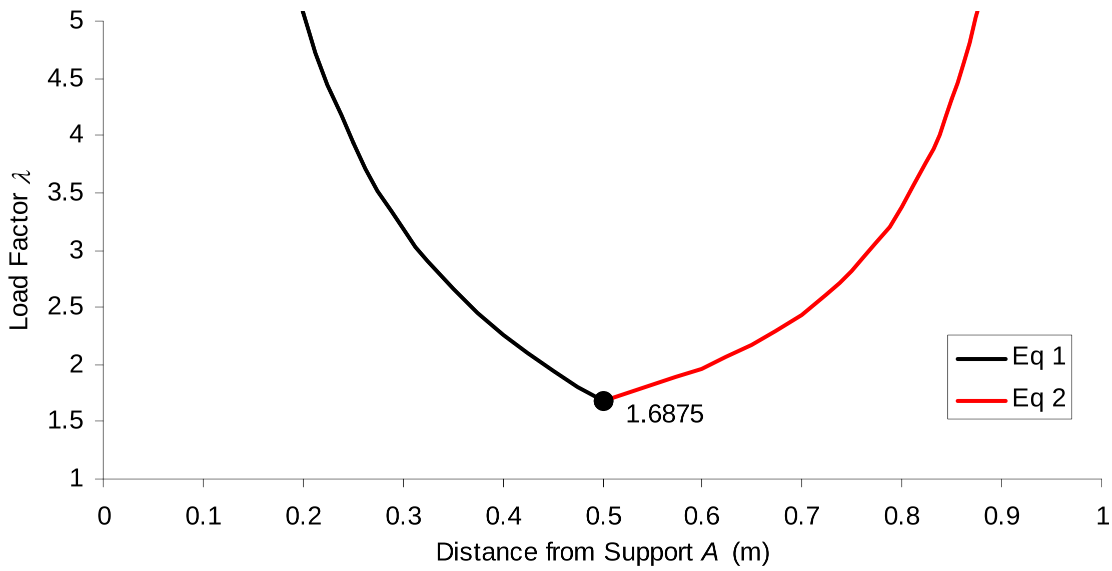
Using $M_P = 9$ kNm:

$$\lambda_{0.5<a\leq 0} = \frac{9}{16}\left[\frac{1+a}{a}\right] \qquad \textbf{Eq. (2)}$$

And again we see that the load factor depends on the position of the hinge.

**Summary**

Plotting how the collapse load factor changes with the position of the hinge, we get:




This tells us that when the load reaches 1.6875 times the working load (i.e. 54 kN) a hinge will form underneath the load, at point *C*, 0.5 m from support *A*. It also tells us that it would take more than 54 kN for a hinge to form at any other place, once it hadn't already formed at *C*. Thus the actual collapse load factor is the smallest of all the possible load factors. Hence we can see that in analysing proposed collapse mechanisms, we are either correct ($\lambda_C = 1.6875$) or we are unsafe ($\lambda > \lambda_C$). This is why plastic analysis is an *upperbound* method.

## *2.7 Theorems of Plastic Analysis*

### Criteria

In Plastic Analysis to identify the correct load factor, there are three criteria of importance:

1. Equilibrium: the internal bending moments must be in equilibrium with the external loading.

2. Mechanism: at collapse the structure, or a part of, can deform as a mechanism.

3. Yield: no point in the structure can have a moment greater than the plastic moment capacity of the section it is applied to.

Based on these criteria, we have the following theorems.

**The Upperbound (Unsafe) Theorem**

*If a bending moment diagram is found which satisfies the conditions of equilibrium and mechanism (but not necessarily yield), then the corresponding load factor is either greater than or equal to the true load factor at collapse.*

This is called the unsafe theorem because for an arbitrarily assumed mechanism the load factor is either exactly right (when the yield criterion is met) or is wrong and is too large, *leading a designer to think that the frame can carry more load than is actually possible*.

Think of it like this: unless it's exactly right, it's dangerous.

Since a plastic analysis will generally meet the equilibrium and mechanism criteria, by this theorem a plastic analysis is either right or dangerous. This is why plastic analyses are not used as often in practice as one might suppose.

The above theorem can be easily seen to apply to the *Illustrative Example*. When we varied the position of the hinge we found a collapse load factor that was either correct ($\lambda = \lambda_C = 1.6875$) or was too big ($\lambda > \lambda_C$).

### The Lowerbound (Safe) Theorem

*If a bending moment diagram is found which satisfies the conditions of equilibrium and yield (but not necessarily that of mechanism), then the corresponding load factor is either less than or equal to the true load factor at collapse.*

This is a safe theorem because the load factor will be less than (or at best equal to) the collapse load factor once equilibrium and yield criteria are met leading the designer to think that the structure can carry less than or equal to its actual capacity.

Think of it like this: you’re either wrong and safe, or you’re right and safe.

Since an elastic analysis will always meet equilibrium and yield conditions, an elastic analysis will always be safe. This is the main reason that it is elastic analysis that is used, in spite of the extra capacity that plastic analysis offers.

### The Uniqueness Theorem

*If a bending moment distribution can be found which satisfies the three conditions of equilibrium, mechanism, and yield, then the corresponding load factor is the true load factor at collapse.*

So to have identified the correct load factor (and hence collapse mode) for a structure we need to meet all three of the criteria:

1. Equilibrium;
2. Mechanism;
3. Yield.

The permutations of the three criteria and the three theorems are summarized in the following table:

<table>
<tr><th>Criterion</th><th>Upperbound (Unsafe) Theorem</th><th>Lowerbound (Safe) Theorem</th><th>Unique Theorem</th></tr>
<tr><td>Mechanism</td><td rowspan="2">$\lambda \ge \lambda_C$</td><td></td><td rowspan="3">$\lambda = \lambda_C$</td></tr>
<tr><td>Equilibrium</td><td rowspan="2">$\lambda \le \lambda_C$</td></tr>
<tr><td>Yield</td><td></td></tr>
</table>

### Corollaries of the Theorems

Some other results immediately apparent from the theorems are the following:

1. If the collapse loads are determined for all possible mechanisms, then the actual collapse load will be the lowest of these (Upperbound Theorem);
2. The collapse load of a structure cannot be decreased by increasing the strength of any part of it (Lowerbound Theorem);
3. The collapse load of a structure cannot be increased by decreasing the strength of any part of it (Upperbound Theorem);
4. The collapse load is independent of initial stresses and the order in which the plastic hinges form (Uniqueness Theorem);

The first point above is the basis for using virtual work in plastic analysis. However, in doing so, it is essential that the designer considers the actual collapse more. To not do so would lead to an unsafe design by the Upperbound Theorem.

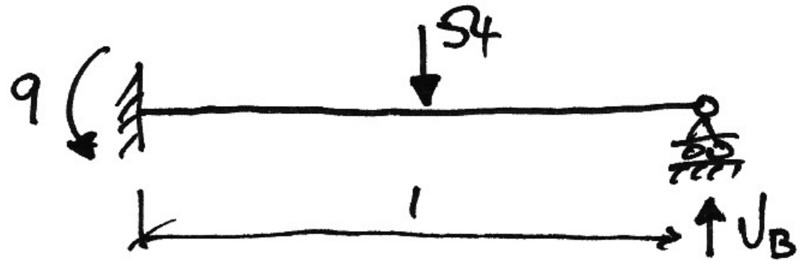
Note that the Uniqueness Theorem does not claim that the BMD at collapse is unique – only that the collapse load factor is unique. Although rare, it is possible for more than one BMD to satisfy the Uniqueness Theorem, but they will have the same load factor.

**Illustrative Example Cont'd**

**Plastic Hinge Under the Load**

We discovered previously that the collapse load factor was 1.6875 and this occurred when the hinge was under the point load. Therefore, this collapse mode should meet all three criteria of the Uniqueness Theorem:

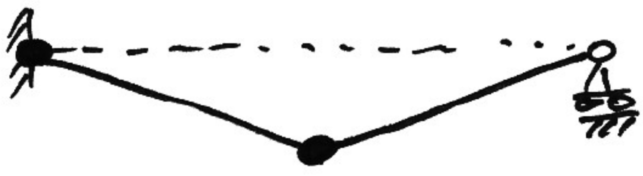
1. Equilibrium: check on the moment at *C* say:




$$\sum M \text{ about } A = 0 \quad 54 \cdot 0.5 - 9 - V_B = 0 \Rightarrow V_B = 18 \text{ kN}$$

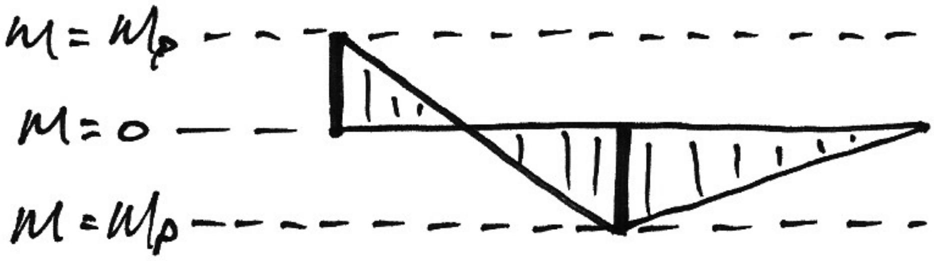
Thus, from a free-body diagram of $|CB|$, $M_C = 18 \cdot 0.5 = 9$ kNm as expected. Thus the equilibrium condition is met.

2. Mechanism: Given the number of hinges it is obvious the structure collapses:



$$R - C - 3 = 4 - 2 - 3 = -1$$

3. Yield: Check that there is no moment greater than $M_P = 9$ kNm:

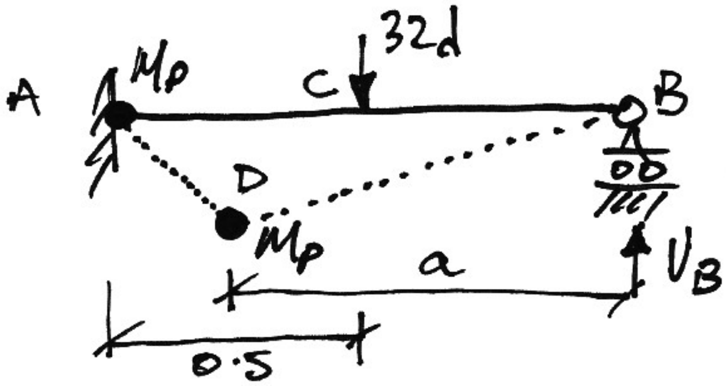


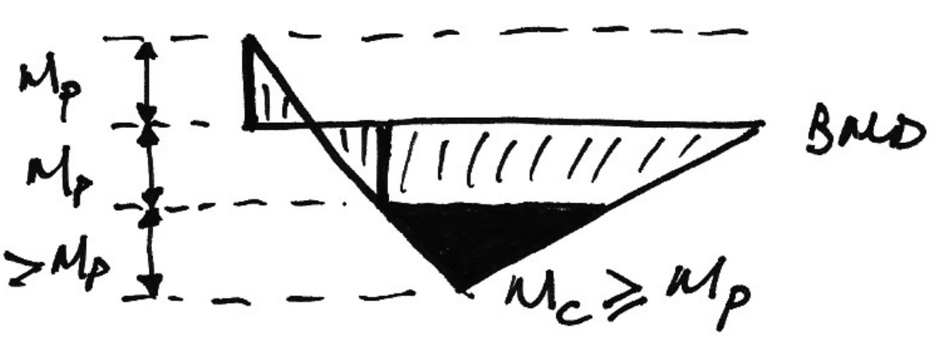

And so the yield criterion is met.

Since all three conditions are met we are assured that the have the actual collapse load factor by the Uniqueness Theorem.

**Other Collapse Modes**

Using the analyses carried out previously for different positions of the plastic hinge, we can check these collapse modes against the Uniqueness Theorem. For the case of the hinge between *A* and *C*:



To determine this BMD, we calculate the reaction $V_B$ by considering the free body diagram *BCD*:

$$\sum M \text{ about } D = 0 \quad \therefore M_P + 32\lambda(a - 0.5) - V_B a = 0$$
$$\therefore V_B = \frac{M_P}{a} + 32\lambda - \frac{16\lambda}{a}$$

Thus the moment under the point load is:

$$M_C = 0.5 \cdot V_B = \frac{M_P}{2a} + 16\lambda - \frac{8\lambda}{a}$$

Substituting in the expression for $\lambda$ from Eq. (1) previously:

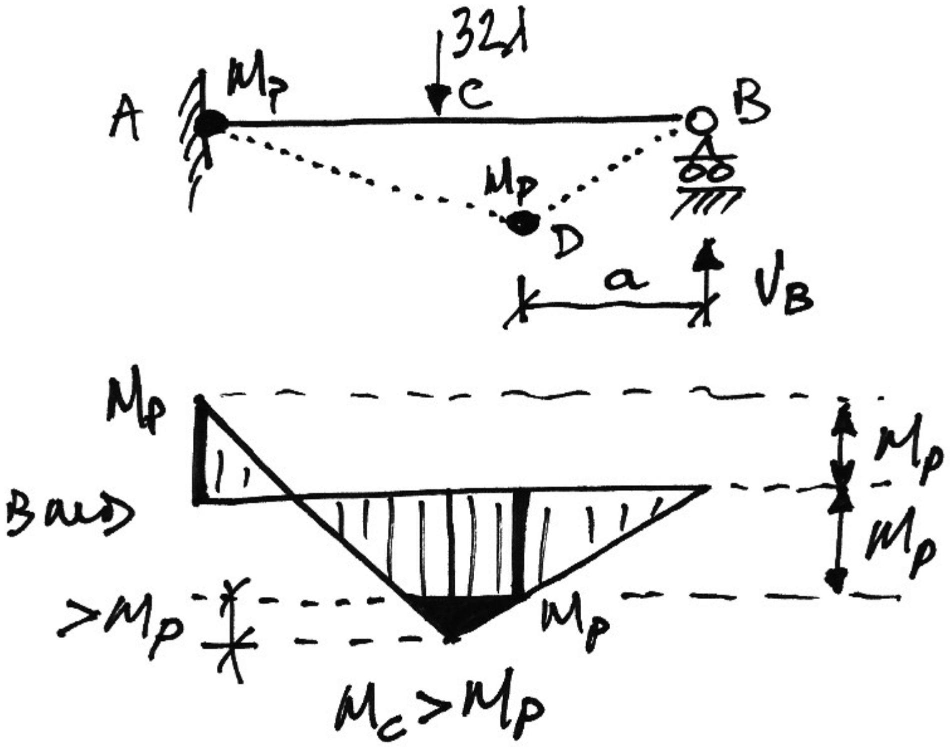
$$M_C = \frac{M_P}{2a} + \left(16 - \frac{8}{a}\right)\left[\frac{M_P}{16}\left(\frac{a+1}{1-a}\right)\right]$$

Which after some algebra becomes:

$$M_C = M_P\left[\frac{a}{1-a}\right]$$

And so because $0.5 \le a \le 1.0$, $M_C \ge M_P$ as shown in the BMD. Only when $a = 0.5$ does $M_C = M_P$, which is of course the correct solution.

For the case of the hinge being between *C* and *B*, we have:




Again, we find the reaction $V_B$ by considering the free body diagram *DB*:

$$\sum M \text{ about } D = 0 \quad \therefore M_P - V_B a = 0 \qquad \therefore V_B = \frac{M_P}{a}$$

Thus the moment under the point load at *C* is:

$$M_C = M_P\left[\frac{1}{2a}\right]$$

And since $0 \le a \le 0.5$ then $\infty \le 1/2a \le 1$ and so $M_C \ge M_P$. Again only when $a = 0.5$ does $M_C = M_P$.

**Summary**

We have seen that for any position of the plastic hinge, other than at exactly *C*, the yield condition is not met. Therefore, in such cases, the Uniqueness Theorem tells us that the solution is not the correct one.

Notice that in these examples the mechanism and equilibrium conditions are always met. Therefore the Upperbound Theorem tells us that our solutions in such cases are either correct (as in when $a = 0.5$) or are unsafe (as in $\lambda > \lambda_C$).

In cases where one of the conditions of the Uniqueness Theorem is not met, we assume a different collapse mode and try again.

## 2.8 Plastic Design

When we come to design a structure using plastic methods, it is the load factor that is known in advance and it is the plastic moment capacity that is the objective. The general virtual work equations for a proposed collapse mode *i* is

$$\delta W_e = \delta W_I$$
$$\lambda_i \cdot \sum P_j \delta_{ji} = \sum M_{Pj} \theta_j$$

In which *j* is an individual load and deflection or plastic moment and rotation pair. If we take the $M_P$ of each member to be some factor, $\phi$, of a nominal $M_P$, then we have:

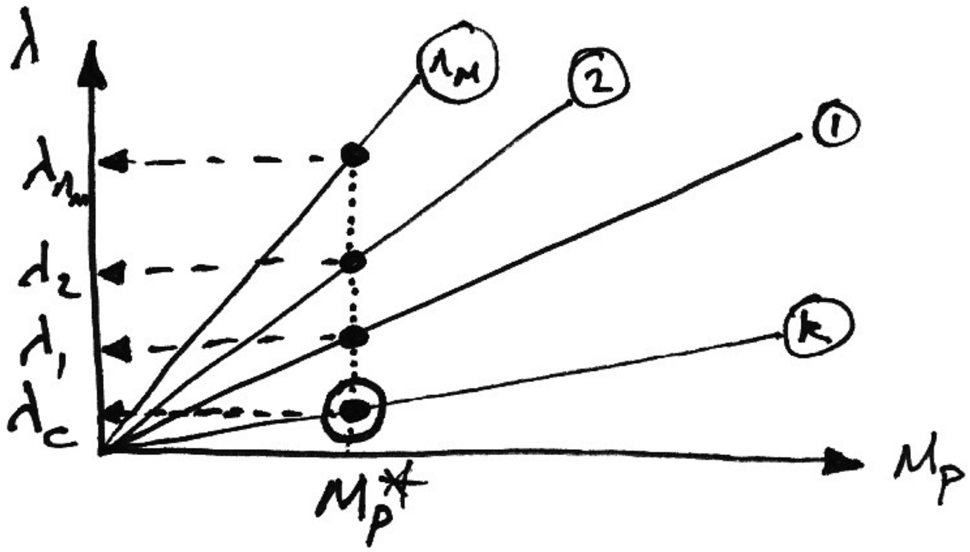
$$\lambda_i \cdot \sum P_j \delta_{ji} = M_{Pj} \cdot \sum \phi_j \theta_j$$

Since work is a scalar quantity, and since the sum of work done on both sides is positive, we can see that the load factor and plastic moment capacity have a linear relationship of slope *m* for each collapse mode *i*:

$$\lambda_i = M_P \cdot \frac{\sum \phi_j \theta_j}{\sum P_j \delta_{ji}}$$
$$\lambda_i = m_i \cdot M_P$$

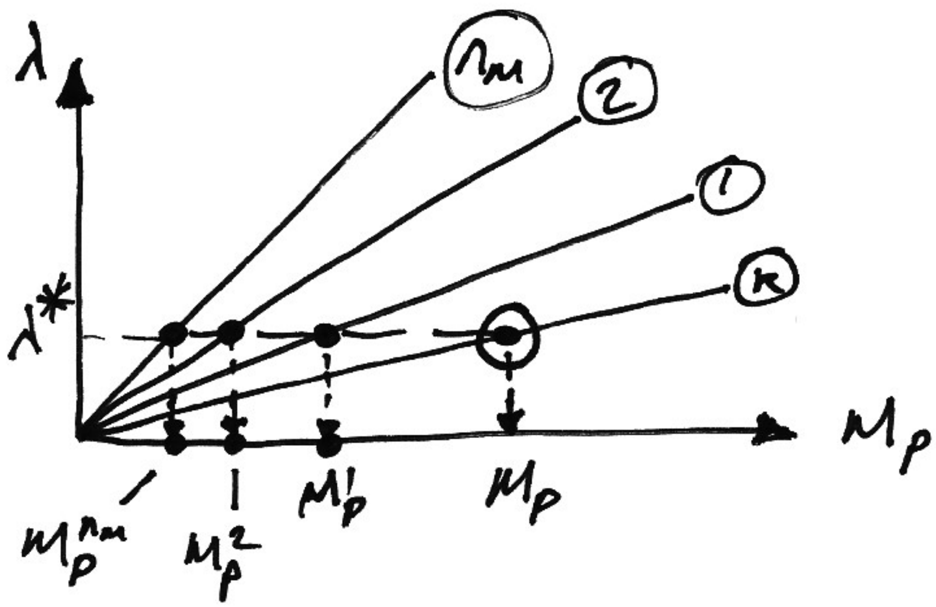
Thus for each collapse mode, $1 \le k \le n_m$, we can plot the load factor against the plastic moment capacity. We do so for two cases:

1. Load Factor Required – Design Plastic Moment Capacity Known:




We can see from this graph that for a particular value of the plastic moment capacity, $M_P$*, collapse mode *k* gives the lowest load factor and so by the Upperbound Theorem is the true collapse mode.

2. Design Load Factor Known –Plastic Moment Capacity Required:



From this graph we can see that for a particular value of the load factor, $\lambda *$, collapse mode *k* gives the highest design plastic moment capacity, $M_P$. However, since by the Upperbound Theorem we know collapse mode *k* to be the true collapse mode, it is therefore the highest value of $M_P$ from each mode that is required.

Mathematically, using the Upperbound Theorem, the above is summarized as:

$$\begin{aligned}\lambda_C &= \min \lambda_i \\ &= \min\left[m_i \cdot M_P\right] \\ &= M_P \min m_i\end{aligned}$$

Hence when the desired $\lambda_C$ is specified:

$$\begin{aligned}M_P &= \frac{\lambda_C}{\min m_i} \\ &= \max\left[\frac{\lambda_C}{m_i}\right] \\ M_P &= \max\left[\frac{\lambda_C \sum P_j \delta_{ji}}{\sum \phi_j \theta_j}\right]\end{aligned}$$

In summary, if:

- Design plastic moment capacity is known – design for <u>lowest</u> load factor;
- Design load factor is known – design for <u>highest</u> plastic moment capacity.

## 2.9 Summary of Important Points

**Number of Hinges Required for Collapse:**

In general we require sufficient hinges to turn the structure into a mechanism, thus:

$$\begin{matrix}\text{No. of Plastic} \\ \text{Hinges Required}\end{matrix} = {}^{\circ}\text{Indet} + 1$$

However, this does not apply in cases of local partial collapses.

**The Three Theorems of Plastic Analysis:**

| *Criterion* | *Upperbound (Unsafe) Theorem* | *Lowerbound (Safe) Theorem* | *Unique Theorem* |
|---|---|---|---|
| Mechanism | $\lambda \geq \lambda_C$ | | $\lambda = \lambda_C$ |
| Equilibrium | $\lambda \geq \lambda_C$ | $\lambda \leq \lambda_C$ | $\lambda = \lambda_C$ |
| Yield | | $\lambda \leq \lambda_C$ | $\lambda = \lambda_C$ |

**Collapse Load Factor**

By the Unsafe Theorem, which applies when the virtual work method is used:

$$\lambda_C = \min_{1 \leq i \leq n_m} \lambda_i$$

**Design Value of Plastic Moment Capacity**

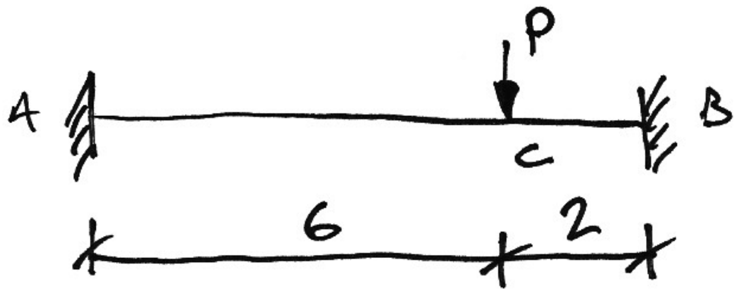
The design value of $M_P$ is the maximum of the design values for $M_P$ from each collapse mode:

$$M_P = \max_{1 \leq i \leq n_m} M_{P,i}$$

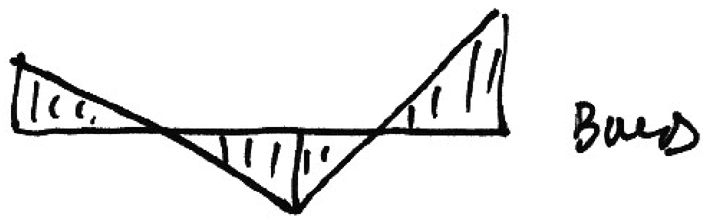
# 3. Beams

### 3.1 Example 1 – Fixed-Fixed Beam with Point Load

For the following beam, find the load at collapse, given that $M_P = 60\text{ kNm}$:



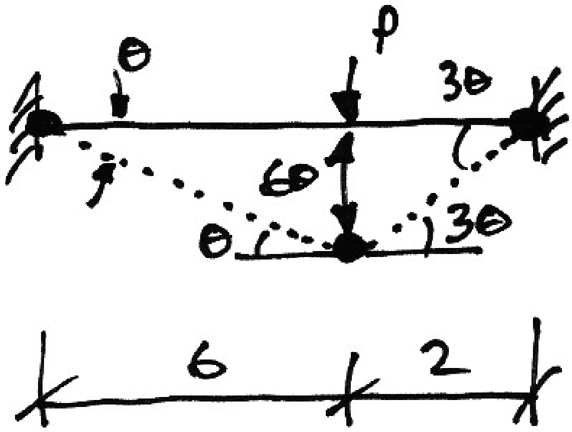

To start the problem, we examine the usual elastic BMD to see where the plastic hinges are likely to form:




We also need to know how many hinges are required. This structure is 3° statically indeterminate and so we might expect the number of plastic hinges required to be 4. However, since one of the indeterminacies is horizontal restraint, removing it would not change the bending behaviour of the beam. Thus for a bending collapse only 2 indeterminacies apply and so it will only take 3 plastic hinges to cause collapse.

So looking at the elastic BMD, we'll assume a collapse mode with the 3 plastic hinges at the peak moment locations: *A*, *B*, and *C*.

Next, we impose a virtual rotation of $\theta$ to the plastic hinge at $A$ and using the $S = R\theta$ rule, relate all other displacements to it, and then apply the virtual work equation:



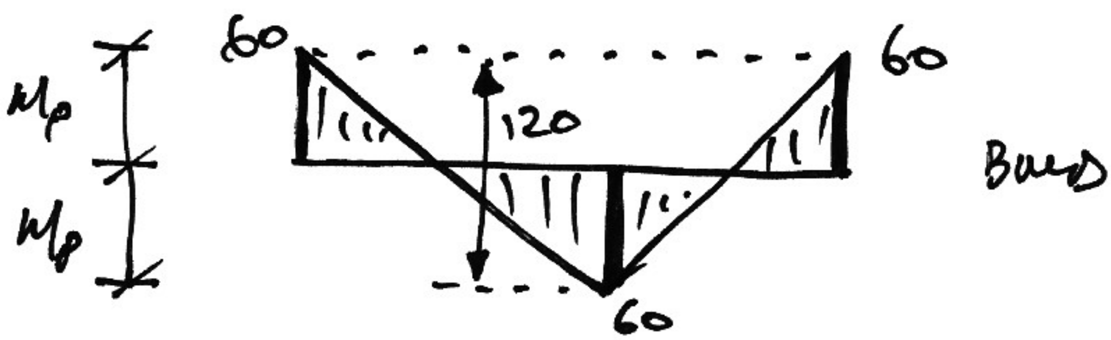

$$\delta W_e = \delta W_I$$

$$P(6\theta) = \underbrace{M_P(\theta)}_{\text{At } A} + \underbrace{M_P(\theta + 3\theta)}_{\text{At } C} + \underbrace{M_P(3\theta)}_{\text{At } B}$$

$$6P\theta = 8M_P\theta$$

$$P = \frac{8}{6}M_P$$

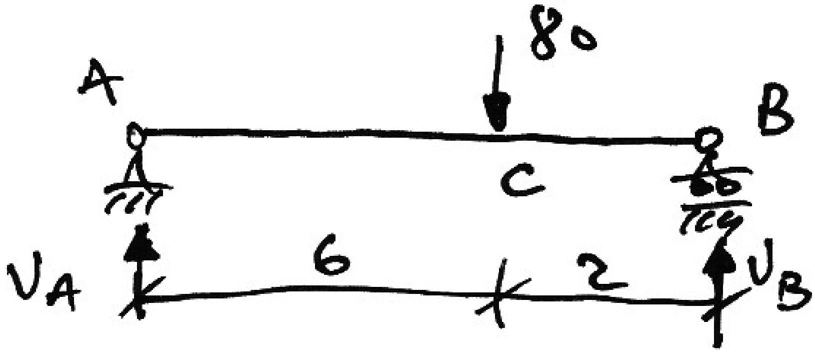
Since $M_P = 60 \text{ kNm}$ the load required for collapse is $P = 80 \text{ kN}$ and so the collapse BMD for this mode is:




We need to check that this is the correct solution using the Uniqueness Theorem:

*1. Equilibrium:*

We'll check that the height of the free BMD is 120 kNm as per the collapse BMD:




$$\sum M \text{ about } A = 0 \quad \therefore 80 \cdot 6 - 8V_B = 0 \quad \therefore V_B = 60 \text{ kN}$$

Thus, using a free body diagram of *CB*:

$$\sum M \text{ about } C = 0 \quad \therefore M_C - 2V_B = 0 \quad \therefore M_C = 120 \text{ kNm}$$

And so the applied load is in equilibrium with the free BMD of the collapse BMD.

*2. Mechanism:*

From the proposed collapse mode it is apparent that the beam is a mechanism.

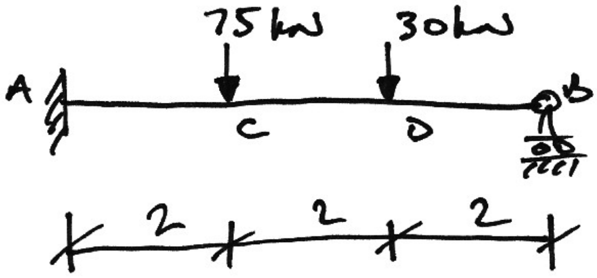
*3. Yield:*

From the collapse BMD it can be seen that nowhere is $M_P$ exceeded.

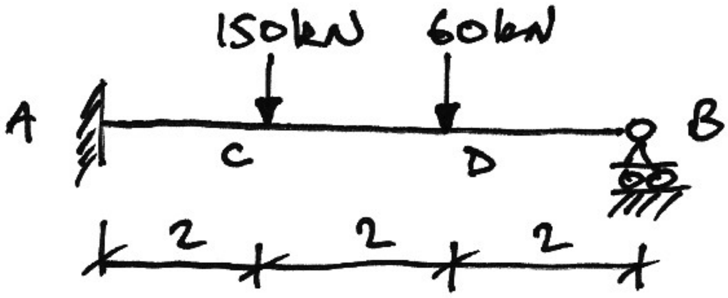
Thus the solution meets the three conditions and so, by the Uniqueness Theorem, is the correct solution.

### 3.2 Example 2 – Propped Cantilever with Two Point Loads

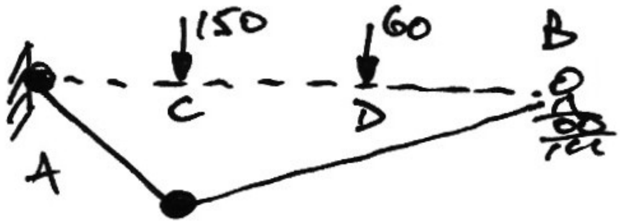
For the following beam, for a load factor of 2.0, find the required plastic moment capacity:

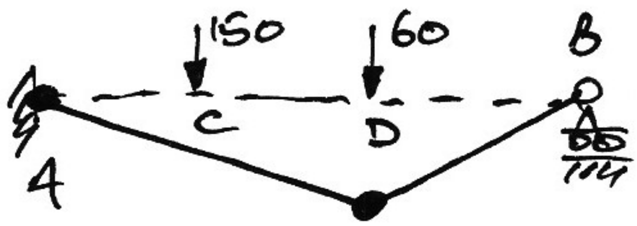



Allowing for the load factor, we need to design the beam for the following loads:




Once again we try to picture possible failure modes. Since maximum moments occur underneath point loads, there are two real possibilities:



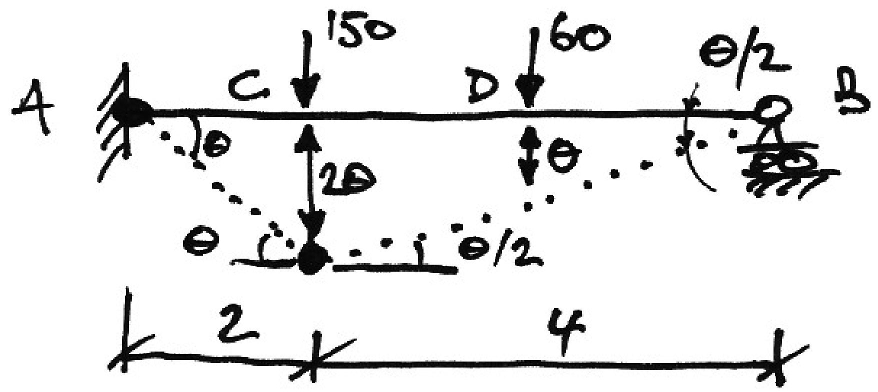




**Mode 1 – Plastic Hinge at *C***

**Mode 2– Plastic Hinge at *D***

Therefore, we analyse both and apply the Upperbound Theorem to find the design plastic moment capacity.

**Mode 1 – Plastic Hinge at *C*:**



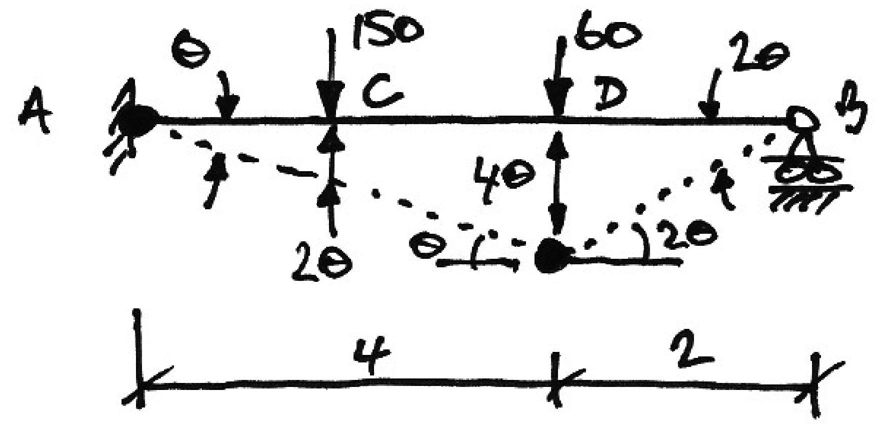

$$\delta W_e = \delta W_I$$

$$150(2\theta) + 60(\theta) = \underbrace{M_P(\theta)}_{\text{At } A} + \underbrace{M_P\left(\theta + \frac{\theta}{2}\right)}_{\text{At } C}$$

$$360\theta = \frac{5}{2} M_P \theta$$

$$M_P = 144 \text{ kNm}$$

**Mode 2 – Plastic Hinge at *D*:**

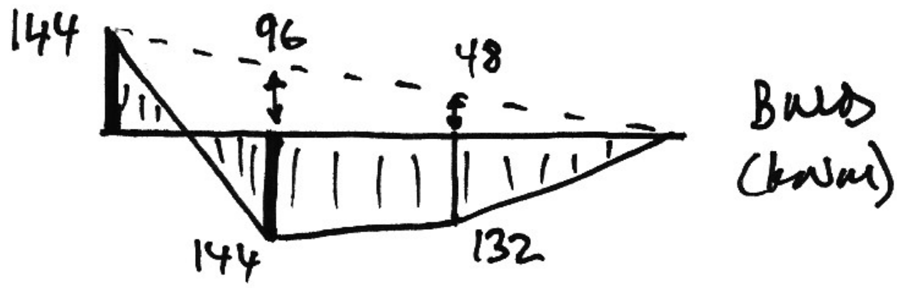


$$\delta W_e = \delta W_I$$
$$150(2\theta) + 60(4\theta) = \underbrace{M_P(\theta)}_{\text{At } A} + \underbrace{M_P(\theta + 2\theta)}_{\text{At } D}$$
$$540\theta = 4M_P\theta$$
$$M_P = 135 \text{ kNm}$$

So by the application of the Upperbound theorem for the design plastic capacity, we choose $M_P = 144$ kNm as the design moment and recognize Mode 1 to be the correct failure mode. We check this by the Uniqueness Theorem:

*1. Equilibrium:*

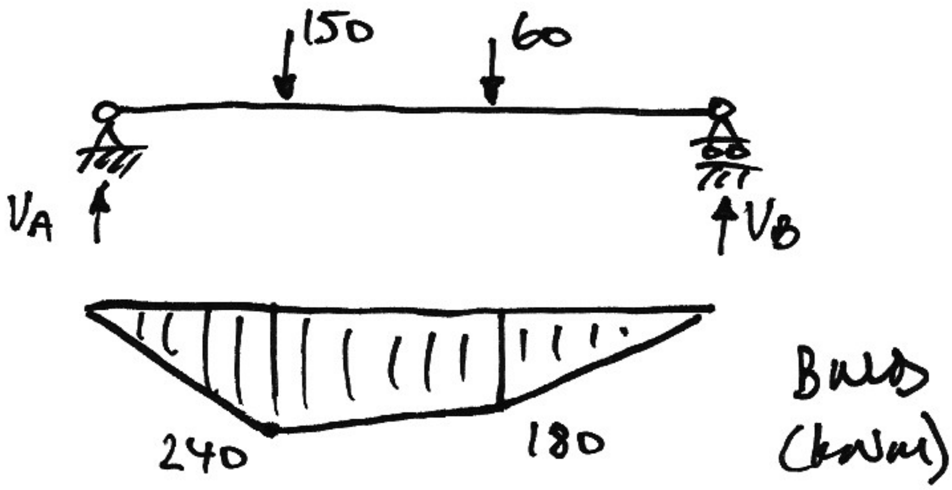
Using the BMD at collapse, we'll check that the height of the free BMD is that of the equivalent simply-supported beam. Firstly the collapse BMD from Mode 1 is:




Hence, the total heights of the free BMD are:

$$M_C = 96 + 144 = 240 \text{ kNm}$$
$$M_D = 48 + 132 = 180 \text{ kNm}$$

Checking these using a simply-supported beam analysis:




$$\sum M \text{ about } A = 0 \quad \therefore 150 \cdot 2 + 60 \cdot 4 - 6V_B = 0 \;\therefore V_B = 90 \text{ kN}$$
$$\sum F_y = 0 \quad \therefore 150 + 60 - 90 - V_A = 0 \quad \therefore V_A = 120 \text{ kN}$$

Thus, using appropriate free body diagrams of *AC* and *DB*:

$$M_C = 120 \cdot 2 = 240 \text{ kNm}$$
$$M_D = 90 \cdot 2 = 180 \text{ kNm}$$

And so the applied load is in equilibrium with the free BMD of the collapse BMD.

*2. Mechanism:*

From the proposed collapse mode it is apparent that the beam is a mechanism. Also, since it is a propped cantilever and thus one degree indeterminate, we require two plastic hinges for collapse, and these we have.

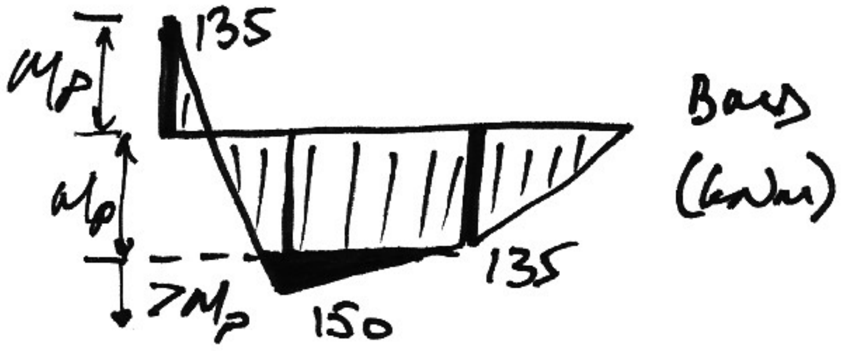
*3. Yield:*

From the collapse BMD it can be seen that nowhere is the design $M_P = 144$ kNm exceeded.

Thus by the Uniqueness Theorem we have the correct solution.

Lastly, we'll examine why the Mode 2 collapse is not the correct solution. Since the virtual work method provides an upperbound, then, by the Uniqueness Theorem, it must not be the correct solution because it must violate the yield condition.

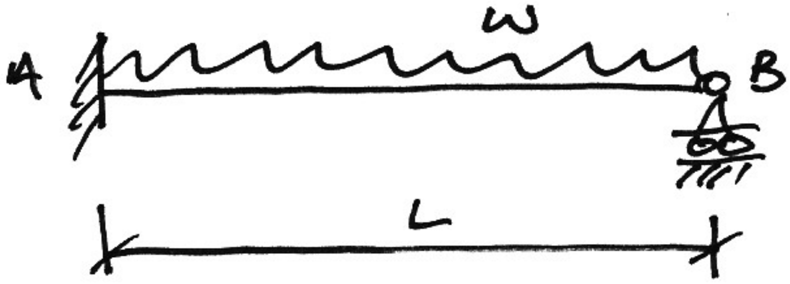
Using the collapse Mode 2 to determine reactions, we can draw the following BM<D for collapse Mode 2:



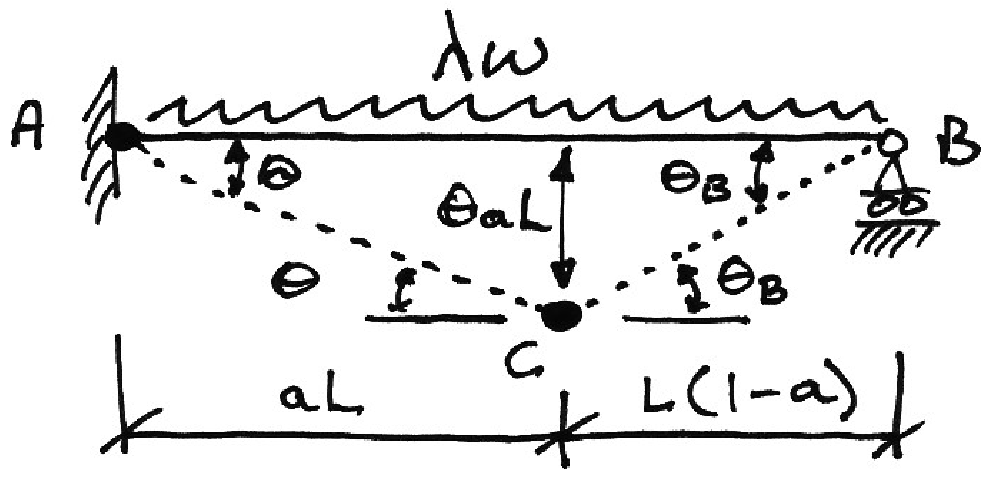

From this it is apparent that Mode 2 is not the unique solution, and so the design plastic moment capacity must be 144 kNm as implied previously from the Upperbound Theorem.

### 3.3 Example 3 – Propped Cantilever under UDL

For the general case of a propped cantilever, find the locations of the plastic hinges at collapse, and express the load at collapse in terms of the plastic moment capacity.




When considering UDLs, it is not readily apparent where the plastic hinge should be located in the span. For this case of a propped cantilever we require 2 hinges, one of which will occur at *A*, as should be obvious. However, we need to keep the location of the span hinge variable at say, *aL*, from *A*:




Using $S = R\theta$, we find the rotation at *B*:

$$\theta aL = L(1-a)\theta_B$$

And so:

$$\theta_B = \theta \cdot \frac{a}{(1-a)}$$

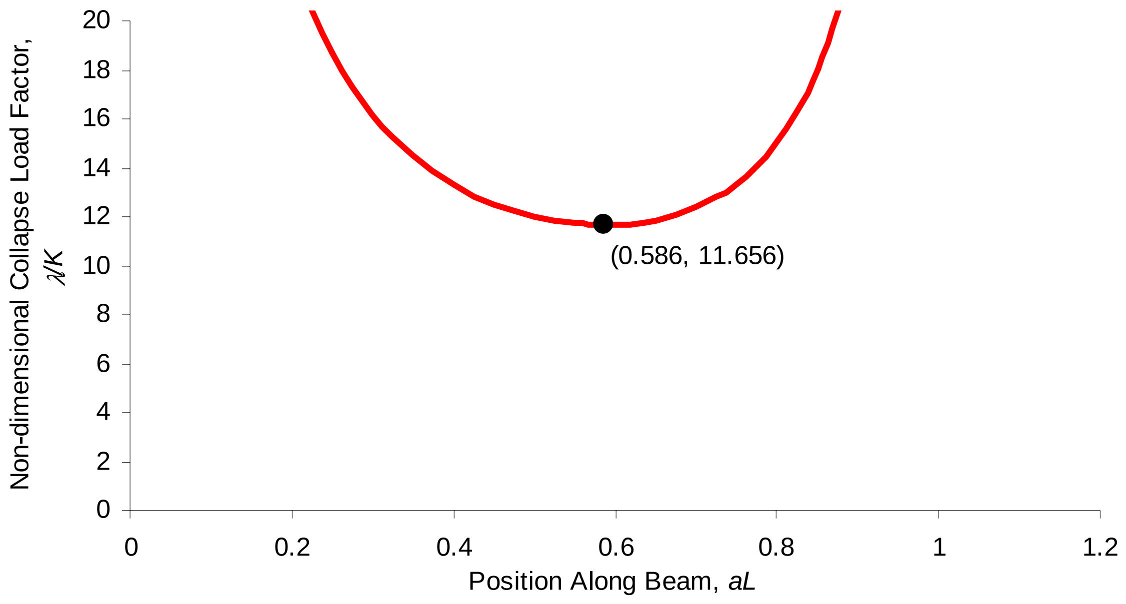
Thus, noting that the external work done by a UDL is the average distance it moves, we have:

$$\begin{aligned}
\delta W_e &= \delta W_I \\
(\lambda wL)\left(\frac{\theta aL}{2}\right) &= \underbrace{M_P(\theta)}_{\text{At } A} + \underbrace{M_P\left(\theta + \theta \cdot \frac{a}{1-a}\right)}_{\text{At } C} \\
\frac{\lambda waL^2}{2}\theta &= M_P\theta\left(2 + \frac{a}{1-a}\right) \\
\frac{\lambda waL^2}{2} &= M_P\left(\frac{2-a}{1-a}\right) \\
\lambda &= \frac{2M_P}{waL^2}\left(\frac{2-a}{1-a}\right)
\end{aligned}$$

If we introduce a non-dimensional quantity, $K \equiv M_P / wL^2$, wehave:

$$\lambda = K \cdot \frac{2}{a}\left(\frac{2-a}{1-a}\right)$$

Thus the collapse load factor is a function of the position of the hinge, *a*, as expected. Also, we can plot the function $\lambda / K$ against *a* to visualize where the minimum might occur:




To determine the critical collapse load factor, suing the Upperbound Theorem, we look for the minimum load factor using:

$$\frac{d\lambda}{da} = 0$$

To do this, we'll expand the fraction:

$$\lambda = K \cdot \frac{2}{a}\left(\frac{2-a}{1-a}\right) = K \cdot \frac{4-2a}{a-a^2}$$

Using the quotient rule for derivates:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d\lambda}{da} = \frac{\left(a-a^2\right)(-2) - (4-2a)(1-2a)}{\left(a-a^2\right)^2} = 0$$

Thus multiplying across by $\left(a-a^2\right)^2$ and simplifying gives:

$$-2a^2+8a-4=0$$

Thus:

$$\begin{aligned} a &= \frac{-8 \pm \sqrt{8^2-4(-2)(-4)}}{2(-2)} \\ &= 2 \pm \sqrt{2} \end{aligned}$$

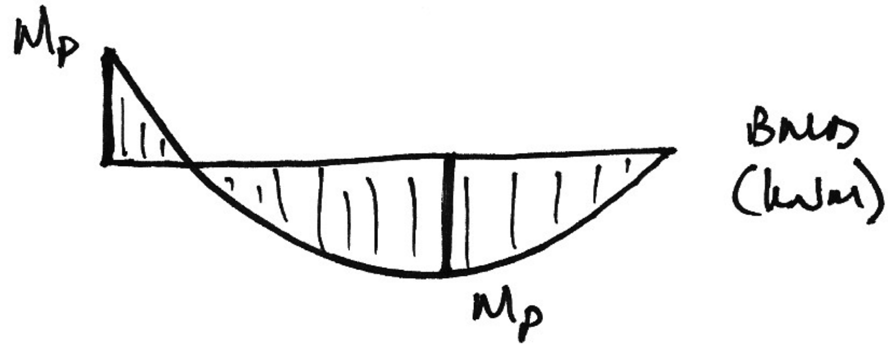
Since we know $0 \le a \le 1$, then:

$$a = 2-\sqrt{2} = 0.586$$

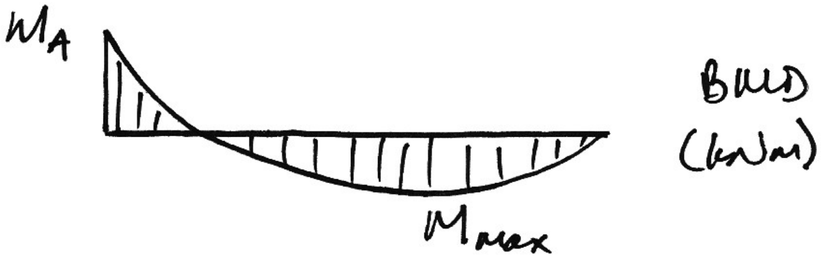
At this value for *a*, the collapse load factor is:

$$\begin{aligned} \lambda_C &= \frac{M_P}{wL^2} \cdot \frac{2}{0.586}\left(\frac{2-0586}{1-0.586}\right) \\ &= 11.656\frac{M_P}{wL^2} \end{aligned}$$

These values are shown in the graph previously. The collapse BMD is:



The propped cantilever is a good structure to illustrate the use of the Lowerbound Theorem. Consider the standard elastic BMD for this structure which meets the equilibrium condition:

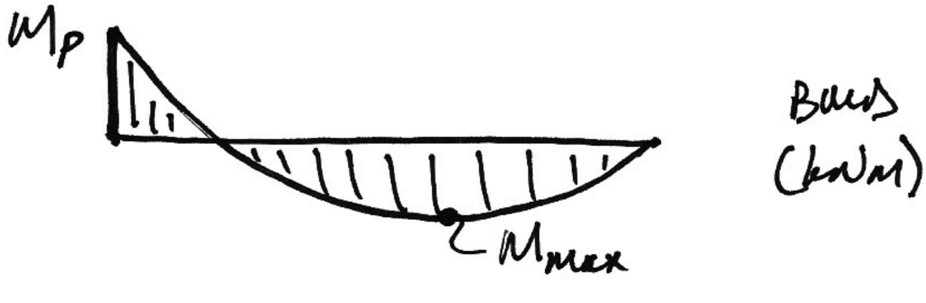



$$M_A = \frac{wL^2}{8} \qquad M_{max} = \frac{9wL^2}{128}$$

If we increase the load by a load factor $\lambda$ so that $M_A = M_P$, and since $M_{\max} < M_A$ we meet the yield condition, then we have:

$$M_P = \frac{\lambda wL^2}{8}$$

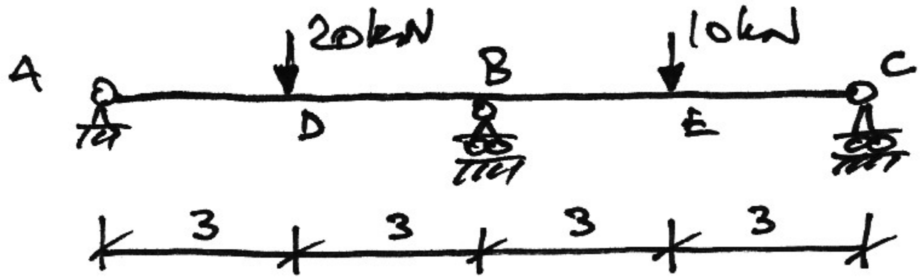
$$\lambda = 8\frac{M_P}{wL^2} < \lambda_C = 11.656\frac{M_P}{wL^2}$$



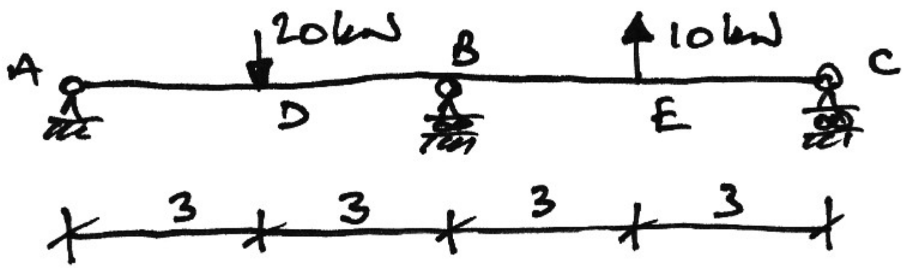

By meeting the equilibrium and yield conditions, but not the mechanism condition, we have a lowerbound on the critical load factor without doing the virtual work analysis. This is one of the main reasons elastic analyses are mostly used in practice.

### 3.4 Problems

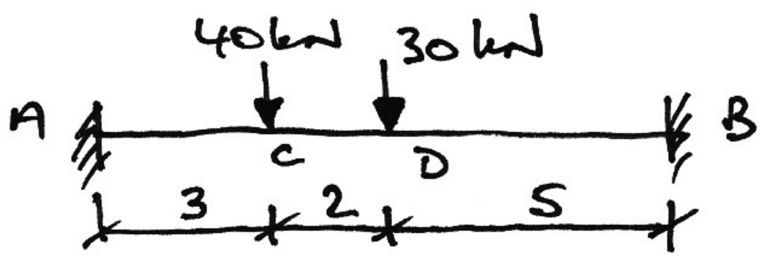
1. For the following prismatic beam of $M_P = 30$ kNm, find the load factor at collapse. (*Ans*. 1.5)




2. For the following prismatic beam of $M_P = 30$ kNm, find the load factor at collapse. (*Ans*. 1.33)




3. For the following prismatic beam of $M_P = 86$ kNm, find the load factor at collapse. (*Ans*. 1.27)



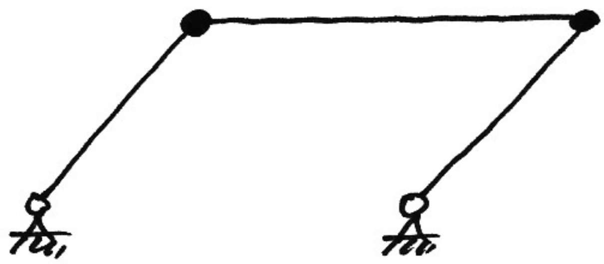
# 4. Frames

## *4.1 Collapse Mechanisms*

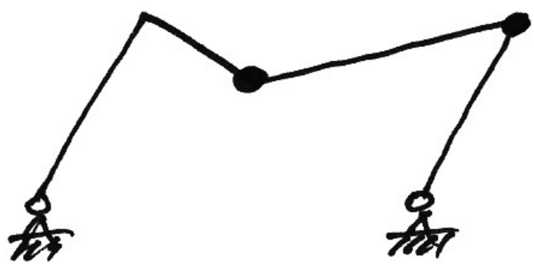
In frames, the basic modes of collapse are:

**Beam-type collapse:**

**Sway Collapse:**



**Combination Collapse:**

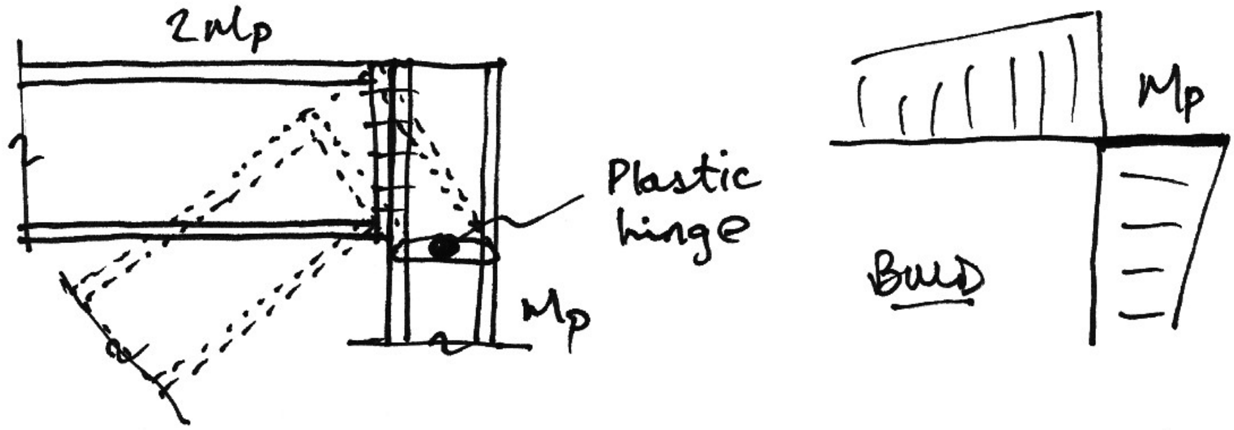


**Combination of Mechanisms**

One of the most powerful tools in plastic analysis is Combination of Mechanisms. This allows us to work out the virtual work equations for the beam and sway collapses separately and then combine them to find the collapse load factor for a combination collapse mode.

**Location of Plastic Hinge at Joints**

In frames where members of different capacities meet at joints, it is the weaker member that develops the plastic hinge. So, for example:

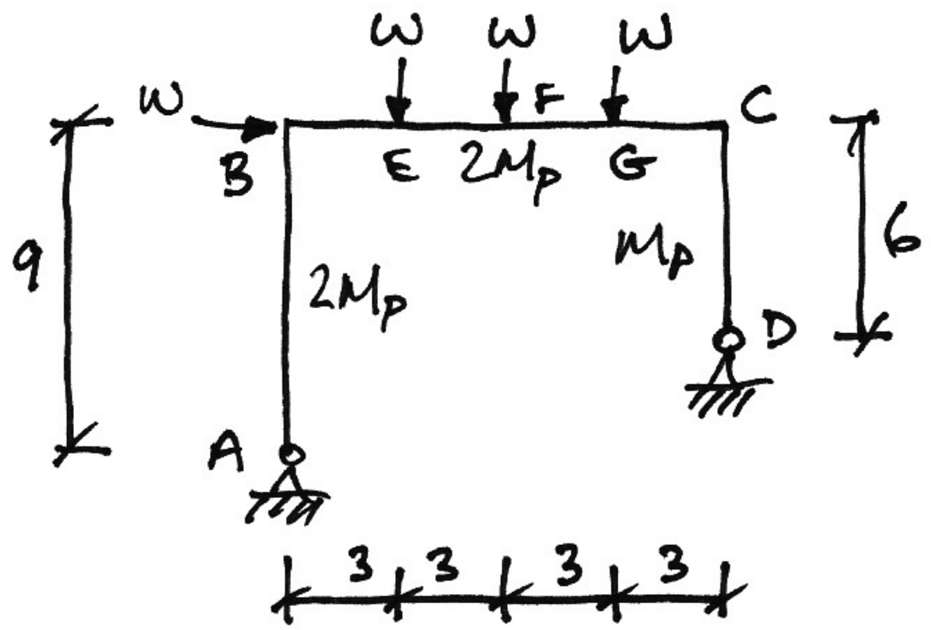



The plastic hinge occurs in the column and not in the beam section since the column section is weaker.

This is important when calculating the external virtual work done.

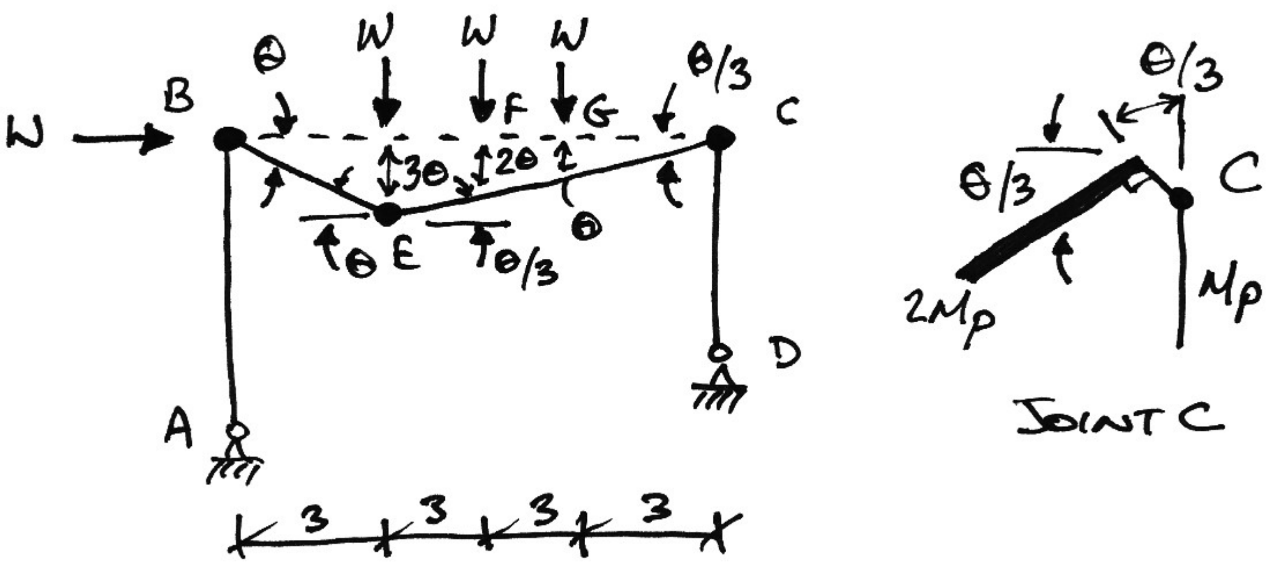
### *4.2 Example 4 – Frame*

Find the collapse load in terms of the plastic moment capacity:




Using the idea of Combination of Mechanisms, we will analyse the beam and sway modes separately, and then combine them in various ways to achieve a solution.

**Beam Collapse Mode:**



Notice that, as previously mentioned, we must take the plastic hinge at joint *C* to be in the column which has the smaller $M_P$. Applying the virtual work equation:

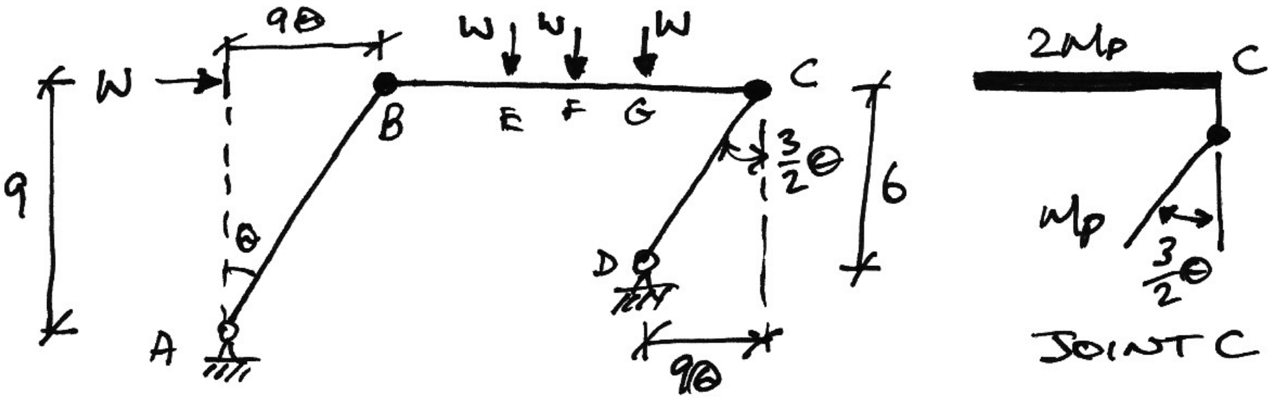
$$\delta W_e = \delta W_I$$

$$\underbrace{W(3\theta)}_{\text{At }E} + \underbrace{W(2\theta)}_{\text{At }F} + \underbrace{W(\theta)}_{\text{At }G} = \underbrace{2M_P(\theta)}_{\text{At }B} + \underbrace{2M_P\left(\frac{4}{3}\theta\right)}_{\text{At }E} + \underbrace{M_P\left(\frac{1}{3}\theta\right)}_{\text{At }C}$$

$$6W\theta = 5M_P\theta$$

$$W = \frac{5}{6}M_P$$

**Sway Collapse Mode:**




Again notice how careful we are of the hinge location at joint *C*.

$$\delta W_e = \delta W_I$$

$$\underbrace{W(9\theta)}_{\text{At }B} = \underbrace{2M_P(\theta)}_{\text{At }B} + \underbrace{M_P\left(\frac{3}{2}\theta\right)}_{\text{At }C}$$

$$9W\theta = 3.5M_P\theta$$

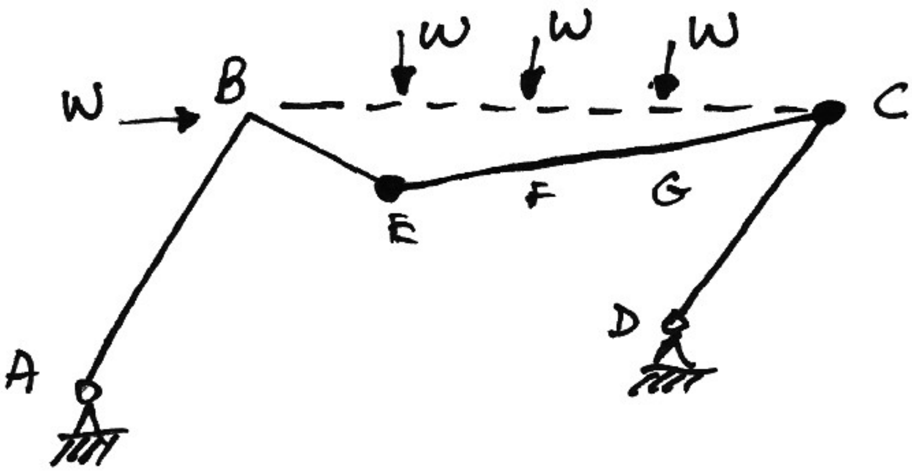
$$W = \frac{7}{18}M_P$$

**Combined Collapse Mechanism**

To arrive at a solution, we want to try to minimize the collapse load value. Examining the previous equations, this means that we should try to maximize the external work done and minimize the internal work done. So:

- To maximize the external work done we need to make every load move through some displacement, unlike the sway mechanism;
- To minimize the internal work done we try to remove a hinge, whilst maintaining a mechanism.

Based on the above try the following:




Instead of using virtual work, we can combine the equations already found:

- External virtual work: Since all forces move through displacements:

$$\delta W_e = \underbrace{6W\theta}_{\text{Beam}} + \underbrace{9W\theta}_{\text{Sway}} = 15W\theta$$

- Internal virtual work: we can add but we must remove the work done by the hinge at *B* for both the beam and sway modes:

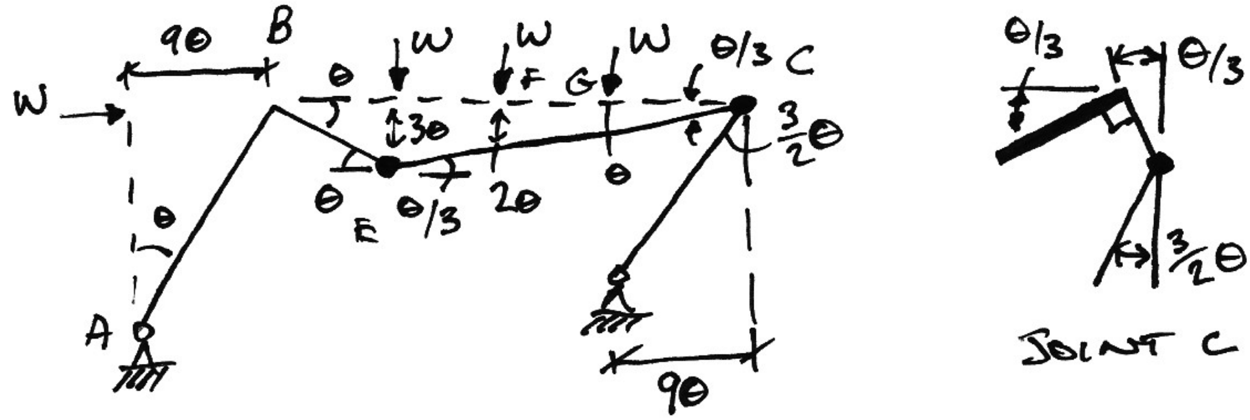
$$\delta W_I = \underbrace{5M_P\theta}_{\text{Beam}} + \underbrace{3.5M_P\theta}_{\text{Sway}} - \underbrace{2M_P\theta}_{\text{Hinge } B \text{ - Beam}} - \underbrace{2M_P\theta}_{\text{Hinge } B \text{ - Sway}} = 4.5M_P\theta$$

Thus we have:

$$\delta W_e = \delta W_I$$
$$15W\theta = 4.5M_P\theta$$
$$W = \frac{3}{10}M_P$$

Since this is lower than either of the previous mechanisms, we think this is the solution, and so check against the three conditions of the Uniqueness Theorem.

To prove that the combination of mechanisms works, we do the virtual work analysis:

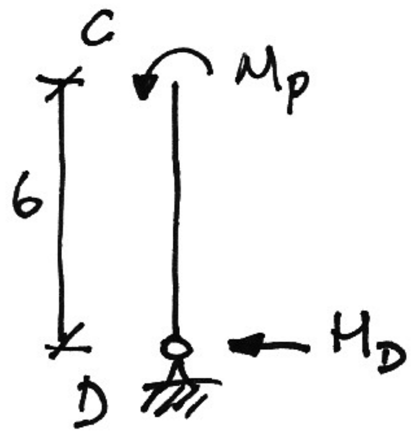



$$\delta W_e = \delta W_I$$
$$\underbrace{W(9\theta)}_{\text{At } B} + \underbrace{W(3\theta)}_{\text{At } E} + \underbrace{W(2\theta)}_{\text{At } F} + \underbrace{W(\theta)}_{\text{At } G} = \underbrace{2M_P\left(\frac{4}{3}\theta\right)}_{\text{At } E} + \underbrace{M_P\left(\frac{3}{2}\theta + \frac{1}{3}\theta\right)}_{\text{At } C}$$
$$15W\theta = 4.5M_P\theta$$
$$W = \frac{3}{10}M_P$$

Check for the three conditions, recognizing that $M_P = \frac{W}{0.3} = 3.33W$

*1. Equilibrium:*

We start by determining the reactions:




$$\sum M \text{ about } C = 0 \quad \therefore 6H_D - M_P = 0$$

$$\therefore H_D = \frac{M_P}{6} = \frac{3.33W}{6} = 0.55W$$

$$\sum F_x = 0 \qquad \therefore H_A = W - 0.55W = 0.45W$$

For the whole frame:

$$\sum M \text{ about } D = 0 \quad \therefore 12V_A + 3H_A + 6W - 9W - 6W - 3W = 0 \qquad \therefore V_A = 0.89W$$

Thus the moment at *E*, from a free-body diagram of *ABE*, is:

$$\sum M \text{ about } E = 0 \quad \therefore 3V_A + 9H_A - M_E = 0 \qquad \therefore M_E = 6.71W$$

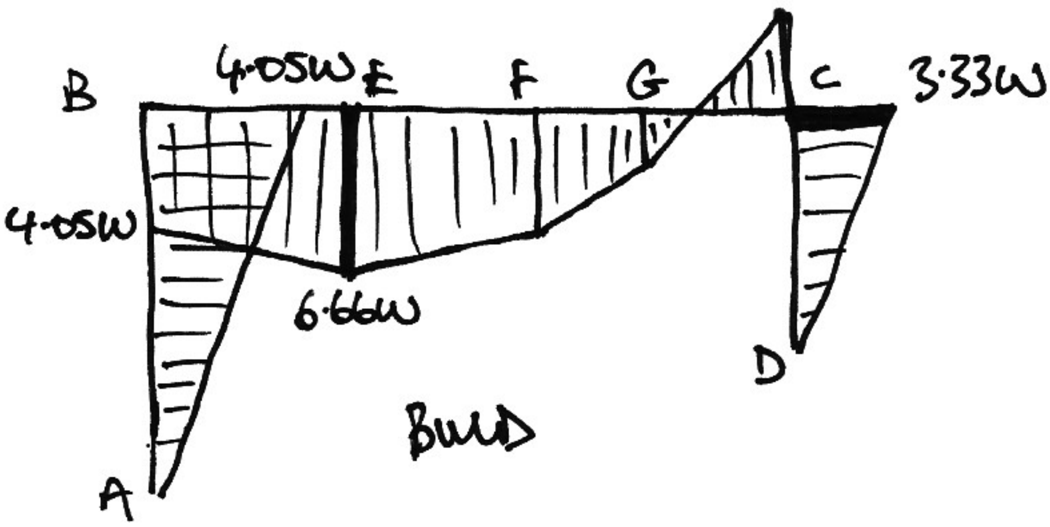
Since there is a plastic hinge at *E* of value $2M_P = 2 \cdot (3.33W) = 6.67W$ we have equilibrium.

*2. Mechanism:*

The frame is obviously a mechanism since $R - C - 3 = 4 - 2 - 3 = -1$.

*3. Yield:*

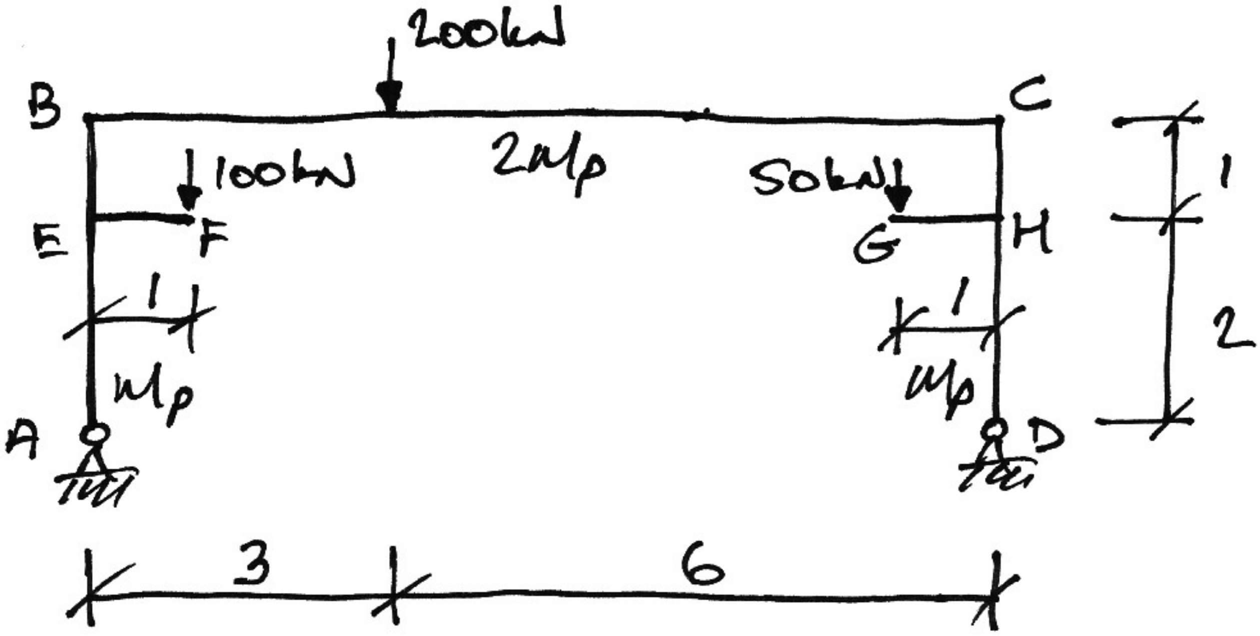
To verify yield we draw the collapse BMD from the reactions:



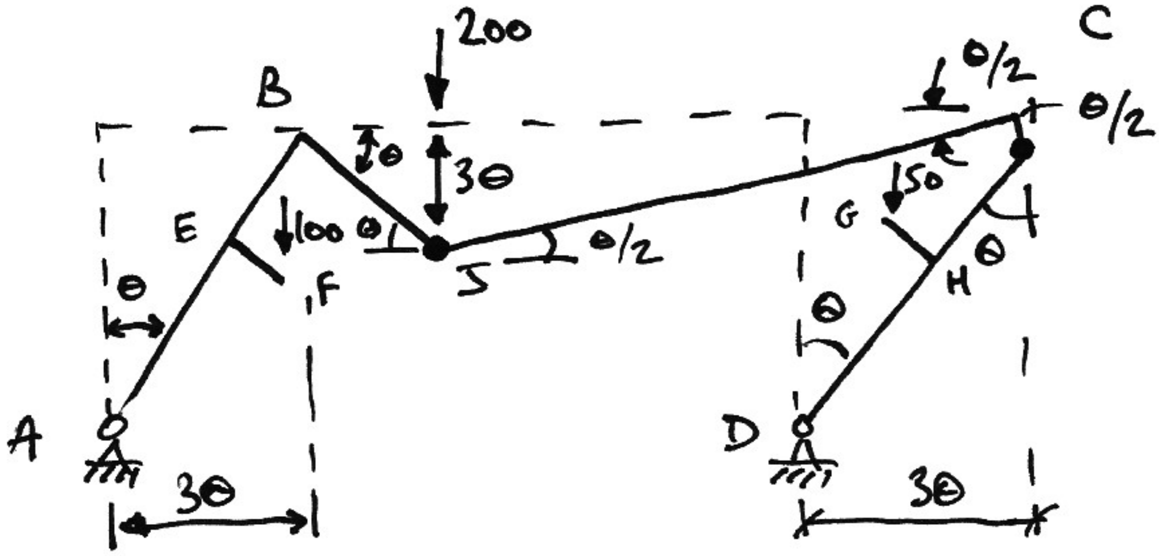

From the diagram we see that there are no moments greater than $2M_P = 6.67W$ in members *AB* and *BC*, and no moments greater than $M_P = 3.33W$ in member *CD*.

### 4.3 Example 5 – Frame, Summer 1997

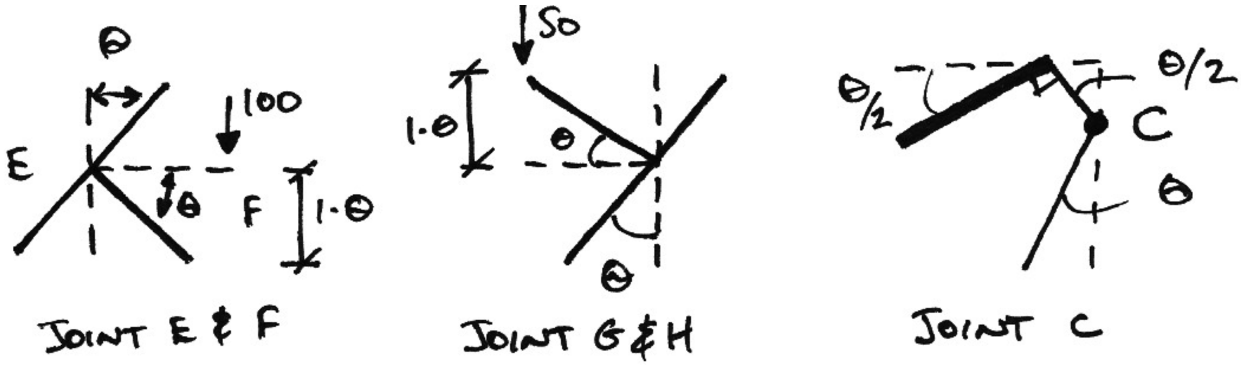
For the following frame, find the plastic moment capacity required for collapse under the loads given.




The structure is 1 degree indeterminate so the number of plastic hinges required is 2. We propose the following collapse mechanism:



Also, looking closely at the relevant joints:




Thus we have:

$$\delta W_e = \delta W_I$$

$$\underbrace{200(3\theta)}_{\text{At } J} + \underbrace{100(\theta)}_{\text{At } F} - \underbrace{50(\theta)}_{\text{At } G} = \underbrace{2M_P\left(\frac{3}{2}\theta\right)}_{\text{At } J} + \underbrace{M_P\left(\frac{3}{2}\theta\right)}_{\text{At } C}$$

$$650\theta = 4.5M_P\theta$$

$$M_P = 144.44 \text{ kNm}$$

Notice that the 50 kN point load at $G$ does negative external work since it moves against its direction of action.

Note also that there are other mechanisms that could be tried, some of which are unreasonable.

Next we check this solution to see if it is unique:

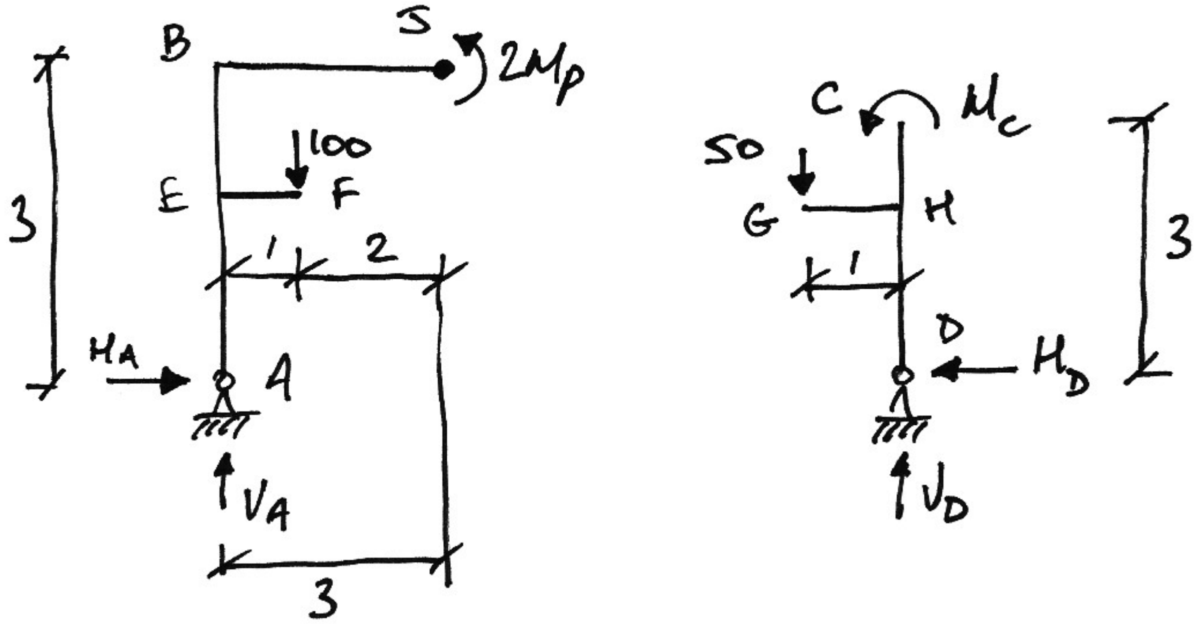
*1. Equilibrium:*

For the whole frame, taking moments about *D* gives:

$$50 \cdot 1 + 200 \cdot 6 + 100 \cdot 8 - 9V_A = 0 \qquad \therefore V_A = 227.8 \text{ kN}$$

Using a free body diagram of *ABJ*, and taking moments about the plastic hinge at *J*:

$$2 \cdot 144.4 + 100 \cdot 2 - 3 \cdot 227.8 - 3H_A = 0 \quad \therefore H_A = 64.9 \text{ kN}$$




So for the whole frame:

$$\sum F_x = 0 \qquad \therefore H_A - H_D = 0 \qquad \therefore H_D = 64.9 \text{ kN}$$

Thus for the free body diagram of *CD*, taking moments about *C*:

$$M_C - 50 \cdot 1 - 3H_D = 0 \qquad \therefore M_C = 144.7 \text{ kNm}$$

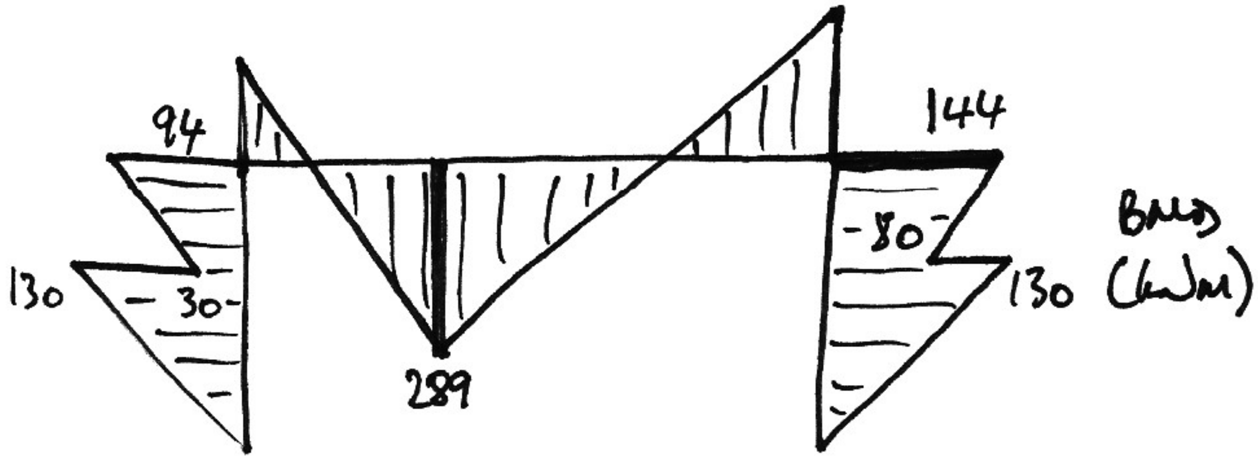
Since this is the value of $M_P$ we have a plastic hinge at *C* as expected. Thus the loads are in equilibrium with the collapse mode.

*2. Mechanism:*

Since $R - C - 3 = 4 - 2 - 3 = -1$ we have a mechanism.

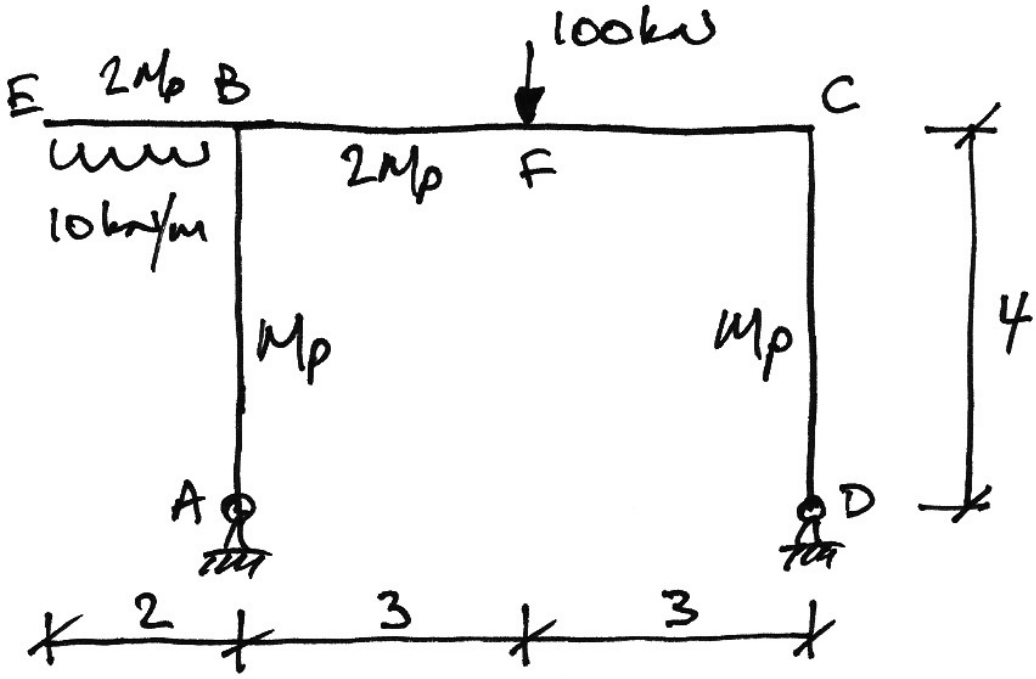
*3. Yield:*

Drawing the bending moment diagram at collapse shows that no section has a moment greater than its moment capacity of either $M_P$ or $2M_P$:



### 4.4 Example 6 – Frame, Sumer 2000

For the following frame, find the collapse load factor when $M_P = 120\ \text{kNm}$:




*To be done in class.*

### 4.5 Problems

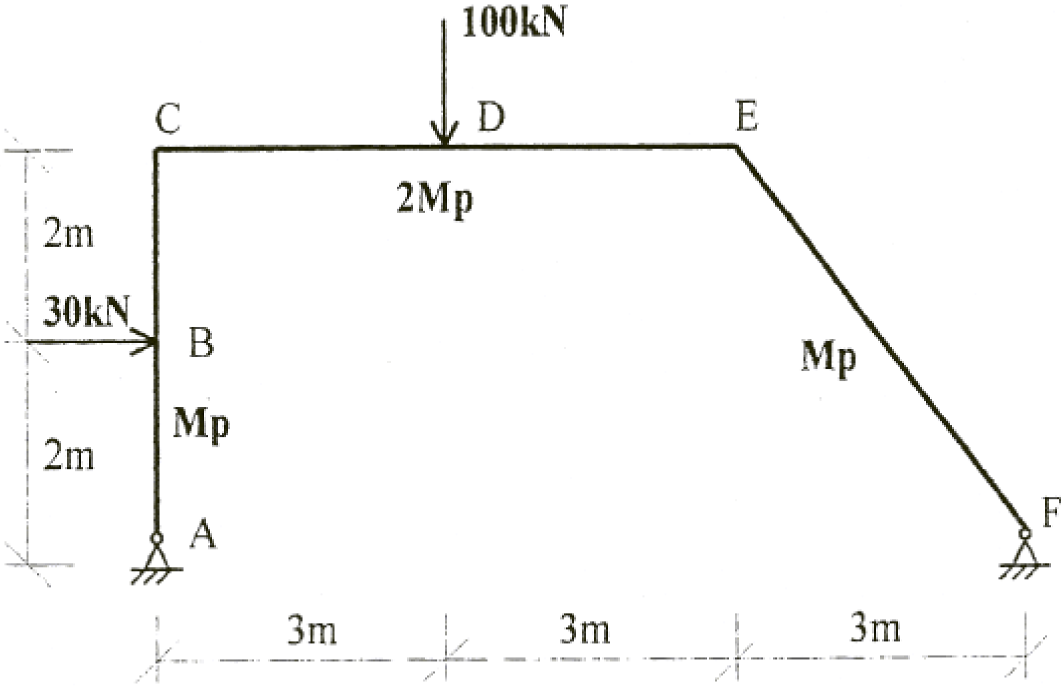
1. (Summer 1999) The following rigid-jointed frame is loaded with working loads as shown:

    (a) Find the value of the collapse load factor when $M_P = 120$ kNm;

    (b) Show that your solution is the unique solution;

    (c) Sketch the bending moment diagram at collapse, showing all important values.

(*Ans.* $\lambda = 2.0$)



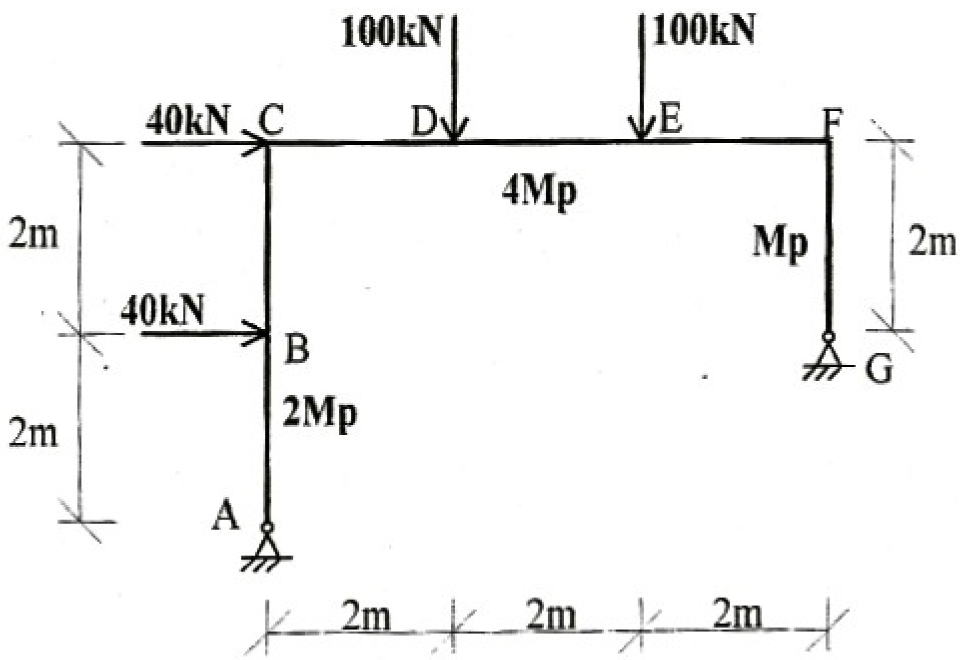
2. (Summer 2001) The following rigid-jointed frame is loaded with working loads as shown:

   (a) Find the value of the collapse load factor when $M_P = 120$ kNm;

   (b) Show that your solution is the unique solution;

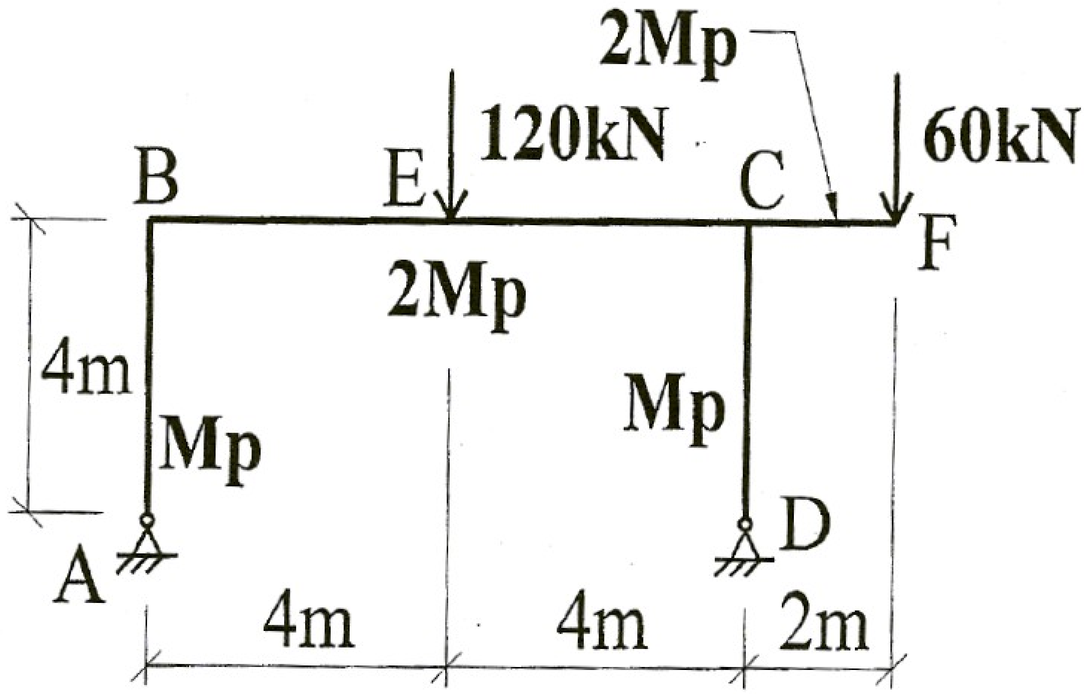
   (c) Sketch the bending moment diagram at collapse, showing all important values.

(*Ans.* $\lambda = 1.89$)



3. (Summer 2004) The following rigid-jointed frame is loaded with working loads as shown:
   (a) Find the value of the collapse load factor when $M_P = 160$ kNm;
   (b) Show that your solution is the unique solution;
   (c) Sketch the bending moment diagram at collapse, showing all important values.

(*Ans.* $\lambda = 2.13$)



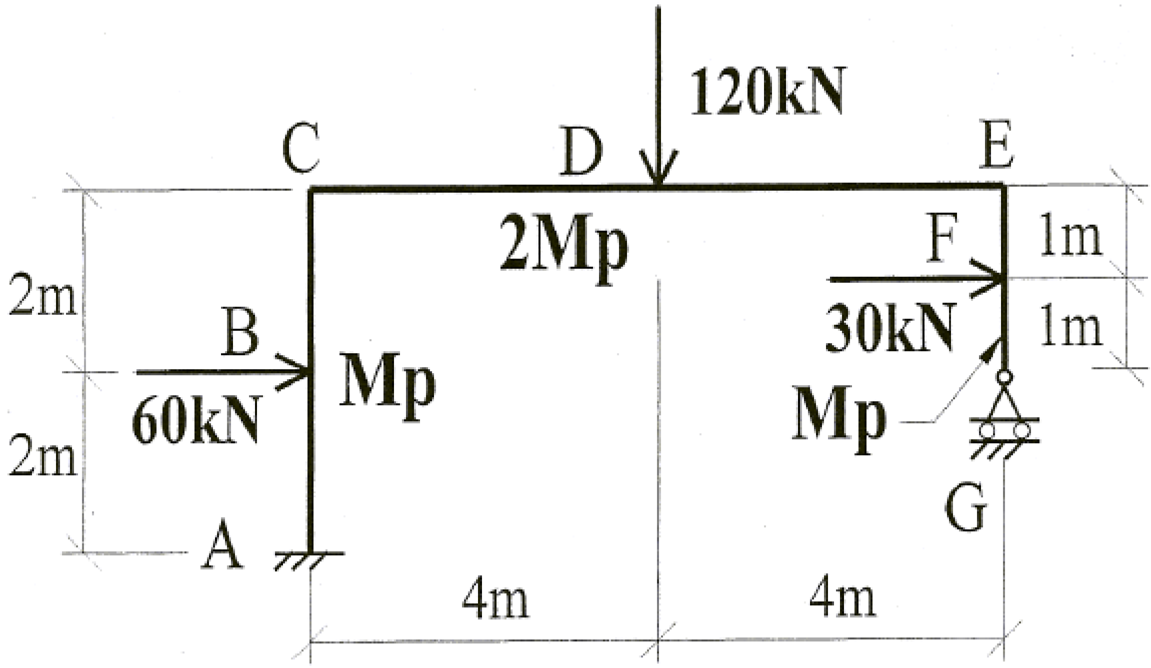
4. (Summer 2005) The following rigid-jointed frame is loaded with working loads as shown:

   (a) Find the value of the collapse load factor when $M_P = 200 \text{ kNm}$;

   (b) Show that your solution is the unique solution;

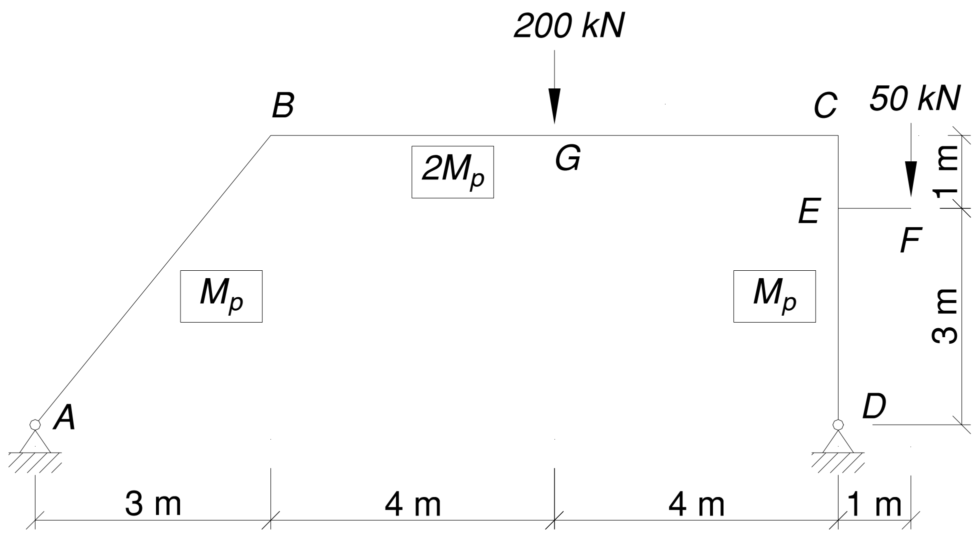
   (c) Sketch the bending moment diagram at collapse, showing all important values.

(*Ans.* $\lambda = 1.33$)



5. (Summer 2007) The following rigid-jointed frame is loaded so that the force system shown is just sufficient to cause collapse in the main frame ABCD:
    (a) Find the value of $M_P$ given that the relative plastic moment capacities are as shown in the figure;
    (b) Show that your solution is the unique solution;
    (c) Sketch the bending moment diagram at collapse, showing all important values.

($Ans.\ M_P = 175.8$ kNm)



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