CHAPTER 10

Method of Virtual Work

Chapter 10 Method of Virtual Work

10.1 Introduction
10.2 Work of a Force
10.3 Principle of Virtual Work
10.4 Applications of the Principle of Virtual Work
10.5 Real Machines. Mechanical Efficiency
10.6 Work of a Force during a Finite Displacement
10.7 Potential Energy
10.8 Potential Energy and Equilibrium
10.9 Stability of Equilibrium

*10.1 INTRODUCTION

In the preceding chapters, problems involving the equilibrium of rigid bodies were solved by expressing that the external forces acting on the bodies were balanced. The equations of equilibrium $\Sigma F_x = 0$, $\Sigma F_y = 0$, $\Sigma M_A = 0$ were written and solved for the desired unknowns. A different method, which will prove more effective for solving certain types of equilibrium problems, will now be considered. This method is based on the *principle of virtual work* and was first formally used by the Swiss mathematician Jean Bernoulli in the eighteenth century.

As you will see in Sec. 10.3, the principle of virtual work states that if a particle or rigid body, or, more generally, a system of connected rigid bodies, which is in equilibrium under various external forces, is given an arbitrary displacement from that position of equilibrium, the total work done by the external forces during the displacement is zero. This principle is particularly effective when applied to the solution of problems involving the equilibrium of machines or mechanisms consisting of several connected members.

In the second part of the chapter, the method of virtual work will be applied in an alternative form based on the concept of *potential energy*. It will be shown in Sec. 10.8 that if a particle, rigid body, or system of rigid bodies is in equilibrium, then the derivative of its potential energy with respect to a variable defining its position must be zero.

In this chapter, you will also learn to evaluate the mechanical efficiency of a machine (Sec. 10.5) and to determine whether a given position of equilibrium is stable, unstable, or neutral (Sec. 10.9).

*10.2 WORK OF A FORCE

Let us first define the terms *displacement* and *work* as they are used in mechanics. Consider a particle which moves from a point A to a neighboring point A' (Fig. 10.1). If $\mathbf{r}$ denotes the position vector corresponding to point A, the small vector joining A and A' may be denoted by the differential $d\mathbf{r}$; the vector $d\mathbf{r}$ is called the *displacement* of the particle. Now let us assume that a force $\mathbf{F}$ is acting on the particle. The *work of the force* $\mathbf{F}$ *corresponding to the displacement* $d\mathbf{r}$ is defined as the quantity

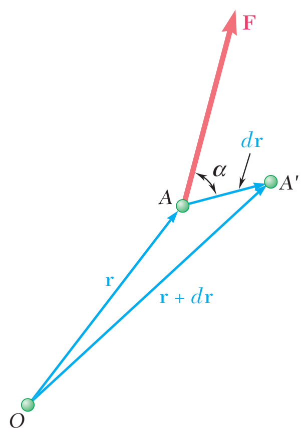


Fig. 10.1

$$dU = \mathbf{F} \cdot d\mathbf{r} \tag{10.1}$$

obtained by forming the scalar product of the force $\mathbf{F}$ and the displacement $d\mathbf{r}$. Denoting respectively by F and ds the magnitudes of the force and of the displacement, and by α the angle formed by $\mathbf{F}$ and $d\mathbf{r}$, and recalling the definition of the scalar product of two vectors (Sec. 3.9), we write

$$dU = F\, ds \cos \alpha \tag{10.1'}$$

Being a *scalar quantity*, work has a magnitude and a sign, but no direction. We also note that work should be expressed in units obtained by multiplying units of length by units of force. Thus, if U.S. customary units are used, work should be expressed in ft · lb or in · lb. If SI units are used, work should be expressed in N · m. The unit of work N · m is called a *joule* (J).†

It follows from (10.1′) that the work dU is positive if the angle α is acute and negative if α is obtuse. Three particular cases are of special interest. If the force $\mathbf{F}$ has the same direction as $d\mathbf{r}$, the work dU reduces to $F\,ds$. If $\mathbf{F}$ has a direction opposite to that of $d\mathbf{r}$, the work is $dU = -F\,ds$. Finally, if $\mathbf{F}$ is perpendicular to $d\mathbf{r}$, the work dU is zero.

The work dU of a force $\mathbf{F}$ during a displacement $d\mathbf{r}$ can also be considered as the product of F and the component $ds \cos \alpha$ of the displacement $d\mathbf{r}$ along $\mathbf{F}$ (Fig. 10.2*a*). This view is particularly

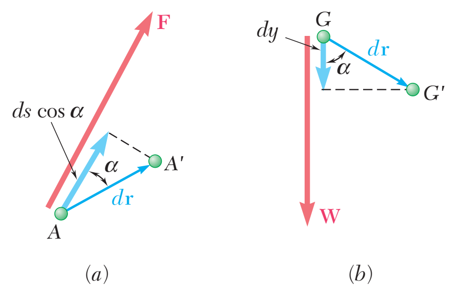


Fig. 10.2

Photo 10.1 The forces exerted by the hydraulic cylinders to position the bucket lift shown can be effectively determined using the method of virtual work since a simple relation exists among the displacements of the points of application of the forces acting on the members of the lift.

useful in the computation of the work done by the weight $\mathbf{W}$ of a body (Fig. 10.2*b*). The work of $\mathbf{W}$ is equal to the product of W and the vertical displacement dy of the center of gravity G of the body. If the displacement is downward, the work is positive; if it is upward, the work is negative.

A number of forces frequently encountered in statics *do no work:* forces applied to fixed points ($ds = 0$) or acting in a direction perpendicular to the displacement ($\cos \alpha = 0$). Among these forces are the reaction at a frictionless pin when the body supported rotates about the pin; the reaction at a frictionless surface when the body in contact moves along the surface; the reaction at a roller moving along its track; the weight of a body when its center of gravity moves horizontally; and the friction force acting on a wheel rolling without slipping (since at any instant the point of contact does not move). Examples of forces which *do work* are the weight of a body (except in the case considered above), the friction force acting on a body sliding on a rough surface, and most forces applied on a moving body.

†The joule is the SI unit of *energy,* whether in mechanical form (work, potential energy, kinetic energy) or in chemical, electrical, or thermal form. We should note that even though N · m = J, the moment of a force must be expressed in N · m, and not in joules, since the moment of a force is not a form of energy.

In certain cases, the sum of the work done by several forces is zero. Consider, for example, two rigid bodies AC and BC connected at C by a *frictionless pin* (Fig. 10.3*a*). Among the forces acting on AC is the force $\mathbf{F}$ exerted at C by BC. In general, the work of this force will not be zero, but it will be equal in magnitude and opposite in sign to the work of the force $-\mathbf{F}$ exerted by AC on BC, since these forces are equal and opposite and are applied to the same particle. Thus, when the total work done by all the forces acting on AB and BC is considered, the work of the two internal forces at C cancels out. A similar result is obtained if we consider a system consisting of two blocks connected by an *inextensible cord* AB (Fig. 10.3*b*). The work of the tension force $\mathbf{T}$ at A is equal in magnitude to the work of the tension force $\mathbf{T}'$ at B, since these forces have the same magnitude and the points A and B move through the same distance; but in one case the work is positive, and in the other it is negative. Thus, the work of the internal forces again cancels out.

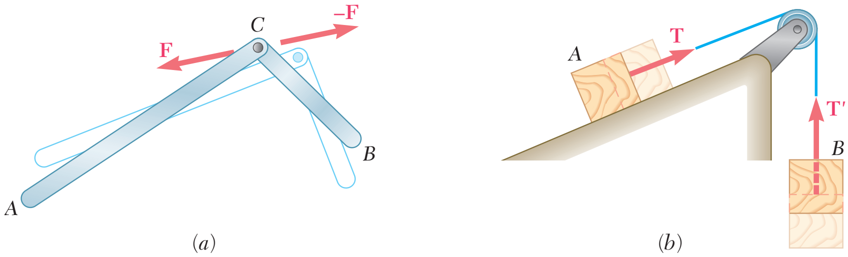


Fig. 10.3

It can be shown that the total work of the internal forces holding together the particles of a rigid body is zero. Consider two particles A and B of a rigid body and the two equal and opposite forces $\mathbf{F}$ and $-\mathbf{F}$ they exert on each other (Fig. 10.4). While, in general, small displacements $d\mathbf{r}$ and $d\mathbf{r}'$ of the two particles are different, the components of these displacements along AB must be equal; otherwise, the particles would not remain at the same distance from each other, and the body would not be rigid. Therefore, the work of $\mathbf{F}$ is equal in magnitude and opposite in sign to the work of $-\mathbf{F}$, and their sum is zero.

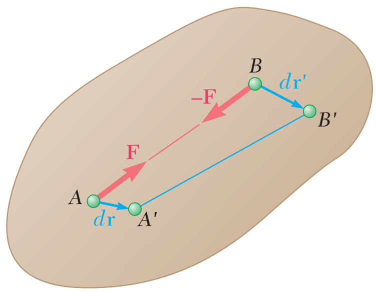


Fig. 10.4

In computing the work of the external forces acting on a rigid body, it is often convenient to determine the work of a couple without considering separately the work of each of the two forces forming the couple. Consider the two forces $\mathbf{F}$ and $-\mathbf{F}$ forming a couple of moment $\mathbf{M}$ and acting on a rigid body (Fig. 10.5). Any small displacement of the rigid body bringing A and B, respectively, into A' and B'' can be divided into two parts, one in which points A and B undergo equal displacements $d\mathbf{r}_1$, the other in which A' remains fixed while B' moves into B'' through a displacement $d\mathbf{r}_2$ of magnitude $ds_2 = r\,d\theta$. In the first part of the motion, the work of $\mathbf{F}$ is equal in magnitude and opposite in sign to the work of $-\mathbf{F}$, and their sum is zero. In the second part of the motion, only force $\mathbf{F}$ works, and its work is $dU = F\,ds_2 = Fr\,d\theta$. But the product Fr is equal to the magnitude M of the moment of the couple. Thus, the work of a couple of moment $\mathbf{M}$ acting on a rigid body is

$$dU = M\,d\theta \tag{10.2}$$

where $d\theta$ is the small angle expressed in radians through which the body rotates. We again note that work should be expressed in units obtained by multiplying units of force by units of length.

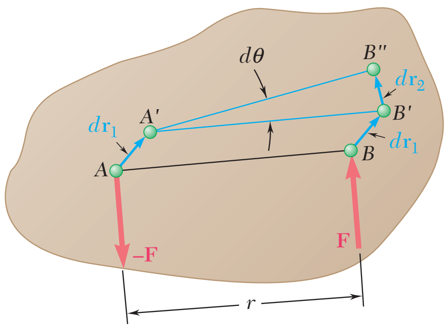


Fig. 10.5

*10.3 PRINCIPLE OF VIRTUAL WORK

Consider a particle acted upon by several forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ (Fig. 10.6). We can imagine that the particle undergoes a small displacement from A to A'. This displacement is possible, but it will not necessarily take place. The forces may be balanced and the particle at rest, or the particle may move under the action of the given forces in a direction different from that of AA'. Since the displacement considered does not actually occur, it is called a *virtual displacement* and is denoted by $\delta\mathbf{r}$. The symbol $\delta\mathbf{r}$ represents a differential of the first order; it is used to distinguish the virtual displacement from the displacement $d\mathbf{r}$ which would take place under actual motion. As you will see, virtual displacements can be used to determine whether the conditions of equilibrium of a particle are satisfied.

The work of each of the forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ during the virtual displacement $\delta\mathbf{r}$ is called *virtual work*. The virtual work of all the forces acting on the particle of Fig. 10.6 is

$$\begin{aligned}\delta U &= \mathbf{F}_1 \cdot \delta\mathbf{r} + \mathbf{F}_2 \cdot \delta\mathbf{r} + \cdots + \mathbf{F}_n \cdot \delta\mathbf{r} \\ &= (\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n) \cdot \delta\mathbf{r}\end{aligned}$$

or

$$\delta U = \mathbf{R} \cdot \delta\mathbf{r} \tag{10.3}$$

where $\mathbf{R}$ is the resultant of the given forces. Thus, the total virtual work of the forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ is equal to the virtual work of their resultant $\mathbf{R}$.

The principle of virtual work for a particle states that *if a particle is in equilibrium, the total virtual work of the forces acting on the particle is zero for any virtual displacement of the particle.* This condition is necessary: if the particle is in equilibrium, the resultant $\mathbf{R}$ of the forces is zero, and it follows from (10.3) that the total virtual work δU is zero. The condition is also sufficient: if the total virtual work δU is zero for any virtual displacement, the scalar product $\mathbf{R} \cdot \delta\mathbf{r}$ is zero for any $\delta\mathbf{r}$, and the resultant $\mathbf{R}$ must be zero.

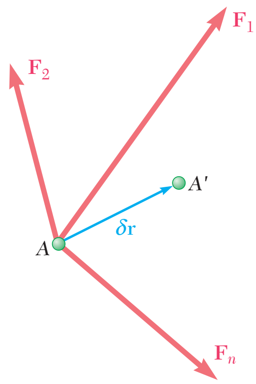


Fig. 10.6

In the case of a rigid body, the principle of virtual work states that *if a rigid body is in equilibrium, the total virtual work of the external forces acting on the rigid body is zero for any virtual displacement of the body.* The condition is necessary: if the body is in equilibrium, all the particles forming the body are in equilibrium and the total virtual work of the forces acting on all the particles must be zero; but we have seen in the preceding section that the total work of the internal forces is zero; the total work of the external forces must therefore also be zero. The condition can also be proved to be sufficient.

The principle of virtual work can be extended to the case of a *system of connected rigid bodies.* If the system remains connected during the virtual displacement, *only the work of the forces external to the system need be considered,* since the total work of the internal forces at the various connections is zero.

*10.4 APPLICATIONS OF THE PRINCIPLE OF VIRTUAL WORK

The principle of virtual work is particularly effective when applied to the solution of problems involving machines or mechanisms consisting of several connected rigid bodies. Consider, for instance, the toggle vise ACB of Fig. 10.7a, used to compress a wooden block. We

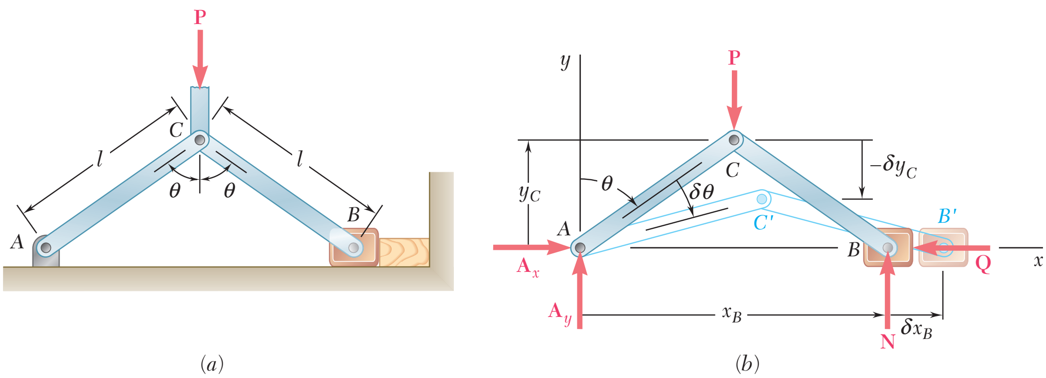


Fig. 10.7

wish to determine the force exerted by the vise on the block when a given force **P** is applied at C, assuming that there is no friction. Denoting by **Q** the reaction of the block on the vise, we draw the free-body diagram of the vise and consider the virtual displacement obtained by giving a positive increment $\delta\theta$ to the angle θ (Fig. 10.7b). Choosing a system of coordinate axes with origin at A, we note that x_B increases while y_C decreases. This is indicated in the figure, where a positive increment δx_B and a negative increment $-\delta y_C$ are shown. The reactions $\mathbf{A}_x$, $\mathbf{A}_y$, and **N** will do no work during the virtual displacement considered, and we need only compute the work of **P** and **Q**. Since **Q** and δx_B have opposite senses, the virtual work of **Q** is $\delta U_Q = -Q\,\delta x_B$. Since **P** and the increment shown $(-\delta y_C)$ have the same sense, the virtual work of **P** is $\delta U_P = +P(-\delta y_C) = -P\,\delta y_C$. The minus signs obtained could have been predicted by simply noting that the forces **Q** and **P** are directed opposite to the positive

x and y axes, respectively. Expressing the coordinates x_B and y_C in terms of the angle θ and differentiating, we obtain

$$\begin{aligned} x_B &= 2l \sin\theta & y_C &= l\cos\theta \\ \delta x_B &= 2l\cos\theta\,\delta\theta & \delta y_C &= -l\sin\theta\,\delta\theta \end{aligned} \tag{10.4}$$

The total virtual work of the forces $\mathbf{Q}$ and $\mathbf{P}$ is thus

$$\begin{aligned} \delta U &= \delta U_Q + \delta U_P = -Q\,\delta x_B - P\,\delta y_C \\ &= -2Ql\cos\theta\,\delta\theta + Pl\sin\theta\,\delta\theta \end{aligned}$$

Making $\delta U = 0$, we obtain

$$2Ql\cos\theta\,\delta\theta = Pl\sin\theta\,\delta\theta \tag{10.5}$$

$$Q = \tfrac{1}{2}P\tan\theta \tag{10.6}$$

The superiority of the method of virtual work over the conventional equilibrium equations in the problem considered here is clear: by using the method of virtual work, we were able to eliminate all unknown reactions, while the equation $\Sigma M_A = 0$ would have eliminated only two of the unknown reactions. This property of the method of virtual work can be used in solving many problems involving machines and mechanisms. *If the virtual displacement considered is consistent with the constraints imposed by the supports and connections, all reactions and internal forces are eliminated and only the work of the loads, applied forces, and friction forces need be considered.*

The method of virtual work can also be used to solve problems involving completely constrained structures, although the virtual displacements considered will never actually take place. Consider, for example, the frame ACB shown in Fig. 10.8*a*. If point A is kept fixed, while B is given a horizontal virtual displacement (Fig. 10.8*b*), we need consider only the work of $\mathbf{P}$ and $\mathbf{B}_x$. We can thus determine

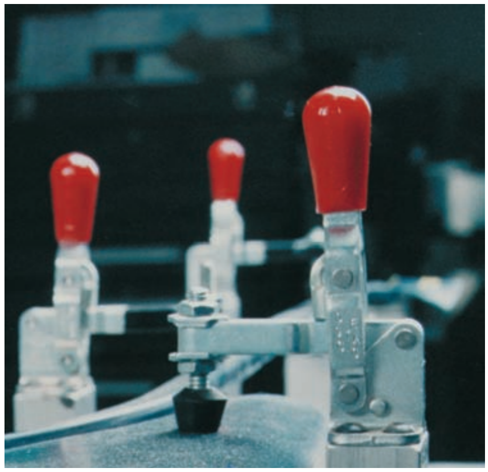


Photo 10.2 The clamping force of the toggle clamp shown can be expressed as a function of the force applied to the handle by first establishing the geometric relations among the members of the clamp and then applying the method of virtual work.

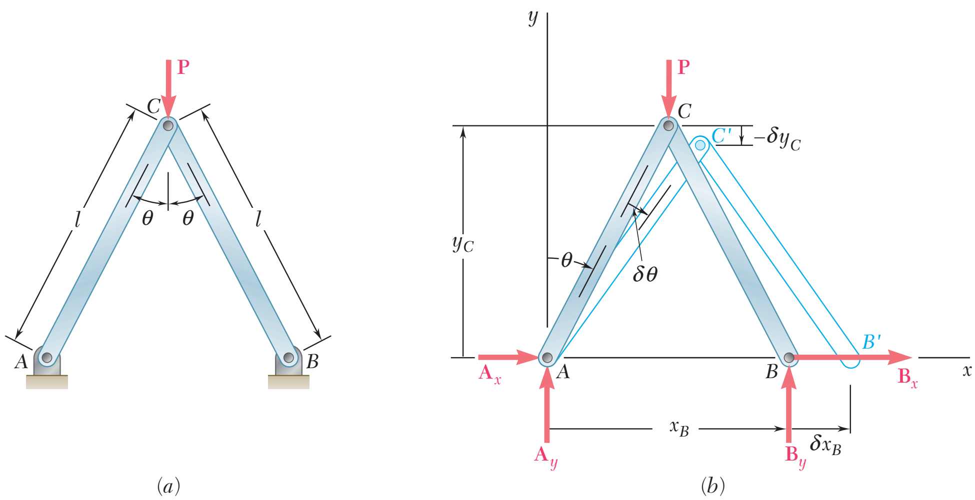


Fig. 10.8

the reaction component $\mathbf{B}_x$ in the same way as the force $\mathbf{Q}$ of the preceding example (Fig. 10.7*b*); we have

$$B_x = -\tfrac{1}{2}P \tan \theta$$

Keeping B fixed and giving to A a horizontal virtual displacement, we can similarly determine the reaction component $\mathbf{A}_x$. The components $\mathbf{A}_y$ and $\mathbf{B}_y$ can be determined by rotating the frame ACB as a rigid body about B and A, respectively.

The method of virtual work can also be used to determine the configuration of a system in equilibrium under given forces. For example, the value of the angle θ for which the linkage of Fig. 10.7 is in equilibrium under two given forces $\mathbf{P}$ and $\mathbf{Q}$ can be obtained by solving Eq. (10.6) for tan θ.

It should be noted, however, that the attractiveness of the method of virtual work depends to a large extent upon the existence of simple geometric relations between the various virtual displacements involved in the solution of a given problem. When no such simple relations exist, it is usually advisable to revert to the conventional method of Chap. 6.

*10.5 REAL MACHINES. MECHANICAL EFFICIENCY

In analyzing the toggle vise in the preceding section, we assumed that no friction forces were involved. Thus, the virtual work consisted only of the work of the applied force $\mathbf{P}$ and of the reaction $\mathbf{Q}$. But the work of the reaction $\mathbf{Q}$ is equal in magnitude and opposite in sign to the work of the force exerted by the vise on the block. Equation (10.5), therefore, expresses that the *output work* $2Ql \cos \theta\, \delta\theta$ is equal to the *input work* $Pl \sin \theta\, \delta\theta$. A machine in which input and output work are equal is said to be an "ideal" machine. In a "real" machine, friction forces will always do some work, and the output work will be smaller than the input work.

Consider, for example, the toggle vise of Fig. 10.7*a*, and assume now that a friction force $\mathbf{F}$ develops between the sliding block B and the horizontal plane (Fig. 10.9). Using the conventional methods of statics and summing moments about A, we find $N = P/2$. Denoting by μ the coefficient of friction between block B and the horizontal

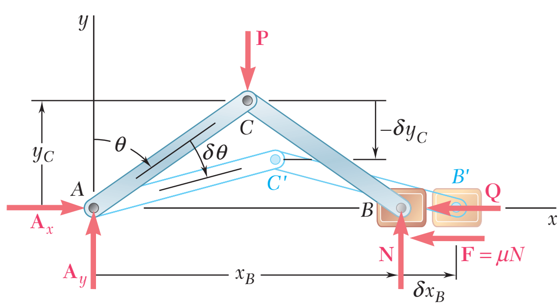


Fig. 10.9

plane, we have $F = \mu N = \mu P/2$. Recalling formulas (10.4), we find that the total virtual work of the forces **Q**, **P**, and **F** during the virtual displacement shown in Fig. 10.9 is

$$\begin{aligned}\delta U &= -Q\,\delta x_B - P\,\delta y_C - F\,\delta x_B \\ &= -2Ql\cos\theta\,\delta\theta + Pl\sin\theta\,\delta\theta - \mu Pl\cos\theta\,\delta\theta\end{aligned}$$

Making $\delta U = 0$, we obtain

$$2Ql\cos\theta\,\delta\theta = Pl\sin\theta\,\delta\theta - \mu Pl\cos\theta\,\delta\theta \tag{10.7}$$

which expresses that the output work is equal to the input work minus the work of the friction force. Solving for Q, we have

$$Q = \tfrac{1}{2}P(\tan\theta - \mu) \tag{10.8}$$

We note that $Q = 0$ when $\tan\theta = \mu$, that is, when θ is equal to the angle of friction ϕ, and that $Q < 0$ when $\theta < \phi$. The toggle vise may thus be used only for values of θ larger than the angle of friction.

The *mechanical efficiency* of a machine is defined as the ratio

$$\boxed{\eta = \frac{\text{output work}}{\text{input work}}} \tag{10.9}$$

Clearly, the mechanical efficiency of an ideal machine is $\eta = 1$, since input and output work are then equal, while the mechanical efficiency of a real machine will always be less than 1.

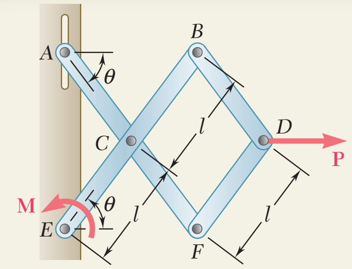
In the case of the toggle vise we have just analyzed, we write

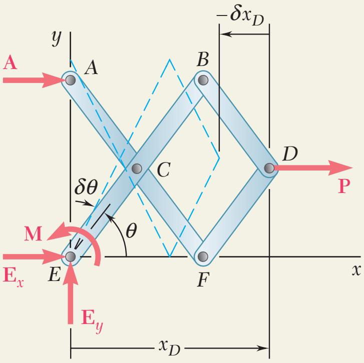
$$\eta = \frac{\text{output work}}{\text{input work}} = \frac{2Ql\cos\theta\,\delta\theta}{Pl\sin\theta\,\delta\theta}$$

Substituting from (10.8) for Q, we obtain

$$\eta = \frac{P(\tan\theta - \mu)l\cos\theta\,\delta\theta}{Pl\sin\theta\,\delta\theta} = 1 - \mu\cot\theta \tag{10.10}$$

We check that in the absence of friction forces, we would have $\mu = 0$ and $\eta = 1$. In the general case, when μ is different from zero, the efficiency η becomes zero for $\mu\cot\theta = 1$, that is, for $\tan\theta = \mu$, or $\theta = \tan^{-1}\mu = \phi$. We note again that the toggle vise can be used only for values of θ larger than the angle of friction ϕ.





SAMPLE PROBLEM 10.1

Using the method of virtual work, determine the magnitude of the couple **M** required to maintain the equilibrium of the mechanism shown.

SOLUTION

Choosing a coordinate system with origin at E, we write

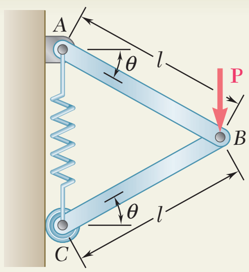
$$x_D = 3l \cos \theta \qquad \delta x_D = -3l \sin \theta \, \delta\theta$$

Principle of Virtual Work. Since the reactions **A**, $\mathbf{E}_x$, and $\mathbf{E}_y$ will do no work during the virtual displacement, the total virtual work done by **M** and **P** must be zero. Noting that **P** acts in the positive x direction and **M** acts in the positive θ direction, we write

$\delta U = 0$: $\qquad +M\,\delta\theta + P\,\delta x_D = 0$

$$+M\,\delta\theta + P(-3l \sin\theta\,\delta\theta) = 0$$

$$M = 3Pl \sin \theta$$



SAMPLE PROBLEM 10.2

Determine the expressions for θ and for the tension in the spring which correspond to the equilibrium position of the mechanism. The unstretched length of the spring is h, and the constant of the spring is k. Neglect the weight of the mechanism.

SOLUTION

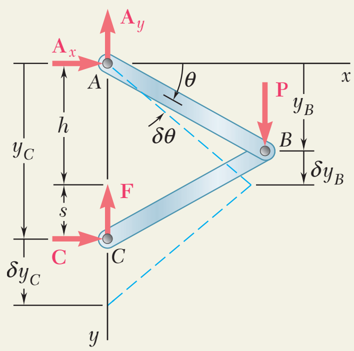
With the coordinate system shown

$$y_B = l \sin\theta \qquad y_C = 2l \sin\theta$$
$$\delta y_B = l \cos\theta\,\delta\theta \qquad \delta y_C = 2l\cos\theta\,\delta\theta$$

The elongation of the spring is $\qquad s = y_C - h = 2l \sin\theta - h$

The magnitude of the force exerted at C by the spring is

$$F = ks = k(2l \sin\theta - h) \tag{1}$$



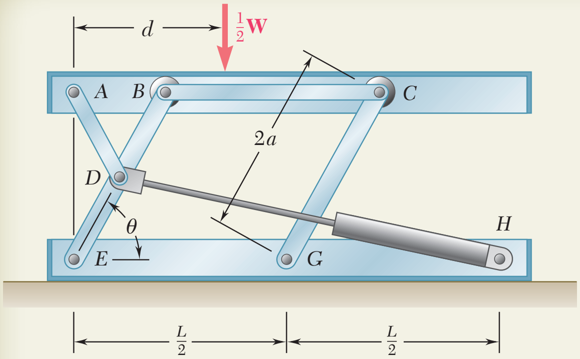
Principle of Virtual Work. Since the reactions $\mathbf{A}_x$, $\mathbf{A}_y$, and **C** do no work, the total virtual work done by **P** and **F** must be zero.

$\delta U = 0$: $\qquad P\,\delta y_B - F\,\delta y_C = 0$

$$P(l\cos\theta\,\delta\theta) - k(2l\sin\theta - h)(2l\cos\theta\,\delta\theta) = 0$$

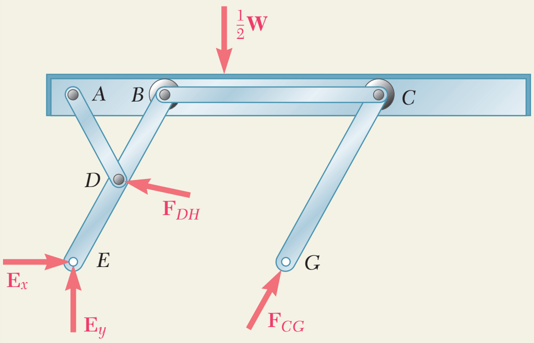
$$\sin\theta = \frac{P + 2kh}{4kl}$$

Substituting this expression into (1), we obtain $\qquad F = \frac{1}{2}P$



SAMPLE PROBLEM 10.3

A hydraulic-lift table is used to raise a 1000-kg crate. It consists of a platform and of two identical linkages on which hydraulic cylinders exert equal forces. (Only one linkage and one cylinder are shown.) Members EDB and CG are each of length $2a$, and member AD is pinned to the midpoint of EDB. If the crate is placed on the table, so that half of its weight is supported by the system shown, determine the force exerted by each cylinder in raising the crate for $\theta = 60°$, $a = 0.70$ m, and $L = 3.20$ m. This mechanism has been previously considered in Sample Prob. 6.7.

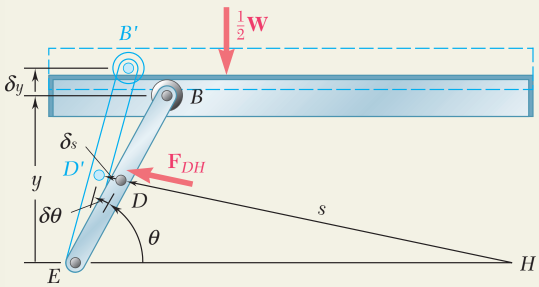


SOLUTION

The machine considered consists of the platform and of the linkage, with an input force $\mathbf{F}_{DH}$ exerted by the cylinder and an output force equal and opposite to $\frac{1}{2}\mathbf{W}$.

Principle of Virtual Work. We first observe that the reactions at E and G do no work. Denoting by y the elevation of the platform above the base, and by s the length DH of the cylinder-and-piston assembly, we write

$$\delta U = 0: \qquad -\tfrac{1}{2}W\,\delta y + F_{DH}\,\delta s = 0 \tag{1}$$



The vertical displacement δy of the platform is expressed in terms of the angular displacement $\delta\theta$ of EDB as follows:

$$y = (EB)\sin\theta = 2a\sin\theta$$
$$\delta y = 2a\cos\theta\,\delta\theta$$

To express δs similarly in terms of $\delta\theta$, we first note that by the law of cosines,

$$s^2 = a^2 + L^2 - 2aL\cos\theta$$

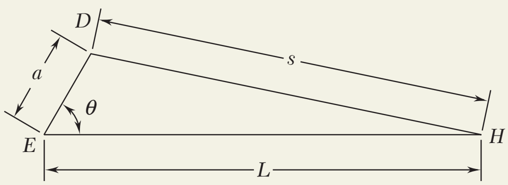
Differentiating,

$$2s\,\delta s = -2aL(-\sin\theta)\,\delta\theta$$
$$\delta s = \frac{aL\sin\theta}{s}\,\delta\theta$$

Substituting for δy and δs into (1), we write

$$(-\tfrac{1}{2}W)2a\cos\theta\,\delta\theta + F_{DH}\frac{aL\sin\theta}{s}\,\delta\theta = 0$$

$$F_{DH} = W\frac{s}{L}\cot\theta$$



With the given numerical data, we have

$$W = mg = (1000\text{ kg})(9.81\text{ m/s}^2) = 9810\text{ N} = 9.81\text{ kN}$$
$$s^2 = a^2 + L^2 - 2aL\cos\theta$$
$$= (0.70)^2 + (3.20)^2 - 2(0.70)(3.20)\cos 60° = 8.49$$
$$s = 2.91\text{ m}$$

$$F_{DH} = W\frac{s}{L}\cot\theta = (9.81\text{ kN})\frac{2.91\text{ m}}{3.20\text{ m}}\cot 60°$$

$$F_{DH} = 5.15\text{ kN} \blacktriangleleft$$

SOLVING PROBLEMS ON YOUR OWN

In this lesson you learned to use the *method of virtual work,* which is a different way of solving problems involving the equilibrium of rigid bodies.

The work done by a force during a displacement of its point of application or by a couple during a rotation is found by using Eqs. (10.1) and (10.2), respectively:

$$dU = F\,ds \cos \alpha \tag{10.1}$$
$$dU = M\,d\theta \tag{10.2}$$

Principle of virtual work. In its more general and more useful form, this principle can be stated as follows: *If a system of connected rigid bodies is in equilibrium, the total virtual work of the external forces applied to the system is zero for any virtual displacement of the system.*

As you apply the principle of virtual work, keep in mind the following:

1. Virtual displacement. A machine or mechanism in equilibrium has no tendency to move. However, *we can cause, or imagine, a small displacement.* Since it does not actually occur, such a displacement is called a *virtual displacement.*

2. Virtual work. The work done by a force or couple during a virtual displacement is called *virtual work.*

3. You need consider only the forces which do work during the virtual displacement.

4. Forces which do no work during a virtual displacement that is consistent with the constraints imposed on the system are:
   - **a.** Reactions at supports
   - **b.** Internal forces at connections
   - **c.** Forces exerted by inextensible cords and cables

None of these forces need be considered when you use the method of virtual work.

5. Be sure to express the various virtual displacements involved in your computations in terms of a *single virtual displacement.* This is done in each of the three preceding sample problems, where the virtual displacements are all expressed in terms of $\delta\theta$.

6. Remember that the method of virtual work is effective only in those cases where the geometry of the system makes it relatively easy to relate the displacements involved.

10.1 Determine the vertical force **P** that must be applied at C to maintain the equilibrium of the linkage.

10.2 Determine the horizontal force **P** that must be applied at A to maintain the equilibrium of the linkage.

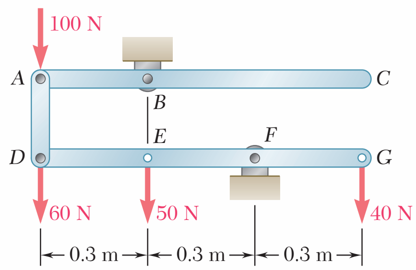


Fig. P10.1 and P10.3

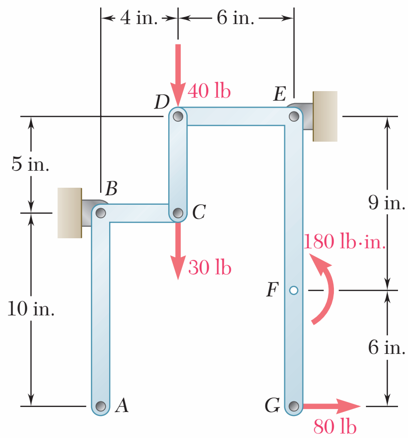


Fig. P10.2 and P10.4

10.3 and 10.4 Determine the couple **M** that must be applied to member ABC to maintain the equilibrium of the linkage.

10.5 Knowing that the maximum friction force exerted by the bottle on the cork is 60 lb, determine (*a*) the force **P** that must be applied to the corkscrew to open the bottle, (*b*) the maximum force exerted by the base of the corkscrew on the top of the bottle.

10.6 The two-bar linkage shown is supported by a pin and bracket at B and a collar at D that slides freely on a vertical rod. Determine the force **P** required to maintain the equilibrium of the linkage.

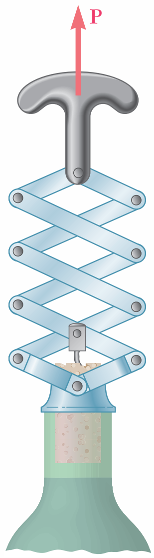


Fig. *P10.5*

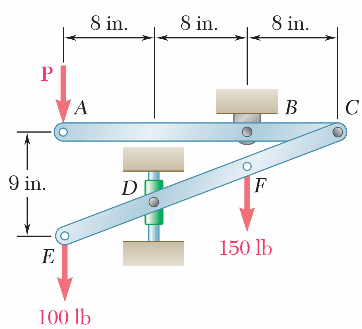


Fig. *P10.6*

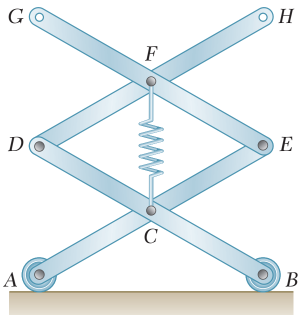


Fig. P10.7 and P10.8

10.7 A spring of constant 15 kN/m connects points C and F of the linkage shown. Neglecting the weight of the spring and linkage, determine the force in the spring and the vertical motion of point G when a vertical downward 120-N force is applied (*a*) at point C, (*b*) at points C and H.

10.8 A spring of constant 15 kN/m connects points C and F of the linkage shown. Neglecting the weight of the spring and linkage, determine the force in the spring and the vertical motion of point G when a vertical downward 120-N force is applied (*a*) at point E, (*b*) at points E and F.

10.9 Knowing that the line of action of the force **Q** passes through point C, derive an expression for the magnitude of **Q** required to maintain equilibrium.

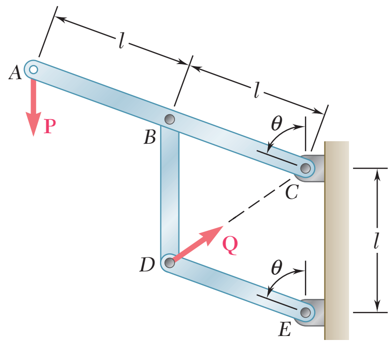


Fig. P10.9

10.10 Solve Prob. 10.9 assuming that the force **P** applied at point A acts horizontally to the left.

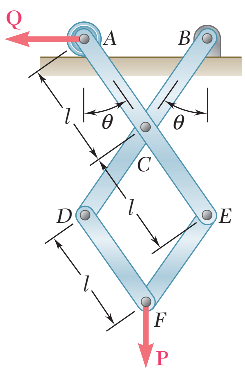


Fig. P10.11

10.11 The mechanism shown is acted upon by the force **P**; derive an expression for the magnitude of the force **Q** required to maintain equilibrium.

10.12 and *10.13* The slender rod AB is attached to a collar A and rests on a small wheel at C. Neglecting the radius of the wheel and the effect of friction, derive an expression for the magnitude of the force **Q** required to maintain the equilibrium of the rod.

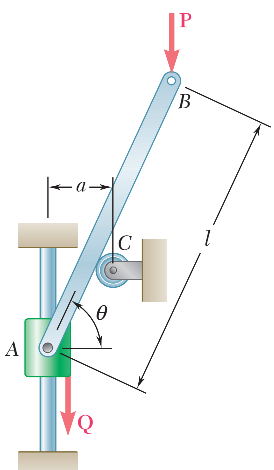


Fig. P10.12

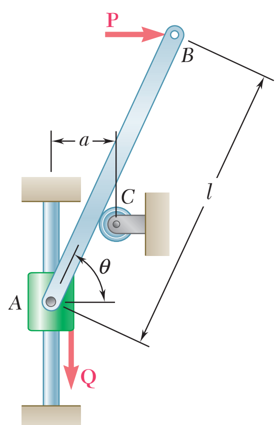


Fig. *P10.13*

10.14 Derive an expression for the magnitude of the force **Q** required to maintain the equilibrium of the mechanism shown.

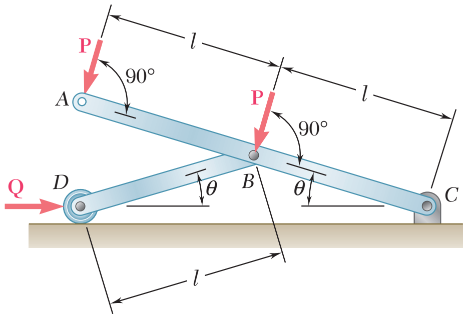


Fig. P10.14

10.15 A uniform rod *AB* of length *l* and weight *W* is suspended from two cords *AC* and *BC* of equal length. Derive an expression for the magnitude of the couple **M** required to maintain equilibrium of the rod in the position shown.

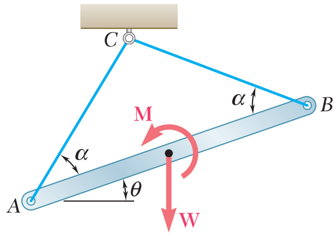


Fig. P10.15

10.16 and 10.17 Derive an expression for the magnitude of the couple **M** required to maintain the equilibrium of the linkage shown.

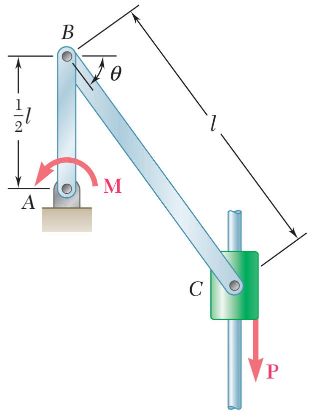


Fig. P10.16

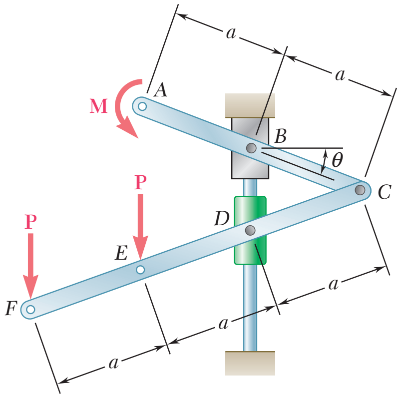


Fig. P10.17

10.18 The pin at *C* is attached to member *BCD* and can slide along a slot cut in the fixed plate shown. Neglecting the effect of friction, derive an expression for the magnitude of the couple **M** required to maintain equilibrium when the force **P** that acts at *D* is directed (*a*) as shown, (*b*) vertically downward, (*c*) horizontally to the right.

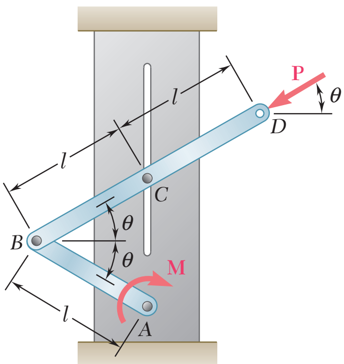


Fig. P10.18

10.19 A 4-kN force **P** is applied as shown to the piston of the engine system. Knowing that $AB = 50$ mm and $BC = 200$ mm, determine the couple **M** required to maintain the equilibrium of the system when (*a*) $\theta = 30°$, (*b*) $\theta = 150°$.

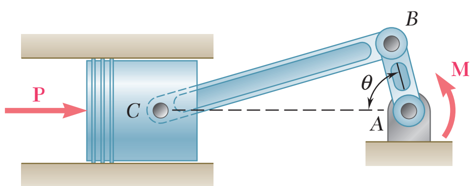


Fig. P10.19 and P10.20

10.20 A couple **M** of magnitude 100 N · m is applied as shown to the crank of the engine system. Knowing that $AB = 50$ mm and $BC = 200$ mm, determine the force **P** required to maintain the equilibrium of the system when (*a*) $\theta = 60°$, (*b*) $\theta = 120°$.

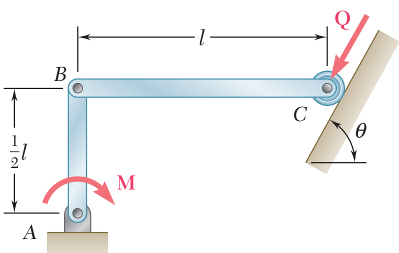


Fig. P10.21 and P10.22

10.21 For the linkage shown, determine the couple **M** required for equilibrium when $l = 1.8$ ft, $Q = 40$ lb, and $\theta = 65°$.

10.22 For the linkage shown, determine the force **Q** required for equilibrium when $l = 18$ in., $M = 600$ lb · in., and $\theta = 70°$.

10.23 Determine the value of θ corresponding to the equilibrium position of the mechanism of Prob. 10.11 when $P = 45$ lb and $Q = 160$ lb.

10.24 Determine the value of θ corresponding to the equilibrium position of the mechanism of Prob. 10.9 when $P = 80$ N and $Q = 100$ N.

10.25 Rod AB is attached to a block at A that can slide freely in the vertical slot shown. Neglecting the effect of friction and the weights of the rods, determine the value of θ corresponding to equilibrium.

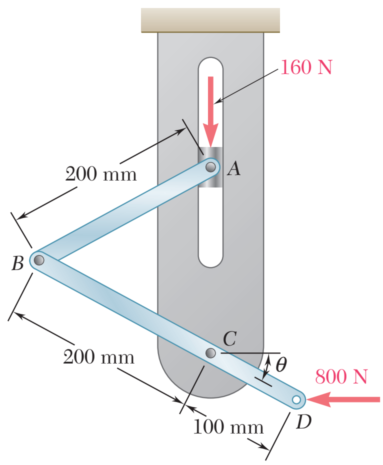


Fig. P10.25

10.26 Solve Prob. 10.25 assuming that the 800-N force is replaced by a 24-N · m clockwise couple applied at *D*.

10.27 Determine the value of θ corresponding to the equilibrium position of the rod of Prob. 10.12 when $l = 30$ in., $a = 5$ in., $P = 25$ lb, and $Q = 40$ lb.

10.28 Determine the values of θ corresponding to the equilibrium position of the rod of Prob. 10.13 when $l = 600$ mm, $a = 100$ mm, $P = 50$ N, and $Q = 90$ N.

10.29 Two rods *AC* and *CE* are connected by a pin at *C* and by a spring *AE*. The constant of the spring is k, and the spring is unstretched when $\theta = 30°$. For the loading shown, derive an equation in P, θ, l, and k that must be satisfied when the system is in equilibrium.

10.30 Two rods *AC* and *CE* are connected by a pin at *C* and by a spring *AE*. The constant of the spring is 1.5 lb/in., and the spring is unstretched when $\theta = 30°$. Knowing that $l = 10$ in. and neglecting the weight of the rods, determine the value of θ corresponding to equilibrium when $P = 40$ lb.

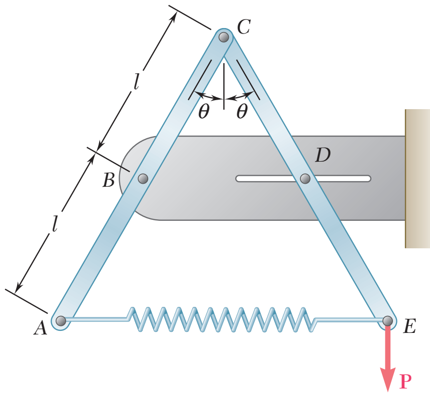


Fig. *P10.29* and P10.30

10.31 Solve Prob. 10.30 assuming that force **P** is moved to *C* and acts vertically downward.

10.32 Rod *ABC* is attached to blocks *A* and *B* that can move freely in the guides shown. The constant of the spring attached at *A* is $k = 3$ kN/m, and the spring is unstretched when the rod is vertical. For the loading shown, determine the value of θ corresponding to equilibrium.

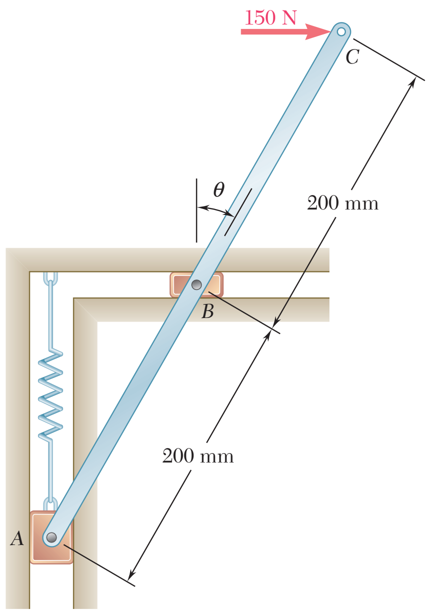


Fig. P10.32

10.33 A load **W** of magnitude 600 N is applied to the linkage at B. The constant of the spring is $k = 2.5$ kN/m, and the spring is unstretched when AB and BC are horizontal. Neglecting the weight of the linkage and knowing that $l = 300$ mm, determine the value of θ corresponding to equilibrium.

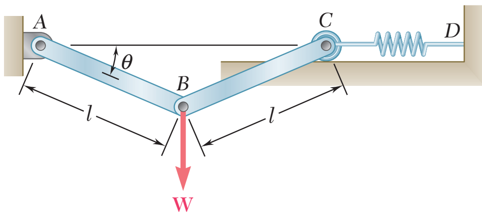


Fig. P10.33 and *P10.34*

10.34 A vertical load **W** is applied to the linkage at B. The constant of the spring is k, and the spring is unstretched when AB and BC are horizontal. Neglecting the weight of the linkage, derive an equation in θ, W, l, and k that must be satisfied when the linkage is in equilibrium.

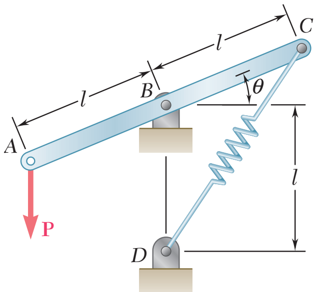


Fig. P10.35 and P10.36

10.35 and 10.36 Knowing that the constant of spring CD is k and that the spring is unstretched when rod ABC is horizontal, determine the value of θ corresponding to equilibrium for the data indicated.

10.35 $P = 300$ N, $l = 400$ mm, $k = 5$ kN/m.
10.36 $P = 75$ lb, $l = 15$ in., $k = 20$ lb/in.

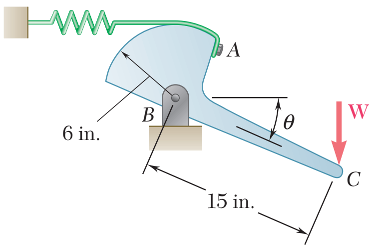


Fig. P10.37

10.37 A load **W** of magnitude 72 lb is applied to the mechanism at C. Neglecting the weight of the mechanism, determine the value of θ corresponding to equilibrium. The constant of the spring is $k = 20$ lb/in., and the spring is unstretched when $\theta = 0$.

10.38 A force **P** of magnitude 240 N is applied to end E of cable CDE, which passes under pulley D and is attached to the mechanism at C. Neglecting the weight of the mechanism and the radius of the pulley, determine the value of θ corresponding to equilibrium. The constant of the spring is $k = 4$ kN/m, and the spring is unstretched when $\theta = 90°$.

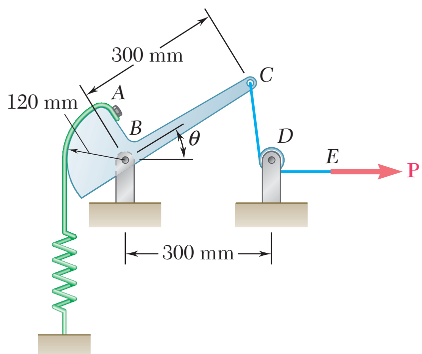


Fig. P10.38

10.39 The lever AB is attached to the horizontal shaft BC that passes through a bearing and is welded to a fixed support at C. The torsional spring constant of the shaft BC is K; that is, a couple of magnitude K is required to rotate end B through 1 rad. Knowing that the shaft is untwisted when AB is horizontal, determine the value of θ corresponding to the position of equilibrium when $P = 100$ N, $l = 250$ mm, and $K = 12.5$ N · m/rad.

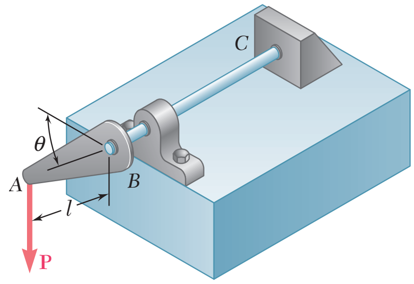


Fig. P10.39

10.40 Solve Prob. 10.39 assuming that $P = 350$ N, $l = 250$ mm, and $K = 12.5$ N · m/rad. Obtain answers in each of the following quadrants: $0 < \theta < 90°$, $270° < \theta < 360°$, $360° < \theta < 450°$.

10.41 The position of boom ABC is controlled by the hydraulic cylinder BD. For the loading shown, determine the force exerted by the hydraulic cylinder on pin B when $\theta = 65°$.

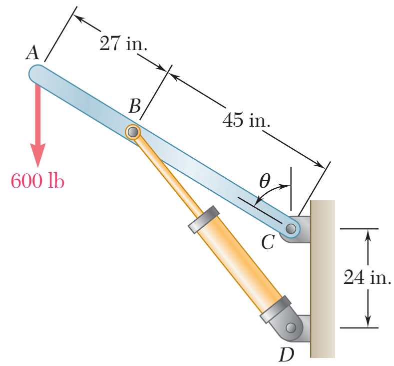


Fig. *P10.41* and *P10.42*

10.42 The position of boom ABC is controlled by the hydraulic cylinder BD. For the loading shown, (*a*) express the force exerted by the hydraulic cylinder on pin B as a function of the length BD, (*b*) determine the smallest possible value of the angle θ if the maximum force that the cylinder can exert on pin B is 2.5 kips.

10.43 The position of member ABC is controlled by the hydraulic cylinder CD. For the loading shown, determine the force exerted by the hydraulic cylinder on pin C when $\theta = 55°$.

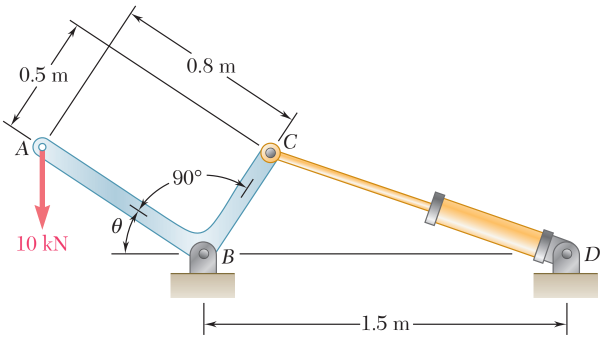


Fig. P10.43 and P10.44

10.44 The position of member ABC is controlled by the hydraulic cylinder CD. Determine the angle θ knowing that the hydraulic cylinder exerts a 15-kN force on pin C.

10.45 The telescoping arm *ABC* is used to provide an elevated platform for construction workers. The workers and the platform together weigh 500 lb and their combined center of gravity is located directly above *C*. For the position when $\theta = 20°$, determine the force exerted on pin *B* by the single hydraulic cylinder *BD*.

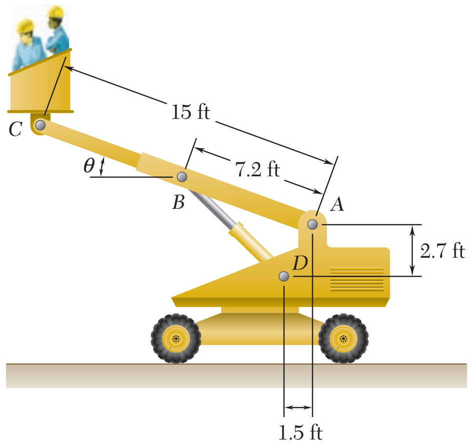


Fig. P10.45

10.46 Solve Prob. 10.45 assuming that the workers are lowered to a point near the ground so that $\theta = -20°$.

10.47 A block of weight *W* is pulled up a plane forming an angle α with the horizontal by a force **P** directed along the plane. If μ is the coefficient of friction between the block and the plane, derive an expression for the mechanical efficiency of the system. Show that the mechanical efficiency cannot exceed $\frac{1}{2}$ if the block is to remain in place when the force **P** is removed.

10.48 Denoting by μ_s the coefficient of static friction between the block attached to rod *ACE* and the horizontal surface, derive expressions in terms of *P*, μ_s, and θ for the largest and smallest magnitude of the force **Q** for which equilibrium is maintained.

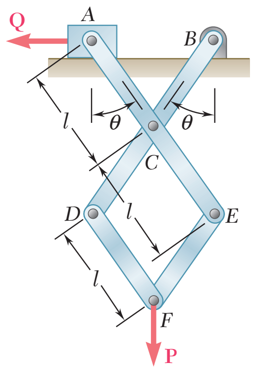


Fig. *P10.48* and P10.49

10.49 Knowing that the coefficient of static friction between the block attached to rod *ACE* and the horizontal surface is 0.15, determine the magnitude of the largest and smallest force **Q** for which equilibrium is maintained when $\theta = 30°$, $l = 0.2$ m, and $P = 40$ N.

10.50 Denoting by μ_s the coefficient of static friction between collar C and the vertical rod, derive an expression for the magnitude of the largest couple **M** for which equilibrium is maintained in the position shown. Explain what happens if $\mu_s \geq \tan\theta$.

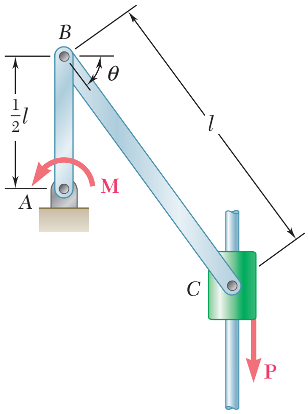


Fig. P10.50 and P10.51

10.51 Knowing that the coefficient of static friction between collar C and the vertical rod is 0.40, determine the magnitude of the largest and smallest couple **M** for which equilibrium is maintained in the position shown, when $\theta = 35°$, $l = 600$ mm, and $P = 300$ N.

10.52 Derive an expression for the mechanical efficiency of the jack discussed in Sec. 8.6. Show that if the jack is to be self-locking, the mechanical efficiency cannot exceed $\frac{1}{2}$.

10.53 Using the method of virtual work, determine the reaction at E.

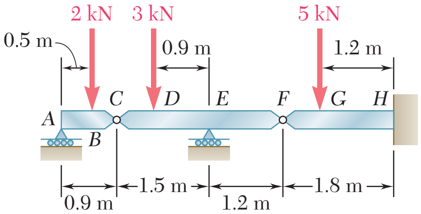


Fig. P10.53 and P10.54

10.54 Using the method of virtual work, determine separately the force and couple representing the reaction at H.

10.55 Referring to Prob. 10.43 and using the value found for the force exerted by the hydraulic cylinder CD, determine the change in the length of CD required to raise the 10-kN load by 15 mm.

10.56 Referring to Prob. 10.45 and using the value found for the force exerted by the hydraulic cylinder BD, determine the change in the length of BD required to raise the platform attached at C by 2.5 in.

10.57 Determine the vertical movement of joint D if the length of member BF is increased by 1.5 in. (*Hint:* Apply a vertical load at joint D, and, using the methods of Chap. 6, compute the force exerted by member BF on joints B and F. Then apply the method of virtual work for a virtual displacement resulting in the specified increase in length of member BF. This method should be used only for small changes in the lengths of members.)

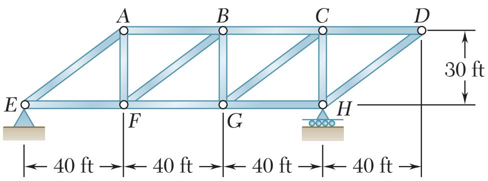


Fig. P10.57 and P10.58

10.58 Determine the horizontal movement of joint D if the length of member BF is increased by 1.5 in. (See the hint for Prob. 10.57.)

*10.6 WORK OF A FORCE DURING A FINITE DISPLACEMENT

Consider a force **F** acting on a particle. The work of **F** corresponding to an infinitesimal displacement $d\mathbf{r}$ of the particle was defined in Sec. 10.2 as

$$dU = \mathbf{F} \cdot d\mathbf{r} \tag{10.1}$$

The work of **F** corresponding to a finite displacement of the particle from A_1 to A_2 (Fig. 10.10*a*) is denoted by $U_{1\to2}$ and is obtained by integrating (10.1) along the curve described by the particle:

$$U_{1\to2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \tag{10.11}$$

Using the alternative expression

$$dU = F\, ds \cos \alpha \tag{10.1'}$$

given in Sec. 10.2 for the elementary work dU, we can also express the work $U_{1\to2}$ as

$$U_{1\to2} = \int_{s_1}^{s_2} (F \cos \alpha)\, ds \tag{10.11'}$$

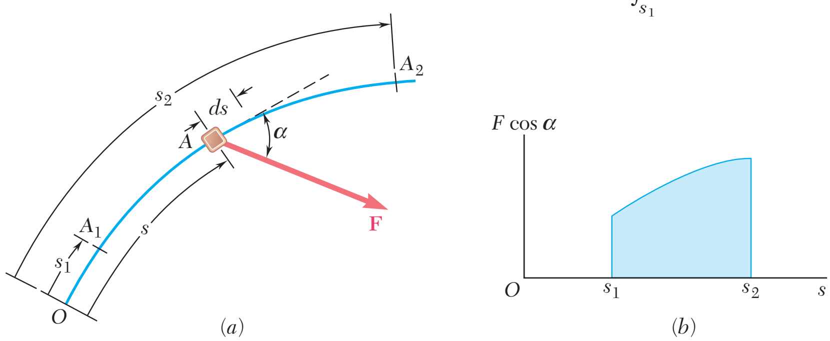


Fig. 10.10

where the variable of integration s measures the distance along the path traveled by the particle. The work $U_{1\to2}$ is represented by the area under the curve obtained by plotting $F \cos \alpha$ against s (Fig. 10.10*b*). In the case of a force **F** of constant magnitude acting in the direction of motion, formula (10.11′) yields $U_{1\to2} = F(s_2 - s_1)$.

Recalling from Sec. 10.2 that the work of a couple of moment **M** during an infinitesimal rotation $d\theta$ of a rigid body is

$$dU = M\, d\theta \tag{10.2}$$

we express as follows the work of the couple during a finite rotation of the body:

$$U_{1\to2} = \int_{\theta_1}^{\theta_2} M\, d\theta \tag{10.12}$$

In the case of a constant couple, formula (10.12) yields

$$U_{1\to2} = M(\theta_2 - \theta_1)$$

Work of a Weight. It was stated in Sec. 10.2 that the work of the weight **W** of a body during an infinitesimal displacement of the body is equal to the product of W and the vertical displacement of the center of gravity of the body. With the y axis pointing upward, the work of **W** during a finite displacement of the body (Fig. 10.11) is obtained by writing

$$dU = -W\,dy$$

Integrating from A_1 to A_2, we have

$$U_{1\rightarrow 2} = -\int_{y_1}^{y_2} W\,dy = Wy_1 - Wy_2 \tag{10.13}$$

or

$$U_{1\rightarrow 2} = -W(y_2 - y_1) = -W\,\Delta y \tag{10.13$'$}$$

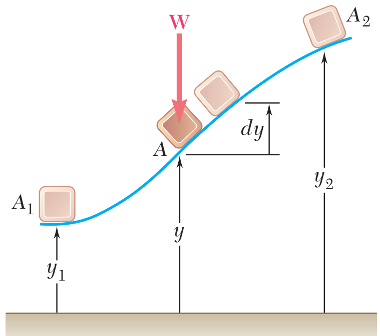


Fig. 10.11

where Δy is the vertical displacement from A_1 to A_2. The work of the weight **W** is thus equal to *the product of* W *and the vertical displacement of the center of gravity of the body.* The work is *positive* when $\Delta y < 0$, that is, *when the body moves down.*

Work of the Force Exerted by a Spring. Consider a body A attached to a fixed point B by a spring; it is assumed that the spring is undeformed when the body is at A_0 (Fig. 10.12*a*). Experimental evidence shows that the magnitude of the force **F** exerted by the spring on a body A is proportional to the deflection x of the spring measured from the position A_0. We have

$$F = kx \tag{10.14}$$

where k is the *spring constant,* expressed in N/m if SI units are used and expressed in lb/ft or lb/in. if U.S. customary units are used. The work of the force **F** exerted by the spring during a finite displacement of the body from $A_1(x = x_1)$ to $A_2(x = x_2)$ is obtained by writing

$$dU = -F\,dx = -kx\,dx$$

$$U_{1\rightarrow 2} = -\int_{x_1}^{x_2} kx\,dx = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \tag{10.15}$$

Care should be taken to express k and x in consistent units. For example, if U.S. customary units are used, k should be expressed in lb/ft and x expressed in feet, or k in lb/in. and x in inches; in the first case, the work is obtained in ft · lb; in the second case, in in · lb. We note that the work of the force **F** exerted by the spring on the body is *positive* when $x_2 < x_1$, that is, *when the spring is returning to its undeformed position.*

Since Eq. (10.14) is the equation of a straight line of slope k passing through the origin, the work $U_{1\rightarrow 2}$ of **F** during the displacement from A_1 to A_2 can be obtained by evaluating the area of the trapezoid shown in Fig. 10.12*b*. This is done by computing the values F_1 and F_2 and multiplying the base Δx of the trapezoid by its mean height $\frac{1}{2}(F_1 + F_2)$. Since the work of the force **F** exerted by the spring is positive for a negative value of Δx, we write

$$U_{1\rightarrow 2} = -\tfrac{1}{2}(F_1 + F_2)\,\Delta x \tag{10.16}$$

Formula (10.16) is usually more convenient to use than (10.15) and affords fewer chances of confusing the units involved.

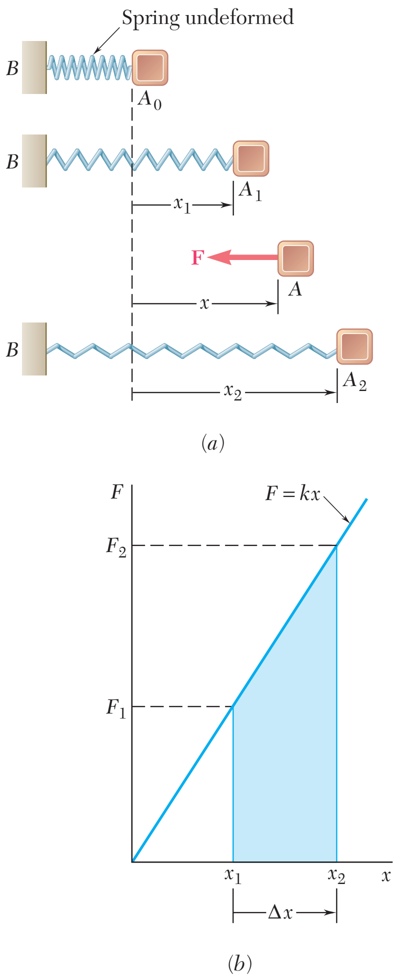


Fig. 10.12

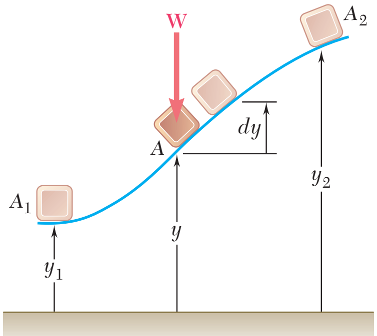


Fig. 10.11 (*repeated*)

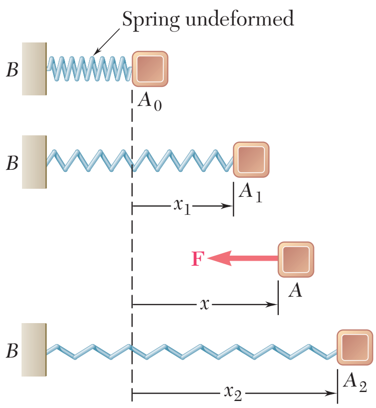


Fig. 10.12a (*repeated*)

*10.7 POTENTIAL ENERGY

Considering again the body of Fig. 10.11, we note from Eq. (10.13) that the work of the weight **W** during a finite displacement is obtained by subtracting the value of the function Wy corresponding to the second position of the body from its value corresponding to the first position. The work of **W** is thus independent of the actual path followed; it depends only upon the initial and final values of the function Wy. This function is called the *potential energy* of the body with respect to the *force of gravity* **W** and is denoted by V_g. We write

$$U_{1\rightarrow 2} = (V_g)_1 - (V_g)_2 \qquad \text{with } V_g = Wy \tag{10.17}$$

We note that if $(V_g)_2 > (V_g)_1$, that is, *if the potential energy increases* during the displacement (as in the case considered here), *the work* $U_{1\rightarrow 2}$ *is negative.* If, on the other hand, the work of **W** is positive, the potential energy decreases. Therefore, the potential energy V_g of the body provides a measure of *the work which can be done* by its weight **W**. Since only the *change* in potential energy, and not the actual value of V_g, is involved in formula (10.17), an arbitrary constant can be added to the expression obtained for V_g. In other words, the level from which the elevation y is measured can be chosen arbitrarily. Note that potential energy is expressed in the same units as work, i.e., in joules (J) if SI units are used† and in ft · lb or in · lb if U.S. customary units are used.

Considering now the body of Fig. 10.12*a*, we note from Eq. (10.15) that the work of the elastic force **F** is obtained by subtracting the value of the function $\frac{1}{2}kx^2$ corresponding to the second position of the body from its value corresponding to the first position. This function is denoted by V_e and is called the *potential energy* of the body with respect to the *elastic force* **F**. We write

$$U_{1\rightarrow 2} = (V_e)_1 - (V_e)_2 \qquad \text{with } V_e = \tfrac{1}{2}kx^2 \tag{10.18}$$

and observe that during the displacement considered, the work of the force **F** exerted by the spring on the body is negative and the potential energy V_e increases. We should note that the expression obtained for V_e is valid only if the deflection of the spring is measured from its undeformed position.

The concept of potential energy can be used when forces other than gravity forces and elastic forces are involved. It remains valid as long as the elementary work dU of the force considered is an *exact differential.* It is then possible to find a function V, called potential energy, such that

$$dU = -dV \tag{10.19}$$

Integrating (10.19) over a finite displacement, we obtain the general formula

$$U_{1\rightarrow 2} = V_1 - V_2 \tag{10.20}$$

which expresses that *the work of the force is independent of the path followed and is equal to minus the change in potential energy.* A force which satisfies Eq. (10.20) is said to be a *conservative force.*‡

†See footnote, page 559.

‡A detailed discussion of conservative forces is given in Sec. 13.7 of *Dynamics.*

*10.8 POTENTIAL ENERGY AND EQUILIBRIUM

The application of the principle of virtual work is considerably simplified when the potential energy of a system is known. In the case of a virtual displacement, formula (10.19) becomes $\delta U = -\delta V$. Moreover, if the position of the system is defined by a single independent variable θ, we can write $\delta V = (dV/d\theta)\,\delta\theta$. Since $\delta\theta$ must be different from zero, the condition $\delta U = 0$ for the equilibrium of the system becomes

$$\frac{dV}{d\theta} = 0 \qquad (10.21)$$

In terms of potential energy, therefore, the principle of virtual work states that *if a system is in equilibrium, the derivative of its total potential energy is zero.* If the position of the system depends upon several independent variables (the system is then said to possess *several degrees of freedom*), the partial derivatives of V with respect to each of the independent variables must be zero.

Consider, for example, a structure made of two members AC and CB and carrying a load W at C. The structure is supported by a pin at A and a roller at B, and a spring BD connects B to a fixed point D (Fig. 10.13*a*). The constant of the spring is k, and it is assumed that the natural length of the spring is equal to AD and thus that the spring is undeformed when B coincides with A. Neglecting the friction forces and the weight of the members, we find that the only forces which work during a displacement of the structure are the weight **W** and the force **F** exerted by the spring at point B (Fig. 10.13*b*). The total potential energy of the system will thus be obtained by adding the potential energy V_g corresponding to the gravity force **W** and the potential energy V_e corresponding to the elastic force **F**.

Choosing a coordinate system with origin at A and noting that the deflection of the spring, measured from its undeformed position, is $AB = x_B$, we write

$$V_e = \tfrac{1}{2}kx_B^2 \qquad V_g = Wy_C$$

Expressing the coordinates x_B and y_C in terms of the angle θ, we have

$$\begin{aligned} x_B &= 2l \sin\theta & y_C &= l\cos\theta \\ V_e &= \tfrac{1}{2}k(2l\sin\theta)^2 & V_g &= W(l\cos\theta) \\ V &= V_e + V_g = 2kl^2\sin^2\theta + Wl\cos\theta \end{aligned} \qquad (10.22)$$

The positions of equilibrium of the system are obtained by equating to zero the derivative of the potential energy V. We write

$$\frac{dV}{d\theta} = 4kl^2\sin\theta\cos\theta - Wl\sin\theta = 0$$

or, factoring $l\sin\theta$,

$$\frac{dV}{d\theta} = l\sin\theta(4kl\cos\theta - W) = 0$$

There are therefore two positions of equilibrium, corresponding to the values $\theta = 0$ and $\theta = \cos^{-1}(W/4kl)$, respectively.†

†The second position does not exist if $W > 4kl$.

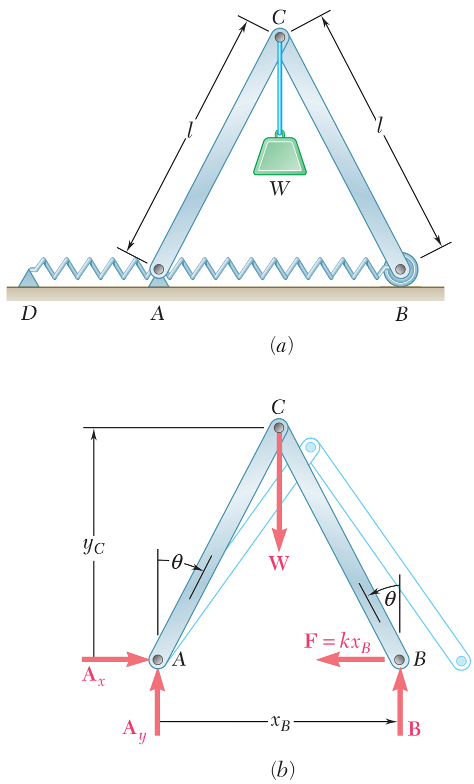
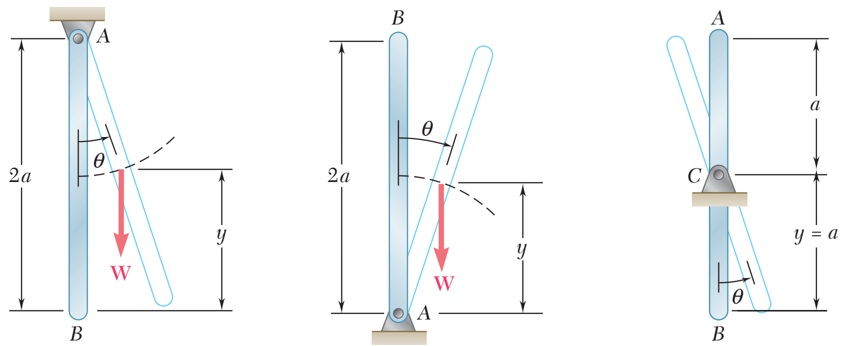


Fig. 10.13

*10.9 STABILITY OF EQUILIBRIUM

Consider the three uniform rods of length $2a$ and weight $\mathbf{W}$ shown in Fig. 10.14. While each rod is in equilibrium, there is an important difference between the three cases considered. Suppose that each rod is slightly disturbed from its position of equilibrium and then released: rod *a* will move back toward its original position, rod *b* will keep moving away from its original position, and rod *c* will remain in its new position. In case *a*, the equilibrium of the rod is said to be *stable;* in case *b*, it is said to be *unstable;* and, in case *c*, it is said to be *neutral*.



(*a*) Stable equilibrium (*b*) Unstable equilibrium (*c*) Neutral equilibrium

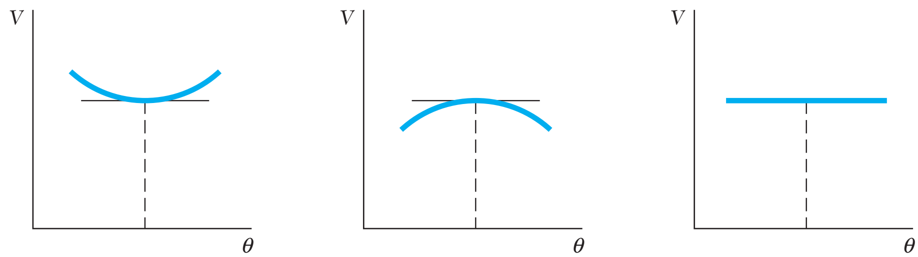
Fig. 10.14

Recalling from Sec. 10.7 that the potential energy V_g with respect to gravity is equal to Wy, where y is the elevation of the point of application of $\mathbf{W}$ measured from an arbitrary level, we observe that the potential energy of rod *a* is minimum in the position of equilibrium considered, that the potential energy of rod *b* is maximum, and that the potential energy of rod *c* is constant. Equilibrium is thus *stable, unstable,* or *neutral* according to whether the potential energy is *minimum*, *maximum,* or *constant* (Fig. 10.15).

That the result obtained is quite general can be seen as follows: We first observe that a force always tends to do positive work and thus to decrease the potential energy of the system on which it is applied. Therefore, when a system is disturbed from its position of equilibrium, the forces acting on the system will tend to bring it back to its original position if V is minimum (Fig. 10.15*a*) and to move it farther away if V is maximum (Fig. 10.15*b*). If V is constant (Fig. 10.15*c*), the forces will not tend to move the system either way.

Recalling from calculus that a function is minimum or maximum according to whether its second derivative is positive or negative, we can summarize the conditions for the equilibrium of a system with one degree of freedom (i.e., a system the position of which is defined by a single independent variable θ) as follows:

$$\frac{dV}{d\theta} = 0 \qquad \frac{d^2V}{d\theta^2} > 0\text{: stable equilibrium}$$
$$\frac{dV}{d\theta} = 0 \qquad \frac{d^2V}{d\theta^2} < 0\text{: unstable equilibrium} \tag{10.23}$$



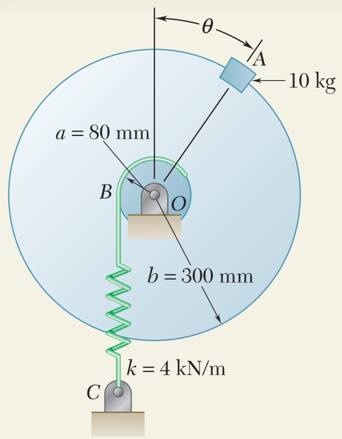
(*a*) Stable equilibrium (*b*) Unstable equilibrium (*c*) Neutral equilibrium

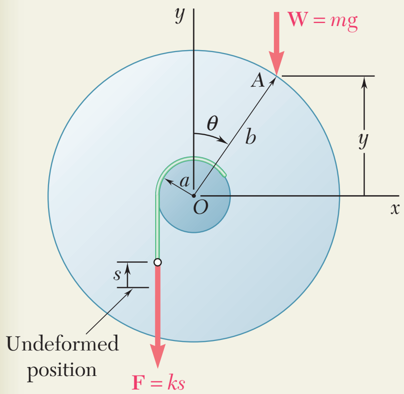
Fig. 10.15

If both the first and the second derivatives of V are zero, it is necessary to examine derivatives of a higher order to determine whether the equilibrium is stable, unstable, or neutral. The equilibrium will be neutral if all derivatives are zero, since the potential energy V is then a constant. The equilibrium will be stable if the first derivative found to be different from zero is of even order and positive. In all other cases the equilibrium will be unstable.

If the system considered possesses *several degrees of freedom*, the potential energy V depends upon several variables, and it is thus necessary to apply the theory of functions of several variables to determine whether V is minimum. It can be verified that a system with 2 degrees of freedom will be stable, and the corresponding potential energy $V(\theta_1, \theta_2)$ will be minimum, if the following relations are satisfied simultaneously:

$$\frac{\partial V}{\partial \theta_1} = \frac{\partial V}{\partial \theta_2} = 0$$
$$\left(\frac{\partial^2 V}{\partial \theta_1 \partial \theta_2}\right)^2 - \frac{\partial^2 V}{\partial \theta_1^2}\frac{\partial^2 V}{\partial \theta_2^2} < 0 \tag{10.24}$$
$$\frac{\partial^2 V}{\partial \theta_1^2} > 0 \quad \text{or} \quad \frac{\partial^2 V}{\partial \theta_2^2} > 0$$





SAMPLE PROBLEM 10.4

A 10-kg block is attached to the rim of a 300-mm-radius disk as shown. Knowing that spring BC is unstretched when $\theta = 0$, determine the position or positions of equilibrium, and state in each case whether the equilibrium is stable, unstable, or neutral.

SOLUTION

Potential Energy. Denoting by s the deflection of the spring from its undeformed position and placing the origin of coordinates at O, we obtain

$$V_e = \tfrac{1}{2}ks^2 \qquad V_g = Wy = mgy$$

Measuring θ in radians, we have

$$s = a\theta \qquad y = b\cos\theta$$

Substituting for s and y in the expressions for V_e and V_g, we write

$$V_e = \tfrac{1}{2}ka^2\theta^2 \qquad V_g = mgb\cos\theta$$
$$V = V_e + V_g = \tfrac{1}{2}ka^2\theta^2 + mgb\cos\theta$$

Positions of Equilibrium. Setting $dV/d\theta = 0$, we write

$$\frac{dV}{d\theta} = ka^2\theta - mgb\sin\theta = 0$$

$$\sin\theta = \frac{ka^2}{mgb}\theta$$

Substituting $a = 0.08$ m, $b = 0.3$ m, $k = 4$ kN/m, and $m = 10$ kg, we obtain

$$\sin\theta = \frac{(4\text{ kN/m})(0.08\text{ m})^2}{(10\text{ kg})(9.81\text{ m/s}^2)(0.3\text{ m})}\theta$$
$$\sin\theta = 0.8699\,\theta$$

where θ is expressed in radians. Solving by trial and error for θ, we find

$$\theta = 0 \quad \text{and} \quad \theta = 0.902\text{ rad}$$
$$\theta = 0 \quad \text{and} \quad \theta = 51.7° \quad ◀$$

Stability of Equilibrium. The second derivative of the potential energy V with respect to θ is

$$\frac{d^2V}{d\theta^2} = ka^2 - mgb\cos\theta$$
$$= (4\text{ kN/m})(0.08\text{ m})^2 - (10\text{ kg})(9.81\text{ m/s}^2)(0.3\text{ m})\cos\theta$$
$$= 25.6 - 29.43\cos\theta$$

For $\theta = 0$: $\quad \dfrac{d^2V}{d\theta^2} = 25.6 - 29.43\cos 0° = -3.83 < 0$

The equilibrium is unstable for $\theta = 0$ ◀

For $\theta = 51.7°$: $\quad \dfrac{d^2V}{d\theta^2} = 25.6 - 29.43\cos 51.7° = +7.36 > 0$

The equilibrium is stable for $\theta = 51.7°$ ◀

SOLVING PROBLEMS ON YOUR OWN

In this lesson we defined the *work of a force during a finite displacement* and the *potential energy* of a rigid body or a system of rigid bodies. You learned to use the concept of potential energy to determine the *equilibrium position* of a rigid body or a system of rigid bodies.

1. The potential energy V of a system is the sum of the potential energies associated with the various forces acting on the system that *do work* as the system moves. In the problems of this lesson you will determine the following:

a. Potential energy of a weight. This is the potential energy due to *gravity*, $V_g = Wy$, where y is the elevation of the weight W measured from some arbitrary reference level. Note that the potential energy V_g may be used with any vertical force $\mathbf{P}$ of constant magnitude directed downward; we write $V_g = Py$.

b. Potential energy of a spring. This is the potential energy due to the *elastic* force exerted by a spring, $V_e = \frac{1}{2}kx^2$, where k is the constant of the spring and x is the deformation of the spring *measured from its unstretched position.*

Reactions at fixed supports, internal forces at connections, forces exerted by inextensible cords and cables, and other forces which do no work do not contribute to the potential energy of the system.

2. Express all distances and angles in terms of a single variable, such as an angle θ, when computing the potential energy V of a system. This is necessary, since the determination of the equilibrium position of the system requires the computation of the derivative $dV/d\theta$.

3. When a system is in equilibrium, the first derivative of its potential energy is zero. Therefore:

a. To determine a position of equilibrium of a system, once its potential energy V has been expressed in terms of the single variable θ, compute its derivative and solve the equation $dV/d\theta = 0$ for θ.

b. To determine the force or couple required to maintain a system in a given position of equilibrium, substitute the known value of θ in the equation $dV/d\theta = 0$ and solve this equation for the desired force or couple.

4. Stability of equilibrium. The following rules generally apply:

a. Stable equilibrium occurs when the potential energy of the system is *minimum,* that is, when $dV/d\theta = 0$ and $d^2V/d\theta^2 > 0$ (Figs. 10.14*a* and 10.15*a*).

b. Unstable equilibrium occurs when the potential energy of the system is *maximum,* that is, when $dV/d\theta = 0$ and $d^2V/d\theta^2 < 0$ (Figs. 10.14*b* and 10.15*b*).

c. Neutral equilibrium occurs when the potential energy of the system is *constant;* $dV/d\theta$, $dV^2/d\theta^2$, and all the successive derivatives of V are then equal to zero (Figs. 10.14*c* and 10.15*c*).

See page 583 for a discussion of the case when $dV/d\theta$, $dV^2/d\theta^2$ but *not all* of the successive derivatives of V are equal to zero.

PROBLEMS

10.59 Using the method of Sec. 10.8, solve Prob. 10.29.

10.60 Using the method of Sec. 10.8, solve Prob. 10.30.

10.61 Using the method of Sec. 10.8, solve Prob. 10.33.

10.62 Using the method of Sec. 10.8, solve Prob. 10.34.

10.63 Using the method of Sec. 10.8, solve Prob. 10.35.

10.64 Using the method of Sec. 10.8, solve Prob. 10.36.

10.65 Using the method of Sec. 10.8, solve Prob. 10.31.

10.66 Using the method of Sec. 10.8, solve Prob. 10.38.

10.67 Show that the equilibrium is neutral in Prob. 10.1.

10.68 Show that the equilibrium is neutral in Prob. 10.6.

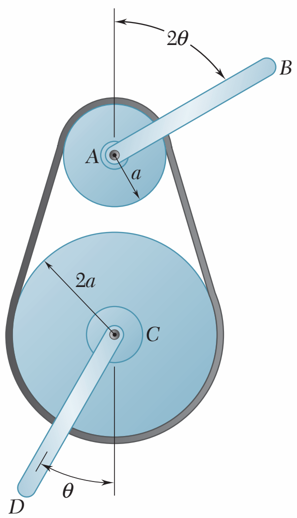


Fig. P10.69

10.69 Two uniform rods, each of mass m and length l, are attached to drums that are connected by a belt as shown. Assuming that no slipping occurs between the belt and the drums, determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

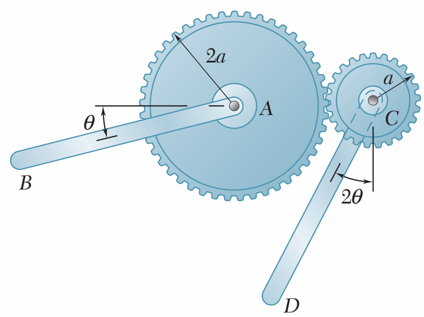


Fig. P10.70

10.70 Two uniform rods AB and CD, of the same length l, are attached to gears as shown. Knowing that rod AB weighs 3 lb and that rod CD weighs 2 lb, determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

10.71 Two uniform rods, each of mass m, are attached to gears of equal radii as shown. Determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

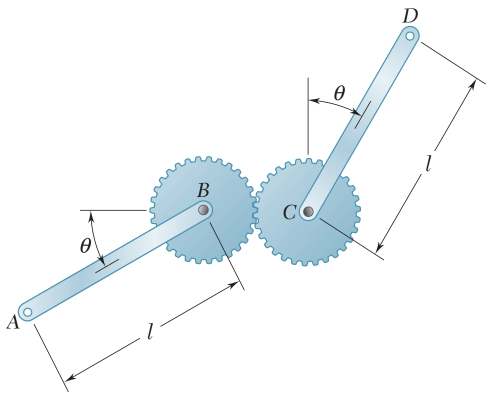


Fig. P10.71 and P10.72

10.72 Two uniform rods, AB and CD, are attached to gears of equal radii as shown. Knowing that $W_{AB} = 8$ lb and $W_{CD} = 4$ lb, determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

10.73 Using the method of Sec. 10.8, solve Prob. 10.39. Determine whether the equilibrium is stable, unstable, or neutral. (*Hint:* The potential energy corresponding to the couple exerted by a torsion spring is $\frac{1}{2}K\theta^2$, where K is the torsional spring constant and θ is the angle of twist.)

10.74 In Prob. 10.40, determine whether each of the positions of equilibrium is stable, unstable, or neutral. (See hint for Prob. 10.73.)

10.75 A load $\mathbf{W}$ of magnitude 100 lb is applied to the mechanism at C. Knowing that the spring is unstretched when $\theta = 15°$, determine that value of θ corresponding to equilibrium and check that the equilibrium is stable.

10.76 A load $\mathbf{W}$ of magnitude 100 lb is applied to the mechanism at C. Knowing that the spring is unstretched when $\theta = 30°$, determine that value of θ corresponding to equilibrium and check that the equilibrium is stable.

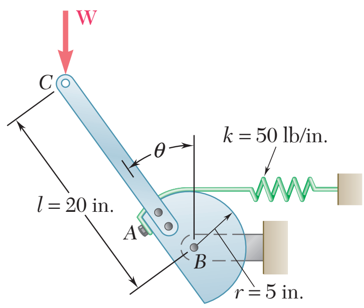


Fig. *P10.75* and *P10.76*

10.77 A slender rod AB, of weight W, is attached to two blocks A and B that can move freely in the guides shown. The constant of the spring is k, and the spring is unstretched when AB is horizontal. Neglecting the weight of the blocks, derive an equation in θ, W, l, and k that must be satisfied when the rod is in equilibrium.

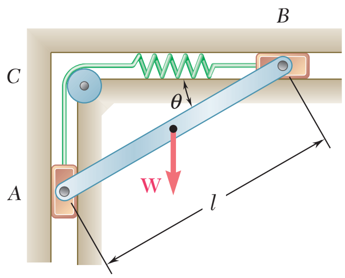


Fig. *P10.77* and P10.78

10.78 A slender rod AB, of weight W, is attached to two blocks A and B that can move freely in the guides shown. Knowing that the spring is unstretched when AB is horizontal, determine three values of θ corresponding to equilibrium when $W = 300$ lb, $l = 16$ in., and $k = 75$ lb/in. State in each case whether the equilibrium is stable, unstable, or neutral.

10.79 A slender rod AB, of weight W, is attached to two blocks A and B that can move freely in the guides shown. Knowing that the spring is unstretched when $y = 0$, determine the value of y corresponding to equilibrium when $W = 80$ N, $l = 500$ mm, and $k = 600$ N/m.

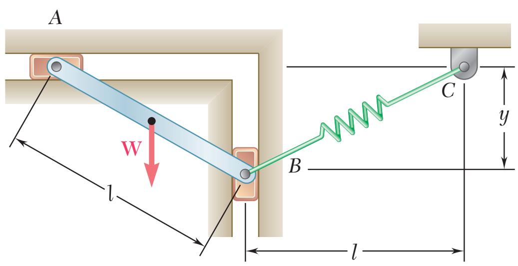


Fig. P10.79

10.80 Knowing that both springs are unstretched when $y = 0$, determine the value of y corresponding to equilibrium when $W = 80$ N, $l = 500$ mm, and $k = 600$ N/m.

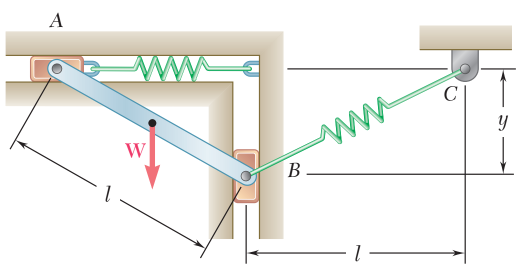


Fig. P10.80

10.81 A spring AB of constant k is attached to two identical gears as shown. Knowing that the spring is undeformed when $\theta = 0$, determine two values of the angle θ corresponding to equilibrium when $P = 30$ lb, $a = 4$ in., $b = 3$ in., $r = 6$ in., and $k = 5$ lb/in. State in each case whether the equilibrium is stable, unstable, or neutral.

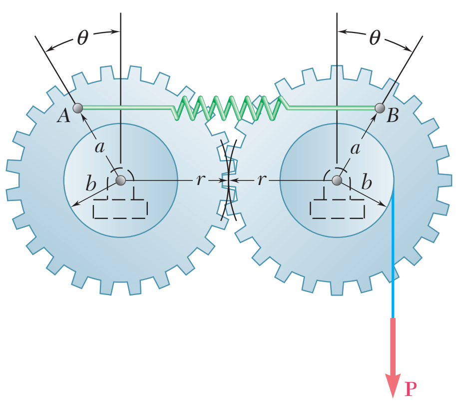


Fig. P10.81 and *P10.82*

10.82 A spring AB of constant k is attached to two identical gears as shown. Knowing that the spring is undeformed when $\theta = 0$, and given that $a = 60$ mm, $b = 45$ mm, $r = 90$ mm, and $k = 6$ kN/m, determine (a) the range of values of P for which a position of equilibrium exists, (b) two values of θ corresponding to equilibrium if the value of P is equal to half the upper limit of the range found in part a.

10.83 A slender rod AB is attached to two collars A and B that can move freely along the guide rods shown. Knowing that $\beta = 30°$ and $P = Q = 400$ N, determine the value of the angle θ corresponding to equilibrium.

10.84 A slender rod AB is attached to two collars A and B that can move freely along the guide rods shown. Knowing that $\beta = 30°$, $P = 100$ N, and $Q = 25$ N, determine the value of the angle θ corresponding to equilibrium.

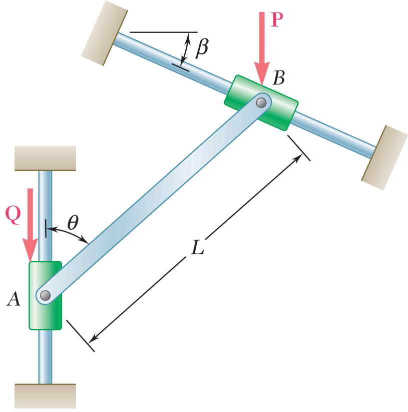


Fig. P10.83 and *P10.84*

10.85 and 10.86 Collar A can slide freely on the semicircular rod shown. Knowing that the constant of the spring is k and that the unstretched length of the spring is equal to the radius r, determine the value of θ corresponding to equilibrium when $W = 50$ lb, $r = 9$ in., and $k = 15$ lb/in.

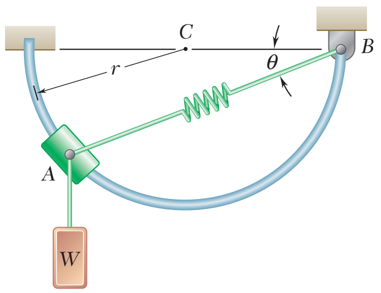


Fig. P10.85

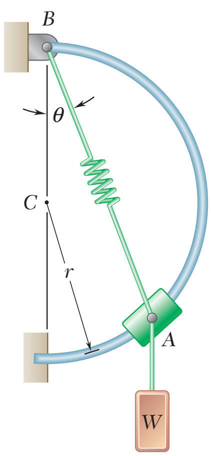


Fig. P10.86

***10.87* and 10.88** Cart B, which weighs 75 kN, rolls along a sloping track that forms an angle β with the horizontal. The spring constant is 5 kN/m, and the spring is unstretched when $x = 0$. Determine the distance x corresponding to equilibrium for the angle β indicated.

10.87 Angle $\beta = 30°$
10.88 Angle $\beta = 60°$

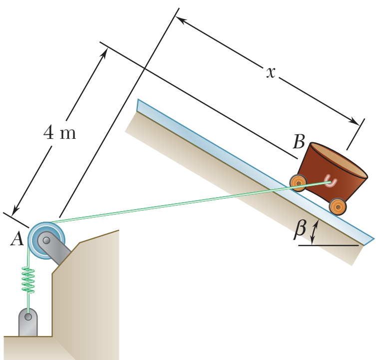


Fig. *P10.87* and P10.88

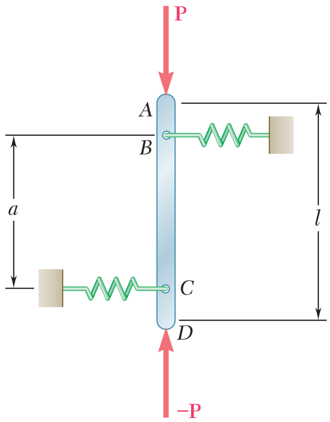


Fig. *P10.89*

10.89 A vertical bar AD is attached to two springs of constant k and is in equilibrium in the position shown. Determine the range of values of the magnitude P of two equal and opposite vertical forces **P** and **−P** for which the equilibrium position is stable if (*a*) $AB = CD$, (*b*) $AB = 2CD$.

10.90 Rod AB is attached to a hinge at A and to two springs, each of constant k. If $h = 25$ in., $d = 12$ in., and $W = 80$ lb, determine the range of values of k for which the equilibrium of the rod is stable in the position shown. Each spring can act in either tension or compression.

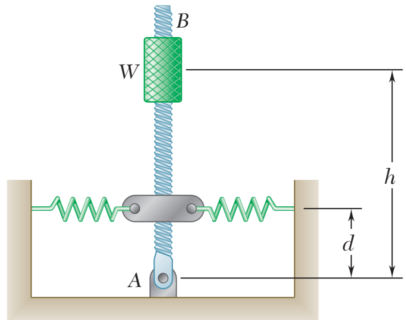


Fig. P10.90 and P10.91

10.91 Rod AB is attached to a hinge at A and to two springs, each of constant k. If $h = 45$ in., $k = 6$ lb/in., and $W = 60$ lb, determine the smallest distance d for which the equilibrium of the rod is stable in the position shown. Each spring can act in either tension or compression.

10.92 and 10.93 Two bars are attached to a single spring of constant k that is unstretched when the bars are vertical. Determine the range of values of P for which the equilibrium of the system is stable in the position shown.

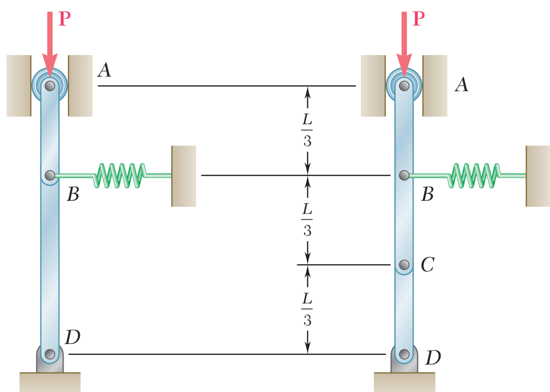


Fig. P10.92 and P10.93

10.94 Two bars AB and BC are attached to a single spring of constant k that is unstretched when the bars are vertical. Determine the range of values of P for which the equilibrium of the system is stable in the position shown.

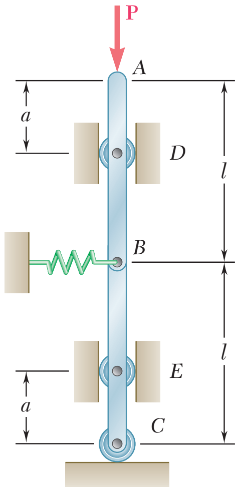


Fig. P10.94

10.95 The horizontal bar BEH is connected to three vertical bars. The collar at E can slide freely on bar DF. Determine the range of values of Q for which the equilibrium of the system is stable in the position shown when $a = 24$ in., $b = 20$ in., and $P = 150$ lb.

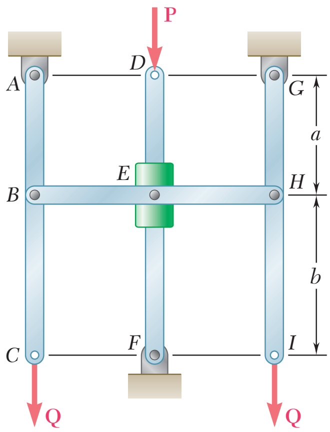


Fig. *P10.95* and P10.96

10.96 The horizontal bar BEH is connected to three vertical bars. The collar at E can slide freely on bar DF. Determine the range of values of P for which the equilibrium of the system is stable in the position shown when $a = 150$ mm, $b = 200$ mm, and $Q = 45$ N.

****10.97*** Bars AB and BC, each of length l and of negligible weight, are attached to two springs, each of constant k. The springs are undeformed, and the system is in equilibrium when $\theta_1 = \theta_2 = 0$. Determine the range of values of P for which the equilibrium position is stable.

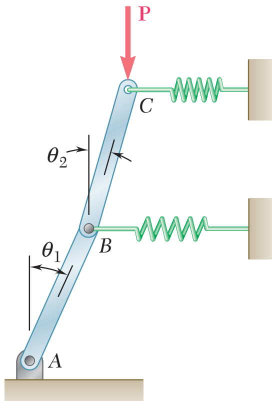


Fig. *P10.97*

***10.98** Solve Prob. 10.97 knowing that $l = 800$ mm and $k = 2.5$ kN/m.

****10.99*** Two rods of negligible weight are attached to drums of radius r that are connected by a belt and spring of constant k. Knowing that the spring is undeformed when the rods are vertical, determine the range of values of P for which the equilibrium position $\theta_1 = \theta_2 = 0$ is stable.

***10.100** Solve Prob. 10.99 knowing that $k = 20$ lb/in., $r = 3$ in., $l = 6$ in., and (*a*) $W = 15$ lb, (*b*) $W = 60$ lb.

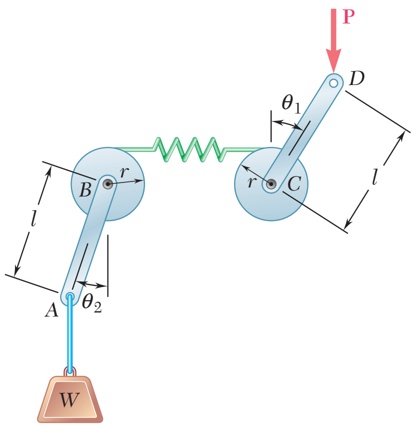


Fig. *P10.99*

REVIEW AND SUMMARY

Work of a force

The first part of this chapter was devoted to the *principle of virtual work* and to its direct application to the solution of equilibrium problems. We first defined the *work of a force* **F** *corresponding to the small displacement* $d\mathbf{r}$ [Sec. 10.2] as the quantity

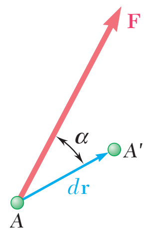


Fig. 10.16

$$dU = \mathbf{F} \cdot d\mathbf{r} \tag{10.1}$$

obtained by forming the scalar product of the force **F** and the displacement $d\mathbf{r}$ (Fig. 10.16). Denoting respectively by F and ds the magnitudes of the force and of the displacement, and by α the angle formed by **F** and $d\mathbf{r}$, we wrote

$$dU = F\, ds \cos \alpha \tag{10.1′}$$

The work dU is positive if $\alpha < 90°$, zero if $\alpha = 90°$, and negative if $\alpha > 90°$. We also found that the *work of a couple of moment* **M** acting on a rigid body is

$$dU = M\, d\theta \tag{10.2}$$

where $d\theta$ is the small angle expressed in radians through which the body rotates.

Virtual displacement

Considering a particle located at A and acted upon by several forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ [Sec. 10.3], we imagined that the particle moved to a new position A' (Fig. 10.17). Since this displacement did not actually take place, it was referred to as a *virtual displacement* and denoted by $\delta\mathbf{r}$, while the corresponding work of the forces was called *virtual work* and denoted by δU. We had

$$\delta U = \mathbf{F}_1 \cdot \delta\mathbf{r} + \mathbf{F}_2 \cdot \delta\mathbf{r} + \cdots + \mathbf{F}_n \cdot \delta\mathbf{r}$$

Principle of virtual work

The *principle of virtual work* states that *if a particle is in equilibrium, the total virtual work δU of the forces acting on the particle is zero for any virtual displacement of the particle.*

The principle of virtual work can be extended to the case of rigid bodies and systems of rigid bodies. Since it involves *only forces which do work,* its application provides a useful alternative to the use of the equilibrium equations in the solution of many engineering problems. It is particularly effective in the case of machines and mechanisms consisting of connected rigid bodies, since the work of the reactions at the supports is zero and the work of the internal forces at the pin connections cancels out [Sec. 10.4; Sample Probs. 10.1, 10.2, and 10.3].

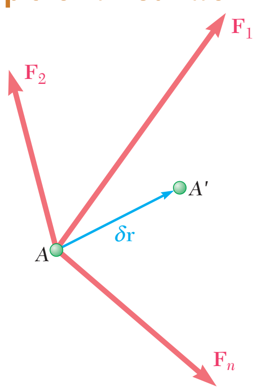


Fig. 10.17

In the case of *real machines*, however [Sec. 10.5], the work of the friction forces should be taken into account, with the result that the *output work will be less than the input work.* Defining the *mechanical efficiency* of a machine as the ratio

Mechanical efficiency

$$\eta = \frac{\text{output work}}{\text{input work}} \tag{10.9}$$

we also noted that for an ideal machine (no friction) $\eta = 1$, while for a real machine $\eta < 1$.

Work of a force over a finite displacement

In the second part of the chapter we considered the *work of forces corresponding to finite displacements* of their points of application. The work $U_{1\rightarrow 2}$ of the force $\mathbf{F}$ corresponding to a displacement of the particle A from A_1 to A_2 (Fig. 10.18) was obtained by integrating the right-hand member of Eq. (10.1) or (10.1′) along the curve described by the particle [Sec. 10.6]:

$$U_{1\rightarrow 2} = \int_{A_1}^{A_2} \mathbf{F} \cdot d\mathbf{r} \tag{10.11}$$

or

$$U_{1\rightarrow 2} = \int_{s_1}^{s_2} (F \cos \alpha)\, ds \tag{10.11′}$$

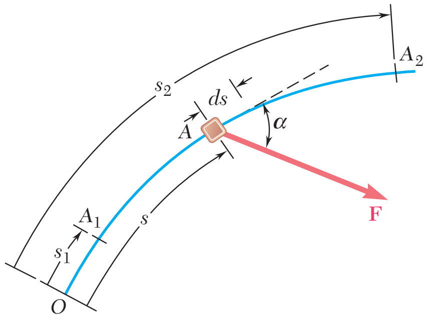


Fig. 10.18

Similarly, the work of a couple of moment $\mathbf{M}$ corresponding to a finite rotation from θ_1 to θ_2 of a rigid body was expressed as

$$U_{1\rightarrow 2} = \int_{\theta_1}^{\theta_2} M\, d\theta \tag{10.12}$$

Work of a weight

The *work of the weight* $\mathbf{W}$ *of a body* as its center of gravity moves from the elevation y_1 to y_2 (Fig. 10.19) can be obtained by making $F = W$ and $\alpha = 180°$ in Eq. (10.11′):

$$U_{1\rightarrow 2} = -\int_{y_1}^{y_2} W\,dy = Wy_1 - Wy_2 \tag{10.13}$$

The work of $\mathbf{W}$ is therefore positive *when the elevation* y *decreases.*

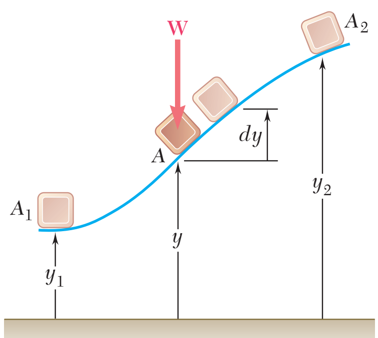


Fig. 10.19

Work of the force exerted by a spring

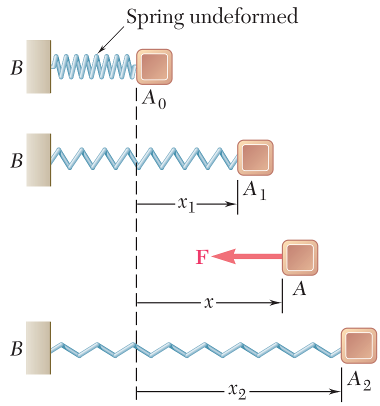


Fig. 10.20

The *work of the force* **F** *exerted by a spring* on a body A as the spring is stretched from x_1 to x_2 (Fig. 10.20) can be obtained by making $F = kx$, where k is the constant of the spring, and $\alpha = 180°$ in Eq. (10.11′):

$$U_{1\rightarrow 2} = -\int_{x_1}^{x_2} kx\,dx = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2 \qquad (10.15)$$

The work of **F** is therefore positive *when the spring is returning to its undeformed position.*

Potential energy

When the work of a force **F** is independent of the path actually followed between A_1 and A_2, the force is said to be a *conservative force,* and its work can be expressed as

$$U_{1\rightarrow 2} = V_1 - V_2 \qquad (10.20)$$

where V is the *potential energy* associated with **F**, and V_1 and V_2 represent the values of V at A_1 and A_2, respectively [Sec. 10.7]. The potential energies associated, respectively, with the *force of gravity* **W** and the *elastic force* **F** exerted by a spring were found to be

$$V_g = Wy \qquad \text{and} \qquad V_e = \tfrac{1}{2}kx^2 \qquad (10.17,\ 10.18)$$

Alternative expression for the principle of virtual work

When the position of a mechanical system depends upon a single independent variable θ, the potential energy of the system is a function $V(\theta)$ of that variable, and it follows from Eq. (10.20) that $\delta U = -\delta V = -(dV/d\theta)\,\delta\theta$. The condition $\delta U = 0$ required by the principle of virtual work for the equilibrium of the system can thus be replaced by the condition

$$\frac{dV}{d\theta} = 0 \qquad (10.21)$$

When all the forces involved are conservative, it may be preferable to use Eq. (10.21) rather than apply the principle of virtual work directly [Sec. 10.8; Sample Prob. 10.4].

Stability of equilibrium

This approach presents another advantage, since it is possible to determine from the sign of the second derivative of V whether the equilibrium of the system is *stable, unstable,* or *neutral* [Sec. 10.9]. If $d^2V/d\theta^2 > 0$, V is *minimum* and the equilibrium is *stable;* if $d^2V/d\theta^2 < 0$, V is *maximum* and the equilibrium is *unstable;* if $d^2V/d\theta^2 = 0$, it is necessary to examine derivatives of a higher order.

REVIEW PROBLEMS

10.101 Determine the vertical force **P** that must be applied at G to maintain the equilibrium of the linkage.

10.102 Determine the couple **M** that must be applied to member $DEFG$ to maintain the equilibrium of the linkage.

10.103 Derive an expression for the magnitude of the couple **M** required to maintain the equilibrium of the linkage shown.

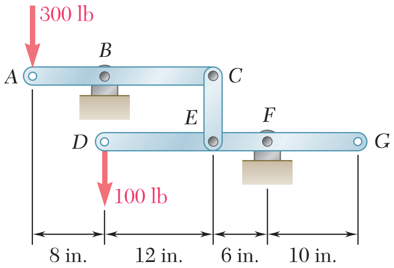


Fig. P10.101 and P10.102

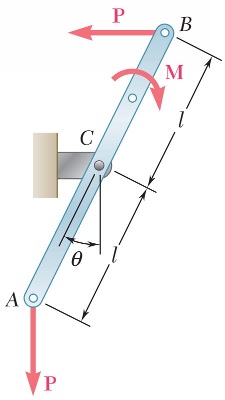


Fig. *P10.103*

10.104 Collars A and B are connected by the wire AB and can slide freely on the rods shown. Knowing that the length of the wire is 440 mm and that the weight W of collar A is 90 N, determine the magnitude of the force **P** required to maintain equilibrium of the system when (*a*) $c = 80$ mm, (*b*) $c = 280$ mm.

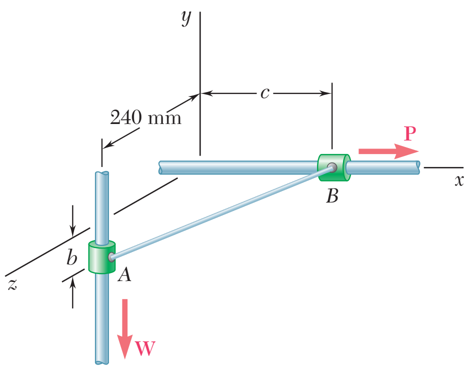


Fig. P10.104

10.105 Collar B can slide along rod AC and is attached by a pin to a block that can slide in the vertical slot shown. Derive an expression for the magnitude of the couple **M** required to maintain equilibrium.

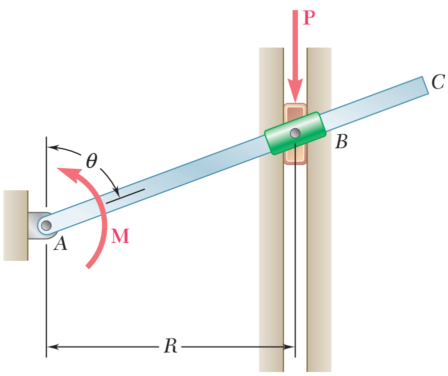


Fig. *P10.105*

10.106 A slender rod of length l is attached to a collar at B and rests on a portion of a circular cylinder of radius r. Neglecting the effect of friction, determine the value of θ corresponding to the equilibrium position of the mechanism when $l = 200$ mm, $r = 60$ mm, $P = 40$ N, and $Q = 80$ N.

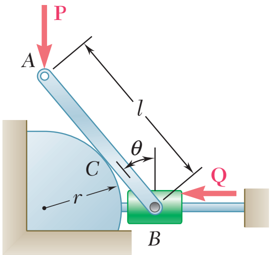


Fig. P10.106

10.107 A horizontal force **P** of magnitude 40 lb is applied to the mechanism at C. The constant of the spring is $k = 9$ lb/in., and the spring is unstretched when $\theta = 0$. Neglecting the weight of the mechanism, determine the value of θ corresponding to equilibrium.

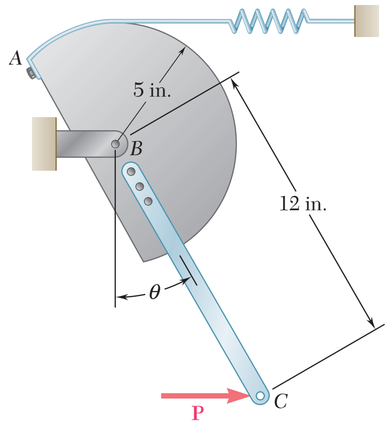


Fig. P10.107

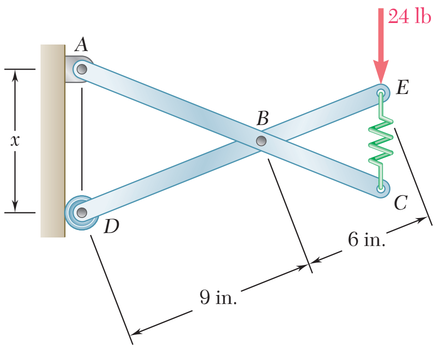


Fig. P10.108

10.108 Two identical rods ABC and DBE are connected by a pin at B and by a spring CE. Knowing that the spring is 4 in. long when unstretched and that the constant of the spring is 8 lb/in., determine the distance x corresponding to equilibrium when a 24-lb load is applied at E as shown.

10.109 Solve Prob. 10.108 assuming that the 24-lb load is applied at C instead of E.

10.110 Two uniform rods, each of mass m and length l, are attached to gears as shown. For the range $0 \le \theta \le 180°$, determine the positions of equilibrium of the system and state in each case whether the equilibrium is stable, unstable, or neutral.

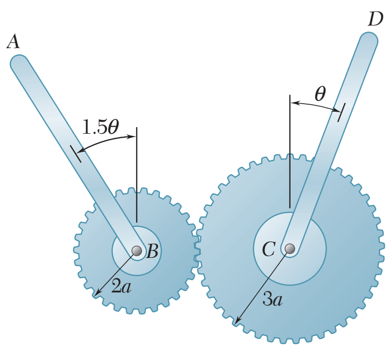


Fig. P10.110

10.111 A homogeneous hemisphere of radius r is placed on an incline as shown. Assuming that friction is sufficient to prevent slipping between the hemisphere and the incline, determine the angle θ corresponding to equilibrium when $\beta = 10°$.

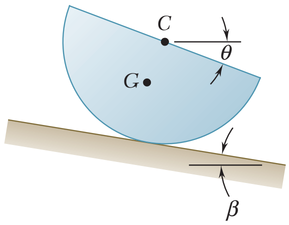


Fig. *P10.111* and P10.112

10.112 A homogeneous hemisphere of radius r is placed on an incline as shown. Assuming that friction is sufficient to prevent slipping between the hemisphere and the incline, determine (*a*) the largest angle β for which a position of equilibrium exists, (*b*) the angle θ corresponding to equilibrium when the angle β is equal to half the value found in part *a*.

COMPUTER PROBLEMS

10.C1 A couple **M** is applied to crank AB in order to maintain the equilibrium of the engine system shown when a force **P** is applied to the piston. Knowing that $b = 2.4$ in. and $l = 7.5$ in., write a computer program that can be used to calculate the ratio M/P for values of θ from 0 to 180° using 10° increments. Using appropriate smaller increments, determine the value of θ for which the ratio M/P is maximum, and the corresponding value of M/P.

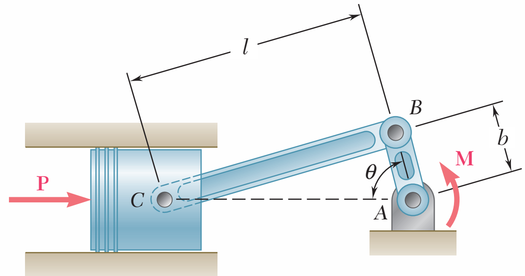


Fig. P10.C1

10.C2 Knowing that $a = 500$ mm, $b = 150$ mm, $L = 500$ mm, and $P = 100$ N, write a computer program that can be used to calculate the force in member BD for values of θ from 30° to 150° using 10° increments. Using appropriate smaller increments, determine the range of values of θ for which the absolute value of the force in member BD is less than 400 N.

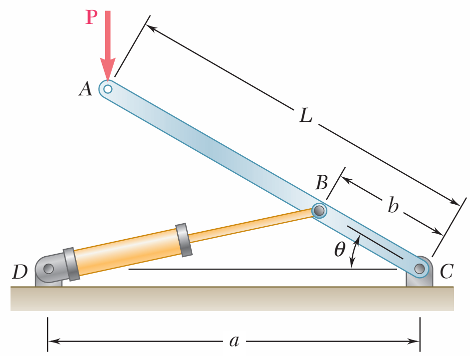


Fig. P10.C2

10.C3 Solve Prob. 10.C2 assuming that the force **P** applied at A is directed horizontally to the right.

10.C4 The constant of spring AB is k, and the spring is unstretched when $\theta = 0$. (a) Neglecting the weight of the member BCD, write a computer program that can be used to calculate the potential energy of the system and its derivative $dV/d\theta$. (b) For $W = 150$ lb, $a = 10$ in., and $k = 75$ lb/in., calculate and plot the potential energy versus θ for values of θ from 0 to 165° using 15° increments. (c) Using appropriate smaller increments, determine the values of θ for which the system is in equilibrium and state in each case whether the equilibrium is stable, unstable, or neutral.

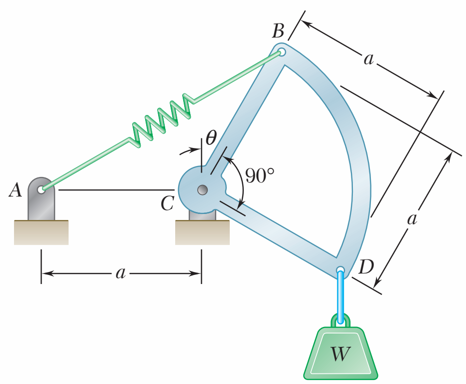


Fig. P10.C4

10.C5 Two rods, AC and DE, each of length L, are connected by a collar that is attached to rod AC at its midpoint B. (a) Write a computer program that can be used to calculate the potential energy V of the system and its derivative $dV/d\theta$. (b) For $W = 75$ N, $P = 200$ N, and $L = 500$ mm, calculate V and $dV/d\theta$ for values of θ from 0 to 70° using 5° increments. (c) Using appropriate smaller increments, determine the values of θ for which the system is in equilibrium and state in each case whether the equilibrium is stable, unstable, or neutral.

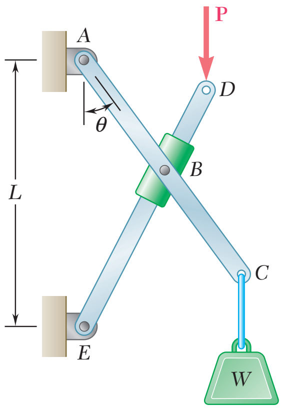


Fig. P10.C5

10.C6 A slender rod ABC is attached to blocks A and B that can move freely in the guides shown. The constant of the spring is k, and the spring is unstretched when the rod is vertical. (a) Neglecting the weights of the rod and of the blocks, write a computer program that can be used to calculate the potential energy V of the system and its derivative $dV/d\theta$. (b) For $P = 150$ N, $l = 200$ mm, and $k = 3$ kN/m, calculate and plot the potential energy versus θ for values of θ from 0 to 75° using 5° increments. (c) Using appropriate smaller increments, determine any positions of equilibrium in the range $0 \leq \theta \leq 75°$ and state in each case whether the equilibrium is stable, unstable, or neutral.

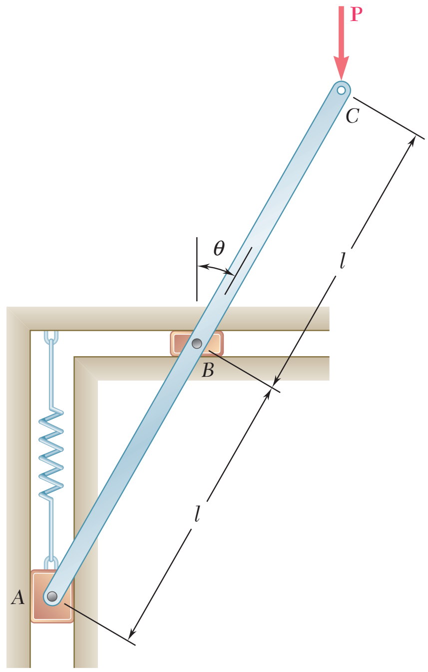


Fig. P10.C6

10.C7 Solve Prob. 10.C6 assuming that the force **P** applied at C is directed horizontally to the right.

The motion of the space shuttle can be described in terms of its *position, velocity,* and *acceleration*. When landing, the pilot of the shuttle needs to consider the wind velocity and the *relative motion* of the shuttle with respect to the wind. The study of motion is known as *kinematics* and is the subject of this chapter.