CHAPTER TWO

METHOD OF LEAST WORK

The method of least work is used for the analysis of statically indeterminate beams, frames and trusses. Indirect use of the Castigliano's 2nd theorem is made and the following steps are taken.

(1) The structure is considered under the action of applied loads and the redundants. The redundants can be decided by choosing a particular basic determinate structure and the choice of redundants may vary within a problem.

(2) Moment expressions for the entire structure are established in terms of the applied loads and the redundants, which are assumed to act simultaneously for beams and frames.

(3) Strain energy stored due to direct forces and in bending etc. is calculated and is partially differentiated with respect to the redundants.

(4) A set of linear equations is obtained, the number of which is equal to that of the redundants.Solution of these equations evaluates the redundants.

NOTE:–

Special care must be exercised while partially differentiating the strain energy expressions and compatibility requirements of the chosen basic determinate structure should also be kept in mind. For the convenience of readers, Castigliano's theorem are given below:

2.1. CASTIGLIANO'S FIRST THEOREM:–

"The partial derivative of the total strain energy stored with respect to a particular deformation gives the corresponding force acting at that point."

Mathematically this theorem is stated as below:

$$\frac{\partial U}{\partial \Delta} = P$$

and

$$\frac{\partial U}{\partial \theta} = M$$

It suggests that displacements correspond to loads while rotations correspond to moments.

2.2. CASTIGLIANO'S SECOND THEOREM :–

"The partial derivative of the total strain energy stored with respect to a particular force gives the corresponding deformation at that point."

Mathematically,

$$\frac{\partial U}{\partial P} = \Delta$$

and

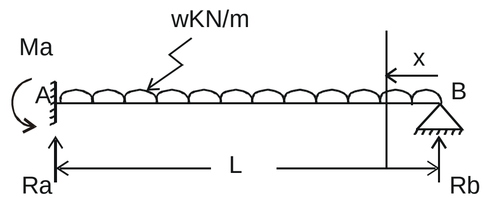
$$\frac{\partial U}{\partial M} = \theta$$

2.3. STATEMENT OF THEOREM OF LEAST WORK.

"In a statically indeterminate structure, the redundants are such that the internal strain energy stored is minimum." This minima is achieved by partially differentiating strain energy and setting it to zero or to a known value. This forms the basis of structural stability and of Finite Element Method.

2.4. Example No.1: **<u>1st Degree Indeterminacy of Beams.</u>**

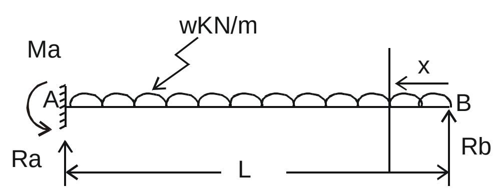
Analyze the following loaded beam by the method of least work.



Number of reactions = 3
Number of equations = 2

The beam is redundant to first degree.

In case of cantilever, always take free end as the origin for establishing moment expressions. Choosing cantilever with support at A and Rb as redundant. Apply loads and redundant simultaneously to BDS.



Taking B as origin (for variation of X)

$$M_X = \left(RbX - \frac{wX^2}{2}\right) \qquad 0 < X < L$$

$$U = \frac{1}{2EI}\int_o^L M^2\,dX. \qquad \text{A generalized strain energy expression due to moments.}$$

Therefore, partially differentiating the strain energy stored w.r.t. redundant, the generalized form is:

$$\frac{\partial U}{\partial R} = \frac{1}{EI}\int_o^L M\left(\frac{\partial M}{\partial R}\right) dX \qquad \text{Where R is a typical redundant.}$$

Putting moment expression alongwith its limits of validity in strain energy expression.

$$U = \frac{1}{2EI}\int_o^L \left(RbX - \frac{wX^2}{2}\right)^2 dX$$

Partially differentiate strain energy U w.r.t. redundant Rb, and set equal to zero.

So $$\frac{\partial U}{\partial Rb} = \Delta b = 0 = \frac{1}{EI}\int_o^L \left(RbX - \frac{wX^2}{2}\right)(X)\,dX,$$ because at B, there should be no deflection.

$$0 = \frac{1}{EI}\int_{o}^{L}\left[RbX^2 - \frac{wX^3}{2}\right]dX$$

$$0 = \frac{1}{EI}\left[\frac{RbX^3}{3} - \frac{wX^4}{8}\right]_o^L$$

Or $\frac{RbL^3}{3} = \frac{wL^4}{8}$

and

$$\boxed{Rb = \frac{+3}{8}wL}$$

The (+ve) sign with Rb indicates that the assumed direction of redundant Rb is correct. Now calculate Ra.

$\sum Fy = 0$

$Ra + Rb = wL$

$Ra = wL - Rb$

$= wL - \frac{3}{8}wL$

$= \frac{8\ wL - 3\ wL}{8}$

$$\boxed{Ra = \frac{5}{8}wL}$$

Put $X = L$ and $Rb = \frac{3}{8}$ wL in moment expression for M_X already established before to get Ma.

$Ma = \frac{3}{8}wL\ .L - \frac{wL^2}{2}$

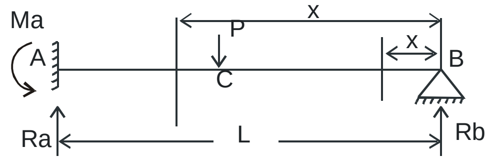
$= \frac{3}{8}wL^2 - \frac{wL^2}{2}$

$= \frac{3\ wL^2 - 4\ wL^2}{8}$

$$\boxed{Ma = -\frac{wL^2}{8}}$$

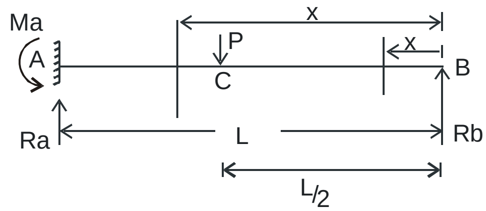
The (–ve) sign with Ma indicates that this reactive moment should be applied such that it gives us tension at the top at point A.

Example No.2: Solve the following propped cantilever loaded at its centre as shown by method of least work.



B.D.S.is a cantiever supported at A.

Rb is a redundant as shown.



BDS under loads and redundant. Taking point B as origin.

$$M_{bc} = R_bX \qquad 0 < X < \frac{L}{2}$$

and $$M_{ac} = R_bX - P\left(x - \frac{L}{2}\right) \qquad \frac{L}{2} < X < L.$$ Now write strain energy expression.

$$U = \frac{1}{2EI}\int_o^{L/2} (R_bX)^2\,dX + \frac{1}{2EI}\int_{L/2}^{L}\left[R_bX - P\left(X - \frac{L}{2}\right)\right]^2 dX.$$ Partially differentiate w.r.t redundant Rb.

$$\frac{\partial U}{\partial R_b} = \Delta b = 0 = \frac{1}{EI}\int_o^{L/2}[R_bX]\,[X]\,dX + \frac{1}{EI}\int_{L/2}^{L}\left[R_bx - P\left(X - \frac{L}{2}\right)\right][X]\,dX$$

$$0 = \frac{1}{EI}\int_o^{L/2} R_bX^2\,dX + \frac{1}{EI}\int_{L/2}^{L}\left(R_bX^2 - PX^2 + P\frac{L}{2}X\right)dX$$

$$0 = \frac{1}{EI}\left[R_b.\frac{X^3}{3}\right]_o^{L/2} + \frac{1}{EI}\left[\frac{R_bX^3}{3} - \frac{PX^3}{3} + \frac{PL}{4}X^2\right]_{L/2}^{L}.$$ Put limits

$$0 = \frac{1}{EI}\left[\frac{R_bL^3}{24} - 0\right] + \frac{1}{EI}\left[\frac{R_bL^3}{3} - \frac{PL^3}{3} + \frac{PL^3}{4} - \frac{R_bL^3}{24} + \frac{PL^3}{24} - \frac{PL^3}{16}\right]$$

$$0 = \frac{1}{EI}\left[\frac{R_bL^3}{24} + \frac{R_bL^3}{3} - \frac{R_bL^3}{24} - \frac{PL^3}{3} + \frac{PL^3}{4} + \frac{PL^3}{24} - \frac{PL^3}{16}\right]$$

$$0 = \frac{1}{EI}\left[\frac{R_bL^3}{3} + \left(\frac{-16PL^3 + 12PL^3 + 2PL^3 - 3PL^3}{48}\right)\right]$$

$$0 = \frac{RbL^3}{3} - \frac{5PL^3}{48}$$

Or $$\frac{RbL^3}{3} = \frac{5PL^3}{48}$$

$$\boxed{Rb = \frac{+5P}{16}}$$

The (+ve) sign with Rb indicates that the assumed direction of redundant Rb is correct. Now Ra can be calculated.

$\sum Fy = 0$

$Ra + Rb = P$

$Ra = P - Rb$

$$Ra = P - \frac{5P}{16} = \frac{16P - 5P}{16}$$

$$\boxed{Ra = \frac{11P}{16}}$$

Put X = L and $Rb = \frac{5P}{16}$ in expression for Mac to get Ma.

$$Ma = \frac{5P}{16} L - P\frac{L}{2}$$

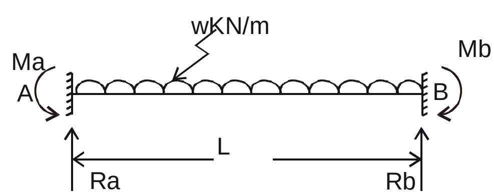
$$= \frac{5 PL - 8 PL}{16}$$

$$\boxed{Ma = \frac{-3 PL}{16}}$$

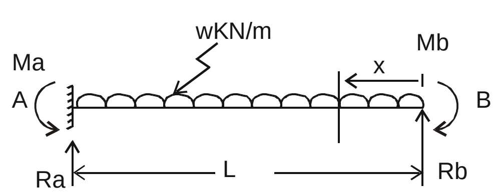
The (−ve) sign with Ma indicates that this reactive moment should be acting such that it gives us tension at the top.

2.5. 2ND DEGREE INDETERMINACY:–

EXAMPLE NO. 3: Analyze the following fixed ended beam loaded by Udl by least work method.



B.D.S. is chosen as a cantilever supported at A. Rb and Mb are chosen as redundants.



BDS UNDER LOADS AND REDUNDANTS

$$Mx = RbX - \frac{wX^2}{2} - Mb \qquad 0 < X < L \qquad \text{Choosing B as origin.}$$

Write strain energy expression.

$$U = \frac{1}{2EI}\int_o^L \left[RbX - \frac{wX^2}{2} - Mb\right]^2 dX$$

Differentiate strain energy partially w.r.t. redundant Rb and use castigations theorem alongwith boundary condition.

$$\frac{\partial U}{\partial Rb} = \Delta b = 0 = \frac{1}{EI}\int_o^L \left[RbX - \frac{wX^2}{2} - Mb\right][X]\, dX$$

$$0 = \frac{1}{EI}\int_o^L \left[RbX - \frac{wX^2}{2} - Mb\right] dX$$

$$0 = \frac{1}{EI}\left[Rb\,\frac{X^3}{3} - \frac{wX^4}{8} - \frac{MbX^2}{2}\right]_o^L$$

$$0 = \frac{1}{EI}\left[Rb\,\frac{L^3}{3} - \frac{wL^4}{8} - \frac{MbL^2}{2}\right]$$

$$0 = Rb\,\frac{L^3}{3} - \frac{wL^4}{8} - \frac{MbL^2}{2} \qquad \rightarrow \quad (1)$$

As there are two redundants, so we require two equations. Now differentiate strain energy expression w.r.t. another redundants Mb. Use castigations theorem and boundary condition.

$$\frac{\partial U}{\partial Mb} = \theta b = 0 = \frac{1}{EI}\int_o^L \left[RbX - \frac{wX^2}{2} - Mb\right](-1)\, dX$$

$$0 = \frac{1}{EI}\int_o^L \left(-RbX + \frac{wX^2}{2} + Mb\right) dX$$

$$0 = \frac{1}{EI}\left[-\frac{RbX^2}{2} + \frac{wX^3}{6} + MbX\right]_o^L$$

$$0 = -\frac{Rb\,L^2}{2} + \frac{wL^3}{6} + MbL.$$

$$\frac{Rb\,L^2}{2} - \frac{wL^3}{6} = MbL$$

So $$Mb = \frac{RbL}{2} - \frac{wL^2}{6} \qquad \rightarrow \quad (2)$$ Put Mb in equation 1, we get

$$0 = \frac{RbL^3}{3} - \frac{wL^4}{8} - \left(\frac{RbL}{2} - \frac{wL^2}{6}\right)\frac{L^2}{2}$$

$$0 = \frac{RbL^3}{3} - \frac{wL^4}{8} - \frac{RbL^3}{4} + \frac{wL^4}{12}$$

$$0 = \frac{RbL^3}{12} - \frac{wL^4}{24}$$

$$\boxed{Rb = \frac{wL}{2}}$$

Put Rb value in equation 2, we have

$$Mb = \left(\frac{wL}{2}\right)\frac{L}{2} - \frac{wL^2}{6}$$

$$Mb = \frac{+wL^2}{12}$$

The (+ve) value with Rb and Mb indicates that the assumed directions of these two redundants are correct. Now find other reactions Ra and Mb by using equations of static equilibrium.

$$\sum Fy = 0$$
$$Ra + Rb = wL$$
$$Ra = wL - Rb$$
$$= wL - \frac{wL}{2}$$

$$\boxed{Ra = \frac{wL}{2}}$$

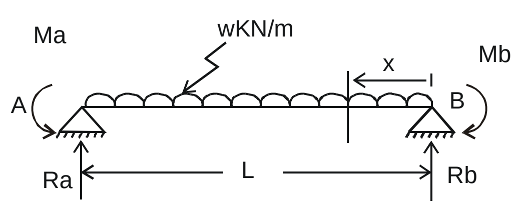
Put $X = L$, $Rb = \frac{wL}{2}$ & $Mb = \frac{wL^2}{12}$ in M_X expression to get Ma

$$Ma = \frac{wL}{2} . L - \frac{wL^2}{2} - \frac{wL^2}{12}$$

$$\boxed{Ma = -\frac{wL^2}{12}}$$

The (–ve) sign with Ma indicates that this moment should be applied in such direction that it gives us tension at the top.

Example No. 4: Solve the same previous fixed ended beam by taking a simple beam as B.D.S.:–
Choosing Ma and Mb as redundants.



BDS UNDER LOADS AND REDUNDANTS

B.D.S. is a simply supported beam , So Ma and Mb are redundants.

$$\sum Ma = 0$$

$$Rb \times L + Ma = Mb + \frac{wL^2}{2}$$

$$Rb \times L = (Mb - Ma) + \frac{wL^2}{2}$$

$$Rb = \left(\frac{Mb - Ma}{L}\right) + \frac{wL}{2}$$ So taking B as origin. Write M_X expression.

$$M_X = RbX - Mb - \frac{wX^2}{2} \qquad 0 < X < L$$

Put Rb value

$$M_X = \left[\left(\frac{Mb - Ma}{L}\right) + \frac{wL}{2}\right] X - \frac{wX^2}{2} - Mb \qquad 0 < X < L.$$ Set up strain energy expression.

$$U = \frac{1}{2EI}\int_o^L \left[\left\{\left(\frac{Mb - Ma}{L}\right) + \frac{wL}{2}\right\} X - \frac{wX^2}{2} - Mb\right]^2 dX.$$ Differentiate w.r.t. Ma first. Use castigations theorem and boundary conditions.

$$\frac{\partial U}{\partial Ma} = \theta a = 0 = \frac{1}{EI}\int_o^L \left[\left\{\left(\frac{Mb - Ma}{L}\right) + \frac{wL}{2}\right\} X - \frac{wX^2}{2} - Mb\right]\left(-\frac{X}{L}\right) dX.$$ In general R.H.S. is $\frac{1}{EI}\int N.m.dX.$

$$0 = \frac{1}{EI}\int_o^L \left(\frac{MbX}{L} - \frac{MaX}{L} + \frac{wL}{2}X - \frac{wX^2}{2} - Mb\right)\left(-\frac{X}{L}\right) dX$$

$$0 = \frac{1}{EI}\int_o^L \left[-\frac{MbX^2}{L^2} + \frac{MaX^2}{L^2} - \frac{wX^2}{2} + \frac{wX^3}{2L} + \frac{MbX}{L}\right] dX \qquad .$$ Integrate it.

$$0 = \frac{1}{EI}\left[-\frac{Mb}{L^2}\frac{X^2}{3} + \frac{Ma}{L^2}\frac{X^3}{3} - \frac{wX^3}{6} + \frac{wX^4}{8L} + \frac{MbX^2}{2L}\right]_o^L \qquad .$$ Simplify it.

$$0 = \frac{MbL}{6} + \frac{MaL}{3} - \frac{wL^3}{24} \qquad \rightarrow \quad (1)$$

Now differentiate U Partially w.r.t. Mb. Use castiglianos theorem and boundary conditions.

$$\frac{\partial U}{\partial Mb} = \theta b = 0 = \frac{1}{EI}\int_o^L \left[\left\{\left(\frac{Mb - Ma}{L}\right) + \frac{wL}{2}\right\} X - \frac{wX^2}{2} - Mb\right]\left(\frac{X}{L} - 1\right) dX$$

$$0 = \frac{1}{EI}\int_o^L \left(\frac{MbX}{L} - \frac{MaX}{L} + \frac{wL}{2}X - \frac{wX^2}{2} - Mb\right)\left(\frac{X}{L} - 1\right) dX$$

$$0 = \int_o^L \left[\frac{MbX^2}{L^2} - \frac{MaX^2}{L^2} + \frac{wLX^2}{2L} - \frac{wX^3}{2L} - \frac{MbX}{L} - \frac{MbX}{L} + \frac{MaX}{L} - \frac{wLX}{2} + \frac{wX^2}{2} + Mb\right] dX$$

$$0 = \left[\frac{MbX^3}{3L^2} - \frac{MaX^3}{3L^2} + \frac{wX^3}{6} - \frac{wX^4}{8L} - \frac{MbX^2}{2L} - \frac{MbX^2}{2L} + \frac{MaX^2}{2L} - \frac{wLX^2}{4} + \frac{wX^3}{6} + MbX\right]_o^L$$

Put limits now.

$$0 = \left[\frac{MbL^3}{3L^2} - \frac{MaL^3}{3L^2} + \frac{wL^3}{6} - \frac{wL^4}{8L} - \frac{MbL^2}{2L} - \frac{MbL^2}{2L} + \frac{MaL^2}{2L} - \frac{wLL^2}{4} + \frac{wL^3}{6} + MbL\right]$$

Simplifying we get.

$$0 = \frac{MbL}{3} + \frac{MaL}{6} - \frac{wL^3}{24}$$

or $$\frac{MbL}{3} = -\frac{MaL}{6} + \frac{wL^3}{24}$$

so $$Mb = \frac{wL^2}{8} - \frac{Ma}{2} \qquad (2),$$ Put Mb in equation (1) we get.

$$0 = \left(\frac{wL^2}{8} - \frac{Ma}{2}\right)\frac{L}{6} + \frac{MaL}{3} - \frac{wL^3}{24}$$ Simplify to get Ma.

$$0 = \frac{wL^3}{48} - \frac{MaL}{12} + \frac{MaL}{3} - \frac{wL^3}{24}$$

$$\boxed{Ma = \frac{wL^2}{12}}$$

Put Ma in equation (2) , we have

$$Mb = \frac{wL^2}{8} - \frac{wL^2}{12} \times \frac{1}{2}$$

or $$Mb = \frac{wL^2}{12};$$ Now $$Rb = \left(\frac{Ma + Mb}{L}\right) + \frac{wL}{2}$$ Putting Ma and Mb we have.

$$Rb = \frac{\left(\frac{wL^2}{12} - \frac{wL^2}{12}\right)}{L} + \frac{wL}{2}$$

$$\boxed{Rb = \frac{wL}{2}}$$, Calculate Ra now.

$\sum Fy = 0$

$Ra + Rb = wL$ Put value of Rb.

$Ra = wL - Rb$

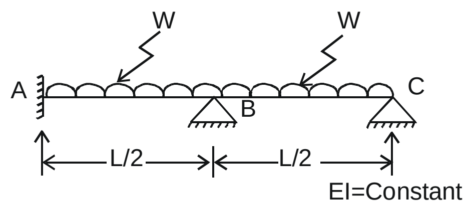
$Ra = wL - \frac{wL}{2}$

$$\boxed{Ra = \frac{wL}{2}}$$

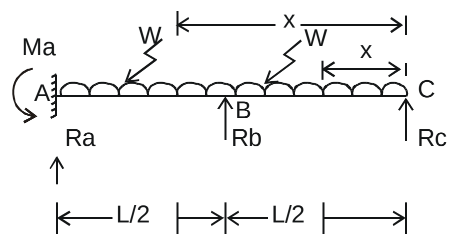
We get same results even with a different BDS. The beam is now statically determinate. SFD and BMD can be drawn. Deflections at can be found by routine methods.

2.6. 2ND DEGREE INDETERMINACY OF BEAMS:–

Exmaple No. 5: Solve the following loaded beam by the method of least work.



B.D.S. is a cantilever supported at A. Rb & Rc are chosen as redundants.



BDS UNDER LOADS AND REDUNDANTS

Choosing C as origin, Set-up moment expressions in different parts of this beam.

$$\text{Mbc} = \text{Rc.X} - \frac{wX^2}{2} \qquad 0 < X < \frac{L}{2}$$

$$\text{Mab} = \text{Rc.X} + \text{Rb}\left(X - \frac{L}{2}\right) - \frac{wX^2}{2} \qquad \frac{L}{2} < X < L$$

. Write strain energy expression for entire structure.

$$U = \frac{1}{2EI}\int_o^{L/2}\left[\text{Rc.X} - \frac{wX^2}{2}\right]^2 dX + \frac{1}{2EI}\int_{L/2}^{L}\left[\text{Rc.X} + \text{Rb}\left(X - \frac{L}{2}\right) - \frac{wX^2}{2}\right]^2 dX$$

Partially differentiate it w.r.t. redundant Rc first. Use castiglianos theorem and boundary conditions.

$$\frac{\partial U}{\partial Rc} = \Delta c = 0 = \frac{1}{EI}\int_o^{L/2}\left[\text{Rc.X} - \frac{wX^2}{2}\right][X]dX + \frac{1}{EI}\int_{L/2}^{L}\left[\text{Rc.X} + \text{Rb}\left(X - \frac{L}{2}\right) - \frac{wX^2}{2}\right][X]\,dX$$

$$0 = \frac{1}{EI}\int_o^{L/2}\left[\text{Rc.X}^2 - \frac{wX^3}{2}\right]dX + \frac{1}{EI}\int_{L/2}^{L}\left[\text{Rc.X}^2 + \text{Rb.X}^2 - \frac{\text{Rb.LX}}{2} - \frac{wX^3}{2}\right]dX$$

. Integrate it.

$$0 = \frac{1}{EI}\left[\text{Rc.}\frac{X^3}{3} - \frac{wX^4}{8}\right]_o^{L/2} + \frac{1}{EI}\left[\text{Rc.}\frac{X^3}{3} + \text{Rb.}\frac{X^3}{3} - \frac{\text{RbLX}^2}{4} . - \frac{wX^4}{8}\right]_{L/2}^{L}.$$

Insert limits and simplify.

$$0 = \frac{\text{Rc.L}^3}{3} + \frac{5\text{Rb.L}^3}{48} - \frac{wL^4}{8} \qquad \rightarrow \quad (1)$$

Now partially differentiate strain energy w.r.t. Rb. Use Castiglianos theorem and boundary conditions.

$$\frac{\partial U}{\partial Rb} = \Delta b = 0 = \frac{1}{EI}\int_{o}^{L/2}\left[Rc.X - \frac{wX^2}{2}\right](0)\,dX + \frac{1}{EI}\int_{L/2}^{L}\left[Rc.X + Rb\left(X - \frac{L}{2}\right) - \frac{wX^2}{2}\right]\left[X - \frac{L}{2}\right]dX$$

$$0 = 0 + \frac{1}{EI}\int_{L/2}^{L}\left[Rc.X^2 + RbX^2 - \frac{RbLX}{2} - \frac{wX^3}{2} - \frac{Rc.L.X}{2} - \frac{RbL.X}{2} + \frac{Rb.L^2}{4} + \frac{wL.X^2}{4}\right]dX.$$

Integrate.

$$0 = \frac{1}{EI}\left[\frac{Rc.X^3}{3} + \frac{Rb.X^3}{3} - \frac{Rb.L.X^2}{4} - \frac{wX^4}{8} - \frac{Rc.L.X^2}{4} - \frac{Rb.LX^2}{4} + \frac{Rb.L^2.X}{4} + \frac{wL.X^3}{12}\right]_{L/2}^{L}.$$

Put limits

$$0 = \frac{Rc.L^3}{3} + \frac{Rb.L^3}{3} - \frac{Rb.L^3}{4} - \frac{wL^4}{8} - \frac{Rc.L^3}{4} - \frac{Rb.L^3}{4} + \frac{Rb.L^3}{4} + \frac{wL^4}{12} - \frac{Rc.L^3}{24} - \frac{Rb.L^3}{24}$$

$$+ \frac{Rb.L^3}{16} + \frac{wL^4}{128} + \frac{Rc.L^3}{16} + \frac{Rb.L^3}{16} - \frac{Rb.L^3}{8} - \frac{wL^4}{96}$$

Simplify to get

$$Rc. = -\frac{2}{5}Rb. + \frac{17}{40}wL \qquad \rightarrow \quad (2)$$ Put this value of Rc in equation (1), to get Rb

$$0 = \left(-\frac{2}{5}Rb. + \frac{17}{40}wL\right)\frac{L^3}{3} + \frac{5}{48}Rb.L^3 - \frac{wL^4}{8} \qquad (1)$$

$$0 = -\frac{2}{15}Rb.L^3 + \frac{17}{120}wL^4 + \frac{5}{48}Rb.L^3 - \frac{wL^4}{8}$$

Simplify to get

$$\boxed{Rb. = \frac{12}{21}wL}$$

Put value of Rb in equation (2) and evaluate Rc,

$$Rc = -\frac{2}{5}\times\frac{12}{21}wL + \frac{17}{40}wL$$

$$\boxed{Rc = \frac{11}{56}wL}$$

The (+ve) signs with Rb & Rc indicate that the assumed directions of these two redundants are correct. Now calculate Ra.

$$\sum Fy = 0$$

$$Ra + Rb + Rc = wL$$

or $$Ra = wL - Rb - Rc$$. Put values of Rb and Rc from above and simplify.

$$= wL - \frac{12}{21} wL - \frac{11\ wL}{56}$$

$$Ra = \frac{373}{1176} wL$$

$$\boxed{Ra = \frac{91}{392} wL}$$

Putting the values of these reactions in Mx expression for span AB and set X = L, we have

$$Ma = Rc.L + Rb.\frac{L}{2} - \frac{wL^2}{2}$$. Put values of Rb and Rc from above and simplify.

$$= \frac{11\ wL}{56}.L + \frac{12}{21} wL \times \frac{L}{2} - \frac{wL^2}{2}$$

$$Ma = -\frac{21}{1176} wL^2$$

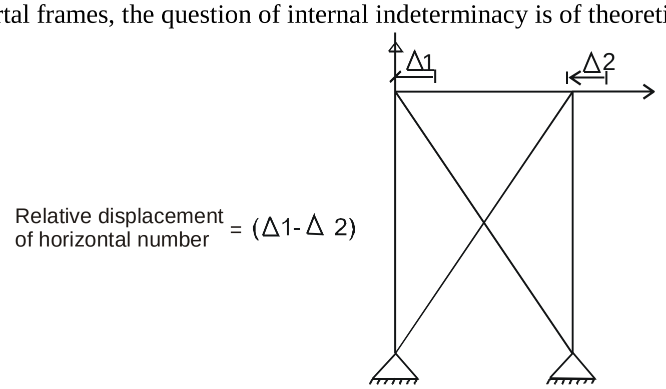
$$\boxed{Ma = -\frac{7}{392} wL^2}$$

The (–ve) sign with Ma indicates that this reactive moment should be applied in such a direction that gives us tension at the top. Now the beam has been analyzed and it is statically determinate now.

2.7. INTERNAL INDETERMINACY OF STRUCTURES BY FORCE METHOD :–

The question of internal indeterminacy relates to the skeletal structures like trusses which have discrete line members connected at the ends. The structures which fall in this category may include trusses and skeletal frames.

For fixed ended portal frames, the question of internal indeterminacy is of theoretical interest only.



Consider he truss shown in the above diagram. If this truss is to be treated as internally indeterminate, more than one members can be considered as redundants. However, the following points should be considered for deciding the redundant members.

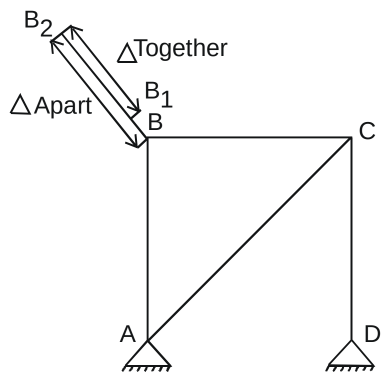
(1) The member which is chosen the redundant member is usually assumed to be removed or cut. The selection of redundant should be such that it should not effect the stability of the remaining structure.

(2) The skeletal redundant members will have unequal elongations at the two ends and in the direction in which the member is located. For example, if a horizontal member is chosen as redundant, then we will be concerned with the relative displacement of that member in the horizontal direction only.

(3) Unequal nodal deflection ($\Delta_1 - \Delta_2$) of a typical member shown above which is often termed as relative displacement is responsible for the self elongation of the member and hence the internal force in that member.

2.7.1. FIRST APPROACH: WHEN THE MEMBER IS REMOVED :–

With reference to the above diagram, we assume that the redundant member (sloping up to left) in the actual structure is in tension due to the combined effect of the applied loads and the redundant itself. Then the member is removed and now the structure will be under the action of applied loads only.

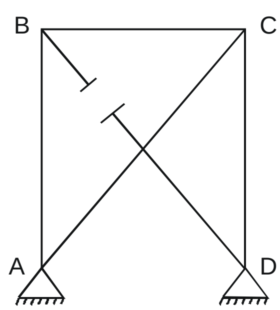


Due to the applied loads, the distance between the points B and D will increase. Let us assume that point B is displaced to its position B_2. This displacement is termed as Δ apart. Now the same structure is considered under the action of redundant force only and let us assume that point B_2 comes to its position B_1 (some of the deflections have been recovered). This displacement is termed as Δ together. The difference of these two displacements (Δapart – Δtogether) is infact the self lengthening of the member BD and the compatibility equation is

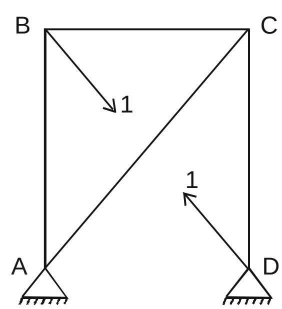
$$\Delta apart - \Delta together = \text{self elongation.}$$

2.7.2. 2ND APPROACH

We assume that the member is infact cut and the distance between the cut ends has to vanish away when the structure is under the action of applied loads and the redundant. In other words, we can say that the deformation produced by the applied loads plus the deformation produced by the redundant should be equal to zero.



F-Diagram



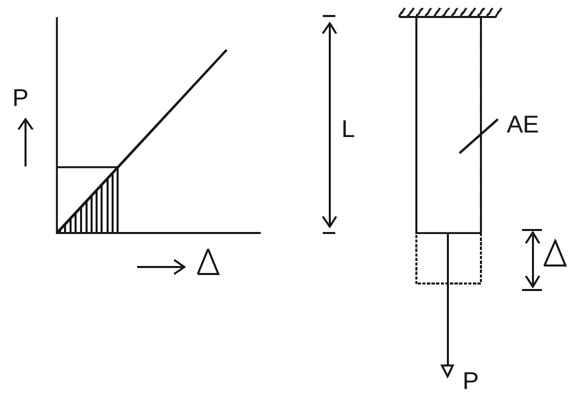
U-Diagram

Total Deflection produced by redundants $\Delta \times R = \sum_{i=1}^{n} \frac{U_i^2 Li}{AiEi} \times X$

Total Deflection produced by loads $\Delta \times L = \sum_{i=1}^{n} \frac{FiUiLi}{AiEi}$

If deflection is (+ve), there is elongation. If deflection is (–ve), there is shortening.

Now $U = \frac{P^2L}{2AE}$ Elastic strain energy stored due to axial forces



PROOF:–

Work done = 1/2 P.Δ = shaded area of P – Δ diagram.

Now $f \; \alpha \; \in$ (Hooke's Law)

or $\frac{P}{A} \alpha \frac{\Delta}{L}$ (For direct stresses)

$\frac{P}{A} = E\frac{\Delta}{L}$ where E is Yung's Modulus of elasticity.

$\Delta = \frac{PL}{AE}$

Therefore work done $= \frac{P\Delta}{2} = \frac{1}{2} P. \frac{PL}{AE}$ (Shaded area under P–Δ line – By putting value of Δ)

Work done $= \frac{P^2L}{2AE}$ (for single member)

Work done $= \sum \frac{P^2L}{2AE}$ (for several members)

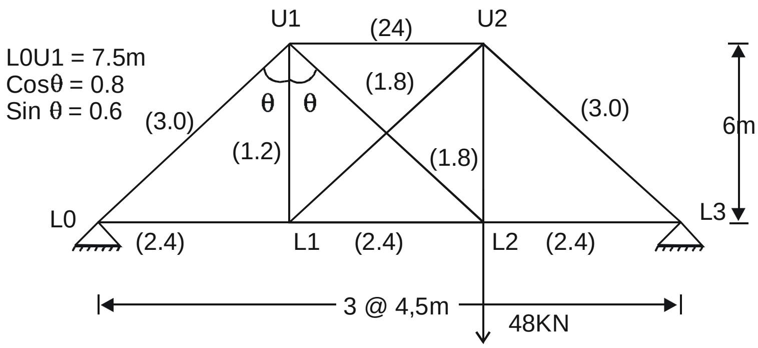
We know that Work done is always equal to strain energy stored.

EXAMPLE NO 6:

Analyze the truss shown below by Method of Least work. Take

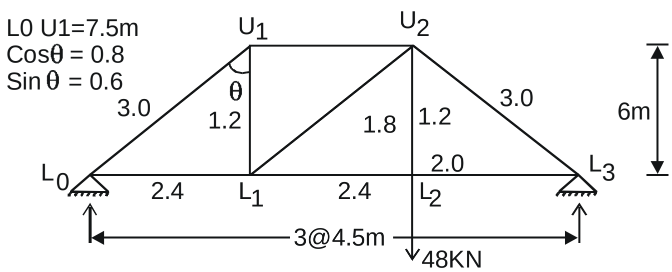
(1) Member U_1L_2 as redundant.

(2) Member U_1U_2 as redundant. Number in brackets ( ) are areas × 10^{-3} m^2. E = 200 × 10^6 KN/m^2



Note: In case of internally redundant trusses, Unit load method (a special case of strain energy method) is preferred over direct strain energy computations followed by their partial differentiation.

SOLUTION: Case 1 – Member U_1L_2 as redundant



F-Diagram

(1) U_1L_2 is redundant: **STEPS**

1 – Remove this member. (See – diagram)

2 – Assume that tensile forces would be induced in this member.

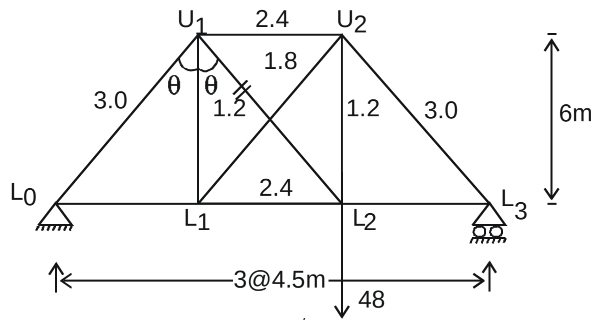
3 – Analyze the structure without U_1L_2 (B.D.S.) or F' diagram.

4 – Displacement of members due to redundant + that due to loads should be equal to zero. OR

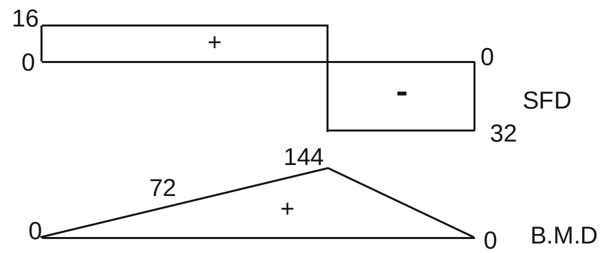
$$\Delta \times L + \Delta \times R = 0$$

5 – Analyze the truss with unit tensile force representing U_1L_2 or U–diagram.

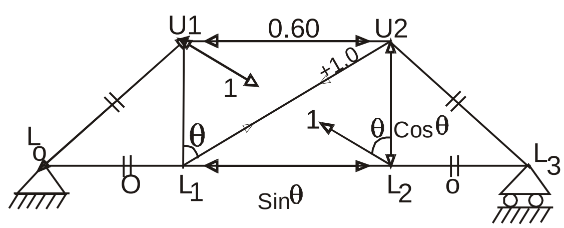
Condition: Δ apart $= \sum_1^8 \dfrac{F'UL}{AE}$ $\qquad$ Δ together$= \sum_1^8 \dfrac{U^2L}{AE} \times P_{U_1 L_2}$



(BDS under loads) F/ - diagram

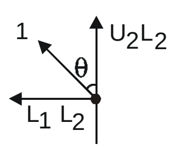


We shall determine member forces for F/ - diagram by method of moments and shears as explained earlier. These are shown in table given in pages to follow. Member forces in U-diagram are determined by the method of joints.



(BDS under) U-diagram redundant unit force.

JOINT (L_2)



$\sum F_X = 0$

$1 \times \text{Sin}\theta + L_1L_2 = 0$

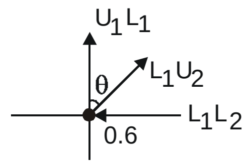
$L_1L_2 = -\text{Sin}\theta = -0.60$

$\sum Fy = 0$

$U_2L_2 + 1 \times \text{Cos}\theta = 0$

$U_2L_2 = -\text{Cos}\theta = -0.80$

Joint (L1)



$\sum F_X = 0$

$L_1U_2 \sin\theta - 0.6 = 0$

$L_1U_2 = \dfrac{0.6}{0.6} = +1$

$\sum Fy = 0$

$L_1U_2 \times 0.80 + UL_1 = 0 \Rightarrow U_1L_1 = -0.80$

Now Book $F^{/}$ forces induced in members as determined by moments and shears method and U forces as determined by method of joints in a tabular form.

Member	$A \times 10^{-3}$ (m^2)	L (m)	Fi′ (KN)	Ui	$\frac{F'UL}{AE} \times 10^{-3}$ (m)	$\frac{U^2L}{AE} \times 10^{-3}$ (m)	Fi=Fi′ +UiX (KN)
U_1U_2	2.4	4.5	– 12	–0.6	+0.0675	3.375×10^{-3}	– 25.15
LoL_1	2.4	4.5	+12	0	0	0	+12
L_1L_2	2.4	4.5	+24	–0.6	– 0.135	3.375×10^{-3}	+10.84
L_2L_3	2.4	4.5	+24	0	0	0	+24
LoU_1	3.0	7.5	– 20	0	0	0	– 20
L_1U_2	4.8	7.5	– 20	+1.0	– 0.416	20.83×10^{-3}	+ 1.93
U_2L_3	3.0	7.5	– 40	0	0	0	– 40
U_1L_1	1.2	6.0	+16	–0.8	– 0.32	16×10^{-3}	– 1.54
U_2L_2	1.2	6.0	+48	–0.8	– 0.96	16×10^{-3}	+30.456
U_1L_2	1.8	7.5	0	+1.0	0	20.83×10^{-3}	+21.96
					$\sum -1.7635\times10^{-3}$	$\sum 80.91 \times 10^{-6}$	

Compatibility equation is

$\Delta \times L + \Delta \times R = 0$

$\Delta \times L = \sum_{1}^{n} \dfrac{F'UL}{AE}$

$\Delta \times R = \sum_{1}^{n} \dfrac{U^2L}{AE} . X$ Putting values from above table in compatibility equation. Where R = X = force in redundant Member U_1L_2

$$-1.7635 \times 10^{-3} + 80.41 \times 10^{-6}. X = 0$$

or $$-1.7635 \times 10^{-3} + 0.08041 \times 10^{-3}. X = 0$$

$$-1.7635 + 0.08041 \times X = 0$$

$$0.08041 X = 1.7635$$

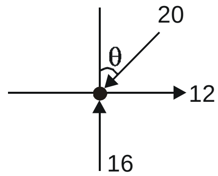
$$X = \frac{1.7635}{0.08041}$$

$X = +21.93$ KN (Force in members U_1L_2)

Now final member forces will be obtained by formula Fi = Fi' + Ui X. These are also given in above table. Apply check on calculated forces.

Check on forces

Joint Lo



Note: Tensile forces in above table carry positive sign and are represented as acting away from joint. Compressive forces carry negative sign and are represented in diagram as acting towards the joint.

$$\sum Fx = 0$$

$$12 - 20 \sin\theta = 0$$

$$12 - 20 \times 0.6 = 0$$

$$0 = 0$$

$$\sum Fy = 0$$

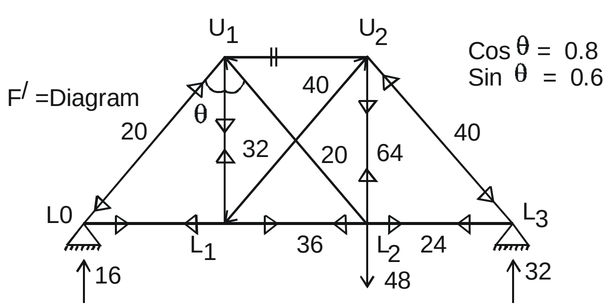
$$16 - 20 \cos\theta = 0$$

$$16 - 20 \times 0.8 = 0$$

$0 = 0$ Checks have been satisfied showing correctness of solution.

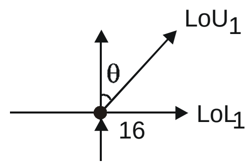
EXMAPLE NO. 7:

CASE 2: Analyze previous loaded Truss by taking $U_1 U_2$ as Redundant



In this case member forces in BDS ($F^{/}$ diagram) have been computed by method of joints due to obvious reasons.)

Joint Lo:-



$\sum Fy = 0$

$16 + LoU_1 \times Cos\theta = 0$

$LoU_1 = -\dfrac{16}{0.8} = -20$

$\sum F_X = 0$

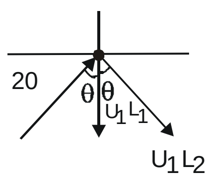
$LoL_1 + LoU_1 \, Sin\theta = 0$

$LoL_1 + LoU_1 \times 0.6 = 0$

$LoL_1 - 20 \times 0.6 = 0$

$LoL_1 = +12$

Joint U_1



$\sum FX = 0$

$20 \, Sin\theta + U_1L_2 \, Sin\theta = 0$

$20 \times 0.6 + U_1L_2 \times 0.6 = 0$

$U_1L_2 = -20$

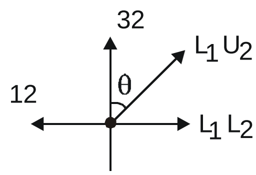
$\sum Fy = 0$

$20 \times 0.8 - U_1L_1 - U_1L_2 \times 0.8 = 0$

$20 \times 0.8 - U_1L_1 + 20 \times 0.8 = 0$

$U_1L_1 = 32$

Joint L_1:



$\sum Fy = 0$

$L_1U_2 \, Cos\theta + 32 = 0$

$$L_1U_2 = -\frac{32}{0.8}$$

$$L_1U_2 = -40$$

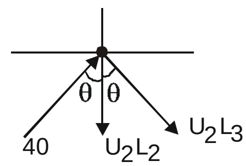
$$\sum FX = 0$$

$$L_1L_2 + L_1U_2 \, \text{Sin}\theta - 12 = 0$$

$$L_1L_2 - 40 \times 0.6 - 12 = 0$$

$$L_1L_2 = 36$$

Joint U_2



$$\sum FX = 0$$

$$40 \, \text{Sin}\theta + U_2L_3 \, \text{Sin}\theta = 0$$

$$40 \times 0.6 + U_2L_3 \times 0.6 = 0$$

$$U_2L_3 = -40$$

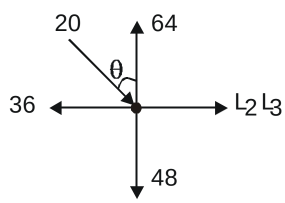
$$\sum Fy = 0$$

$$40 \, \text{Cos}\theta - U_2L_3 \, \text{Cos}\theta - U_2L_2 = 0$$

$$40 \times 0.8 - (-40) \times 0.8 - U_2L_2 = 0$$

$$U_2L_2 = 64$$

Joint L_2



$$\sum FX = 0$$

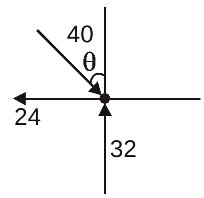
$$L_2L_3 + 20 \, \text{Sin}\theta - 36 = 0$$

$$L_2L_3 + 20 \times 0.6 - 36 = 0$$

$$L_2L_3 - 24 = 0$$

$$L_2L_3 = 24$$

Joint L_3 (Checks)



$\sum FX = 0$

$40\ \text{Sin}\theta - 24 = 0$

$40 \times 0.6 - 24 = 0$

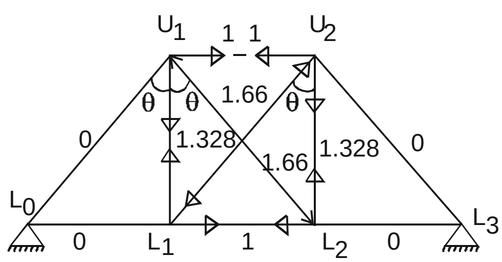
$0 = 0$

$\sum Fy = 0$

$32 - 40\ \text{Cos}\theta = 0$

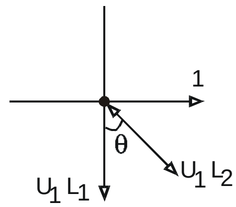
$32 - 40 \times 0.8 = 0$

$0 = 0$ Checks are satisfied. Results are OK and are given in table at page to follow: Now determine member forces in U diagram.



U-Diagram
(BDS under unit redundant force)

Joint U_1



$\sum FX = 0$

$1 + U_1L_2 \times \text{Sin}\theta = 0$

$1 + U_1L_2 \times 0.6 = 0$

$U_1L_2 = -1.66$

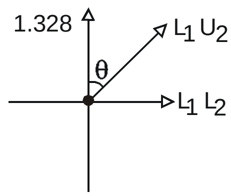
$\sum Fy = 0$

$U_1L_1 + U_1L_2 \times \cos\theta = 0$

$U_1L_1 + (-1.66) \times 0.8 = 0$

$U_1L_1 = 1.328$

Joint L_1 :-



$\sum Fy = 0$

$1.328 + L_1U_2 \times 0.8 = 0$

$L_1U_2 = -\frac{1.328}{0.8} = -1.66$

$\sum FX = 0$

$L_1L_2 + L_1L_2 \times 0.6 = 0$

$L_1L_2 - 1.66 \times 0.6 = 0$

$L_1L_2 = +1$

Entering results of member forces pertaining to F/ diagram and U diagram alongwith member properties in a tabular form.

Member	A × 10^{-3} (m)	L (m)	Fi′ (KN)	U_1	$\frac{F'UL}{AE} \times 10^{-3}$ (m)	$\frac{U^2L}{AE} \times 10^{-3}$ (m)	Fi=Fi+UiX (KN)
U_1U_2	2.4	4.5	0	+1	0	9.375×10^{-3}	–25.34
LoL_1	2.4	4.5	+12	0	0	0	+ 12
L_1L_2	2.4	4.5	+ 36	+ 1	+0.3375	9.375×10^{-3}	+10.66
L_2L_3	2.4	4.5	+24	0	0	0	+ 24
LoU_1	3.0	7.5	– 20	0	0	0	– 20
L_1U_2	1.8	7.5	– 40	–1.66	+1.383	57.4×10^{-3}	+2.06
U_2L_3	3.0	7.5	– 40	0	0	0	– 40
U_1L_1	1.2	6.0	+ 32	1.328	1.0624	44.09×10^{-3}	+ 65.65
U_2L_2	1.2	6.0	+ 64	1.328	2.1248	44.09×10^{-3}	+ 97.65
U_1L_2	1.8	7.5	– 20	–1.66	0.691	57.4×10^{-3}	– 62.06
					$\sum 5.6 \times 10^{-3}$	$\sum 221.73 \times 10^{-6}$	

Compatibility equation is

$\Delta \times L + \Delta \times R = 0$ Putting values of $\Delta \times L$ and $\Delta \times R$ due to redundant from above table.

$56 \times 10^{-3} + 221.73 \times 10^{-6} X = 0$, where X is force in redundant member U_1U_2.

or $5.6 \times 10^{-3} + 0.22173 \times 10^{-3} X = 0$

$$X = \frac{5.6 \times 10^{-3}}{0.22173 \times 10^{-3}}$$

$X = -25.34$ KN. Therefore forces in truss finally are as follows. (by using formula (Fi = Fi' + UiX and are given in the last column of above table)

$FU_1 U_2 = 0 + Ui.x = 0 - 25.34 \times 1 = -25.34$

$FLoL_1 = 12 - 25.34 \times 0 = +12$

$FL_1L_2 = 36 - 25.34 \times 1 = +10.66$

$FL_2L_3 = 24 - 0 = +24$

$FLoU_1 = -20 - 0 \times 25.34 = -20$

$FL_1U_2 = -40 + 1.66 \times 25.34 = +2.06$

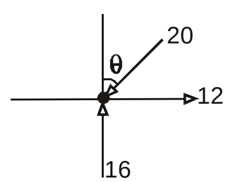
$FU_2L_3 = -40 + 0 \times 25.34 = -40$

$FU_1L_1 = +32 + 1.328 \times 25.34 = +65.65$

$FU_2L_2 = +64 + 1.328 \times 25.34 = +97.65$

$FU_1L_2 = -20 - 1.66 \times 25.34 = -62.06$. Now based on these values final check can be applied.

Joint Lo.



$\sum FX = 0$

$12 - 20 \sin\theta = 0$

$12 - 20 \times 0.6 = 0$

$0 = 0$

$\sum Fy = 0$

$16 - 20 \cos\theta = 0$

$16 - 20 \times 0.8 = 0$

$16 - 16 = 0$

$0 = 0$ Results are OK.

2.8. STEPS FOR TRUSS SOLUTION BY METHOD OF LEAST WORK.

Now instead of Unit load method, we shall solve the previous truss by direct use of method of least work.

(1) Consider the given truss under the action of applied loads and redundant force X in member U_1L_2

(2) The forces in the relevant rectangle will be a function of applied load and redundant force X. (As was seen in previous unit load method solution)

(3) Formulate the total strain energy expression due to direct forces for all the members in the truss.

(4) Partially differentiate the above expressions with respect to X.

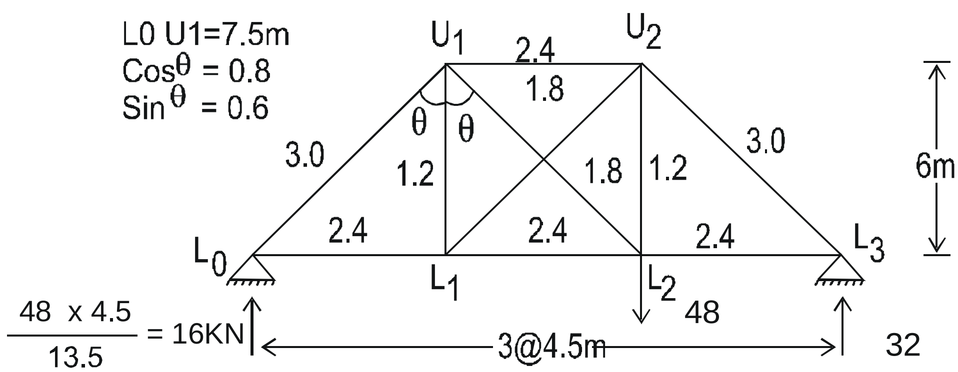
(5) Sum up these expressions and set equal to zero. Solve for X.

(6) With this value of X, find the member forces due to applied loads and redundant acting simultaneously (by applying the principle of super positions).

EXAMPLE NO. 8 :-

Analyze the loaded truss shown below by least work by treating member U_1L_2 as redundant. Numbers in ( ) are areas $\times 10^{-3}$ m^2 . E = 200 $\times 10^6$ KN/m^2.

SOLUTION:-



b = 10

r = 3

j = 6

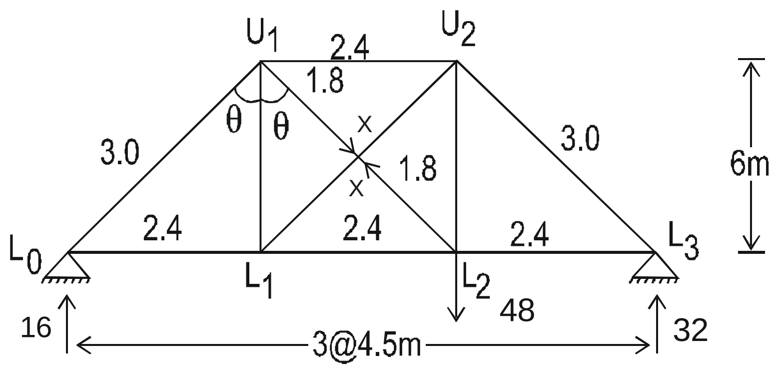
b + r = 2 j

10 + 3 = 2 × 6

13 = 12

D = 13 – 12 = 1

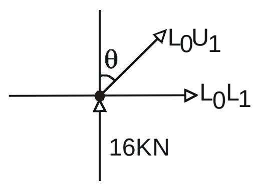
Stable Indeterminate to 1st degree.



F – Diagram (Truss under loads and redundant)

NOTE: Only the rectangle of members containing redundant X contains forces in terms of X as has been seen earlier. Now analyze the Truss by method of joints to get Fi forces.

JOINT L0



$\sum Fy = 0$

$LoU_1 \cos\theta + 16 = 0$

$$LoU_1 = \frac{-16}{\cos\theta}$$

$$= \frac{-16}{0.8}$$

$$\boxed{LoU_1 = -20 \text{ KN}}$$

$\sum FX = 0$

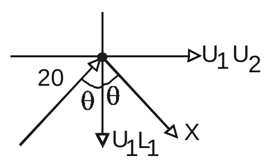
$LoL_1 + LoU_1 \sin\theta = 0$

$LoL_1 + (-20) \times 0.6 = 0$

$LoL_1 - 12 = 0$

$$\boxed{LoL_1 = 12 \text{ KN}}$$

Joint U_1



$\sum FX = 0$

$U_1 U_2 + X \sin\theta + 20 \sin\theta = 0$

$U_1 U_2 + X \times 0.6 + 20 \times 0.6 = 0$

$$\boxed{U_1 U_2 = -(0.6X + 12)}$$

$\sum Fy = 0$

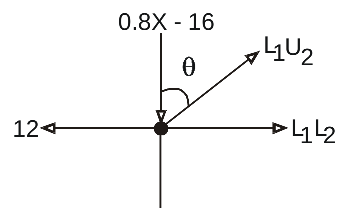
$-U_1 L_1 - X \cos\theta + 20 \cos\theta = 0$

$-U_1 L_1 - X \times 0.8 + 20 \times 0.8 = 0$

$U_1 L_1 = -0.8X + 16$

$$\boxed{U_1 L_1 = -(0.8X - 16)}$$

Joint L_1 :-



$\sum Fy = 0$

$-(0.8X - 16) + L_1 U_2 \cos\theta = 0$

$L_1 U_2 \times 0.8 = 0.8X - 16$

$$\boxed{L_1 U_2 = (X - 20)}$$

$\sum FX = 0$

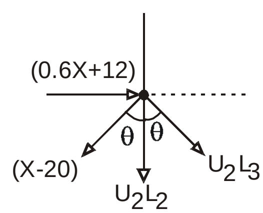
$L_1 L_2 + L1U2 \sin\theta - 12 = 0$ Put value of $L_1 U_2$.

$L_1 L_2 + (X - 20) \times 0.6 - 12 = 0$

$L_1 L_2 + 0.6X - 12 - 12 = 0$

$$\boxed{L_1 L_2 = -(0.6X - 24)}$$

Joint U_2



$\sum FX = 0$

$(0.6X + 12) + U_2 L_3 \sin\theta - (X - 20) \sin\theta = 0$

$0.6X + 12 + U_2 L_3 \times 0.6 - (X - 20) \times 0.6 = 0$

$0.6\ X + 12 + 0.6U_2L_3 - 0.6\ X + 12 = 0$

$U_2L_3 = \dfrac{-24}{0.6}$

$$\boxed{U_2L_3 = -40\ KN}$$

$\sum Fy = 0$

$-U_2L_2 - (X - 20)\ Cos\theta - U_2L_3\ Cos\theta = 0$

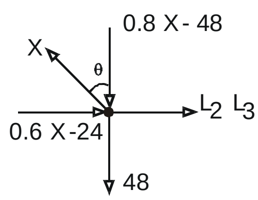
$-U_2L_2 - (X - 20) \times 0.8 - (-40) \times 0.8 = 0$

$-U_2L_2 - 0.8\ X + 16 + 32 = 0$

$-0.8\ X + 48 = U_2L_2$

$$\boxed{U_2L_2 = -(0.8X - 48)}$$

Joint L_2:-



$\sum FX = 0$

$L_2L_3 + 0.6\ X - 24 - X\ Sin\theta = 0$

$L_2L_3 = -0.6\ X + 24 + 0.6\ X$

$$\boxed{L_2L_3 = 24\ KN}$$

$\sum Fy = 0$

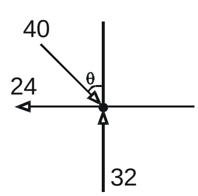
$-(0.8X - 48) - 48 + X\ Cos\theta = 0$

$-0.8X + 48 - 48 + 0.8X = 0$

$0 = 0$ (Check)

Joint L_3 :-

At this joint, all forces have already been calculated. Apply checks for corretness.



$\sum FX = 0$
$40 \text{ Sin}\theta - 24 = 0$
$40 \times 0.6 - 24 = 0$
$24 - 24 = 0$
$0 = 0$ O.K.

$\sum Fy = 0$
$-40 \text{ Cos}\theta + 32 = 0$
$-40 \times 0.8 + 32 = 0$
$-32 + 32 = 0$ O.K. Checks have been satisfied.
$0 = 0$

This means forces have been calculated correctly. We know that strain energy stored in entire Truss is $U = \sum \frac{Fi^2L}{2AE}$

So, $\frac{\partial U}{\partial X} = \Delta = 0 = \frac{\sum Fi \frac{\partial Fi}{\partial X} . Li}{AE}$

$\frac{\sum Fi \frac{\partial Fi}{\partial X} . Li}{AE} = 0 = 80.41 \times 10^{-6}X - 1764.17 \times 10^{-6}$ Values of Fi and Li for various members have been picked up from table annexed.

$$0 = 80.41 X - 1764.17$$

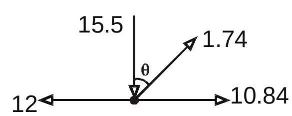
or $80.41 X = 1764.17$

$$X = \frac{1764.17}{80.41}$$

$$X = 21.94 \text{ KN}$$

Now putting this value of X in column S of annexed table will give us member forces.

Now apply equilibrium check on member forces calculated. You may select any Joint say L_1.
Joint L_1 :-

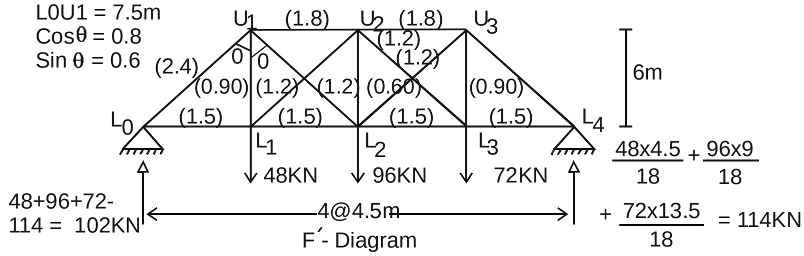


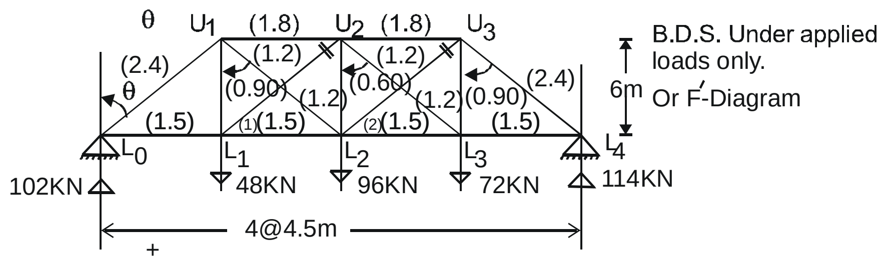
$\sum FX = 0,$
$10.84 - 12 + 1.94 \text{ Sin}\theta = 0$
or $10.84 - 12 + 1.94 \times 0.6 = 0$,
or $0 = 0$ (Check) It means that solution is correct.

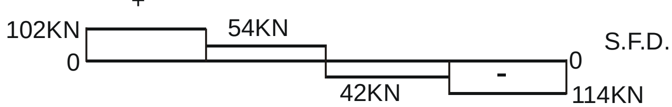
Insert here Page No. 138–A

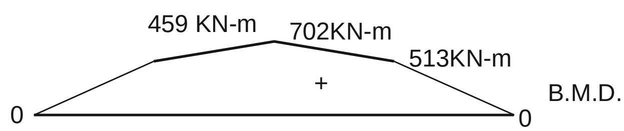
EXAMPLE NO. 9:- By the force method analyze the truss shown in fig. below. By using the forces in members L_1U_2 and L_2U_3 as the redundants. Check the solution by using two different members as the redundants. $E = 200 \times 10^6$ KN/m^2

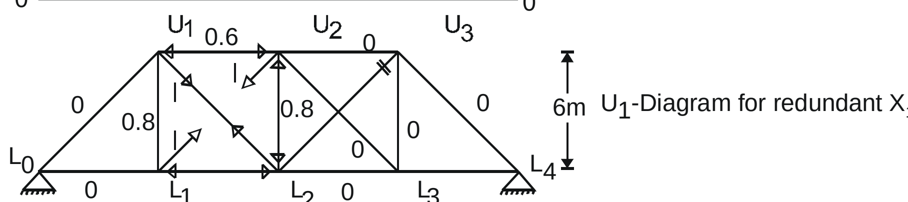
SOLUTION:-

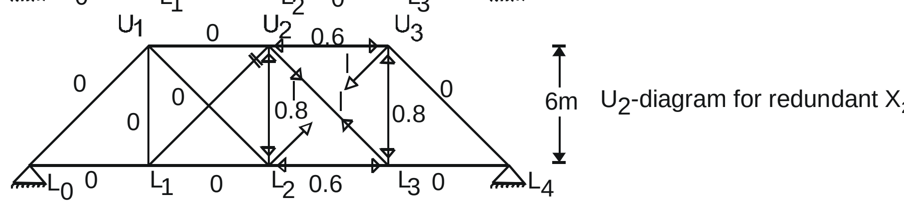












Compatibility equations are:

$\Delta X_1L + \Delta X_1R_1 + \Delta X_1R_2 = 0$ → (1) Change in length in member 1 due to loads and two redundants should be zero.

$\Delta X_2L + \Delta X_2R_1 + \Delta X_2R_2 = 0$ → (2) Change in length in member 2 due to loads and two redundants should be zero.

Here $R_1 = X_1$

$R_2 = X_2$

Where $\Delta X_1L = \frac{\sum .F'U_1 L}{AE}$ = Deflection produced in member (1) due to applied loads.

ΔX_1R_1 = Deflection produced in member (1) due to redundant $R_1 = \sum\left(\frac{U_1^2L}{AE}\right) . X_1$

Δx_1R_2 = Deflection produced in member (1) due to redundant $R_2 = \sum\left(\frac{U_1U_2L}{AE}\right) . X_2$

Δx_2L = Deflection produced in member (2) due to loads = $\sum \frac{F'U_2L}{AE}$

Δx_2R_1 = Deflection produced in member (2) due to redundant $R_1 = \sum\left(\frac{U_1U_2L}{AE}\right) . X_1$

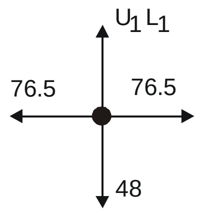
Δx_2R_2 = Deflection produced in member (2) due to redundant $R_2 = \sum\left(\frac{U_2^2L}{AE}\right) . X_2$

From table attached, the above evaluated summations are picked up and final member forces can be seen in the same table. All member forces due to applied loads (Fi' diagram) have been determined by the method of moments and shears and by method of joints for U_1 and U_2 diagrams.

Evaluation of member forces in verticals of F′ – Diagram :-

Forces in verticals are determined from mothod of joints for different trusses shown above.

(Joint L_1)

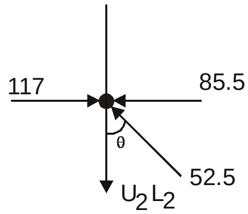


$\sum Fy = 0$

$U_1L_1 - 48 = 0$

$\boxed{U_1L_1 = 48}$

(Joint U_2)



$\sum Fy = 0$

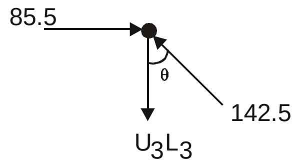
$- U_2L_2 + 52.5 \ \text{Cos}\theta = 0$

$- U_2L_2 + 52.5 \times 0.8 = 0$

$U_2L_2 = 52.5 \times 0.8$

$\boxed{U_2L_2 = + 42}$

(Joint U_3)



$\sum Fy = 0$

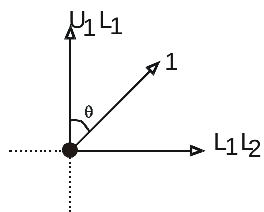
$-U_3L_3 + 142.5 \ \text{Cos}\theta = 0$

$U_3L_3 = 142.5 \times 0.8$

$\boxed{U_3L_3 = +114}$

Evaluation of forces in verticals of U_1 – Diagram:-

(Joint L_1)



$\sum FX = 0$

$L_1L_2 + 1 \ \text{Sin}\ \theta = 0$

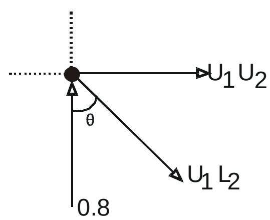
$\boxed{L_1L_2 = -0.6}$

$\sum Fy = 0$

$U_1L_1 + 1 \ \text{Cos}\ \theta = 0$

$\boxed{U_1L_1 = -0.8}$

(Joint U_1)



$\sum FX = 0$

$U_1U_2 + U_1L_2 \ \text{Sin}\theta = 0$

$\sum Fy = 0$

$+ 0.8 - U_1L_2 \cos\theta = 0$

$0.8 = U_1L_2 \times 0.8$

$U_1L_2 = 1$

so $U_1U_2 + 1 \times 0.6 = 0$ Putting value of U_1L_2 in $\sum FX$.

$U_1 U_2 = -0.6$

Now from the table, the following values are taken.

$\Delta X_1L = -0.671 \times 10^{-3}$

$\Delta X_1R_1 = 125.7 \times 10^{-6}X1 = 0.1257 \times 10^{-3}X1$

$\Delta X_1R_2 = 32 \times 10^{-6}\ X2 = 0.032 \times 10^{-3}X2$

$\Delta X_2L = -6.77 \times 10^{-3}$

$\Delta X_2R_1 = 0.032 \times 10^{-3}\ X1$

$\Delta X_2R_2 = 125.6 \times 10^{-6}X2 = 0.1256 \times 10^{-3}X2$

Putting these in compatibility equations, we have.

$$-0.671 \times 10^{-3} + 0.1257 \times 10^{-3}X1 + 0.032 \times 10^{-3}X2 = 0 \quad \rightarrow (1)$$

$$-6.77 \times 10^{-3} + 0.032 \times 10^{-3}\ X1 + 0.1256 \times 10^{-3}X2 = 0 \quad \rightarrow (2)$$

dividing by 10^{-3}

$$-0.671 + 0.1257X1 + 0.032X2 = 0 \quad \rightarrow (1)$$

$$-6.77 + 0.032X1 + 0.1256X2 = 0 \quad \rightarrow (2)$$

From (1), $X_1 = \dfrac{0.671 - 0.032X_2}{0.1257}$ $\rightarrow$ (3)

Put X_1 in (2) & solve for X_2

$$-6.77 + 0.032\left[\frac{0.671 - 0.032X_2}{0.1257}\right] + 0.1256X_2 = 0$$

$$-6.77 + 0.171 - 8.146 \times 10^{-3}X2 + 0.1256X_2 = 0$$

$$-6.599 + 0.1174X_2 = 0$$

$$0.1174X2 = 6.599$$

$$\boxed{X2 = 56.19\ KN}$$

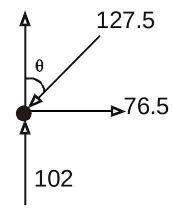
From (3) $X1 = \dfrac{0.671 - 0.032 \times 56.19}{0.1257}$

$X1 = -8.96$ KN

After redundants have been evaluated, final member forces can be calculated by using the formula shown in last column of table. Apply checks on these member forces.

CHECKS:-

(Joint Lo)



$\sum FX = 0$

$76.5 - 127.5 \, Sin\theta = 0$

$76.5 - 127.5 \times 0.6 = 0$

$0 = 0$

$\sum Fy = 0$

$102 - 127.5 \, Cos\theta = 0$

$102 - 127.5 \times 0.8 = 0$

$0 = 0$

The results are O.K. Follow same procedure if some other two members are considered redundant. See example No. 12.

Insert Page No. 143–A

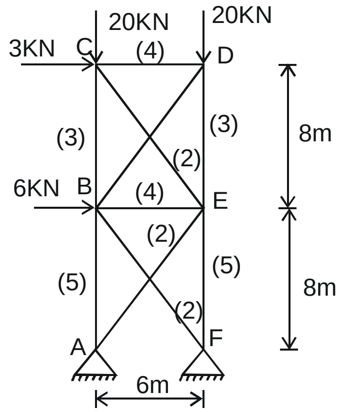
2.9. SIMULTANEIOUS INTERNAL AND EXTERNAL TRUSS REDUNDANCY

EXAMPLE NO. 10: Determine all reactions and member forces of the following truss by using castiglianos theorem or method of least work. Consider it as:

(i) internally redundant;

(ii) internally and externally redundant.

Nos. in ( ) are areas in $\times 10^{-3}m^2$. $E = 200 \times 10^6$ KN/m^2



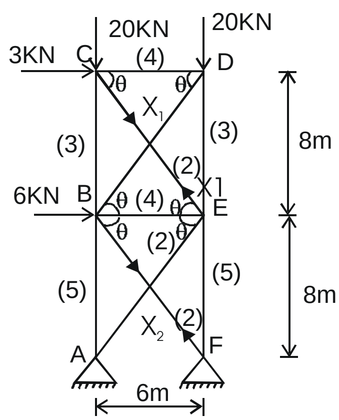
SOLUTION:

DEGREE OF INDETERMINACY :-

$$D = (m + r) - 2j = (10 + 4) - 2 \times 6 = 2$$

Therefore, the truss is internally statically indeterminate to the 2nd degree. There can be two approaches, viz, considering two suitable members as redundants and secondly taking one member and one reaction as redundants for which the basic determinate structure can be obtained by cutting the diagonal CE and replacing it by a pair of forces $X_1 - X_1$ and replacing the hinge at F by a roller support with a horizontal redundant reaction HF = X_2. Applying the first approach and treating inclineds of both storeys sloping down to right as redundants.

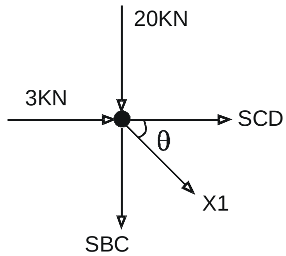
(I) **WHEN THE TRUSS IS CONSIDERED AS INTERNALLY REDUNDANT** :-



Applying method of joints for calculating member forces.

Consider Joint (C) and all unknown forces are assumed to be in tension to begin with , acting away from the joint. Length AE= 10 m , cos θ = 0.6 , sin θ = 0.8

Joint (C)



$\sum FX = 0$

$Scd + 3 + X_1 \cos\theta = 0$

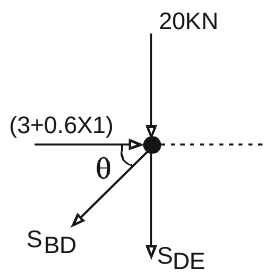
$Scd = -(3 + 0.6 \times X_1)$

$\sum Fy = 0$

$-Sbc - X_1 \sin\theta - 20 = 0$

$Sbc = -(20 + 0.8 X_1)$

Joint (D)



$\sum FX = 0$

$3 + 0.6X_1 - S_{BD} \times 0.6 = 0$

$S_{BD} = (5 + X_1)$

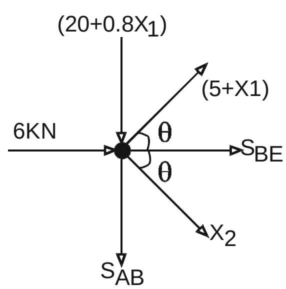
$\sum Fy = 0$

$-S_{DE} - 20 - S_{BD} \sin\theta = 0$

$-S_{DE} - 20 - (5 + X_1) \times 0.80 = 0$

$S_{DE} = -(24 + 0.8X_1)$

Joint (B)



$\sum FX = 0$

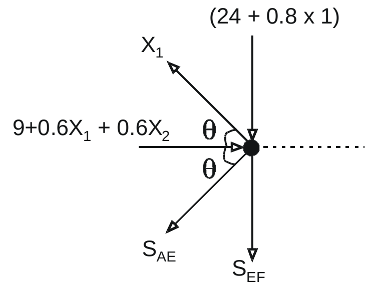
$S_{BE} + (5+X_1) \times 0.6 + X_2 \times 0.6 + 6 = 0$

$S_{BE} = -(9 + 0.6 X_1 + 0.6 X_2)$

$\sum Fy = 0$

$$- S_{AB} - X_2 \, Sin\theta - (20 + 0.8\, X_1) + (5+X_1)\, Sin\theta = 0$$
$$- S_{AB} - 0.8\, X_2 - 20 - 0.8\, X_1 + 4 + 0.8\, X_1 = 0$$
$$S_{AB} = -(16 + 0.8\, X_2)$$

Joint (E)



$$\sum FX = 0$$
$$9 + 0.6\, X_1 + 0.6\, X_2 - X_1 \text{ x } 0.6 - S_{AE} \times 0.6 = 0$$
$$9 + 0.6\, X_2 = S_{AE} \times 0.6$$
$$S_{AE} = (15 + X_2)$$

$$\sum Fy = 0$$
$$- S_{EF} - 24 - 0.8\, X_1 + X_1 \times 0.8 - (15 + X_2) \times 0.8 = 0$$
$$S_{EF} = -24 - 0.8\, X_1 + 0.8\, X_1 - 12 - 0.8\, X_2 = 0$$
$$S_{EF} = -36 - 0.8\, X_2$$
$$S_{EF} = -(36 + 0.8\, X_2)$$

Enter Forces in table. Now applying Catiglianos' theorem and taking values from table attached.

$$\sum S \,.\, \frac{\partial S}{\partial X_1} \,.\, \frac{L}{AE} = 0 = 485.6 + 65.64X_1 + 2.7X_2 = 0 \qquad (1)$$

and

$$\sum S.\, \frac{\partial S}{\partial X_2} \,.\, \frac{L}{AE} = 0 = 748.3 + 2.7X_1 + 62.94\, X_2 = 0 \qquad (2)$$

or $485.6 + 65.64\, X_1 + 2.7\, X_2 = 0 \quad \rightarrow \quad (1)$

$748.3 + 2.7\, X_1 + 62.94\, X_2 = 0 \quad \rightarrow \quad (2)$

From (1)

$$X_2 = -\left(\frac{485.6 + 65.64\, X_1}{2.7}\right) \text{ putting in (2)}$$

$$748.3 + 2.7\, X_1 - 62.94 \left(\frac{485.6 + 65.64\, X_1}{2.7}\right) = 0 \quad \rightarrow \quad (2)$$

$$748.3+2.7X_1-11319.875 - 1530.141X_1 - 10571.575 - 1527.441\, X_1 = 0 \quad \rightarrow \quad (3)$$

$$\boxed{X_1 = -6.921 \text{ KN}}$$

From (3) $X_2 = -\left(\frac{485.6 - 65.64 \times 6.921}{2.7}\right)$

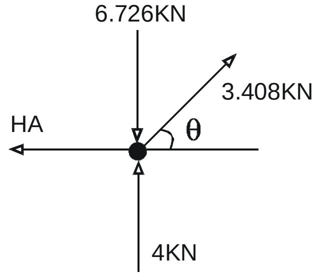
$$\boxed{X_2 = -11.592 \text{ KN}}$$

Now put values of X_1 and X_2 in 5^{th} column of S to get final number forces SF as given in last column of table. Apply equilibrium check to verify correctness of solution.

Insert Page No. 148–A

EQUILIBRIUM CHECKS :-

Joint (A)



$\sum FX = 0$

$3.408 \text{ Cos}\theta - H_A - 0$

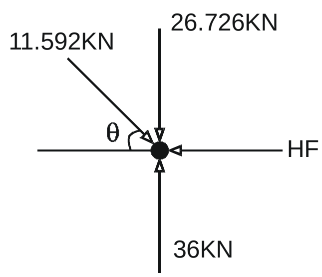
$\boxed{H_A = 2.045 \text{ KN}}$

$\sum Fy = 0$

$-6.726 + 4 + 3.408 \text{ Sin}\theta = 0$

$0 = 0$ Check is OK.

Joint (F)



$\sum FX = 0$

$- H_F + 11.592 \text{ Cos}\theta = 0$

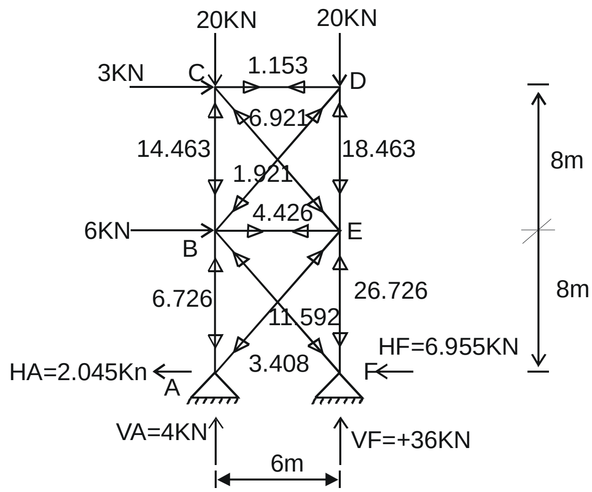
$\boxed{H_F = + 6.955 \text{ KN}}$

$\sum Fy = 0$

$36 - 27.726 - 11.592 \times \text{Sin}\theta = 0$

$0 = 0$ (check)

It means solution is correct. Now calculate vertical reactions and show forces in diagram.



ANALYZED TRUSS

$\sum M_A = 0$

$V_F \times 6 - 20 \times 6 - 3 \times 16 - 6 \times 8 = 0$

$\boxed{V_F = +36\ KN}$

$\sum Fy = 0$

$V_A + V_F = 40\ KN$

$\boxed{V_A = +4\ KN}$

EXAMPLE NO. 11:

CASE II : When the Truss is considered as both externally & internally redundant.

Taking S_{CE} & H_F as redundants. Now Truss is determinate and calculate vertical reactions.

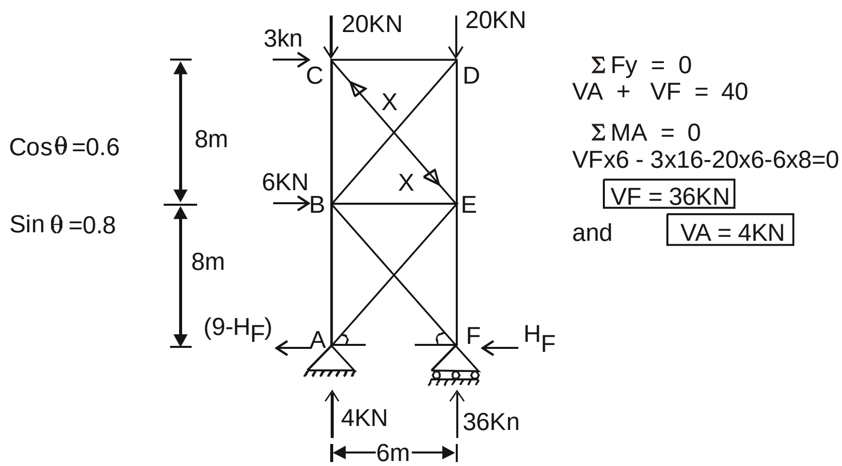


Fig. 2.51

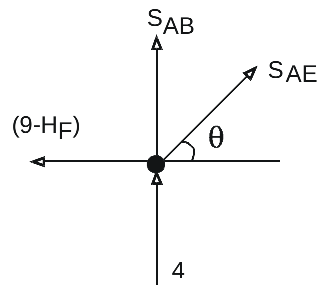
Compatibility Equations are:

$\sum S.\frac{\partial S}{\partial HF} \cdot \frac{L}{AE} = 0$ (1) Partial differentiation of strain energy w.r.t. HF = ΔH = 0. (Pin support)

$\sum S.\frac{\partial S}{\partial X} \cdot \frac{L}{AE} = 0$ (2) Partial differentiation of strain energy w.r.t. X = elongation of member CE due to X = 0.

As before determine member forces Si in members by method of joints.

Joint (A)



$\sum FX = 0$

$S_{AE}\,Cos\theta - (9 - H_F) = 0$

$S_{AE} \times 0.6 - (9 - H_F) = 0$

$S_{AE} = \left(\frac{9 - H_F}{0.6}\right)$

$\boxed{S_{AE} = 15 - 1.67\,H_F}$

$\sum Fy = 0$

$4 + S_{AB} + S_{AE}\,Sin\theta = 0$

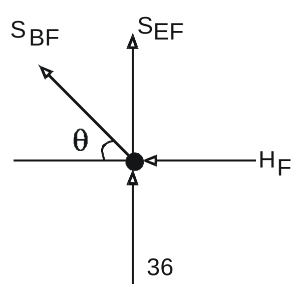
$4 + S_{AB} + (15 - 1.670\,H_F) \times 0.8 = 0$

$4 + S_{AB} + 12 - 1.33\,H_F = 0$

$S_{AB} = -16 + 1.33\,H_F$

$\boxed{S_{AB} = -(16 - 1.33\,H_F)}$

Joint (F)



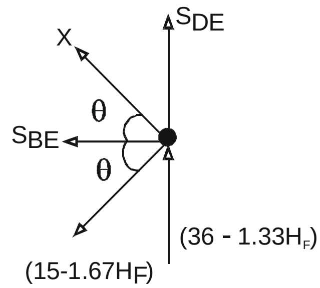
$\sum FX = 0$
$- H_F - S_{BF} \cos\theta = 0$
$- H_F - 0.6\, S_{BF} = 0$
$- H_F = 0.6\, S_{BF}$

$$\boxed{S_{BF} = -1.67\, H_F}$$

$\sum Fy = 0$
$36 + S_{EF} + S_{BF} \sin\theta = 0$
$36 + S_{EF} - 1.67\, H_F \times 0.8 = 0$

$$\boxed{S_{EF} = -(36 - 1.33\, H_F)}$$

Joint (E)



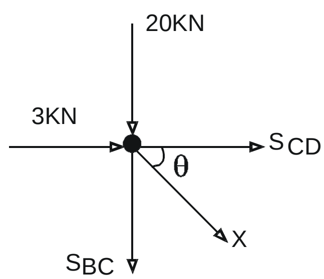
$\sum FX = 0$
$- S_{BE} - X \cos\theta - (15 - 1.67\, H_F) \cos\theta = 0$
$- S_{BE} - 0.6X - (15 - 1.67\, H_F) \times 0.6 = 0$
$- S_{BE} - 0.6X - 9 + H_F = 0$
$H_F - 0.6X - 9 = S_{BE}$

$$\boxed{S_{BE} = (H_F - 0.6\, X - 9)}$$

$\sum Fy = 0$
$S_{DE} + 36 - 1.33\, H_F + X \sin\theta - (15 - 1.67 H_F) \sin\theta = 0$ by putting $\sin\theta = 0.08$
$S_{DE} + 36 - 1.33\, H_F + 0.8X - 12 + 1.33\, H_F = 0$
$S_{DE} = -0.8X - 24$

$$\boxed{S_{DE} = -(24 + 0.8X)}$$

Joint (C)



$\sum FX = 0$

$S_{CD} + 3 + X \cos\theta = 0$

$$\boxed{S_{CD} = -(3 + 0.6X)}$$

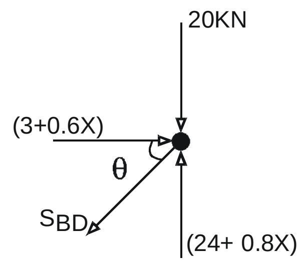
$\sum Fy = 0$

$-20 - S_{BC} - X \sin\theta = 0$

$-20 - S_{BC} - 0.8X = 0$

$$\boxed{S_{BC} = -(20 + 0.8X)}$$

Joint (D)



$\sum FX = 0$

$3 + 0.6X - S_{BD} \cos\theta = 0$

$3 + 0.6X - 0.6\, S_{BD} = 0$

$$\boxed{S_{BD} = (5 + X)}$$

$\sum Fy = 0$

$-20 + 24 + 0.8X - S_{BD} \sin\theta = 0$

$-20 + 24 + 0.8X - (5 + X)\, 0.8 = 0$

$-20 + 24 + 0.8X - 4 - 0.8X = 0$

$0 = 0$ (check)

Calculation of H_F & X_ :–

From the attached table, picking up the values of summations, we have.

$$\sum . S. \frac{\partial S}{\partial H_F} . \frac{L}{AE} = 0 = (-1247.03 + 175.24\, H_F - 4.5 \times X)\, 10^{-6}$$

and $\sum . S. \frac{\partial S}{\partial X} \cdot \frac{L}{AE} = 0 = (460.6 - 4.5\, H_F + 65.64X)\, 10^{-6}$

$$-1247.03 + 175.24\, H_F - 4.5X = 0 \quad \rightarrow (1)$$

$$+ 460.6 - 4.5\, H_F + 65.64X = 0 \quad \rightarrow (2)$$

From (1)

$$X = \left(\frac{-1247.03 + 175.24\, H_F}{4.5}\right) \quad \rightarrow (3)$$

Put in (2) to get H_F

$$460.6 - 4.5\, H_F + 65.64 \left(\frac{-1247.03 + 175.24\, H_F}{4.5}\right) = 0$$

$$460.6 - 4.5\, H_F - 18190.01 + 2556.17\, H_F = 0$$

$$-17729.41 + 2551.67\, H_F = 0$$

$\boxed{H_F = 6.948 \text{ KN}}$ Put this value in 3 to get X.

$$X = \left(\frac{-1247.03 + 175.24 \times 6.948}{4.5}\right) \quad (3)$$

or $\boxed{X = -6.541 \text{ KN}}$ Now calculate number Forces by putting the values of X and H_F in S expressions given in column 5 of the attached table. These final forces appear in last column for S_F.

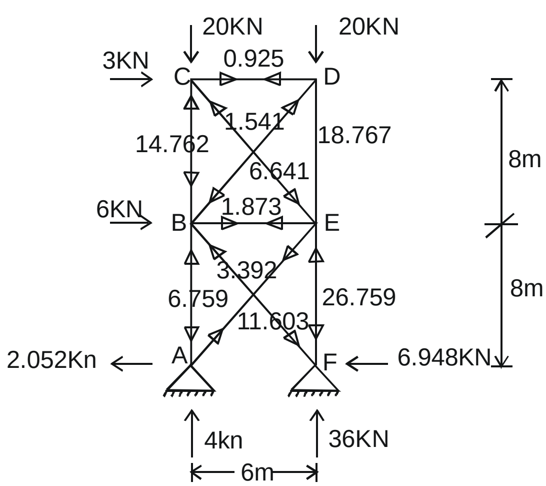
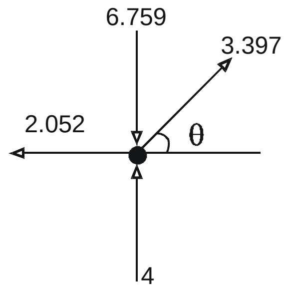


Fig 2.52 ANALYZED TRUSS

Insert Page No. 153–A

Equilibrium checks for the accuracy of calculated member Forces:-

Joint (A)



$\sum FX = 0$

$3.397 \cos\theta - 2.052 = 0$

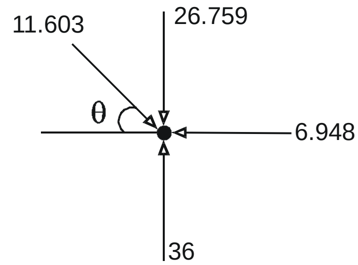
$0 = 0$ Check

$\sum Fy = 0$

$-6.759 + 4 + 3.397 \times 0.8 = 0$

$0 = 0$ Check

Joint (F)



$\sum FX = 0$

$-6.948 + 11.603 \times 0.6 = 0$

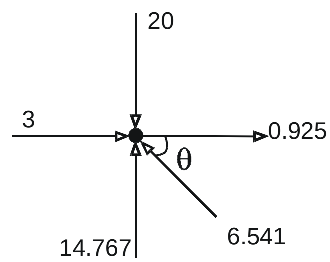
$0 \cong 0$ Check

$\sum Fy = 0$

$36 - 26.759 - 11.603 \times 0.8 = 0$

$0 \cong 0$ Check

Joint (C)



$\sum FX = 0$

$0.925 - 6.541 \times 0.6 + 3 = 0$

$0 = 0$ Check

$\sum Fy = 0$

$14.767 - 20 + 6.541 \times 0.8 = 0$

$0 = 0$ Check. This verifies correctness of solution.

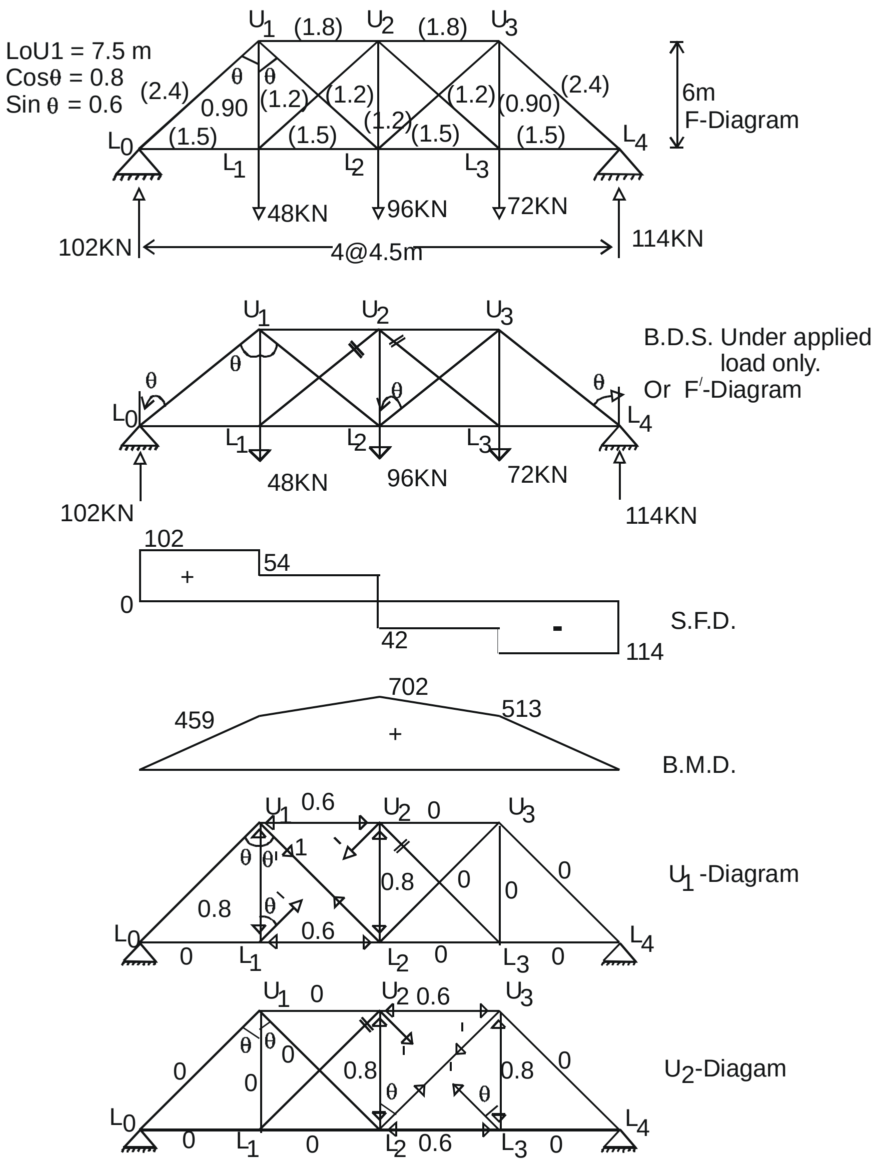
EXAMPLE NO. 12:-

By the unit load–method analyze the internally indeterminate truss shown below. Take the forces in members L_1U_2 and U_2L_3 as the redundants.

Note: The same truss has already been solved in Example No. 9, by taking L_1U_2 and L_2U_3 as redundants.

$E = 200 \times 10^6$ KN/m^2

SOLUTION:-



Compatibility equations are :

$\Delta X_1 + \Delta X_1 R_1 + \Delta X_1 R_2 = 0 \quad \rightarrow (1)$ Here $X_1 = R_1$, $X_2 = R_2$

Deflection created by applied loads and redundants shall be zero.

$\Delta X_2 L + \Delta X_2 R_1 + \Delta X_2 R_2 = 0 \quad \rightarrow (2)$

$\Delta X_1 L = \Sigma . \frac{F' U_1 L}{AE}$ (Change in length of first redundant member by applied loads)

$\Delta X_1 R_1 = \Sigma\left(\frac{U_1^2 L}{AE}\right) X_1$ (Change in length in first redundant member due to first redundant force)

$\Delta X_1 R_2 = \Sigma\left(\frac{U_1 U_2 L}{AE}\right) . X_2$ (Change in length in first redundant member due to second redundant force)

$\Delta X_2 L = \Sigma \frac{F' U_2 L}{AE}$ (Change in second redundant member due to applied load.)

$\Delta X_2 R_1 = \Sigma\left(\frac{U_1 U_2 L}{AE}\right) . X_1$ (Change in length of second redundant member due to first redundant force.)

$\Delta X_2 R_2 = \Sigma\left(\frac{U_2^2 L}{AE}\right) . X_2$ (Change in length of second redundant member due to redundant force in it.)

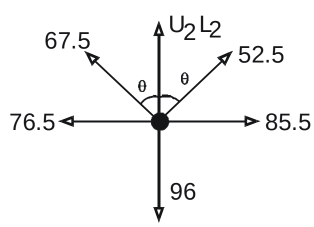
Picking up the above deformations from the table (158–A) and calculate final member forces by following formula.

$F = F' + U_1 X_1 + U_2 X_2$

Forces in chord members and inclineds are determined by the method of moments and shears as explained already, while for verticals method of joints has been used.

Evaluation of force in verticals of F′ – Diagram

(Joint L_2)



$\Sigma FX = 0$

$85.5 - 76.5 + 52.5 \sin\theta - 67.5 \sin\theta = 0$

$85.5 - 76.5 + 52.5 \times 0.6 - 67.5 \times 0.6 = 0$

$0 = 0$ (Check)

$\Sigma Fy + 0$

$U_2L_2 + 52.5 \cos\theta + 67.5 \cos\theta - 96 = 0$

$U_2L_2 = -52.5 \times 0.8 - 67.5 \times 0.8 + 96 = 0$

$\boxed{U_2L_2 = 0}$

Insert Page No. 158–A

Picking the following values from attached table (Table for example No.12)

$\Delta X_1 L = + 1.009 \times 10^{-3}$

$\Delta X_1 R_1 = + 125.7 \times 10^{-6} X_1 = + 0.1257 \times 10^{-3} X_1$

$\Delta X_1 R_2 = + 32 \times 10^{-6} X_2 = + 0.032 \times 10^{-3} X_2$

$\Delta X_2 L = - 0.171 \times 10^{-3}$

$\Delta X_2 R_1 = + 32 \times 10^{-6} X_1 = + 0.032 \times 10^{-3} X_1$

$\Delta X_2 R_2 = + 125.7 \times 10^{-6} X_2 = + 0.1257 \times 10^{-3} X_2$

Putting these in compatibility equals.

$1.009 \times 10^{-3} + 0.1257 \times 10^{-3} X_1 + 0.032 \times 10^{-3} X_2 = 0$ (1)

$- 0.171 \times 10^{-3} + 0.032 \times 10^{-3} X1 + 0.1257 \times 10^{-3} X2 = 0$ (2)

Simplify

$1.009 + 0.1257 X_1 + 0.032 X_2 = 0$ → (1)

$- 0.171 + 0.032 X_1 + 0.1257 X_2 = 0$ → (2)

From (1) $X_1 = \left(\dfrac{-1.009 - 0.032 X_2}{0.1257}\right)$ → (3)

Put in (2) & solve for X_2

$$- 0.171 + 0.032 \left(\frac{-1.009 - 0.032 X_2}{0.1257}\right) + 0.1257 X_2 = 0$$

$- 0.171 - 0.257 - 8.146 \times 10^{-3} X_2 + 0.1257 X_2 = 0$

$- 0.428 + 0.1176 X_2 = 0$

$$X_2 = \frac{0.428}{0.1176}$$

$\boxed{X_2 = 3.64 \text{ KN}}$ Put this in equation (3) to get X_1

$$(3) \Rightarrow X_1 = \left(\frac{-1.009 - 0.032 \times 3.64}{0.1257}\right)$$

$\boxed{X_1 = - 8.95 \text{ KN}}$

So final forces in members are calculated by the following given formula.

$F = F' + U_1 X_1 + U_2 X_2$

$FL_oL_1 = 76.5 + 0 + 0 = + 76.5$ KN

$FL_1 L_2 = 76.5 + (- 0.6) (- 8.95) + 0 = + 81.87$ KN

$FL_2 L_3 = 85.5 + 0 + 3.64) (- 0.6) = + 83.32$ KN

$FL_3 L_4 = 85.5 + 0 + 0 = + 85.5$ KN

$FU_1 U_2 = -117 + (- 0.6) (- 8.95) + 0 = - 111.63$ KN

$FU_2 U_3 = -117 + o + (- 0.6) (3.64) = - 119.18$ KN

$FU_1 L_1 = + 48 + (- 0.8) (- 8.95) + 0 = + 55.16$ KN

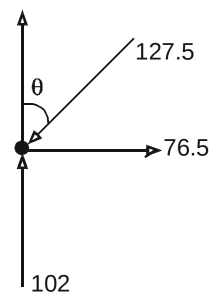
$FU_2 L_2 = 0 + (- 0.8) (- 8.95) + (- 0.8) (3.64) = + 4.25$ KN

$FU_3 L_3$ = + 72 + 0 + (– 0.8) (3.64) = + 69.09 KN
Flo U_1 = – 127.5 + 0 + 0 = – 127.5 KN
$FU_1 L_2$ = + 67.5 + (1) (– 8.95) + 0 = 58.55 KN
$FL_1 U_2$ = 0 + (1) (– 8.95) + 0 = – 8.95 KN
$FU_2 L_3$ = 0 + 0 + (1) (3.64) = + 3.64 KN
$FL_2 U_3$ = 52.5 + 0 + (1) (3.64) = + 56.14 KN
$FU_3 L_4$ = – 142.5 + 0 + 0 = – 142.5 KN

CHECK ON FORCE VALUES

We may apply check at random at any joint. If solution is correct, equilibrium checks will be satisfied at all joint.

Joint Lo.



$\sum FX = 0$
$76.5 - 127.5 \sin\theta = 0$
$76.5 - 127.5 \times 0.6 = 0$
$0 = 0$

$\sum Fy = 0$
$102 - 127.5 \times 0.8 = 0$
$0 = 0$ OK. Results seem to be correct.

The credit for developing method of least work goes to Alberto Castiglianos who worked as an engineer in Italian Railways. This method was presented in a thesis in partial fulfillment of the requirement for the award of diploma engineering of associate engineer. He published a paper for finding deflections which is called Castiglianos first theorem and in consequence thereof, method of least work which is also known as Castiglianos second theorem. Method of least work also mentioned earlier in a paper by an Italian General Menabrea who was not able to give a satisfactory proof. Leonard Euler had also used the method about 50 years ago for derivation of equations for buckling of columns wherein, Daniel Bernolli gave valuable suggestion to him.

Method of least work or Castiglianos second theorem is a very versatile method for the analysis of indeterminate structures and specially to trussed type structures. The method does not however, accounts for erection stresses, temperature stresses or differential support sinking. The reader is advised to use some other method for the analysis of such indeterminate structures like frames and continuos beams.

It must be appreciated in general, for horizontal and vertical indeterminate structural systems, carrying various types of loads, there are generally more than one structural actions present at the same time including direct forces, shear forces, bending moments and twisting moments. In order to have a precise analysis all redundant structural actions and hence strain energies must be considered which would make the method laborious and cumbersome. Therefore, most of engineers think it sufficient to consider only the significant strain energy. The reader should know that most of structural analysis approaches whether classical or matrix methods consider equilibrium of forces and displacement/strain compatibility of members of a system.

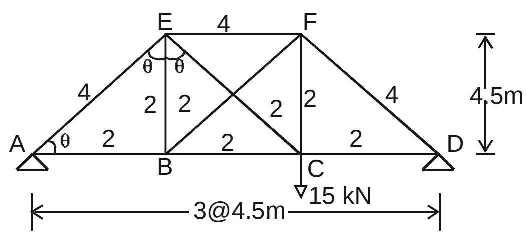
The basis of the method of consistent deformation and method of least work are essentially the same. In consistent deformation method, the deformation produced by the applied loads are equated to these produced by the redundants. This process usually results in the evolution of redundants. However, in the method of least work, total strain energy expression of a structural system in terms of that due to known applied loads and due to redundants is established. Then the total strain energy is partially differentiated with respect to redundant which ultimately result in the evolution of the redundant. It must be appreciated that, for indeterminate structural system like trusses, the unknown redundants maybe external supports reaction or the internal forces or both. And it may not be very clear which type of redundants should be considered as the amount of work involved in terms of requisite calculation may vary. Therefore, a clever choice of redundants (or a basic determinate structure as was the case with consistent deformation method) may often greatly reduce the amount of work involved.

There is often a debate going on these days regarding the utility or justification of classical structural analysis in comparison to the computer method of structural analysis. It is commented that in case of classical methods of structural analysis the student comes across basic and finer points of structural engineering after which a computer analysis of a complex structure maybe undertaken.

In the absence of basic knowledge of classical structural analysis, the engineer maybe in a difficult position to justify to computer results which are again to be checked against equilibrium and deformation compatibility only.

EXAMPLE NO. 13:

The procedure for analysis has already been given. Utilizing that procedure, analyze the following truss by the method of least work. Areas in ( ) carry the units of 10^{-3} m^2 while the value of E can be taken as 200×10^6 KN/m^2.



where i = total degree of indeterminacy

b = number of bars.

r = total number of reactive components which the support can provide.

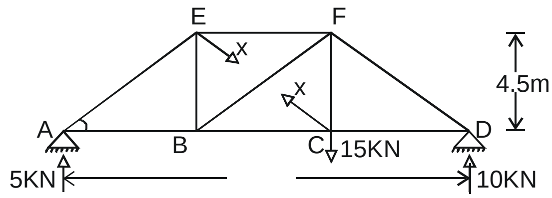
$b + r = 2j$

$10 + 3 > 2 \times 6$ $13 > 12$ so $i = 1$. First degree internal indeterminancy.

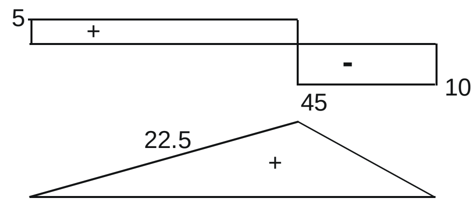
$U = \frac{F^2L}{2\,AE}$ Strain energy due to direct forces induced due to applied loads in a BDS Truss.

$$\frac{\partial U}{\partial X} = F.\frac{\partial F}{\partial X}\,.\,\frac{L}{AE} = 0$$

Note:– We select the redundants in such a way that the stability of the structure is not effected. Selecting member EC as redundant.



F-diagram B.D.S. under the action of applied loads & redundant.



S.F.D. due to applied load only.

B.M.D. due to applied load only.

Method of moments and shears has been used to find forces in BDS due to applied loads. A table has been made. Forces vertical in members in terms of redundant X may be determined by the method of joints as before. From table.

$$\sum F.\frac{\partial F}{\partial x}\cdot\frac{L}{AE} = 0 \qquad = -331.22\times10^{-6} + 51.49\times10^{-6}X$$

$$\text{or} \quad -331.22 + 51.49X = 0$$

$$\boxed{X = +6.433\text{ KN}}$$

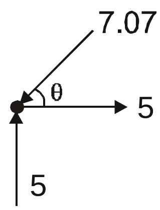
The final member forces are obtained as below by putting value of X in column 5 of the table.

Member	Force (KN)
AB	+ 5
BC	+5.45
CD	+ 10
EF	– 9.55
BE	+ 0.45
CF	+ 10.45
CE	+ 6.43
BF	– 0.64
AE	– 7.07
DF	– 14.14

Insert page No. 164–A

CHECK.

Joint A.



$\sum FX = 0$

$5 - 7.07 \text{ Cos}\theta = 0$

$5 - 7.07 \times 0.707 = 0$

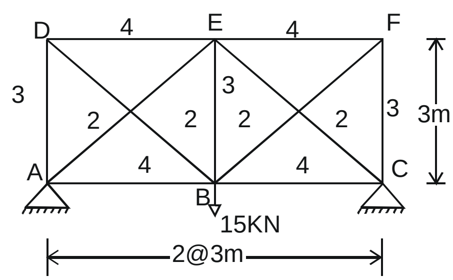
$0 = 0$

$\sum Fy = 0$

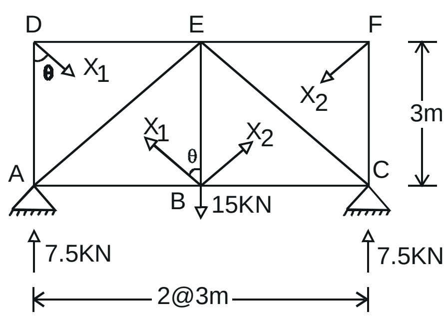
$- 7.07 \times 0.707 + 5 = 0$

$0 = 0$ Check is OK.

EXAMPLE NO. 14:– Analyze the following symmetrically loaded second degree internally indeterminate truss by the method of least work. Areas in ( ) are $10^{-3}m^2$. The value of E can be taken as 200×10^6 KN/m^2



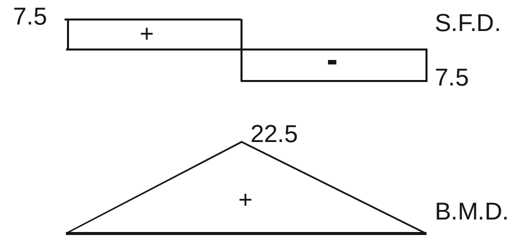
Selecting member BD and Befor e as redundants.



BDS under loads and redundants.

SOLUTION:

Note **:–** By virtue of symmetry, we can expect to have same values for X_1 and X_2. It is known before hand.



SFD and BMD in BDS due to applied loads are shown above.

As in previous case determine member forces in BDS due to applied loads by the method of moments shears while method of joints may be used to determine member forces due to redundants acting separately. Apply super position principal. Then these are entered in a table given.

Summation of relavant columns due to X1 and X2 gives two equations from which these can be calculated. Putting values from table and solving for X1 and X2.

$$[-2.65\times10^{-3}(7.5-0.707X_1)-2.65\times10^{3}(-0.707X_1)-3.53\times10^{-3}(-0.707X_1)$$
$$-3.53\times10^{-3}(15-0.707X_1-0.707X_2)+10.6\times10^{-3}(-10.6+X_1)+10.6\times10^{-3}(X_2)]10^{-3}=0$$

$$-19.875+1.874X_1+1.874X_1+2.450X_1-52.45+2.50X_1+2.5X_2-112.36+10.6X_1+10.6X_1=0$$

$29.898X_1+2.50X_2-185.185=0$ $\rightarrow$ (1) **(Σ col 8)**

$$-2.65\times10^{-3}(7.5-0.707X_2)-2.65\times10^{-3}(-0.707X_2)-3.53\times10^{-3}(15-0.707X_1-0.707X_2)$$
$$-3.53\times10^{-3}(-0.707X_2)+10.6\times10^{-3}(-10.6+X_2)+10.6\times10^{-3}X_2=0$$

$$-19.875+1.874X_2+1.874X_2-52.95+2.50X_1+2.50X_2+2.450X_2-112.36+10.6X_2+10.6X_2=0$$

$2.50X_1+29.898X_2-185.185=0$ $\rightarrow$ (2) **(Σ col 9)**

From (1), $X_1=\left(\dfrac{185.185-2.50X_2}{29.898}\right)$ $\rightarrow$ (3) Put in 2 above

(2) $\Rightarrow$ $2.50\left(\dfrac{185.185-2.50X_2}{29.898}\right)+29.898X_2-185.185=0$

$15.465-0.21X_2+29.898X_2-185.185=0$

$29.689X_2-169.7=0$

$$\boxed{X_2=+5.716\text{ KN}}$$

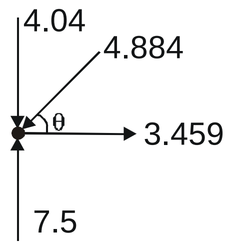
Put X_2 in equation 3 to get X_1. The final member forces are given in last column. These are obtained by putting values of X_1 and X_2, whichever is applicable, in column 5 of the table.

Insert Page No. 166–A

$$\text{Then } X_1 = \left(\frac{185.185 - 2.50 \times 5.716}{29.898}\right)$$

$$\boxed{X_1 = +5.716 \text{ KN}}$$

Equilibrium Check.



$\sum FX = 0$
$3.459 - 4.884 \times \cos\theta = 0$
$3.459 - 4.884 \times 0.707 = 0$
$0 = 0$

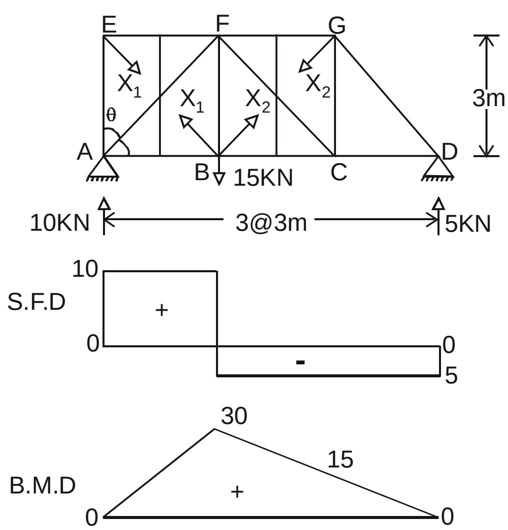
$\sum Fy = 0$
$7.5 - 4.04 - 4.884 \times 0.707 = 0$
$0 = 0$ Checks are satisfied. Results are OK.

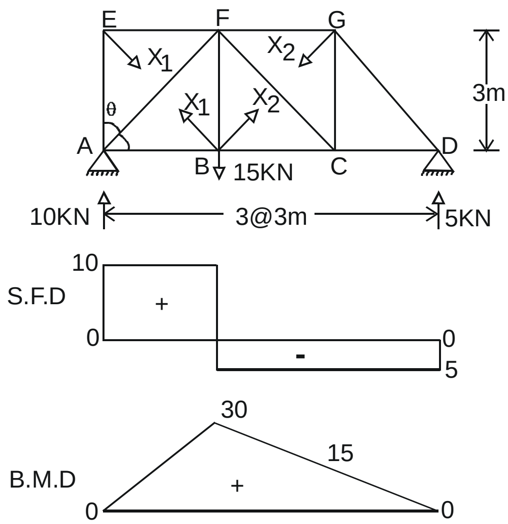
EXAMPLE NO. 15:– Analyze the following internally indeterminate truss by the method of least work. Areas in ( ) are $10^{-3}m^2$. The value of E can be taken as 200×10^6 KN/m^2.

SOLUTION:–

b = 13 , r = 3 , j = 7 so degree of indeterminacy I =(b + r) –2j =2

Choosing members EB and BG as redundants, forces due to loads have been determined by the method of moments and shears for the BDS and are entered in a table. While forces due to redundants X_1 and X_2.

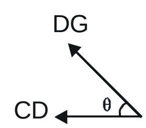




Member Forces Due to Redundants Only.

Please number that due to separate action of redundants X_1 and X_2 member forces will be induced only in the square whose inclineds are X_1 and X_2. There will be no reaction at supports.

Joint D:–



$\sum Fy = 0$

$DG \sin\theta - 0$

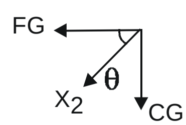
$\boxed{DG = 0}$

$\sum FX = 0$

$DG \cos\theta + CD = 0$

$\boxed{CD = 0}$

Joint G :–



$\sum FX = 0$

$- FG - X_2 \cos\theta = 0$

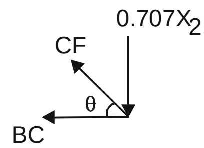
$\boxed{FG = -0.707 X_2}$

$\sum Fy = 0$

$- CG - X_2 \sin\theta = 0$

$\boxed{CG = -0.707 X_2}$

Joint C :–



$\sum Fy = 0$

$CF \sin\theta - 0.707\, X_2 = 0$

$CF = \dfrac{0.707\, X_2}{0.707}$

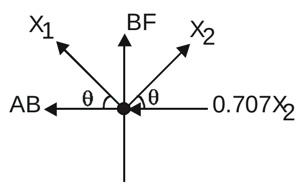
$\boxed{CF = + X_2}$

$\sum FX = 0$

$- BC - CF \cos\theta = 0$

$\boxed{BC = -0.707\, X_2}$

Joint B.



$\sum FX = 0$

$- 0.707\, X_2 - AB + X_2 \cos\theta - X_1 \cos\theta = 0$

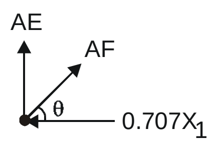
$\boxed{AB = - 0.707\, X_1}$

$\sum Fy = 0$

$X_1 \sin\theta + X_2 \sin\theta + BF = 0$

$\boxed{BF = - 0.707X_1 - 0.707X_2}$

Joint A.



$\sum FX = 0$

$- 0.707\, X_1 + AF \cos\theta = 0$

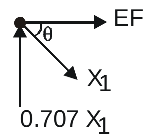
$\boxed{AF = X_1}$

$\sum Fy = 0$

$AE + AF \sin\theta = 0$

$\boxed{AE = -0.707X_1}$

Joint E.



$\sum FX = 0$

$EF + X_1 \cos\theta = 0$

$\boxed{EF = -0.707\ X_1}$

$\sum Fy = 0$

$0.707\ X_1 - 0.707\ X_1 = 0$

$0 = 0$ (Check)

Entering the values of summations from attached table, we have.

$$\sum F.\ \frac{\partial F}{\partial X_1}\ .\ \frac{L}{AE} = 0 = -229.443 \times 10^{-6} + 29.848 \times 10^{-6}\ X_1 + 2.45 \times 10^{-6} X2$$

$$\sum F.\ \frac{\partial F}{\partial X_2}\ .\ \frac{L}{AE} = 0 = -168.9 \times 10^{-6} + 2.45 \times 10^{-6}\ X_1 + 29.848 \times 10^{-6}\ X2$$

Simplifying

$$-229.443 + 29.848\ X_1 + 2.45\ X_2 = 0 \qquad \rightarrow (1)$$

$$-168.9 + 2.45\ X_1 + 29.848\ X_2 = 0 \qquad \rightarrow (2)$$

From (1)

$$X_1 = \left(\frac{-2.45\ X_2 + 229.443}{29.848}\right) \qquad \rightarrow (3)$$

Put in (2) & solve for X_2

$$-168.9 + 2.45\left(\frac{-2.45\ X_2 + 229.443}{29.848}\right) + 29.848\ X_2 = 0$$

$$-168.9 - 0.201\ X_2 + 18.833 + 29.848\ X_2 = 0$$

$$-150.067 + 29.647\ X_2 = 0$$

$$X_2 = \frac{150.067}{29.647}$$

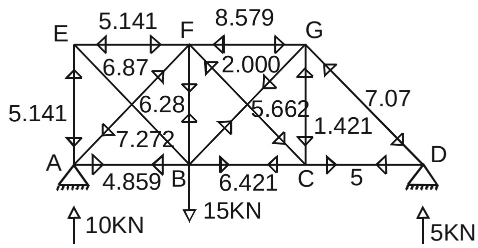
$\boxed{X_2 = +5.062\ KN}$

Insert page No. 170–A

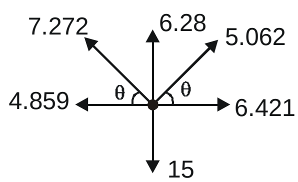
So $X_1 = \dfrac{-2.45 \times 5.062 + 229.443}{29.848}$ by putting value of X_2 in (3)

$$\boxed{X_1 = +7.272 \text{ KN}}$$

EQUILIBRIUM CHECKS :–



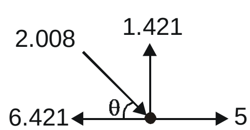
Joint B:–



$\sum FX = 0$
$6.421 + 5.062 \cos\theta - 7.272 \cos\theta - 4.859 = 0$
$0 = 0$

$\sum Fy = 0$
$6.28 - 15 + 5.062 \sin\theta + 7.272 \sin\theta = 0$
$0 = 0$ The results are OK.

Joint C:–



$\sum FX = 0$
$5 + 2.008 \cos\theta - 6.421 = 0$
$0 = 0$

$\sum Fy = 0$
$1.421 - 2.008 \sin\theta = 0$
$0 = 0$ Results are OK.