Introduction to Statically Indeterminate Analysis

Support reactions and internal forces of **statically determinate structures** can be determined using only the **equations of equilibrium**. However, the analysis of **statically indeterminate structures requires additional equations based on the geometry of deformation of the structure**.

1

Additional equations come from ***compatibility relationships***, which **ensure continuity of displacements throughout the structure**. The **remaining equations** are constructed from member ***constitutive equations***, i.e., **relationships between stresses and strains and the integration of these equations over the cross section**.

2

Design of an indeterminate structure is carried out in an iterative manner, whereby the (relative) sizes of the structural members are initially assumed and used to analyze the structure. Based on the computed results (displacements and internal member forces), the member sizes are adjusted to meet governing design criteria. This iteration process continues until the member sizes based on the results of an analysis are close to those assumed for that analysis.

3

Another **consequence of statically indeterminate structures** is that the **relative variation of member sizes influences the magnitudes of the forces that the member will experience**. Stated in another way, **stiffness** (**large member size and/or high modulus materials**) **attracts force**.

Despite these difficulties with statically indeterminate structures, an **overwhelming majority of structures being built today are statically indeterminate**.

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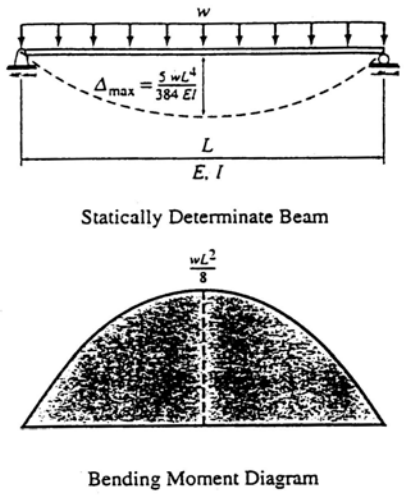
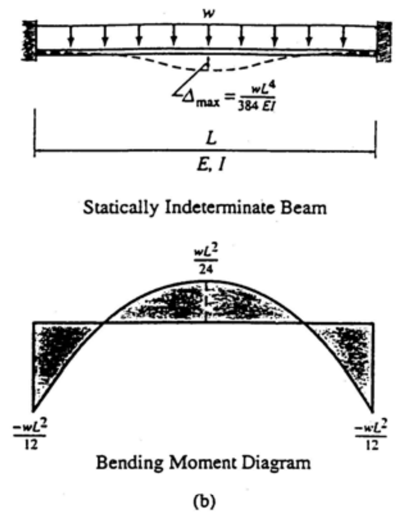
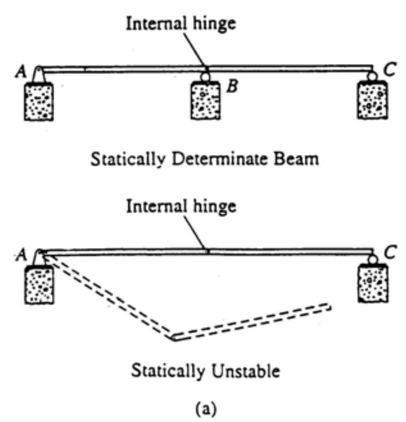
Advantages Statically Indeterminate Structures

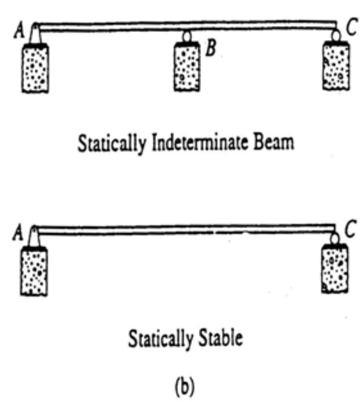




Statically indeterminate structures typically result in smaller stresses and greater stiffness (smaller deflections) as illustrated for this beam.

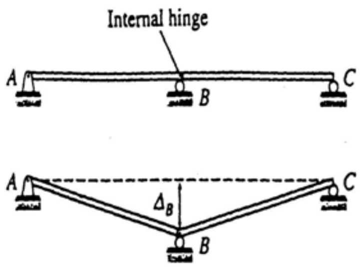


Determinate beam is unstable if middle support is removed or knocked off!

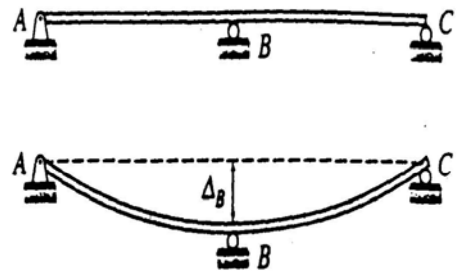


Statically indeterminate structures introduce redundancy, which may insure that failure in one part of the structure will not result in catastrophic or collapse failure of the structure.

Disadvantages of Statically Indeterminate Structures

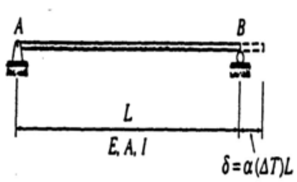


(a) Statically Determinate Beam

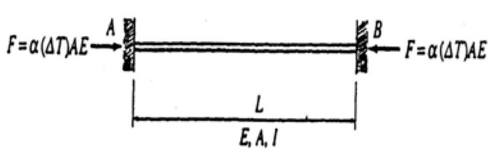


(b) Statically Indeterminate Beam

Statically indeterminate structure is **self-strained** due to support settlement, which produces stresses, as illustrated above.



(a) Statically Determinate Beam



Statically indeterminate struc-tures are also self-strained due to temperature changes and fabrication errors.

Indeterminate Structures: Influence Lines

Influence lines for statically indeterminate structures provide the same information as influence lines for statically determinate structures, i.e. it represents the magnitude of a response function at a particular location on the structure as a unit load moves across the structure.

Our goals in this chapter are:

1. To become familiar with the shape of influence lines for the support reactions and internal forces in continuous beams and frames.
2. To develop an ability to sketch the appropriate shape of influence functions for indeterminate beams and frames.
3. To establish how to position distributed live loads on continuous structures to maximize response function values.

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Qualitative Influence Lines for Statically Inde-terminate Structures: Muller-Breslau's Principle

In many practical applications, it is usually sufficient to draw only the qualitative influence lines to decide where to place the live loads to maximize the response functions of interest. The Muller-Breslau Principle pro-vides a convenient mechanism to construct the qualitative influence lines, which is stated as:

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The influence line for a force (or moment) response function is given by the deflected shape of the released structure by removing the displacement constraint corresponding to the response function of interest from the original structure and giving a unit displacement (or rotation) at the location and in the direction of the response function.

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Procedure for constructing qualitative influence lines for indeterminate structures is: **(1)** remove from the structure the restraint corresponding to the response function of interest, **(2)** apply a unit displacement or rotation to the released structure at the release in the desired response function direction, and **(3)** draw the qualitative deflected shape of the released structure consistent with all remaining support and continuity conditions.

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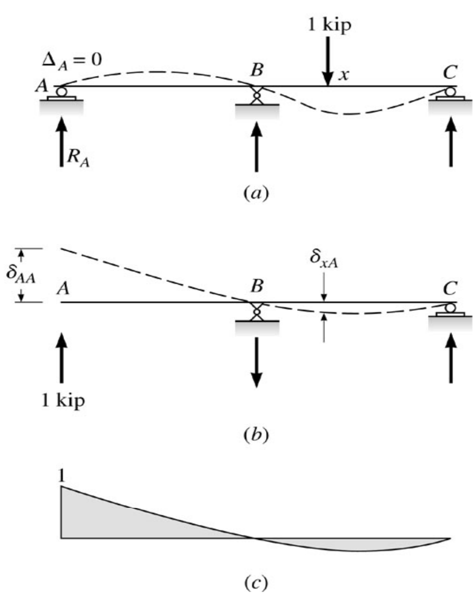
Notice that this procedure is identical to the one discussed for statically determinate structures.

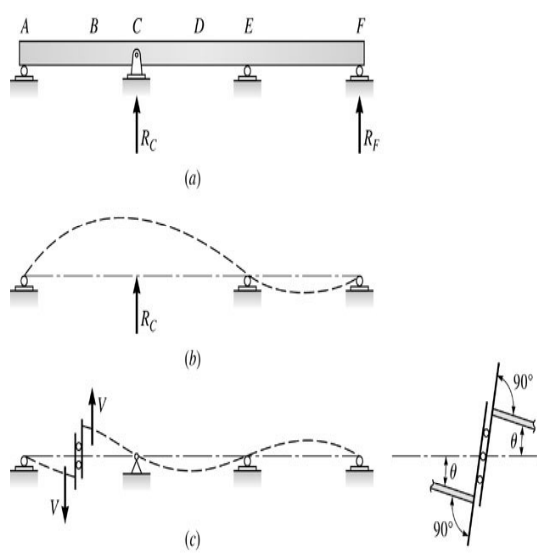
However, unlike statically determinate structures, the influence lines for statically indeterminate structures are typically curved.

Placement of the live loads to maximize the desired response function is obtained from the qualitative ILD.

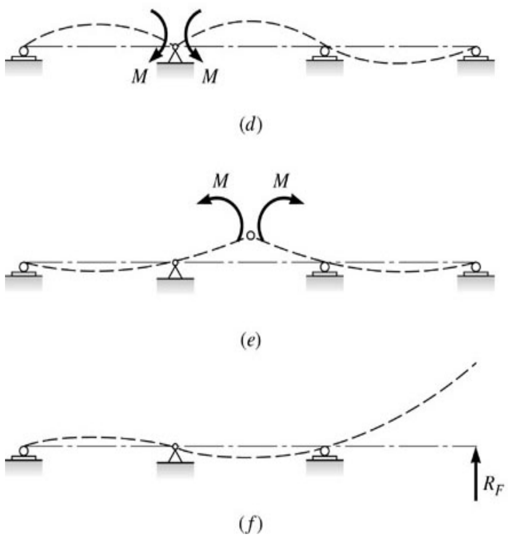
Uniformly distributed live loads are placed over the positive areas of the ILD to maximize the drawn response function values. Because the influence line ordinates tend to diminish rapidly with distance from the response function location, live loads placed more than three span lengths away can be ignored. Once the live load pattern is known, an indeterminate analysis of the structure can be performed to determine the maximum value of the response function.



QILD for R_A



QILD's for R_C and V_B



QILD's for $(M_C)^-$, $(M_D)^+$ and R_F

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Live Load Pattern to Maximize Forces in Multistory Buildings

Building codes specify that members of multistory buildings be designed to support a uniformly distributed live load as well as the dead load of the structure. Dead and live loads are normally considered separately since the dead load is fixed in position whereas the live load must be varied to maximize a particular force at each section of the structure. Such

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maximum forces are typically produced by patterned loading.

Qualitative Influence Lines:

1. Introduce appropriate unit displacement at the desired response function location.
2. Sketch the displacement diagram along the beam or column line (axial force in column) appropriate for the unit displacement and assume zero axial deformation.

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3. Axial column force (do not consider axial force in beams):

   (a) Sketch the beam line qualitative displacement diagrams.

   (b) Sketch the column line qualitative displacement diagrams maintaining equality of the connection geometry before and after deformation.

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4. Beam force:

(a) Sketch the beam line qualitative displacement diagram for which the release has been introduced.

(b) Sketch all column line qualitative displacement diagrams maintaining connection geometry before and after deformation. Start the column line qualitative displacement diagrams from the beam line diagram of (a).

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(c) Sketch remaining beam line qualitative displacement diagrams maintaining connection geometry before and after deformation.

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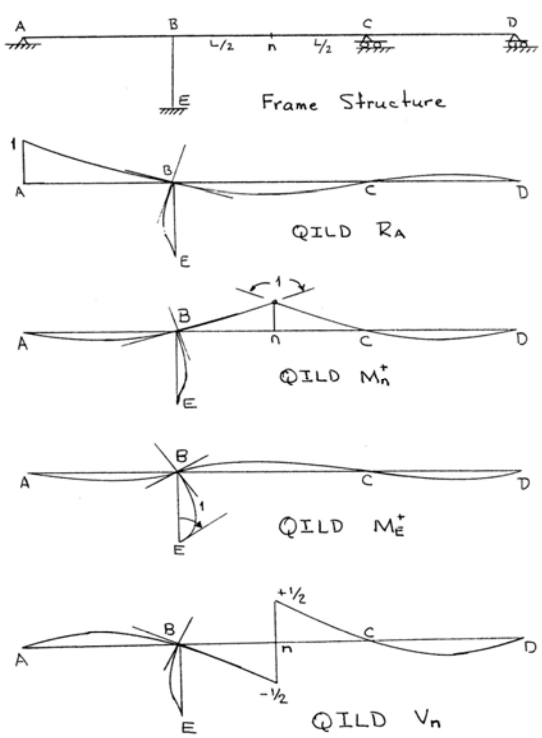
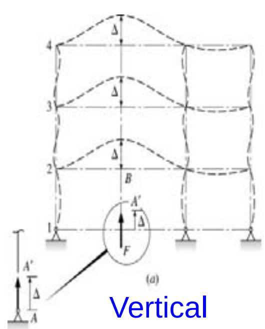
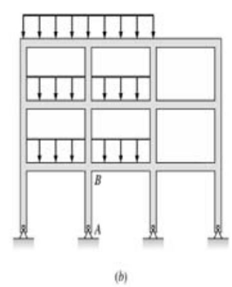




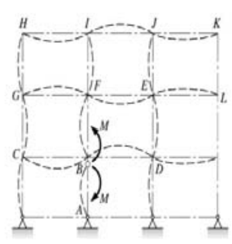
(a)

Vertical Reaction F



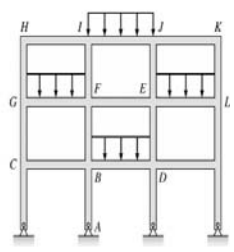
(b)

Load Pattern to Maximize F



(c)

Column Moment M



(d)

Load Pattern to Maximize M

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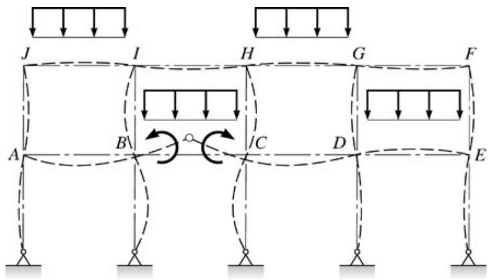

(*a*)

QILD and Load Pattern for Center Beam Moment M

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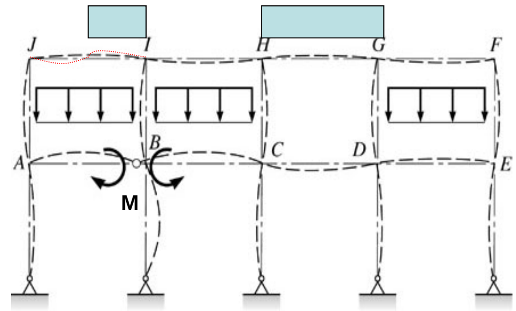
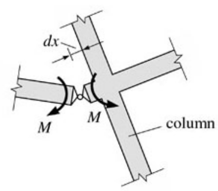


(*b*)

QILD and Load Pattern for End Beam Moment M



(*c*)

Expanded Detail for Beam End Moment

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Envelope Curves

Design engineers often use influence lines to construct shear and moment envelope curves for continuous beams in buildings or for bridge girders. An envelope curve defines the extreme boundary values of shear or bending moment along the beam due to critical placements of design live loads. For example, consider a three-span continuous beam.

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Qualitative influence lines for positive moments are given, shear influence lines are presented later. Based on the qualitative influence lines, critical live load placement can be determined and a structural analysis computer program can be used to calculate the member end shear and moment values for the dead load case and the critical live load cases.

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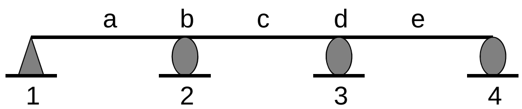
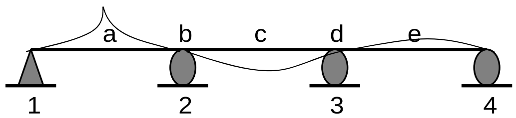
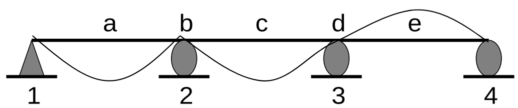


Three-Span Continuous Beam

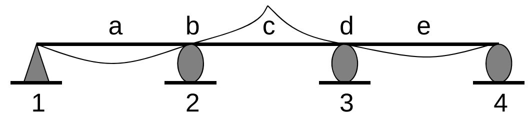


QILD for $(M_a)^+$

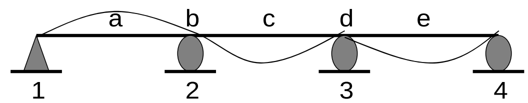


QILD for $(M_b)^+$

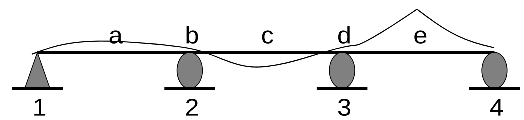
33



QILD for $(M_c)^+$

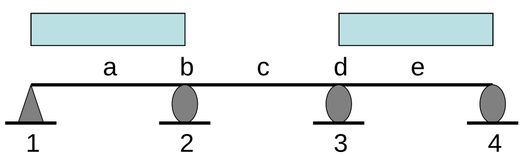


QILD for $(M_d)^+$

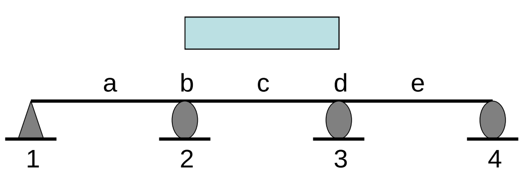


QILD for $(M_e)^+$

34

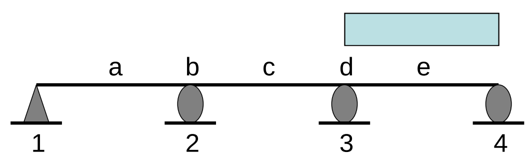


Critical Live Load Placement
for $(M_a)^+$

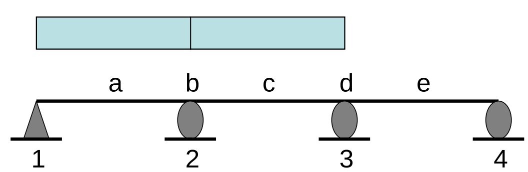


Critical Live Load Placement
for $(M_a)^-$

35

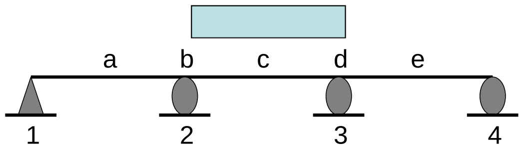


Critical Live Load Placement
for $(M_b)^+$

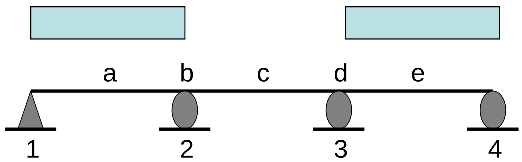


Critical Live Load Placement
for $(M_b)^-$

36

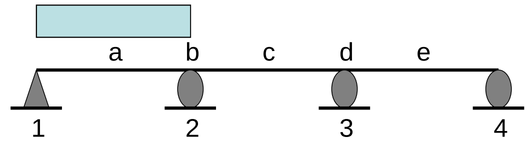


Critical Live Load Placement
for $(M_c)^+$

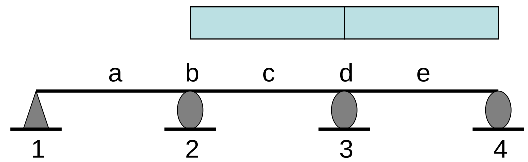


Critical Live Load Placement
for $(M_c)^-$

37

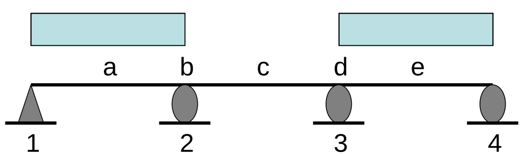


Critical Live Load Placement
for $(M_d)^+$

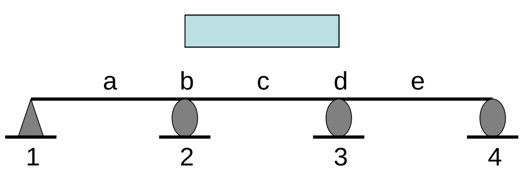


Critical Live Load Placement
for $(M_d)^-$

38



Critical Live Load Placement
for $(M_e)^+$



Critical Live Load Placement
for $(M_e)^-$

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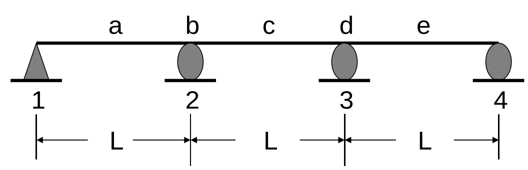
Calculate the moment envelope curve for the three-span continuous beam.



L = 20’ = 240”
E = 3,000 ksi
A = 60 in^2
I = 500 in^4
w_{DL} = 1.2 k/ft – dead load
w_{LL} = 4.8 k/ft – live load

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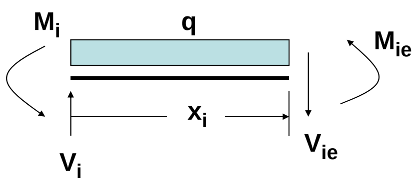
Shear and Moment Equations for a Loaded Span



$$V_{ie} = V_i - q\,x_i$$

$$M_{ie} = -M_i + V_i\,x_i - 0.5q\,(x_i)^2$$

Shear and Moment Equations for an Unloaded Span

(set q = 0 in equations above)

$$V_{ie} = V_i$$

$$M_{ie} = -M_i + V_i\,x_i$$

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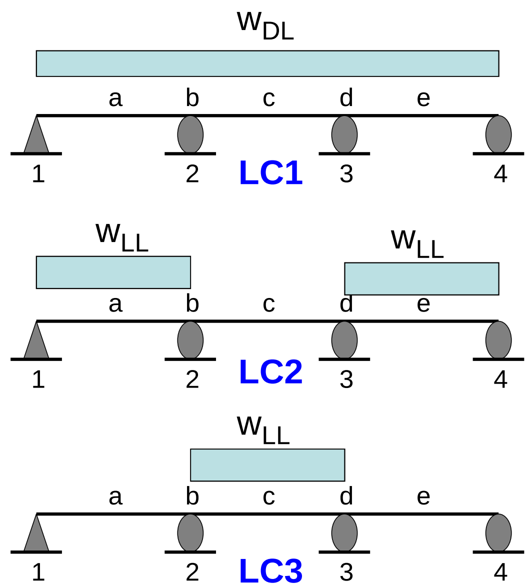
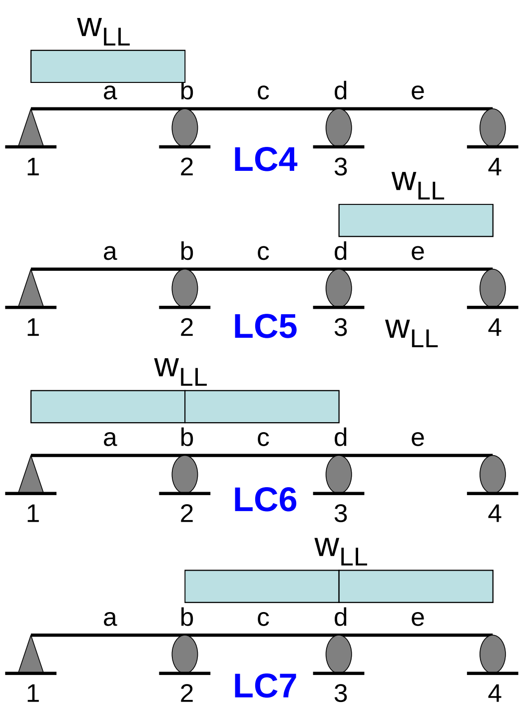
Load Cases



42



A summary of the results from the statically indeterminate beam analysis for each of the seven load cases are given in your class notes.

----- RESULTS FOR LOAD SET: 1
***** M E M B E R   F O R C E S *****

MEMBER	MEMBER NODE	AXIAL FORCE (kip)	SHEAR FORCE (kip)	BENDING MOMENT (ft-k)
1	1	0.00	9.60	0.00
	2	-0.00	14.40	-48.00
2	2	0.00	12.00	48.00
	3	-0.00	12.00	-48.00
3	3	0.00	14.40	48.00
	4	-0.00	9.60	0.00

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The equations for the internal shear and bending moments for each span and each load case are:

Load Case 1

$V_{12} = 9.6 – 1.2x_1$
$M_{12} = 9.6x_1 – 0.6(x_1)^2$

$V_{23} = 12 – 1.2x_2$
$M_{23} = -48 + 12x_2 – 0.6(x_2)^2$

$V_{34} = 14.4 – 1.2x_3$
$M_{34} = -48 + 14.4x_3 – 0.6(x_3)^2$

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Load Case 2

$V_{12} = 43.2 – 4.8x_1$
$M_{12} = 43.2x_1 – 2.4(x_1)^2$

$V_{23} = 0$
$M_{23} = -96$

$V_{34} = 52.8 – 4.8x_3$
$M_{34} = -96 + 52.8x_3 – 2.4(x_3)^2$

Load Case 3

$V_{12} = -4.8$
$M_{12} = -4.8x_1$

$V_{23} = 48 – 4.8x_2$
$M_{23} = -96 + 48x_2 – 2.4(x_2)^2$

$V_{34} = 4.8$
$M_{34} = -96 + 4.8x_3$

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Load Case 4

$V_{12} = 41.6 – 4.8x_1$
$M_{12} = 41.6x_1 – 2.4(x_1)^2$

$V_{23} = 8$
$M_{23} = -128 + 8x_2$

$V_{34} = -1.60$
$M_{34} = 32 - 1.6x_3$

Load Case 5

$V_{12} = 1.6$
$M_{12} = 1.6x_1$

$V_{23} = -8$
$M_{23} = 32 - 8x_2$

$V_{34} = 54.4 – 4.8x_3$
$M_{34} = -128 + 54.4x_3 – 2.4(x_3)^2$

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Load Case 6

$V_{12} = 36.8 – 4.8x_1$
$M_{12} = 36.8x_1 – 2.4(x_1)^2$

$V_{23} = 56 – 4.8x_2$
$M_{23} = -224 + 56x_2 – 2.4(x_2)^2$

$V_{34} = 3.2$
$M_{34} = -64 + 3.2x_3$

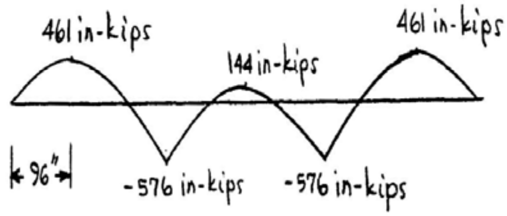
Load Case 7

$V_{12} = -3.2$
$M_{12} = -3.2x_1$

$V_{23} = 40 – 4.8x_2$
$M_{23} = -64 + 40x_2 – 2.4(x_2)^2$

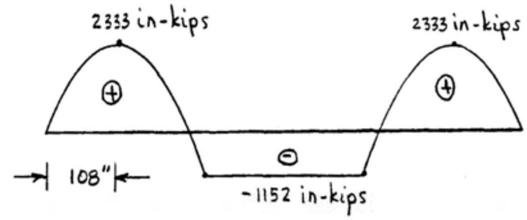
$V_{34} = 59.2 – 4.8x_3$
$M_{34} = -224 + 59.2x_3 – 2.4(x_3)^2$

48

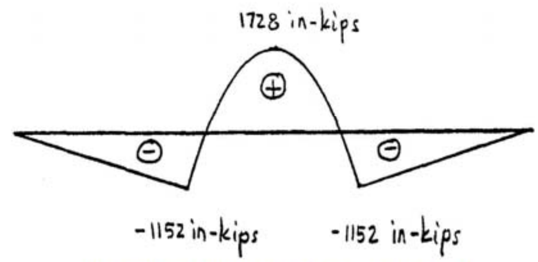


Bending Moment Diagram LC1

49

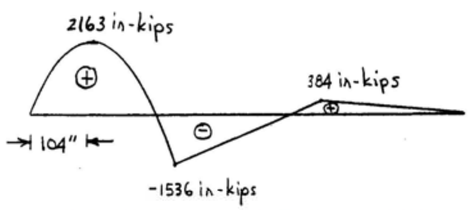


Bending Moment Diagram LC2

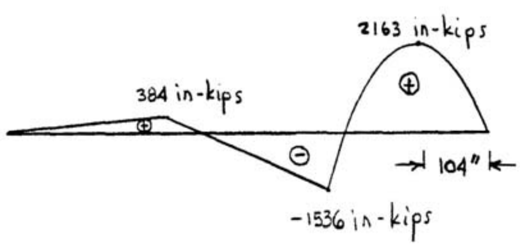


Bending Moment Diagram LC3

50

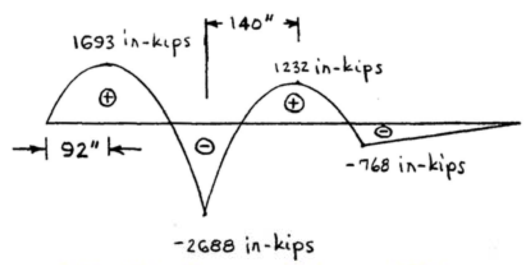


Bending Moment Diagram LC4

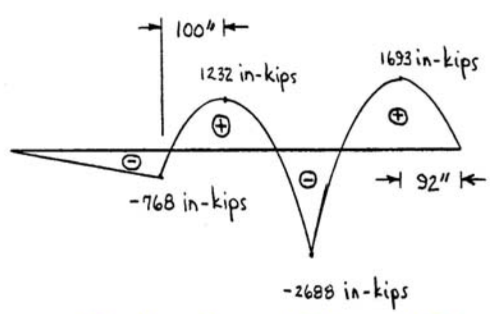


Bending Moment Diagram LC5

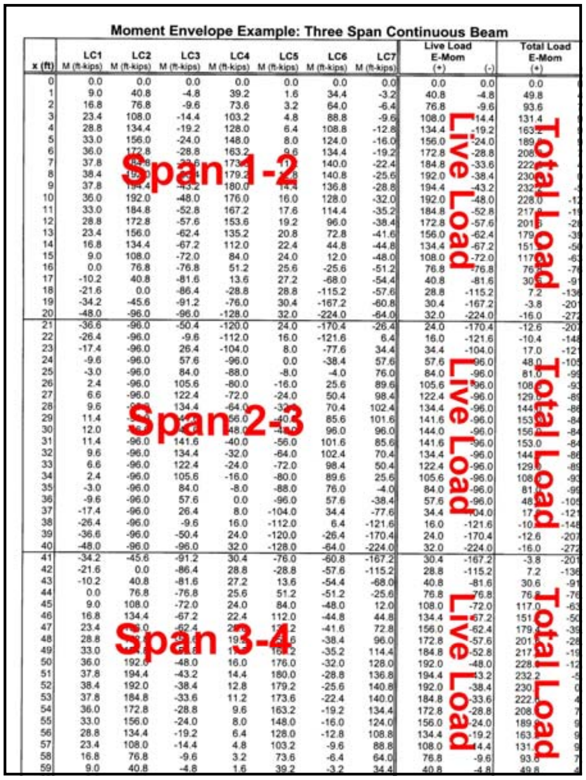
51



Bending Moment Diagram LC6



Bending Moment Diagram LC7 52



Moment Envelope Example: Three Span Continuous Beam

x (ft)	LC1 M (ft-kips)	LC2 M (ft-kips)	LC3 M (ft-kips)	LC4 M (ft-kips)	LC5 M (ft-kips)	LC6 M (ft-kips)	LC7 M (ft-kips)	Live Load E-Mom (+)	Live Load E-Mom (-)	Total Load E-Mom (+)	Total Load E-Mom (-)
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0
1	9.0	40.8	-4.8	39.2	1.6	34.4	-3.2	40.8	-4.8	49.8	4
2	16.8	76.8	-9.6	73.6	3.2	64.0	-6.4	76.8	-9.6	93.6	7
3	23.4	108.0	-14.4	103.2	4.8	88.8	-9.6	108.0	-14.4	131.4	9
4	28.8	134.4	-19.2	128.0	6.4	108.8	-12.8	134.4	-19.2	163.2	9
5	33.0	156.0	-24.0	148.0	8.0	124.0	-16.0	156.0	-24.0	[illegible]	9
6	36.0	172.8	-28.8	163.2	9.6	134.4	-19.2	172.8	-28.8	[illegible]	7
7	37.8	[illegible]	[illegible]	[illegible]	[illegible]	140.0	-22.4	184.8	-33.6	[illegible]	4
8	38.4	[illegible]	[illegible]	179.2	[illegible]	140.8	-25.6	192.0	-38.4	[illegible]	0
9	37.8	[illegible]	-43.2	180.0	[illegible]	136.8	-28.8	194.4	-43.2	[illegible]	-5
10	36.0	192.0	-48.0	176.0	16.0	128.0	-32.0	192.0	-48.0	228.0	-12
11	33.0	184.8	-52.8	167.2	17.6	114.4	-35.2	184.8	-52.8	[illegible]	-19
12	28.8	172.8	-57.6	153.6	19.2	96.0	-38.4	172.8	-57.6	[illegible]	-28
13	23.4	156.0	-62.4	135.2	20.8	72.8	-41.6	156.0	-62.4	[illegible]	-39
14	16.8	134.4	-67.2	112.0	22.4	44.8	-44.8	134.4	-67.2	[illegible]	-50
15	9.0	108.0	-72.0	84.0	24.0	12.0	-48.0	108.0	-72.0	[illegible]	-63
16	0.0	76.8	-76.8	51.2	25.6	-25.6	-51.2	76.8	-76.8	[illegible]	-76
17	-10.2	40.8	-81.6	13.6	27.2	-68.0	-54.4	40.8	-81.6	[illegible]	-91
18	-21.6	0.0	-86.4	-28.8	28.8	-115.2	-57.6	28.8	-115.2	7.2	-136
19	-34.2	-45.6	-91.2	-76.0	30.4	-167.2	-60.8	30.4	-167.2	-3.8	-201
20	-48.0	-96.0	-96.0	-128.0	32.0	-224.0	-64.0	32.0	-224.0	-16.0	-272
21	-36.6	-96.0	-50.4	-120.0	24.0	-170.4	-26.4	24.0	-170.4	-12.6	-207
22	-26.4	-96.0	-9.6	-112.0	16.0	-121.6	6.4	16.0	-121.6	-10.4	-148
23	-17.4	-96.0	26.4	-104.0	8.0	-77.6	34.4	34.4	-104.0	17.0	-121
24	-9.6	-96.0	57.6	-96.0	0.0	-38.4	57.6	57.6	-96.0	48.0	-105
25	-3.0	-96.0	84.0	-88.0	-8.0	-4.0	76.0	84.0	-96.0	81.0	-99
26	2.4	-96.0	105.6	-80.0	-16.0	25.6	89.6	105.6	-96.0	[illegible]	-93
27	6.6	-96.0	122.4	-72.0	-24.0	50.4	98.4	122.4	-96.0	129.0	-89
28	9.6	[illegible]	134.4	-64.0	[illegible]	70.4	102.4	134.4	-96.0	[illegible]	-86
29	11.4	[illegible]	[illegible]	-56.0	[illegible]	85.6	101.6	141.6	-96.0	[illegible]	-84
30	12.0	[illegible]	[illegible]	-48.0	[illegible]	96.0	96.0	144.0	-96.0	156.0	-84
31	11.4	-96.0	141.6	-40.0	-56.0	101.6	85.6	141.6	-96.0	153.0	-84
32	9.6	-96.0	134.4	-32.0	-64.0	102.4	70.4	134.4	-96.0	[illegible]	-86
33	6.6	-96.0	122.4	-24.0	-72.0	98.4	50.4	122.4	-96.0	[illegible]	-89
34	2.4	-96.0	105.6	-16.0	-80.0	89.6	25.6	105.6	-96.0	[illegible]	-93
35	-3.0	-96.0	84.0	-8.0	-88.0	76.0	-4.0	84.0	-96.0	[illegible]	-99
36	-9.6	-96.0	57.6	0.0	-96.0	57.6	-38.4	57.6	-96.0	[illegible]	-105
37	-17.4	-96.0	26.4	8.0	-104.0	34.4	-77.6	34.4	[illegible]	[illegible]	-121
38	-26.4	-96.0	-9.6	16.0	-112.0	6.4	-121.6	16.0	-121.6	[illegible]	-148
39	-36.6	-96.0	-50.4	24.0	-120.0	-26.4	-170.4	24.0	-170.4	-12.6	-207
40	-48.0	-96.0	-96.0	32.0	-128.0	-64.0	-224.0	32.0	-224.0	-16.0	-272
41	-34.2	-45.6	-91.2	30.4	-76.0	-60.8	-167.2	30.4	-167.2	-3.8	-201
42	-21.6	0.0	-86.4	28.8	-28.8	-57.6	-115.2	28.8	-115.2	7.2	-136
43	-10.2	40.8	-81.6	27.2	13.6	-54.4	-68.0	40.8	-81.6	30.6	-91
44	0.0	76.8	-76.8	25.6	51.2	-51.2	-25.6	76.8	-76.8	[illegible]	-76
45	9.0	108.0	-72.0	24.0	84.0	-48.0	12.0	108.0	-72.0	117.0	-63
46	16.8	134.4	-67.2	22.4	112.0	-44.8	44.8	134.4	-67.2	[illegible]	-50
47	23.4	[illegible]	-62.4	[illegible]	[illegible]	-41.6	72.8	156.0	-62.4	[illegible]	-39
48	28.8	[illegible]	[illegible]	19.2	[illegible]	-38.4	96.0	172.8	-57.6	[illegible]	-28
49	33.0	[illegible]	[illegible]	[illegible]	[illegible]	-35.2	114.4	184.8	-52.8	[illegible]	-19
50	36.0	192.0	-48.0	16.0	176.0	-32.0	128.0	192.0	-48.0	[illegible]	-12
51	37.8	194.4	-43.2	14.4	180.0	-28.8	136.8	194.4	-43.2	232.2	-5
52	38.4	192.0	-38.4	12.8	179.2	-25.6	140.8	192.0	-38.4	[illegible]	0
53	37.8	184.8	-33.6	11.2	173.6	-22.4	140.0	184.8	-33.6	[illegible]	4
54	36.0	172.8	-28.8	9.6	163.2	-19.2	134.4	172.8	-28.8	[illegible]	7
55	33.0	156.0	-24.0	8.0	148.0	-16.0	124.0	156.0	-24.0	[illegible]	9
56	28.8	134.4	-19.2	6.4	128.0	-12.8	108.8	134.4	-19.2	[illegible]	9
57	23.4	108.0	-14.4	4.8	103.2	-9.6	88.8	108.0	-14.4	[illegible]	9
58	16.8	76.8	-9.6	3.2	73.6	-6.4	64.0	76.8	-9.6	93.6	7
59	9.0	40.8	-4.8	1.6	39.2	-3.2	34.4	40.8	-4.8	49.8	4

A spreadsheet program listing is included in your class notes that gives the moment values along the span lengths and is used to graph the moment envelope curves.

In the spreadsheet:

Live Load E-Mom (+)
= max (LC2 through LC7)
Live Load E-Mom (-)
= min (LC2 through LC7)

Total Load E-Mom (+) = LC1
+ Live Load E-Mom (+)
Total Load E-Mom (-) = LC1
+ Live Load E-Mom (-)

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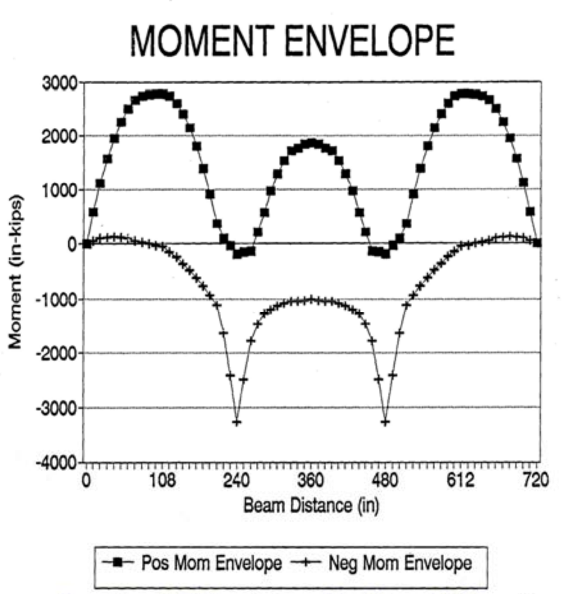
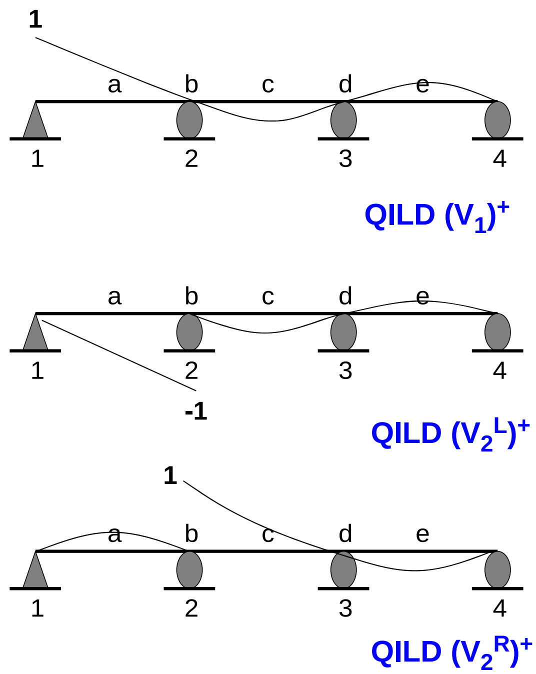


Total Load E-Mom (+) **Total Load E-Mom (-)**

55

Construction of the shear envelope curve follows the same procedure. However, just as is the case with a bending moment envelope, a complete analysis should also load increasing/ decreasing fractions of the span where shear is being considered.

56



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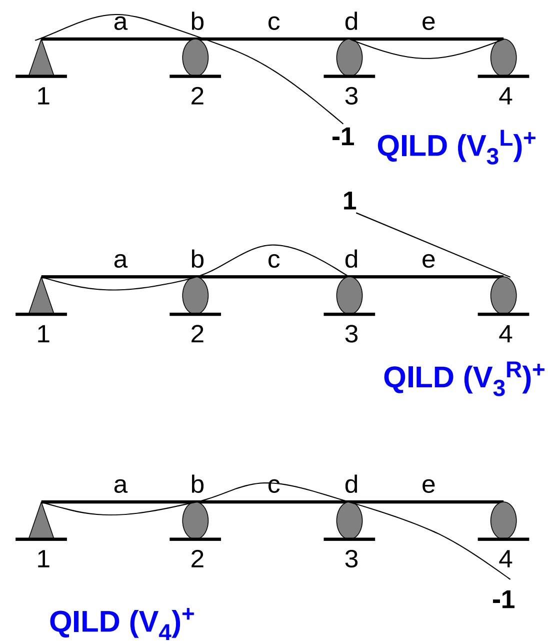

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Shear ILD Notation:

Superscript L = just to the left of the subscript point

Superscript R = just to the right of the subscript point

To obtain the negative shear qualitative influence line diagrams simply flip the drawn positive qualitative influence line diagrams.

59

In practice, the construction of the exact shear envelope is usually unnecessary since an approximate envelope obtained by connecting the maximum possible shear at the reactions with the maximum possible value at the center of the spans is sufficiently accurate. Of course, the dead load shear must be added to the live load shear envelope.

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