CHAPTER THREE

INTRODUCTION TO TWO-HINGED ARCHES

3.0. TWO-HINGED ARCHES:-

The following issues should be settled first.

- Definition.
- Types.
- Basic Principle and B.M.
- Linear Arch.
- Mathematical Generalized Expressions.
- Segmental Arches.

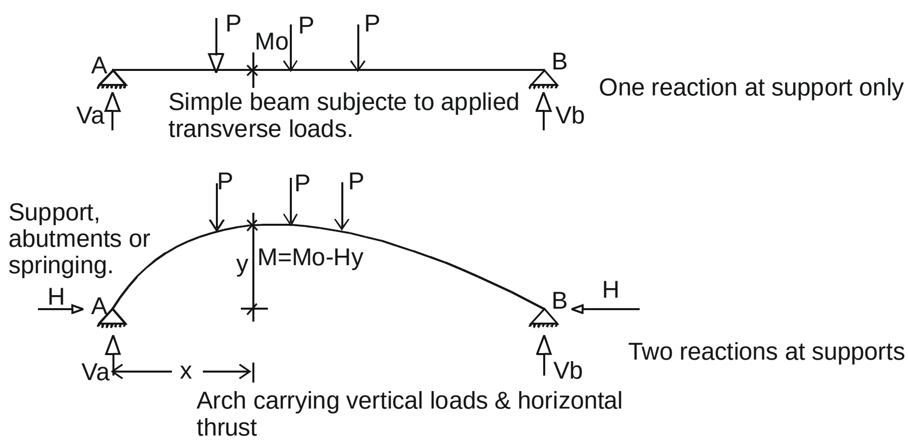
Some information is contained elsewhere where determinate arches have been dealt.

3.1. DEFINITION OF AN ARCH.

"An arch can be defined as a humped or curved beam subjected to transverse and other loads as well as the horizontal thrust at the supports." An efficient use of an arch can be made only if full horizontal restraint is developed at the supports. If either of the support allows some movement in the horizontal direction, it will tend to increase the B.M. to which an arch is subjected and arch would become simply a curved beam.

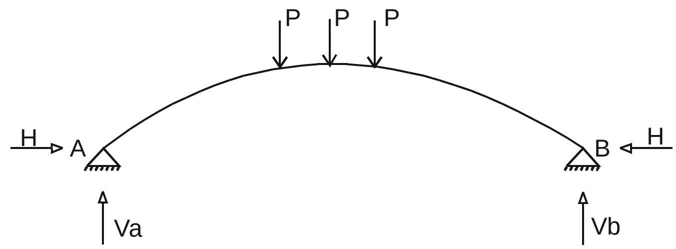
The B.M., in arches due to the applied loads is reduced due to the inward thrust. Analysis is carried out to find the horizontal thrust and also to find the B.M., to which an arch is subjected.

Beam action Vs arch action :



The above beam and arch carry similar loadings.
If Mo = B.M. due to applied loads at a distance X on the simple span of a simple beam where rise is y.
then bending moment in the arch is, $M_X = Mo \pm Hy$

where M_X is the B.M., in the arch at a distance x . H is the horizontal thrust at the springings & y is the rize of the arch at a distance. 'x' as shown in the diagram. The (±) sign is to be used with care and a **(–) sign will be used if the horizontal thrust is inwards or vice versa**. In later case it will behave as a beam.



Under transverse loads, the horizontal thrust at either of the springings abutments is equal. In the arch shown above, the degree of indeterminacy is one and let us consider the horizontal thrust at support B as the redundant. The above loaded arch can be considered equal to the following two diagrams wherein a BDS arch is under the action of loads plus the same BDS arch under the action of inward unit horizontal load at the springings.

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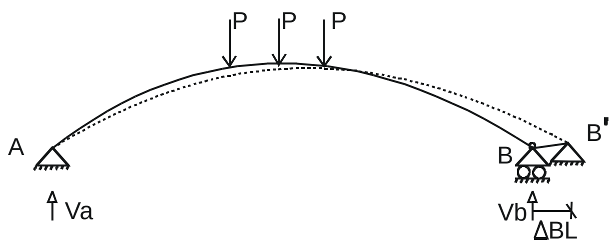

B.D.S. under applied loads (loads try to flatten the arch)

ΔBL stands for displacement of point B due to applied loads in a BDS arch..

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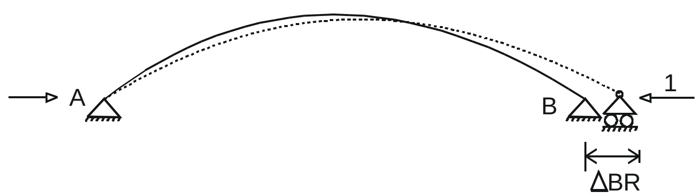


(Flattened arch recovers some of horizontal displacement at B due to unit horizontal loads and will recover fully if full horizontal thirst is applied at B.)
(Arch flattens out under the action of applied loads because freedom in the horizontal direction has been provided at point B.) and all due to full redundant value. This forces the basis of compatibility.

ΔBR stands for displacement of point. B (in the direction of force) due to unit horizontal redundant force at B. Remember that a horizontal reactive component cannot be realized at the roller support. However, we can always apply a horizontal force at the roller.

3.2. Compatibility equation.

$\Delta BL - (\Delta BR) H = 0$ (If unit load is applied in opposite sense so that it also produces flettening, +ve sign may be used in the equation and the final sign with H will be self adjusting.)

$$\text{or } H = \frac{\Delta BL}{\Delta BR} = \frac{\text{displacement at B due to loads}}{\text{displacement at B due to unit horizontal redundant}}$$

We will be considering strain energy stored in bending only.The modified expression for that for curved structural members is as follows.

$$\boxed{U = \int \frac{M^2 ds}{2EI}}$$

Where ds is the elemental length along the centre line of the arch and U is the strain energy stored in bending along centre-line of arch. The bending moment at a distance x from support is

$$M_X = Mo - Hy \text{ (Horizontal thrust is inwards).} \qquad (1)$$

Where Mo = Simple span bending moment (S.S.B.M.) in a similar loaded simple beam.

$$U = \int \frac{M^2\, ds}{2EI}$$

If H is chosen as redundant, then differentiating U w.r.t. H , we have

$$\frac{\partial U}{\partial H} = \Delta BH = 0 = \int \frac{1}{EI} \cdot M. \left(\frac{\partial M}{\partial H}\right) ds \qquad \text{Put M= Mo – Hy and then differentiate.}$$

$$\frac{\partial U}{\partial H} = \Delta BH = 0 = \int \frac{1}{EI} \cdot (Mo - Hy)(-y)\, ds \qquad \text{by putting M from (1)}$$

$$0 = \int \frac{(Hy^2 - Mo\ y)\, ds}{EI} \qquad \text{Simlifying}$$

$$\int \frac{H\, y^2\, ds}{EI} - \int \frac{Mo\ y\, ds}{EI} = 0$$

$$\int \frac{H\, y^2\, ds}{EI} = \int \frac{Mo\ y\, ds}{EI}$$

or

$$\boxed{H = \frac{\int \frac{Moy.ds}{EI}}{\int \frac{y^2 ds}{EI}}}$$

Applying Castigliano's 2nd theorem, ΔBL becomes = $\int \frac{Mo\ y\, ds}{EI}$

$$\text{and } \Delta BR = \int \frac{y^2\, ds}{EI}$$

The algebraic integration of the above integrals can also be performed in limited number of cases when EI is a suitable function of S (total curved arch length), otherwise, go for numerical integration.

For prismatic (same cross section) members which normally have EI constant, the above expression can be written as follows:

$$H = \frac{\int Mo\ y\ ds}{\int y^2 ds}$$

3.3. TYPES OF ARCHES :–

The arches can be classified into a variety of ways depending mainly upon the material of construction and the end conditions.

(1) **Classification Of Arches Based On Material of Construction** :–

The following arches fall in this particular category:

a) Brick masonary arches.
b) Reinforced concrete arches.
c) Steel arches.

The span of the arches which can be permitted increases as we approach steel arches from the brick masonary arches.

(2) **Classification Of Arches Based On End Conditions :–**

The following arches fall in this particular category:

a) Three hinged arches.
b) Two hinged arches.
c) Fixed arches.

In the ancient times, three hinged arches have been used to support wide spans roofs. However, their use is very rare in bridge construction since the discontinuity at the crown hinge is communicated to the main deck of the bridge. In three hinged arches, all reactive components are found by statical considerations without considering the deformations of the arch rib. Therefore, they are insensitive to foundation movements and temperature changes etc., and are statically determinate. These are covered as a separate chapter in this book.

The Romans exploited the potential of arches to a great extent. However, their emperical analysis approach became available in the early 18th century.

3.4. LINEAR ARCH :–

This is just a theoretical arch at every X–section of which the B.M. is zero.

$$M = M_o - Hy = 0$$

or $M_o = Hy$ (The B.M. due to applied loads is balanced by Hy).

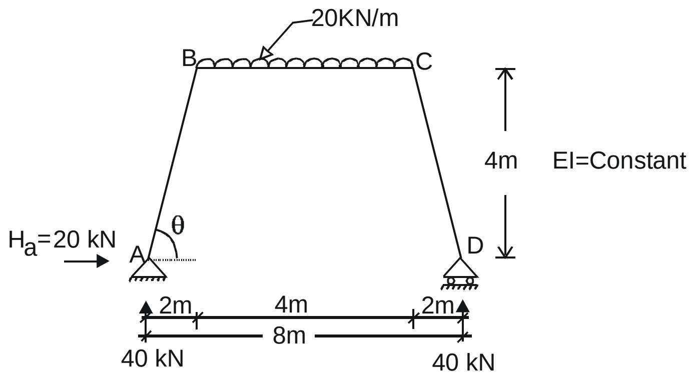
therefore, $y = \frac{M_o}{H}$

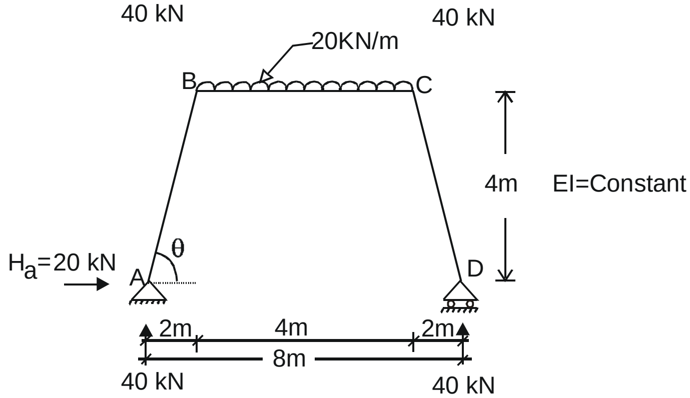
This is the equation for the centre line of a linear arch. With the change in position and the number of loads on the arch, the corresponding linear arch would also change as M_o keeps on changing. Therefore, there are infinite number of such arches for every load pattern and position on the actual arch.

EXAMPLE NO. 1:

3.5. ANALYSIS OF TWO – HINGED SEGMENTAL ARCHES

We develop the method for indeterminate arches starting with the simplest cases of segmental arches. Solve the following segmental arch by using the basic principles of consistent deformation method and by treating horizontal thrust at support D as the redundant. The segmental arches could be used in tunnels and in water ways.





(Ha will occur only point D is a hinge support)

M – Diagram. Due to applied loads. Similarly reactions due to supermetrical loading.

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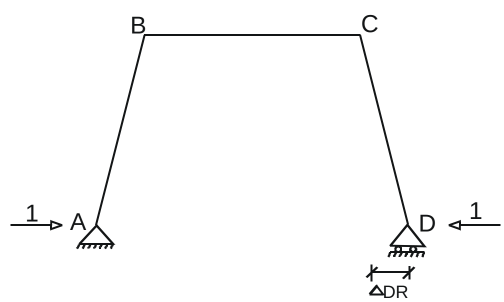


m – Diagram. Due to unit redundant at D.

(X is varied along length of members). Find Cosθ and Sinθ.

cos θ = 0.4472 , sin θ = 0.8944.

Sab sin θ + 40 =0 so Sab= $\frac{-40}{0.8944}$ = – 44.722. Consider equilibrium of joint A and project forces in y-direction. (M-diagram)

Consider same diagram with roller at D. Now consider joint A and Project forces in X direction to evaluate Ha. Sab Cosθ + Ha = 0 or –44.722 x 0.4472 + Ha = 0

or Ha= 20KN

Compatibility equation

$$\Delta DL - \Delta DR.\ H = 0$$

$$\text{Or } H = \frac{\Delta DL}{\Delta DR} = \frac{\text{Horizontal displacement of D due to loads}}{\text{Horizontal displacement of D due to redundants}}$$

$$\Delta DL = \int \frac{Mmdx}{EI}$$

Applying Unit load method concepts,

$$\Delta DR = \int \frac{m^2\,dx}{EI}$$

Now we attempt the evaluation of these integrals in a tabular form. X is measured along member axis.

Member	Origin.	Limits.	M	m
AB	A	0 – 4.472	40 X Cosθ =40X0.477= 17.88X	+1.XSinθ=+0.894X
BC	B	0 – 4	$40(2+X)-10X^2=$ $80 + 40X - 10X^2$	+ 4
CD	D	0 – 4.472	17.88 X	+ 0.894 X

$$\Delta DL = \int \frac{MmdX}{EI} = \frac{1}{EI}\int_o^{4.472} (17.88X)(+0.894X)dX + \frac{1}{EI}\int_o^{4} 80+40X - 10X)(+4)\,dX$$

$$+ \frac{1}{EI}\int_o^{4.472} (17.88\,X)(+0.894\,X)\,dX$$

$$= \frac{2}{EI}\int_o^{4.472} (+15.985\,X^2)dX + \frac{1}{EI}\int_o^{4}(+320+160X - 40X^2)\,dX \qquad \text{Integrate and put limits}$$

$$= \frac{+31.969}{EI}\left|\frac{X^3}{3}\right|_o^{4.472} + \frac{1}{EI}\left|+320X + \frac{160X^2}{2} - \frac{40X^2}{3}\right|_o^{4}$$

$$= \frac{+10.656}{EI}(\,4.472^3 - 0) + \frac{1}{EI}\left(+320\times 4 + 80\times 16 - \frac{40}{3}\times 16\right)$$

$$\Delta DL = \frac{+2659.72}{EI}$$

$$\Delta DR = \int \frac{m^2 dX}{EI} = \frac{1}{EI}\int_o^{4.472} (+0.894X)^2\, dX + \frac{1}{EI}\int_o^{4} 16\, dX + \frac{1}{EI}\int_o^{4.472} (+0.894X)^2\, dX$$

$$= \frac{2}{EI}\int_o^{4.472} 0.799\, X^2\, dX + \frac{16}{EI}\int_o^{4} dX$$

$$= \frac{1.598}{EI}\left|\frac{X^3}{3}\right|_o^{4.472} + \frac{16}{EI}\left| X \right|_o^{4}$$

$$= \frac{0.533}{EI}[(4.472)^3 - 0] + \frac{16}{EI}(4 - 0)$$

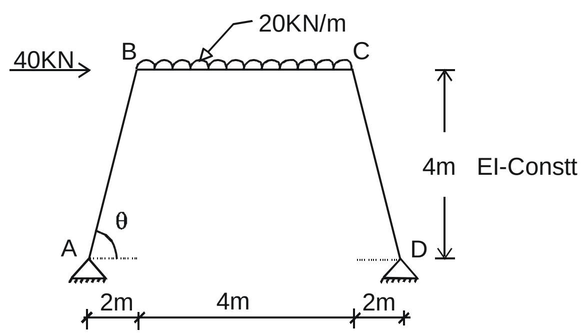
$$\Delta DR = \frac{111.653}{EI}$$

$$H = \frac{\Delta DL}{\Delta DR}$$

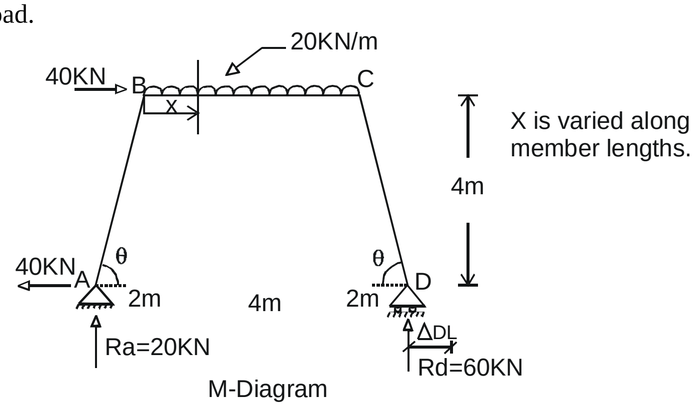
$$= \frac{2659.72/EI}{111.653/EI}$$

$$\boxed{H = 23.82\ KN}$$

EXAMPLE NO. 2:– Solve the following arch by using consistent deformation method.



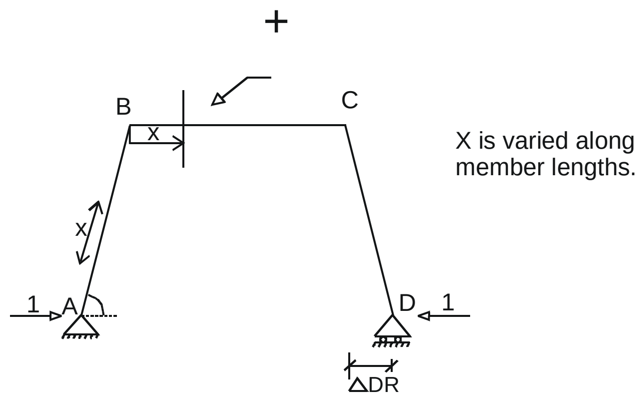
The above redundant / segmental arch can be replaced by the following similar arches carrying loads and redundant unit load.



BDS UNDER LOADS

$\Sigma Ma = 0$; $Rd \times 8 = 20 \times 4 \times 4 + 40 \times 4$

$\therefore$ $Rd = 60$ KN so $Ra = 20$ KN



m – Diagram

BDS UNDER UNIT REDUNDANT AT D

Compatibility equation is

$\Delta DL - \Delta DR.H = 0$

Where ΔDL = Horizontal deflection of D in BDS due to applied loads.

ΔDR = Horizontal deflection at D due to Unit redundant.

H = Total Horizontal redundant.

Or $H = \dfrac{\Delta DL}{\Delta DR}$

and $\Delta DL = \int \dfrac{MmdX}{EI}$

$\Delta DR = \int \dfrac{m^2\, dX}{EI}$

Member	Origin	Limits	M	m	EI
AB	A	0–4.472	$20X \cos\theta + 40X \sin\theta$ $20X \times 0.447 + 40X \times 0.894$ $= 44.72X$	$X\sin\theta = 0.894X$	Constt.
BC	B	0 – 4	$20(2+X) + 40 \times 4 - 10X^2$ $40 + 20X + 160 - 10X^2 =$ $-10X^2 + 20X + 200$	+ 4	Constt.
CD	D	0–4.472	$60X \cos\theta = 60X \times 0.447$ $= 26.82\ X$	0.894X	Constt.

$$\Delta DL = \int \frac{MmdX}{EI} = \frac{1}{EI}\int_{o}^{4.472} (+44.72X)(0.894X)\, dX + \frac{1}{EI}\int_{o}^{4} (-10X^2 + 20X - 200)\ 4\, dX$$

$$+ \frac{1}{EI}\int_o^{4.472} (26.82X)(0.894X)dX$$

$$1.33X = \frac{2 \times 23.977}{EI}\left|\frac{X^3}{3}\right|_o^{4.472} + \frac{4}{EI}\left|\frac{-10X^3}{3} + \frac{20X^2}{2} + 200X\right|_o^4$$

$$\Delta DL = \frac{63.97}{EI}\left[\frac{4.472^3}{3}\right] + \frac{4}{EI}\left[\frac{-10}{3} \times 4^3 + 10 \times 4^2 + 200 \times 4\right] = \frac{+4893.8}{EI}$$

$$\Delta DR = \int \frac{m^2\,dX}{EI} = \frac{1}{EI}\int_o^{4.472} (0.894X)^2 + \frac{1}{EI}\int_o^4 16dX + \frac{1}{EI}\int_o^{4.472} (0.894X)^2$$

$$= \frac{2}{EI}\int_o^{4.472} 0.799X^2\,dX + \frac{16}{EI}\int_o^4 dX$$

$$= \frac{1.598}{EI}\left|\frac{X^3}{3}\right|_o^{4.472} + \frac{16}{EI}\left|X\right|_o^4$$

$$= \frac{0.533}{EI}\left[(4.472)^3 - 0\right] + \frac{16}{EI}(4 - 0)$$

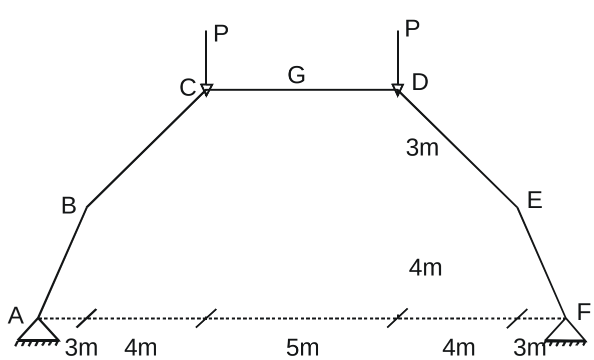
$$\Delta DR = \frac{111.653}{EI}$$

$$\therefore \quad H = \frac{\Delta DL}{\Delta DR}$$

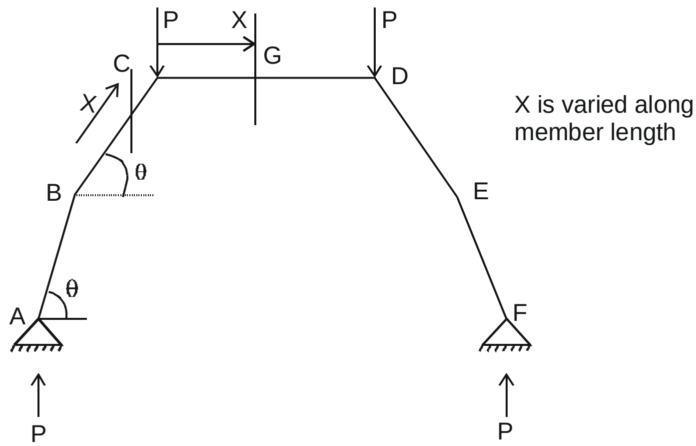
$$= \frac{+4893.8/EI}{111.653/EI}$$

So $\boxed{H = +43.83 \text{ KN}}$

EXAMPLE NO. 3:- Determine the horizontal thrust for the for following loaded segmental arch. Take EI equal to constant.

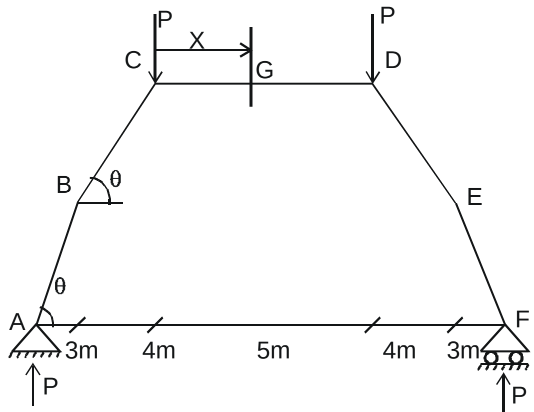


SOLUTION :–

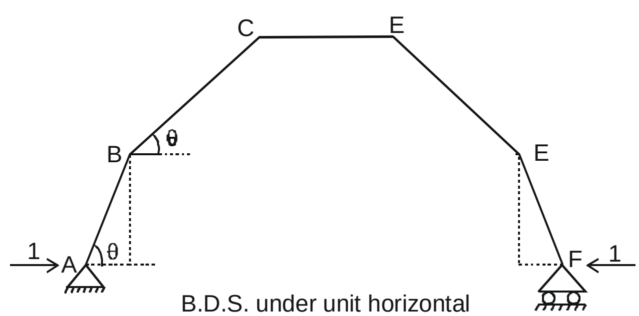


Now consider a BDS under Loads and redundant separately for the same arch and evaluate integrals.

An inspection of the arch indicates that it is symmetrical about point G and is indeterminate to the first degree choosing horizontal reaction at F as the redundant, we draw two basic determinate structures under the action of applied loads and the redundant horizontal thirst at support F.



M-Diagram (BDS under loads)



B.D.S. under unit horizontal redundant load at F.
m-Diagram

Because of symmetry, Moments and hence strain energy is computed for half frame.

Portion	Origin	Limits	M	m
AB	A	0 – 5	PX Cosθ = X0.6 PX	0.8 X
BC	B	0 – 5	P (3+0.8X)	4 + 0.6X
CG	C	0 – 2.5	P (7+X) – PX = 7 P	7

$$\Delta FL = 2\int_o^5 \frac{(0.6\ PX)(0.8X)}{EI}dX + 2\int_o^5 \frac{P(3+0.8X)(4+0.6X)}{EI}dX + 2\int_o^{2.5}\frac{49\ P}{EI}dX$$

$$= \frac{2\ P}{EI}\left[\int_o^5 0.48\ X^2\ dX + \int_o^5 (0.48\ X^2+5X+12)dX + \int_o^{2.5} 49\ dX\right]$$

$$= \frac{2\ P}{EI}\left[\left|\frac{0.48X^3}{3}\right|_o^5 + \left|\frac{0.48\ X^3}{3} + \frac{5\ X^2}{2} + 12X\right|_o^5 + \left|49X\right|_o^{2.5}\right]$$

$$= \frac{2\ P}{EI}\left[\frac{0.48}{3}\times 5^3 + \frac{0.48\times 5^3}{3} + \frac{5\times 5^2}{2} + 12\times 5 + 49\times 2.5\right]$$

$$\Delta FL = \frac{570\ P}{EI}$$ (deflection of point F due to loads)

$$\Delta FR = \frac{2}{EI}\int_o^5 (0.8X)^2 dX + \frac{2}{EI}\int_o^5 (16 + 0.36X^2 + 4.8X)\ dX + \frac{2}{EI}\int_o^{2.5} 49dX$$

$$= \frac{2}{EI}\left[\left|\frac{0.64X^3}{3}\right|_o^5 + \left|16X + \frac{0.36X^3}{3} + \frac{4.8X^2}{2}\right|_o^5 + \left|49X\right|_o^{2.5}\right]$$

$$= \frac{2}{EI}\left[\frac{0.64\times 5^3}{3} + 16\times 5 + \frac{0.36}{3}\times 5^3 + \frac{4.8\times 5^2}{2} + 49\times 2.5\right]$$

$$\Delta FR = \frac{608.33}{EI}, \quad H = \frac{\Delta FL}{\Delta FR}$$

$$H = \frac{570\ P}{608.32}$$

So $\boxed{H = 0.937\ P}$

NOTE :– Compatibility equation is

$$\Delta FL - \Delta FR \times H = 0$$

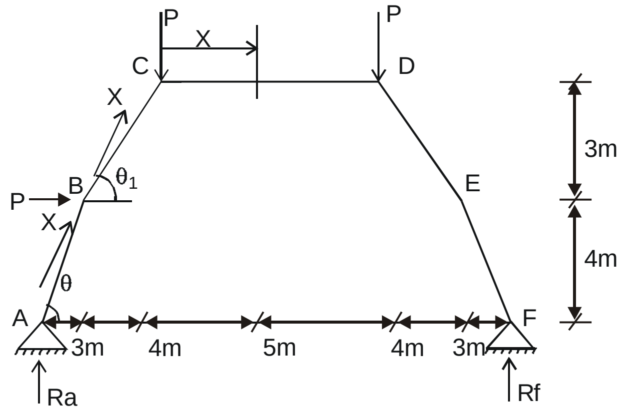
$$\Delta FL = \Delta FR \times H$$

$$H = \frac{\Delta FL}{\Delta FR}$$

We take compression on outer side & tension on inner side +ve in case of M and m-diagram.

EXAMPLE NO. 4 :– Determine the horizontal thrust provided that EI = Constt for the following loaded segmental arch.:

SOLUTION:



Taking horizontal reaction at F as redundant. ΣMa=0

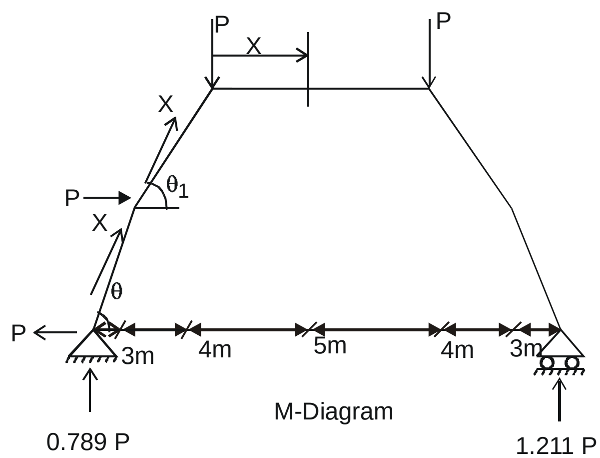
Rf. 19 = P . 12 + P .7 + 4. P , So

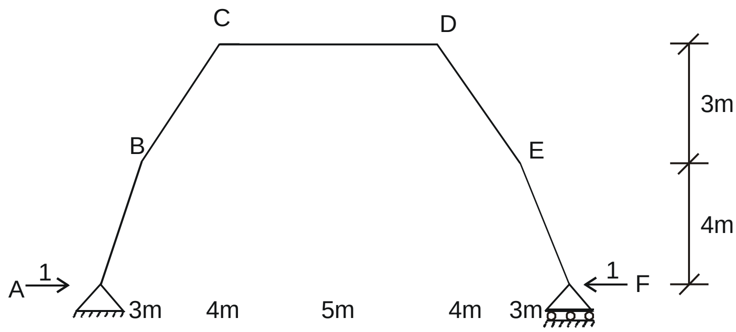
$$\boxed{Rf = 1.211\ P}$$

and therefore Ra is,

$$Ra = 2P - 1.211\ P$$

$$\boxed{Ra = +0.789\ P}$$





m-diagram (Unit redundant at F)

Portion	Origin	Limits	M	m
AB	A	0 – 5	0.789 PX Cosθ+PX Sinθ = 0.4734 PX + 0.8 PX = 1.2734 PX	1 × X Cos θ = 0.8X
BC	B	0 – 5	0.789 P(3 + XCosθ_1) +P(4 + XSinθ_1) – PX Sinθ_1 = 0.6312 PX+6.367 P	1(4 + X Sinθ_1) = 4 + 0.6X
CD	C	0 – 5	0.789P (7+X)+P×7–P×3–PX = – 0.211 PX + 9.523 P	+ 7
DE	E	0 – 5	1.211 P(3+X Cosθ_1) = 3.633 P + 0.9688 PX	1(4 + X Sinθ_1) = 4 + 0.6X
EF	F	0 – 5	1.211 PX Cos θ= 0.7266 PX	X Sin θ = 0.8X

Determine Sines and Cosines of θ and θ_1.

$$\Delta FL = \frac{1}{EI}\Big[\int_o^5 (1.2734\ PX)(0.8\ X)dX + \int_o^5 (0.6312\ PX + 6.367\ P)$$

$$(4 + 0.6X)\ dX + \int_o^5 (-\ 0.211\ PX + 9.523\ P)(7)dX$$

$$+ \int_o^5 (3.633\ P + 0.9688\ PX)(4 + 0.6X) + \int_o^5 (0.7266PX\ (0.8X)\ dX\Big]$$

$$= \frac{P}{EI}\Big[\int_o^5 1.01872X^2\ dX + \int_o^5 (2.5248X + 0.37872X^2 + 25.468 + 3.8202\ X)\ dX$$

$$+ \int_o^5 (-\ 1.477X + 66.661)\ dX + \int_o^5 (14.532 + 2.1798X$$

$$+ 3.8752X + 0.58128X^2\ )\ dX + \int_o^5 0.58128X\ dX \quad . \qquad \text{Simplifying we get.}$$

$$= \frac{P}{EI}\int_o^5 (1.97872X^2 + 11.50428X + 106.661)\ dX$$

$$\Delta FL = \frac{P}{EI}\left| 1.97972\frac{X^3}{3} + 11.50428\frac{X^2}{2} + 106.661X \right|_o^5$$

$$= \frac{P}{EI}\left[1.97872 \times \frac{5^3}{3} + 11.50428 \times \frac{5^2}{2} + 106.661 \times 5\right]$$

$$\Delta FL = \frac{759.56\ P}{EI}$$

$$\Delta FR = \frac{1}{EI}\left[\int_o^5 (0.8X)^2 dX + \int_o^5 (16+0.36X^2+4.8X)\,dX\right.$$

$$\left. + \int_o^5 49\,dX + \int_o^5 (16+0.36X^2+4.8X)\,dX + \int_o^5 0.64\,X^2\,dX\right]$$

$$= \frac{1}{EI}\left|\frac{0.64X^3}{3} + 16X + 0.36\frac{X^3}{3} + 4.8\frac{X^2}{2} + 49X + 16X + 0.36\frac{X^3}{3} + \frac{4.8X^2}{2} + \frac{0.64X^3}{3}\right|_o^5$$

$$= \frac{1}{EI}\left[\frac{0.64}{3} \times 5^3 + 16 \times 5 + \frac{0.36 \times 5^3}{3} + \frac{4.8 \times 5^2}{2} + 49 \times 5\right.$$

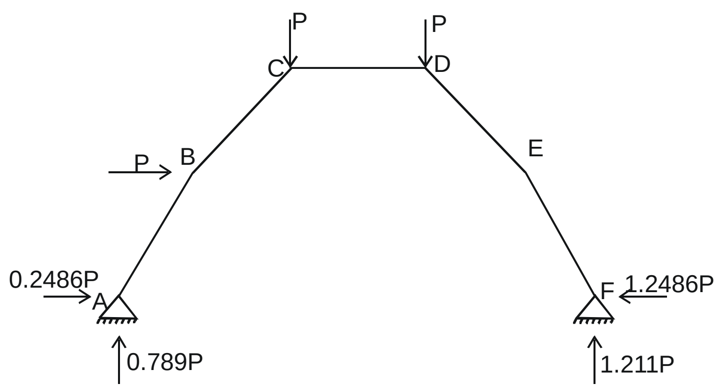
$$\left. + 16 \times 5 + \frac{0.36}{3} \times 5^3 + \frac{4.8}{2} \times 5^2 + \frac{0.64}{3} \times 5^3\right] \quad . \qquad \text{Simplifying}$$

$\Delta FR = \frac{608.33}{EI}$. Compatibility equation remains the same. Putting values of integrals, we have

$$H = \frac{\Delta FL}{\Delta FR}$$

$$= \frac{759.56\ P}{EI} \bigg/ \frac{608.33}{EI}$$

$\boxed{H = 1.2486\ P}$ Now all reactions are shown.



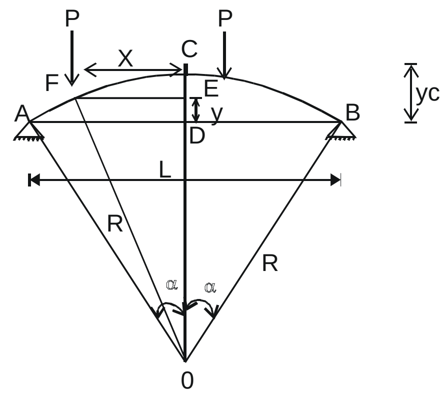
ANALYZED SEGMENTAL ARCH

Check : $\sum Mc = 0$

$0.789P \times 7 - 0.2486\ P \times 7 - P \times 3 + P \times 5 + 1.2486\ P \times 7 - 1.211P \times 12 = 0$

$0 = 0$ O.K.

3.6. ANALYSIS OF TWO HINGED CIRCULAR ARCHES :–



The circular arches are infact a portion of the circle and are commonly used in bridge construction. From the knowledge of determinate circular arches, it is known that the maximum thrust and the vertical reactions occur at the springings. Therefore, logically there should be a greater moment of inertia near the springings rather than that near the mid–span of the arch. The approach is called the secant variation of inertia and is most economical. However, to establish the basic principles, we will first of all consider arches with constant EI. The following points are normally required to be calculated in the analysis.

(1) Horizontal thrust at the springings.

(2) B.M. & the normal S.F. at any section of the arch.

Usually, the span and the central rise is given and we have to determine;

(i) the radius of the arch;

(ii) the equation of centre line of the circular arch.

Two possible analysis are performed.

(1) Algebraic integration.

(2) Numerical integration.

After solving some problems, it will be amply demonstrated that algebraic integration is very laborious and time consuming for most of the cases. Therefore, more emphasis will be placed on numerical integration which is not as exact but gives sufficiently reliable results. Some researches have shown that if arch is divided in sixteen portions, the results obtained are sufficiently accurate. In general, the accuracy increases with the increase or more in number of sub–divisions of the arch.

We will be considering two triangles.
1 – Δ ADO
2 – ΔEFO

By considering Δ ADO

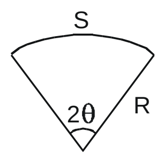
$OB^2 = OD^2 + BD^2$

$R^2 = (R-yc)^2 + (L/2)^2$
$R^2 = R^2 - 2Ryc + yc^2 + L^2/4$
$0 = yc(yc - 2R) + L^2/4$
or $yc(yc - 2R) = -L^2/4$
$-yc(yc - 2R) = L^2/4$

$$\boxed{yc(2R - yc) = \frac{L^2}{4}} \qquad (1)$$

By considering Δ EFO
$OF^2 = OE^2 + EF^2$
$R^2 = (R - yc + y)^2 + X^2$
$R^2 - X^2 = (R - yc + y)^2$

$$R - yc + y = \sqrt{R^2 - x^2}$$



$$\boxed{y = \sqrt{R^2 - X^2} - (R - yc)} \qquad (2)$$

The detailed derivation of this equation can be found in some other Chapter of this book.

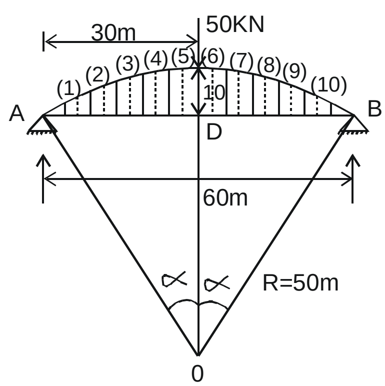
In this case, S = R (2 θ) where θ is in radiains. S is the total length along centre line of the arch.

$H = \frac{\int Myds}{\int y^2 ds}$ as before obtained By eliminating EI as we are considering EI = Constt

EXAMPLE NO. 5:–

A two– hinged circular arch carries a concentrated force of 50 KN at the centre. The span & the rise of the arch are 60m & 10m respectively. Find the horizontal thrust at the abutments.

SOLUTION :– The arch span is divided in ten equal segments and ordinates are considered at the centre of each segment.



$R = \frac{L^2}{8yc} + \frac{yc}{2}$, where R = Redious, yc = Central rise and L = Span of arch.

$= \frac{(60)^2}{8 \times 10} + \frac{10}{2}$

$R = 50$ m

$\text{Sin}\alpha = \frac{30}{50} = 0.6$. Now compute angle α is radians.

$\alpha = 36.87^{\circ}$, we know π rad = 180°

$180° = \pi$ rad

$1° = \frac{\pi}{180}$ rad

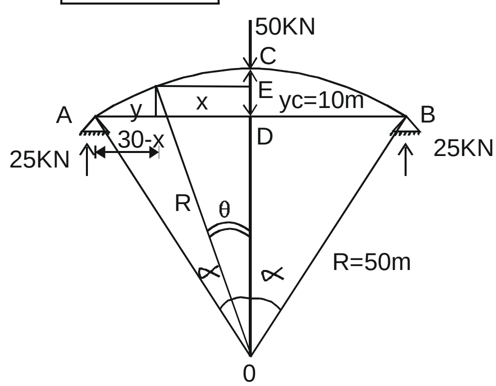
So $36.87° = \frac{\pi}{180} \times 36.87$ radians

$36.87° = 0.6435 \text{ rad} = \alpha$

$\alpha = 0.6435$ rad

$S = R(2\alpha) = 50(2 \times 0.6435)$, Where S is length of arch along its centre-line
For circular arches. X is varied from centre to abutments.

S = 64.35 m



$H = \frac{\int M y ds}{\int y^2 ds}$

where M = Simple span (S.S) B.M. in the arch due to applied loads only.

$M_{bc} = M_{ac} = 25(30 - X)$ in two portions at a distance X from mid span.

$OE = R \cos\theta$

$OD = R - yc = 50 - 10 = 40$ m

$y = OE - OD$ [Since OC = OD + CD = 50 and CD = 10 = Yc]

$y = R\cos\theta - 40$

and $ds = Rd\theta$

$X = R\sin\theta$

Evaluation of Numerator :–

$Mx = 25(30 - X)$, $ds = Rd\theta$, $y = R\cos\theta - 40$

$\int Myds = 2\int_o^{\alpha}[25(30 - R\sin\theta)][R\cos\theta - 40][Rd\theta]$, By putting X, y and ds from above. Also put value of α which is in radians.

$$= 50R\int_o^{0.6435}(30 - R\sin\theta)(R\cos\theta - 40)\,d\theta, \qquad \text{we know, } 2\sin\theta\cos\theta = \sin 2\theta.$$

$$= 50R\int_o^{0.6435}(30R\cos\theta\cos\theta - R^2\sin\theta\cos\theta - 1200 + 40R\sin\theta)\,d\theta$$

$$= 50R\left|30R\sin\theta + \frac{R^2}{2}\cdot\frac{\cos 2\theta}{2} - 1200\,\theta - 40R\cos\theta\right|_o^{0.6435} \quad \text{Put limits now}$$

$$= 50\times 50\left[30\times 50\times 0.6 + \frac{2500}{4}\times 0.28 - 1200\times 0.6435 - 40\times 50\times 0.8 - \frac{50^2}{4}\times 1 + 40\times 50\times 1\right]$$

$= +194500$

$\int Myds = 194.5\times 10^3$

Evaluation of Denominator :–

We know $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

and $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$$\int y^2 ds = 2\int_o^{0.6435}(R\cos\theta - 40)^2(Rd\theta)$$

$$= 2R\int_o^{0.6435}(R^2\cos^2\theta - 80R\cos\theta + 1600)\,d\theta$$

$$= 2R\int_o^{0.6435}\left[\frac{R^2}{2}(1 + \cos 2\theta) - 80R\cos\theta + 1600\right]d\theta \quad \text{Integrate}$$

$$= 2R\left|\frac{R^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) - 80R\sin\theta + 1600\,\theta\right|_o^{0.6435} \quad \text{Put limits now}$$

$$= 2 \times 50\left[\frac{50^2}{2}\left(0.6435 + \frac{0.96}{2}\right) - 80 \times 50 \times 0.6 + 1600 \times 0.6435\right]$$

$$= 3397.5$$

$$\int y^2 ds = +3.3975 \times 10^3$$

$$H = \frac{194.5 \times 10^3}{3.3975 \times 10^{-3}}$$

$$\boxed{H = 57.2 \text{ KN}}$$

EXAMPLE NO. 5: BY NUMERICAL INTEGRATION :–

The values of X, y and M are determined at the mid ordinates of the segments. The basic philosophy is that if we consider a very small arc length that would be regarded as a straight line and therefore we tend to average out these values.

$$y = \sqrt{R^2 - X^2} - (R - yc)$$

or $$y = \sqrt{50^2 - X^2} - (50 - 10)$$

or $$y = \sqrt{50^2 - X^2} - (40) \qquad (1)$$ See segments of Example 5 about 4 page before.

For section (1)

$X_1 = 27$, from (1), $y_1 = \sqrt{50^2 - 27^2} - (40) = 2.08$ m

For section (2)

$X_2 = 21$ from (1), $y_2 = \sqrt{50^2 - 21^2} - (40) = 5.738$ m and so on.

M = 25 (30 – X) = (750 – 25X) 0 < X < 30 as before

Now do numerical integration in a tabular form as under.

Section.	X	y.	M	My	y^2
1	27	2.08	75	156.00	4.33
2	21	5.380	225	1210.50	28.94
3	15	7.69	375	3883.75	59.14
4	9	9.18	525	4819.50	84.27
5	3	9.91	675	6689.25	98.21
6	3	9.91	675	6689.25	98.21
7	9	9.18	525	4819.50	84.27
8	15	7.69	375	2883.75	59.14
9	21	5.380	225	1210.50	28.94
10	27	2.08	75	156.00	4.33
				∑31518	∑549.78

$S = 64.35$ m

and $ds = \dfrac{64.35}{10}$

$ds = 6.435$ m

$$H = \frac{\int Myds}{\int y^2 ds} = \frac{\sum Myds}{\sum y^2 ds}$$

$$= \frac{31518 \times 6.435}{549.78 \times 6.435} \quad (\textit{\textbf{Note:-}} \text{ ds cancels out })$$

$$\boxed{H = 57.33 \text{ KN}}$$

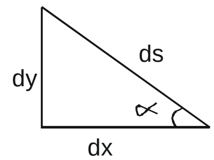
A result similar to that already obtained from algebraic solution

3.7. ARCHES WITH SECANT VARIATION OF INERTIA :–

If Io is the second moment of area of arch rib at the crown: Then secant variation of inertia means.

$I = Io \sec. \alpha$ and

$ds \cos \alpha = dX$



Or $ds = dX \sec \alpha$

$$H = \frac{\int \frac{Myds}{EI}}{\int \frac{y^2 ds}{EI}}$$

If it is built of the same material, then E would cancel out:

$$H = \frac{\int \frac{Myds}{I}}{\int \frac{y^2 ds}{I}} \quad \text{Put } I = I_o \sec \alpha$$

$$H = \frac{\int \frac{My\, dX \sec \alpha}{Io \sec\alpha}}{\int \frac{y2dX \sec \alpha}{Io \sec \alpha}}$$

$$\boxed{H = \frac{\int MydX}{\int y^2\, dX}}$$

If we utilize the above expression for horizontal thrust, it may be kept in mind that integration can now take place in the Cartesian coordinate system instead of the polar coordinate system.

3.8. BY SECANT VARIATION USING ALGEBRAIC INTEGRATION :–

EXAMPLE NO. 6: Analyze the arch in Example No. 5:

We know, $y = \sqrt{R^2 - X^2} - (R - yc)$

$$y = \sqrt{50^2 - X^2} - 40$$

$$Mac = Mbc = 25\,(30 - X) \qquad 0 < X < 30$$

$$\int MydX = 2\int_o^{30} 25\,(30 - X)\left[\sqrt{50^2 - X^2} - 40\right] dX$$

$$= 50\left[30\int_o^{30}\sqrt{50^2 - X^2}\,.\,dX - \int_o^{30} 1200dX - \int_o^{30}\sqrt{50^2 - X^2}\,.\,XdX + 40\int_o^{30} XdX\right]$$

$$= 1500\int_o^{30}\sqrt{50^2 - X^2}\,dX - 1200 \times 50\int_o^{30} dX - 50\int_o^{30}\sqrt{50^2 - X^2}\,XdX + 2000\int_o^{30} XdX$$

Put $X = 50\,Sin\,\theta = R\,sin\theta$

$dX = 50\,Cos\theta\,d\theta$

At $X = 0 \quad \theta = 0$

At $X = 30 \quad \theta = 0.6435$

Now Evaluate integrals

Substitutions

$$Cos^2\theta = 1 + \frac{Cos2\theta}{2}$$

$$\int Cos^2\theta = \frac{\theta}{2} + \frac{Sin^2\theta}{4}$$

$$\int Cos^2\theta\,Sin\theta d\theta = -\frac{Cos^3\theta}{3}$$

by letting $X = Cos\theta$

$dX = -Sin\,\theta d\theta$

$$\int MydX = 1500\int_o^{0.6435}\sqrt{50^2\,(1 - Sin^2\theta)}\,(50\,Cos\theta d\theta) - 60000\,\Big|\,X\,\Big|_o^{30}$$

$$+ 25\left|\frac{(50^2 - X^2)^{3/2}}{3/2}\right|_o^{30} + 2000\left|\frac{X^2}{2}\right|_o^{30}$$

$$= 1500 \times 50^2\int_o^{0.6435}\frac{(1 + Cos2\theta)}{2}\,d\theta - 6 \times 10^4\,(30)$$

$$+\frac{50}{3}\left[(50^2-30^2)^{3/2}-(30^2)+1000\,(30^2)\right]$$

$$= 187.5\times 10^4\left|\theta+\frac{\text{Sin}\,2\theta}{2}\right|_{o}^{0.6435}-180\times 10^4-1016666.666+90\times 10^4$$

$$\int \text{MydX} = 187.5\times 10^4\left[0.6435+\frac{\text{Sin}(2\times 0.6435)}{2}\right]-1916666.666$$

$$= 2106561.918-1916666.666$$

$$\int \text{MydX} = 189895.252$$

$$\int y^2 dX = 2\int_{o}^{30}\left(50^2-X^2+40^2-80\sqrt{50^2-X^2}\right)dX$$

$$= 2\int_{o}^{30}\left(4100-X^2-80\sqrt{50^2-X^2}\right)dX$$

Substitutions:

X = 50 Sin θ

dX = 50 Cosθdθ

$1-\text{Sin}^2\theta = \text{Cos}^2\theta$

$$= 8200\int_{o}^{30} dX-2\int_{o}^{30}X^2\,dX-160\int_{o}^{0.6435}\sqrt{50^2\,\text{Cos}^2\theta}\,d\theta$$

$$= 8200\left|X\right|_{o}^{30}-2\left|\frac{X^3}{3}\right|_{o}^{30}-\frac{160\times 50^2}{2}\int_{o}^{0.6435}(1+\text{Cos}\,2\theta)d\theta$$

$$= 8200\,(30)-\frac{2}{3}(30^3)-\frac{160\times 50^2}{2}\left|\theta+\frac{\text{Sin}\,2\theta}{2}\right|_{o}^{0.6435}$$

$$= 228000-\frac{160\times 50^2}{2}\left[0.6435+\frac{\text{Sin}(2\times 0.6435)}{2}\right]$$

$$= 228000-224699.938$$

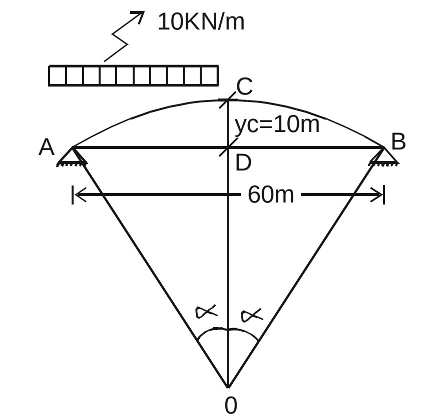
$$\int y^2 dX = 3300.062$$

$$H = \frac{\int \text{MydX}}{\int y^2 dX}$$

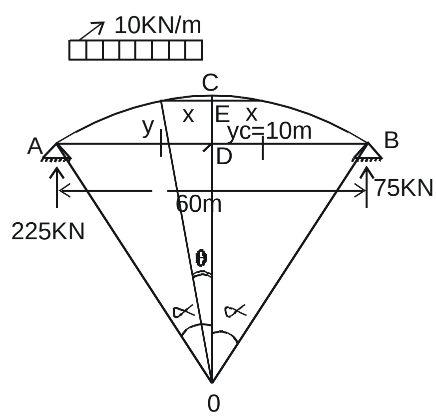
$$= \frac{189895.252}{3300.062}$$

$$\boxed{H = 57.543\text{ KN}}$$

EXAMPLE NO. 7:- A circular arch carries a uniformly distributed load on its left half, calculate the horizontal thrust.



SOLUTION :– Determine Vertical Support reactions as usual and write moment expressions due to applied loads only without considering horizontal thrust.



From diagram, $X = R\,\text{Sin}\theta$

$$\text{Mac} = 225\,(30 - R\,\text{Sin}\theta) - 5\,(30 - R\,\text{Sin}\,\theta)^2$$, in other words. $\text{Mac} = Va\,(30 - X) - w\,X^2/2$

where $X = R\sin\theta$

and $$\text{Mbc} = 75\,(30 - R\,\text{Sin}\theta)$$

$$OD = OC - CD = 50 - 10 = 40\text{ m}$$

$$y = OE - OD = R\,\text{Cos}\theta - 40$$

so $$H = \frac{\int Myds}{\int y^2 ds}$$

Evaluation of Numerator.

$$\int Myds = \int_o^{0.6435} [225\,(30 - R\,Sin\theta) - 5(30 - R\,Sin\theta)^2]\ [R\,Cos\theta - 40]\,(Rd\theta)$$

$$+ \int_o^{0.6435} [75(30 - R\,Sin\theta)]\,[R\,Cos\theta - 40]\,[Rd\theta].$$ This consists of two integrals.

Evaluate First Integral

$$= I_1 = R\int_o^{0.6435} [6750 - 225\,R\,Sin\theta - 4500 - 5\,R^2\,Sin^2\theta + 300\,R\,Sin\theta]\,[R\,Cos\,\theta - 40]$$

$$I_1 = R\int_o^{0.6435} \Big[2250 + 75\,R\,Sin\theta - 5\,R^2\,Sin^2\theta][R\,Cos\,\theta - 40\Big]\,d\theta$$

$$= R\int_o^{0.6435} \Big[2250\,R\,Cos\,\theta + 75\,R^2\,Sin\,\theta\,Cos\theta - 5\,R^3\,Sin^2\theta\,Cos\theta$$

$$- 90000 - 3000\,R\,Sin\,\theta + 200\,R^2\,Sin^2\theta\Big]\,d\theta$$

$$= R\int_o^{0.6435} \Big[2250\,R\,Cos\,\theta + 75\,R^2\,Sin\,\theta\,Cos\,\theta - 5\,R^3 Sin^2\theta\,Cos\,\theta$$

Let X = Sinθ

dX = cosθ dθ

$$\text{So} \int Sin^2\theta\,Cos\theta\,d\theta = \int X^2\,dX = \frac{X^3}{3} = \frac{Sin^3\theta}{3}$$

$$- 90000 - 3000\,R\,Sin\theta + 200\,R^2\left(\frac{1 - Cos2\theta}{2}\right)\Big]\,d\theta$$

$$= R\,\Big|\,2250\,R\,Sin\,\theta - \frac{75}{2}R^2\frac{Cos^2\theta}{2} - 5\frac{R^3\,Sin^3\theta}{3} - 90000\,\theta$$

$$+ 3000\,R\,Cos\theta + \frac{200}{2}R^2\left(\theta - \frac{Sin2\,\theta}{2}\right)\Big|_o^{0.6435}$$

$$= 50\,\Big[\,2250 \times 50 \times 0.6 - \frac{75}{4} \times 2500 \times 0.28 - 5 \times 50^3 \times \frac{0.216}{3}$$

$$- 90000 \times 0.6435 + 3000 \times 50 \times 0.8 + \frac{200}{2} \times 50^2\left(0.6435 - \frac{0.96}{2}\right)$$

$$+ \frac{75}{4} \times 2500 \times 1 - 3000 \times 50 \times 1\Big]$$

$$= 50 \Big[67500 - 13125 - 45000 - 57915 + 120000 + 160875$$
$$- 120000 + 46875 - 150000 \Big]$$
$$= 50\,(9210)$$
$$I_1 = 460.5 \times 10^3$$

Now Evaluate

2nd Integral $= I_2 = R\int_{o}^{0.6435} (2250 - 75\,R\,\text{Sin}\,\theta)(R\,\text{Cos}\,\theta - 40)\,(d\theta)$ multiply two expressions.

$I_2 = R\int_{o}^{0.6435} 2250\,R\,\text{Cos}\,\theta - 75\,R^2\,\text{Sin}\,\theta\,\text{Cos}\,\theta - 90000 + 3000\,R\,\text{Sin}\,\theta)\,d\theta$ Integrate now.

$$= R\,\Big|\,2250R\,\text{Sin}\,\theta + \frac{75}{2}R^2\frac{\text{Cos}\,2\theta}{2} - 90000\,\theta - 3000R\,\text{Cos}\theta\,\Big|_{o}^{0.6435}$$

$$= 50\,(2250 \times 50 \times 0.6 + \frac{75}{4} \times 2500 \times 0.28 - 90000 \times 0.6435$$
$$- 3000 \times 50 \times 0.8 - \frac{75}{4} \times 2500 \times 1 + 3000 \times 50 \times 1)$$

$$I_2 = 291.75 \times 10^3$$

Add these two integrals (I_1 and I_2) of $\int$Myds.

$$\int \text{Myds} = I_1 + I_2$$
$$= 460.5 \times 10^3 + 291.75 \times 10^3$$

or $\boldsymbol{\int \text{Myds} = 752.25 \times 10^3}$

Now Evaluate

$$\int y^2 ds = 2\int_{o}^{0.6435} (R\,\text{Cos}\theta - 40)^2\,(R\,d\theta)$$

$= 2\,R\int_{o}^{0.6435} (R^2\text{Cos}^2\theta + 1600 - 80\,R\,\text{Cos}\theta)d\theta$; We know that $\text{Cos}2\theta = \dfrac{1+\text{Cos}^2\theta}{2}$

$$= 2\,R\int_{o}^{0.6435} \frac{R^2}{2}(1 + \text{Cos}2\theta) + 1600 - 80\,R\,\text{Cos}\theta\,d\theta$$

$$= 2\,R\,\Big|\,\frac{R^2}{2}\left(\theta + \text{Sin}\frac{2\theta}{2}\right) + 1600\,\theta - 80\,R\,\text{Sin}\theta\,\Big|_{o}^{0.6435}$$

$= 2 \times 50\left[\dfrac{50^2}{2}\left(0.6435 + \dfrac{0.96}{2}\right) + 1600 \times 0.6435 - 80 \times 50 \times 0.6\right]$, So $\int y^2 ds = 3.3975 \times 10^3$

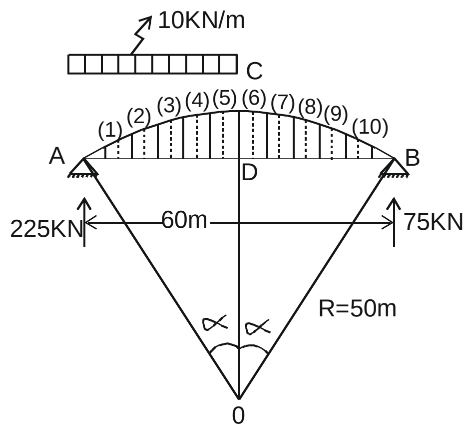
$$H = \frac{\int Myds}{\int y^2 ds}$$

$$H = \frac{752.25 \times 10^3}{3.3975 \times 10^3}$$

$$\boxed{H = 221.42 \text{ KN}}$$

EXAMPLE NO. 8: Analyze the same problem by numerical Integration.
Write moment expression for segments in portions AC and BC due to applied loading only for a simple span.

For segments 1 – 5, Mac = 225 (30 – X) – 5 (30 – X)^2 as before but in Cartesian co-ordinate system.
For segments 6 – 10, Mbc = 75 (30 – X)



Note: X is measured for mid span and y is corresponding rise. Now attempt in a tabular form.

Section	X	y	M	My	y^2
1.	27	2.08	630	1310.4	4.33
2	21	5.38	1620	8715.6	28.94
3	15	7.69	2250	17302.5	59.14
4	9	9.18	2520	23133.6	84.27
5	3	9.91	2430	24081.3	98.21
6	3	9.91	2025	20067.75	98.21
7	9	9.18	1575	14458.5	84.27
8	15	7.69	1125	8651.25	59.14
9	21	5.38	675	3624.75	28.94
10	27	2.08	225	468	4.33
				∑121813.65	∑549.78

S = R (2 α)

= 50 × 2 × 0.6435

S = 64.35 m

so $ds = \frac{64.35}{10} = 6.435$ m , (Because S has been divided in Ten Segments)

$$H = \frac{\int Myds}{\int y^2 ds}$$

$$= \frac{\sum Myds}{\sum y^2 ds}$$

$$= \frac{121813.65 \times 6.435}{549.78 \times 6.435}$$ (***Note:*** ds cancels out)

$$\boxed{H = 221.57 \text{ KN}}$$

Same answer as obtained by algebraic integration.

EXAMPLE NO. 9: Analyze the previous arch for by assuming secant variation of inertia. Integrate along the x – axis by considering arch to be a beam.

$$Mac = 225 (30 - X) - 5 (30 - X)^2 \qquad 0 < X < 30$$

$$Mbc = 75 (30 - X) \qquad 0 < X < 30$$

$$y = \sqrt{50^2 - X^2} - 40$$

$$\int MydX = \int_o^{30} [225 (30 - X) - 5 (30 - X)^2] [\sqrt{50^2 - X^2} - 40] \, dX$$

$$+ \int_o^{30} [75 (30 - X)] [\sqrt{50^2 - X^2} - 40] \, dX$$, By taking y expression common, we have

$$\int MydX = \int_o^{30} [6750 - 225X - 5 (900 - 60X + X^2) + 2250 - 75X] [\sqrt{50^2 - X^2} - 40)] \, dX$$

$$= \int_o^{30} (-5X^2 + 4500)[\sqrt{50^2 - X^2} - 40] \, dX$$ X terms cancel out

Let X = 50 sinθ, then dX = 50 cosθ dθ, So $\sqrt{(50^2 - X^2)} = 50 \text{ Cos}\theta$. Putting these we have.

$$= \int_o^{0.6435} (4500 - 12500 \sin^2\theta) (50 \cos\theta - 40) (50 \cos\theta) \, d\theta$$

Note : In solving the above expression , the following trignometrical relationships are used.

1. $\text{Sin}^2\theta = 1 - \cos^2\theta$ and $\int \cos^2\theta = \theta/2 + \sin 2\theta/4$
2. $\int \cos^3\theta = \sin\theta - \sin^3\theta/3$
3. $\int \cos^4\theta = 3\theta/8 + \sin 2\theta/4 + \sin 4\theta/32$

By using the above formulas and solving the integral, we get the value as follows.

$\int MydX = 730607.23$. Now evaluate $\int y^2 dX$.

$\int y^2 dX = 2\int_{o}^{30} [\sqrt{(50^2 - X^2)} - 40]^2 \, dX$. By evaluating on similar lines as stated above; we have.

$= 3322.0$

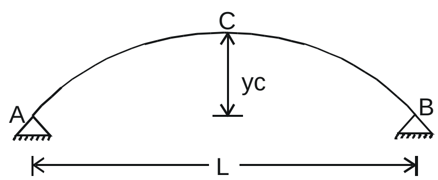
$$H = \frac{\int MydX}{-\int y^2 \, dX}$$

$$= \frac{730607.23}{3322.0}$$

$$\boxed{H = 220.0 \text{ KN}}$$

The same may be solved by numerical integration

3.9. TWO HINGED PARABOLIC ARCHES



Equation of the centre line of a parabolic arch with either abutment as origin is

$$y = CX(L - X) \rightarrow (1)$$

At $X = \frac{L}{2}$ $y = yc$ Putting

$$yc = C \times \frac{L}{2}\left(L - \frac{L}{2}\right)$$

$$yc = C.\frac{L}{2}\left(\frac{L}{2}\right)$$

$$yc = \frac{C.L^2}{4}$$

$$C = \frac{4\,yc}{L^2}$$

Putting the value of ‘C’ in equation (1), we have.

$$y = \frac{4\,yc}{L^2}X(L - X)$$

$y = \frac{4\,yc\,X}{L^2}(L - X)$, rated for $0 < X < L$

and $\frac{dy}{dX} = \frac{4\,yc}{L^2}(L - 2X)$ $\qquad 0 < X < L$

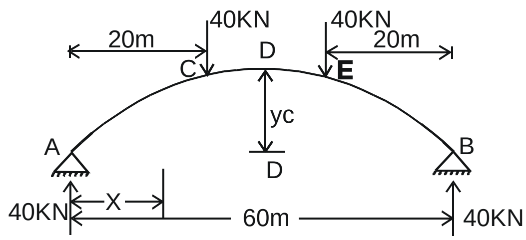
So $H = \frac{\int MydX}{\int y^2dX}$

In parabolic arches, origin for X is usually their supports.

EXAMPLE NO. 10:– A two–hinged parabolic arch with secant variation of inertia is subjected to the loads at 3rd points as shown in the diagram. Determine the horizontal thrust at abutments & plot the B.M.D.

Verify your answer by numerical integration.

SOLUTION:–



It is a symmetrically loaded arch. So moment expression on simple span in portions AC and CD may be found and corresponding integrals may be evaluated and multiplied by 2.

$M_{ac} = 40\,X$ $\qquad 0 < X < 20$

$M_{cd} = 40\,X - 40\,(X - 20) = 800$ $\qquad 20 < X < 30$

$y = \frac{4\,yc\,X}{L^2}(L - X)$, Put value of yc and L for simplification purpose.

$= \frac{4\,.\,10\,.\,X}{60^2}(60 - X)$

or $y = 0.011\,X\,(60 - X) = 0.011 \times 60\,X - 0.011\,X^2$

$$\int MydX = 2\int_o^{20}(40\,X)(0.011 \times 60\,X - 0.011\,X^2)dX$$

$$+2\int_{20}^{30} 800(0.66\,X - 0.011X^2)\,dX$$

Simplifying

$$= 2\int_o^{20}(26.4\,X^2 - 0.44\,X^3)\,dX + 2\int_o^{30}(528X - 8.8X^2)dX$$

$$= 2\left|\frac{26.4\,X^3}{3} - \frac{0.44\,X^4}{4}\right|_o^{20} + 2\left|\frac{528\,X^2}{2} - \frac{8.8\,X^3}{3}\right|_{20}^{30}$$

$$= 2\left(\frac{26.4}{3} \times 20^{3} - \frac{0.44}{4} \times 20^{4}\right) + 2\left(\frac{528}{2} \times 30^{2} - \frac{8.8}{3} \times 30^{3} - \frac{528}{2} \times 20^{2} + \frac{8.8}{3} \times 20^{3}\right)$$

$= 105600 + 152533.33$

$= 258133.33$

$\int MydX = 258.133 \times 10^{3}$. Now evaluate $\int y^2 dX$.

$$\int y^2 dX = \int_o^{60} (0.011 \times 60\,X - 0.011\,X^2)^2\,dX$$

$$= \int_o^{60} [(0.66)^2 X^2 + (0.011)^2 X^4 - 2 \times 0.66 \times 0.011\,X^3]dX$$

$$= \int_o^{60} (0.4356\,X^2 + 1.21 \times 10^{-4}\,X^4 - 0.01452\,X^3)\,dX$$

$$= \left|\frac{0.4356\,X^3}{3} + \frac{1.21 \times 10^{-4}\,X^5}{5} - \frac{0.01452\,X^4}{4}\right|_o^{60}$$

$$= \frac{0.4356}{3} \times 60^3 + \frac{1.21 \times 10^{-4}}{5} \times 60^5 - \frac{0.01452}{4} \times 60^4$$

$= 3136.32$

$\int y^2 dX = 3.136 \times 10^3$

$$H = \frac{\int MydX}{\int y^2 dX}$$

$$= \frac{258.133 \times 10^3}{3.136 \times 10^3}$$

$$\boxed{H = 82.3\text{ KN}}$$

$M = Mo - Hy$, $y = 0.001\ X(60 - X)$, at $X = 20$, $y = y_E$

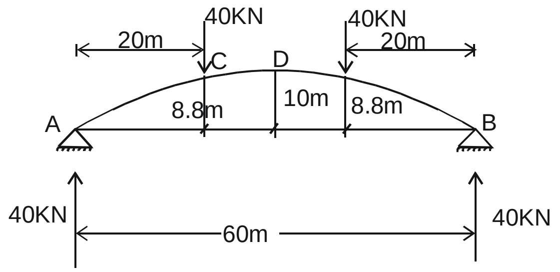
$yc = 0.011 \times 20\,(60 - 20) = 8.8\text{ m} = y_E$

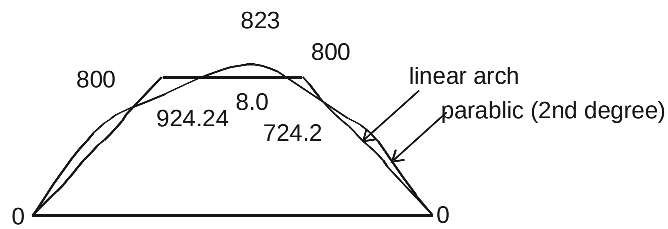
$Mc = 40 \times 20 - 82.3 \times 8.8 = 75.76$ KN–m

$M_D = (40 \times 30 - 40 \times 10) - 82.3 \times 10 = -23$ KN

$M_E = 40 \times 20 - 82.3 \times 8.8 = 75.76$ KN

Now BMD can be plotted.

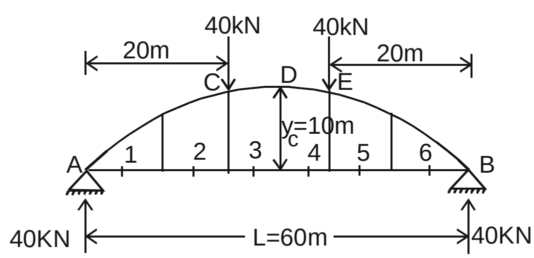




Note: – The length of the segment should be even multiple of span. More than 5 or 6 segments will give slightly improved answer.

3.10. EDDY'S THEOREM:– The difference between the linear arch and the actual arch is the BMD at that point.

EXAMPLE NO. 11:- Analyze the following loaded two hinged arch by numerical integration method.



$M_{ac} = 40X \qquad 0 < X < 20$

$M_{cd} = 40X - 40(X - 20) = 800 \qquad 20 < X < 40$

$M_{eb} = 40X - 40(X - 20) - 40(X - 40) = 2400 - 40X \qquad 40 < X < 60$

and $y = 0.011 \times (60 - X) = 0.66X - 0.011X^2$ (As before) solving in a tabular forces.

Section	X	y	M	My	y^2
1	5	3.025	200	605	9.15
2	15	7.425	600	4455	55.13
3	25	9.625	800	7700	92.64
4	35	9.625	800	7700	92.64
5	45	7.425	600	4455	55.13
6	55	3.025	200	605	9.15
				∑25520	∑313.84

$$L = 60 \text{ m}, \; dX = \frac{60}{6} = 10 \text{ m}$$

$$H = \frac{\sum MydX}{\sum y^2 dX}$$

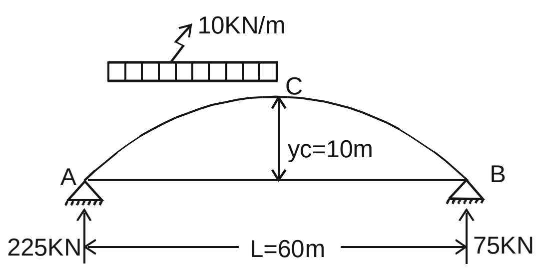
$$= \frac{25520 \times 10}{313.84 \times 10}$$

$$\boxed{H = 81.31 \text{ KN}}$$

Almost similar result was obtained by algebraic integration earlier.

EXAMPLE NO. 12:- A two–hinged parabolic arch with secant variation of inertia is subjected to a uniformly distributed load on its left half. Determine the horizontal thrust at abutments and plot the B.M.D. Verify your answer by numerical integration.

SOLUTION :–



$M_{ac} = 225X - 5X^2 \qquad 0 < X < 30$

$M_{bc} = 75X \qquad 0 < X < 30$

$$y = \frac{4y_c X}{L^2}(L - X)$$

$$= \frac{4 \,.\, 10 \,.\, X}{60^2}(L - X)$$

$$= 0.011\, X(60 - X)$$

$$y = 0.66X - 0.011X^2 \qquad \text{and } \frac{dy}{dX} = 0.66 - 0.022X = \text{Tan}\theta$$

$$\int MydX = \int_o^{30} (225X - 5X^2)(0.66X - 0.011X^2)\,dX + \int_o^{30} 75X(0.66X - 0.011X^2)\,dX$$

$$= \int_o^{30} (148.5X^2 - 2.475X^3 - 3.3X^3 + 0.055X^4)\,dX + \int_o^{30} (49.5X^2 - 0.825X^3)\,dX$$

$$= \left| \frac{148.5X^3}{3} - \frac{2.475X^4}{4} - \frac{3.3X^4}{4} + \frac{0.055X^5}{5} \right|_o^{30} + \left| \frac{49.5X^3}{3} - \frac{0.825X^4}{4} \right|_o^{30}$$

$$= \left[\frac{148.5}{3} \times 30^3 - \frac{2.475 \times 30^4}{4} - \frac{3.3 \times 30^4}{4} + \frac{0.055 \times 30^5}{5} \right] + \left[\frac{49.5 \times 30^3}{3} - \frac{0.825 \times 30^4}{4} \right]$$

$$= 712800.0174$$

$$\int MydX = 712.8 \text{ x } 10^3$$

$$\int y^2 dX = \int_o^{60} (0.66X - 0.011X^2)^2\,dX$$

$$= \int_o^{60} [(0.66^2)X^2 + (0.011)^2X^4 - 2 \,.\, 0.66 \,.\, 0.011X^3]\,dX$$

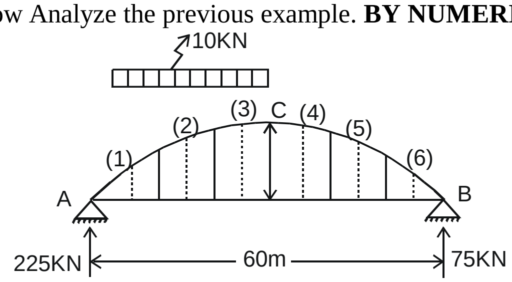
$$= \left| (0.66)^2 \frac{X^3}{3} + (0.011)^2 \frac{X^5}{5} - 2 \,.\, 0.66 \,.\, 0.011 \frac{X^4}{4} \right|_o^{60}$$

$$= 3.136 \times 10^{-3}$$

$$H = \frac{712.8 \;\; 10^3}{3.136 \;\; 10^3}$$

$$\boxed{H = 227.30 \text{ KN}}$$

EXAMPLE NO. 13:- Now Analyze the previous example. **BY NUMERICAL INTEGRATION :–**



Mac = $225X - 5X^2$ $\quad 0 < X < 30$

Mcb = $225X - 300(X - 15)$ $\quad 30 < X < 60$

$y = 0.66X - 0.011X^2$ (same as before). Attempt in a tabular form.

Section	X	y	M	My	y^2
1	5	3.025	1000	3025	9.15
2	15	7.425	2250	16706.25	55.13
3	25	9.625	2500	24062.5	92.64
4	35	9.625	1875	18046.875	92.64
5	45	7.425	1125	8353.125	55.13
6	55	3.05	375	1134.375	9.15
				∑71328.125	∑313.84

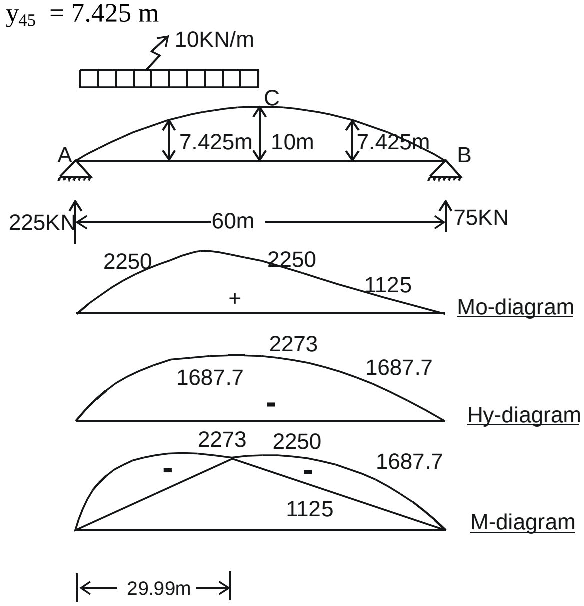
$$H = \frac{71328.125 \,.\, 10}{313.84 \,.\, 10}$$

$$\boxed{H = 227.28 \text{ KN}}$$

WE GET THE SAME ANSWER AS WAS OBTAINED BY ALGEBRAIC INTEGRATION.

y_{15} = 0.66 X 15 – 0.011 (15)2 = 2.425 m

y_{45} = 7.425 m



Point of contraflexure. Write a generalized Mx expression and set that to zero.

$$Mx = 225X - 5X^2 - 227.30 + [0.011 \text{ X } (60 - X)] = 0$$

$$225X - 5X^2 - 150.02X + 2.50X^2 = 0$$

$$-2.5X^2 + 74.98X = 0$$

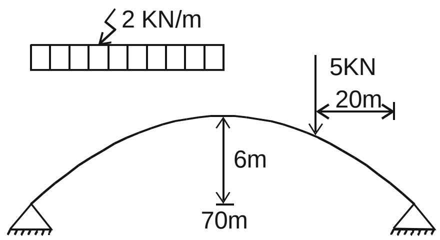
$$-2.5X + 74.98 = 0$$

$$\boxed{X = 29.99 \text{ m}}$$

Insert this value back in Mx expression to find M max in the arch.

EXAMPLE NO. 14:- Analyze the following arch by algebraic and numerical integration. Consider :

A. the arch to be parabolic and then circular.
B. moment of inertia constant and then with secant variation.

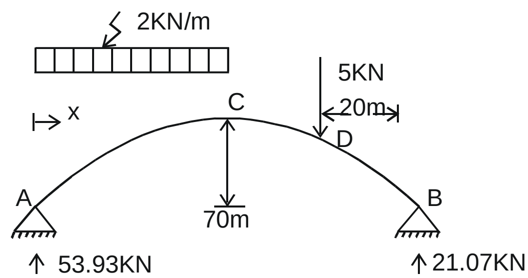


Generally arches have been used by the engineers and architects dating back to old roman buildings, Mughal and Muslim architecture. Main applications are in bridges, churches, mosques and other buildings. Arch behaviour is dependent upon stiffness of supports, commonly called abutments or springings so that horizontal reaction develops.

SOLUTION :–

A. PARABOLIC ARCH AND ALGEBRAIC INTEGRATION

$$\frac{70 \times 52.5}{70} + \frac{5 \times 20}{70} = 53.93$$



Determine simple span bending moments.

Mac $= 53.93\,X - X^2$ $\quad 0 < X < 35$

Mcd $= 53.93X - 70(X - 17.5)$ $\quad 35 < X < 50$

$= 53.93X - 70X + 1225$

$= -16.07X + 1225$

Mdb $= 53.93X - 70(X-17.5) - 5\,(X-50)$ $\quad 0 < X < 70$

$= 53.93X - 70X + 1225 - 5X + 250$

$= -21.07X + 1475$

Y $= \dfrac{4YcX}{L^2}\,(L - X)$

$= \dfrac{4\,.\,6\,.\,X}{70^2}\,(70 - X)$

$= 4.898\,.\,10^{-3}\,X\,(70 - X)$

Y $= 0.343X - 4.898\,.\,10^{-3}\,X^2$

$$\int MydX = \int_{o}^{35} (53.93X - X^2)\,(0.343X - 4.898 \times 10^{-3}\,X^2)\,dX$$

$$+\int_{35}^{50} (-16.07X + 1225)(0.343X - 4.898 \times 10^{-3}\ X^2)\,dX$$

$$+\int_{50}^{70} (-21.07X + 1475)(0.343X - 4.898 \times 10^{-3} X^2)\,dX \quad \text{Multiply the expressions}$$

$$= \int_{o}^{35} (18.498X^2 - 0.264X^3 - 0.343X^3 + 4.898 \times 10^{-3} X^4)\,dX$$

$$+\int_{35}^{50} (-5.512X^2 + 0.079X^3 + 420.175X - 6X^2)\,dX$$

$$+\int_{50}^{70} (-7.227X^2 + 0.103X^3 + 505.925X - 7.225X^2)\,dX \quad \text{re-arranging we get}$$

$$= \int_{o}^{35} (4.898 \times 10^{-3} X^4 - 0.607X^3 + 18.498X^2)\,dX$$

$$+\int_{35}^{50} (0.079X^3 - 11.512X^2 + 420.175X)\,dX$$

$$+\int_{50}^{70} (0.103X^3 - 14.452X^2 + 505.925X)\,dX$$

$$= \left| 4.898 \times 10^{-3}\frac{X^5}{5} - 0.607\frac{X^4}{4} + 18.498\frac{X^3}{3} \right|_{o}^{35} + \left| 0.079\frac{X^4}{4} - 11.512\frac{X^3}{3} + 420.175\frac{X^2}{2} \right|_{35}^{50}$$

$$+ \left| 0.103\frac{X^4}{4} - 14.452\frac{X^3}{3} + 505.925\frac{X^2}{2} \right|_{50}^{70} . \quad \text{Insert limits and simplify}$$

$$= 88097.835 + 46520.7188 + 14251.3336$$

$$\int MydX = 148869.8874 \quad . \qquad \text{Now calculate } \int y^2 dX$$

$$\int y^2 dX = \int_{o}^{70} (0.343X - 4.898 \times 10^{-3} X^2)^2\,dX$$

$$= \int_{o}^{70} (0.118X^2 + 2.399 \times 10^{-5} X^4 - 3.360 \times 10^{-3} X^3)\,dX$$

$$\int y^2 dX = \left| \frac{0.118X^3}{3} + 2.399 \times 10^{-5}\frac{X^5}{5} - 3.360 \times 10^{-3}\frac{X^4}{4} \right|_{o}^{70}$$

$$= 1386.932$$

$$H = \frac{\int MydX}{\int y^2 dX}$$

$$= \frac{148869.8874}{1386.932}$$

$$\boxed{H = 107.34\ KN}$$

B. SOLUTION OF SAME PARABOLIC ARCH BY NUMERICAL INTEGRATION:–

We know $M_{ac} = 53.93X - X^2$ $\quad 0 < X < 35$

$M_{cd} = 53.93X - 70\,(X - 17.5)$ $\quad 35 < X < 50$

$M_{db} = 53.93X - 70\,(X - 17.5) - 5\,(X - 50)$ $\quad 50 < X < 70$

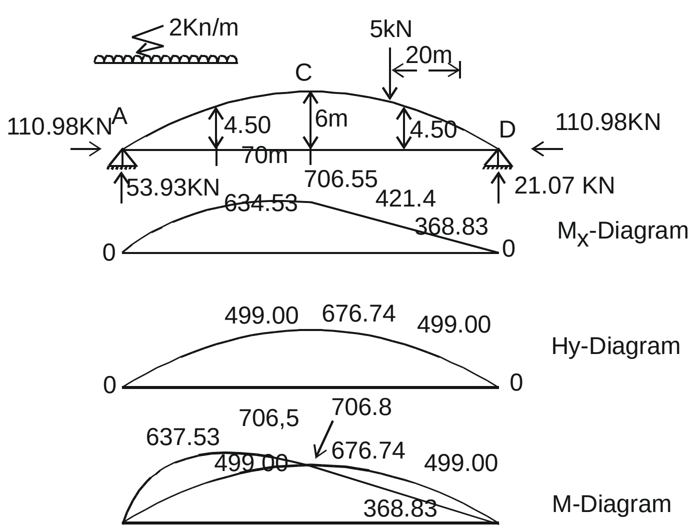
$y = 0.343X - 4.898 \,.\, 10^{-3}\, X^2$. Solve in a tabular form.

SECTION	X	Y	M	MY	Y^2
1	3.5	1.14	176.51	201.22	1.30
2	10.5	3.06	456.02	1395.42	9.36
3	17.5	4.50	637.53	2868.89	20.27
4	24.5	5.46	721.04	3936.88	29.35
5	31.5	5.94	706.55	4196.91	35.34
6	38.5	5.94	606.31	3601.48	35.34
7	45.5	5.46	493.82	2696.26	29.85
8	52.5	4.50	368.83	1659.74	20.27
9	59.5	3.06	221.34	677.29	9.36
10	66.5	1.14	73.85	84.18	1.30
				∑ 21318.27	∑ 192.24

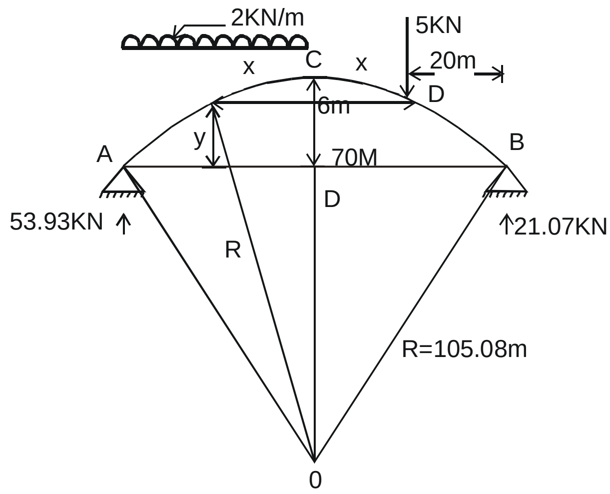
$$H = \frac{\int MydX}{\int y^2 dX}$$

$$= \frac{21318.27 \times 7}{192.24 \times 7}$$

$H = 110.89$ KN Accuracy can be increased by increasing the number of segments. Now BMD is drawn.



C. CONSIDERING IT TO BE A CIRCULAR ARCH WITH ALGEBRAIC INTEGRATION



$$R = \frac{L^2}{8yc} + \frac{yc}{2}$$

$$R = \frac{70^2}{8X6} + \frac{6}{y}$$

$$R = 105.08 \text{ m}$$

$$y = \sqrt{R^2 - X^2} - (h - yc) \text{ and } \frac{dy}{dX} = \tan\theta = \frac{-X}{\sqrt{105.082 - X^2}}$$

$$y = \sqrt{105.08^2 - X^2} - (105.08 - 6)$$

$$y = \sqrt{105.08^2 - X^2} - 99.08$$. Establishment expressions.

$$Mac = 53.93\,(35 - X) - (35 - X)^2 \qquad 0 < X < 35$$

$$Mbd = 21.07\,(35 - X) \qquad 0 < X < 20$$

$$Mdc = 21.07\,(35 - X) - 5\,(15 - X) \qquad 20 < X < 35$$

$$\int My\,dX = \int_o^{35} \left[53.93\,(35 - X) - (35 - X)^2\right]\left[\sqrt{105.08^2 - X^2} - 99.08\right] dX$$

$$+ \int_o^{20} 21.07\,(35 - X)\left[\sqrt{105.08^2 - X^2} - 99.08\right] dX$$

$$+ \int_o^{35} \left[21.07(35 - X) - 5\,(15 - X)\right]\left[\sqrt{105.08^2 - X^2} - 99.08\right] dX$$

$$\int My\,dX = I_1 + I_2 + I_3$$

(Where I_1 , I_2 and I_3 are 1st , 2nd and 3rd integrals of above expression respectively). These are evaluated separately to avoid lengthy simultaneous evaluation of above $\int$ My dX expression.

Evaluation of $I_1 = \int_o^{35} \left[53.93 \times 35 - 53.93X - (35^2 + X^2 - 70X)\right] \left[\sqrt{105.08^2 - X^2} - 99.08\right] dX$

$$= \int_o^{35} (662.55 + 16.07X - X^2) \left[\sqrt{105.08^2 - X^2} - 99.08\right] dX$$

$$= \int_o^{35} \Big[662.55 \sqrt{105.08^2 - X^2} + 16.07\, X \sqrt{105.08^2 - X^2}$$

$$- X^2 \sqrt{105.08^2 - X^2} - 65645.454 - 1592.216X + 99.08X^2\Big] dX$$

$$= 662.55 \int_o^{35} \sqrt{105.08^2 - X^2}\, dX - \frac{16.07}{2} \int_o^{35} \sqrt{105.08^2 - X^2}\, (-2X) dX .$$

Taking constants out.

$$\frac{1}{2} \int_o^{35} X \sqrt{105.08^2 - X^2}\, (-2X) dX - 65645.454 \int_o^{35} dX - 1592.216$$

$$\int_o^{35} X dX + 99.08 \int_o^{35} X^2 dX$$

Put $X = 105.08 \operatorname{Sin}\theta$

and $dX = 105.08 \operatorname{Cos}\theta\, d\theta$

At $X = 0$, $\theta = 0$

At $X = 35$, $\theta = 0.3396$ radians $= 19.4°$

$$I_1 = 662.55 \int_o^{0.3396} \sqrt{105.08^2 - 105.08^2 \times \operatorname{Sin}^2 \theta}\,(105.08)\operatorname{Cos}\theta d\theta$$

$$- \frac{16.07}{2} \left| \left(\frac{105.08^2 - X^2}{3/2}\right)^{3/2} \right|_o^{35} + \frac{1}{2} \left[\left| X \left(\frac{105.08^2 - X^2}{3/2}\right)^{3/2} \right|_o^{35} - \int_o^{35} \left(\frac{105.08^2 - X^2}{3/2}\right)^{3/2} .dX \right]$$

$$- 65645.454 \left| X \right|_o^{35} - 1592.216 \left| \frac{X^2}{2} \right|_o^{35} + 99.08 \left| \frac{X^3}{3} \right|_o^{35}$$

$$= 662.55 \times 105.08^2 \int_o^{0.3396} \operatorname{Cos}^2 \theta\, d\theta - \frac{16.07}{3} [(105.08^2 - 35^2)^{3/2}$$

$$- (105.08^2)^{3/2}] + \frac{1}{3} [^{35} (105.08^2 - 35^2)^{3/2} - \int_o^{35} (105.08^2 - X^2)^{3/2} dX]$$

$$- 65645.454\,(35 - 0) - \frac{1592.216}{2}(35^2\,) + 99.08\left(\frac{35^3}{3}\right)$$

$$I_1 = 7315748.83\int_o^{0.3396}\left(\frac{1+\text{Cos}^2\,\theta}{2}\right)d\theta + 1005048.922 + 11347550.55$$

$$-\frac{1}{3}\int_o^{0.3396} 105.08^4\,\text{Cos}^4\,\theta d\theta - 1856804.857$$

$$= \frac{7315748.83}{2}\left|\theta + \frac{\text{Sin}^2\,\theta}{2}\right|_o^{0.3396} - \frac{1}{3}(105.08)^4$$

$$\int_o^{0.3396} \text{Cos}^2\,\theta\,(1 - \text{Sin}^2\,\theta)d\theta + 10495794.62$$

$$I_1 = \frac{7315748.83}{2}\left[0.3396 + \frac{\text{Sin}\,(2\times 0.3396)}{2}\right] + 10495794.62$$

$$-\frac{1}{3}\times(105.08)^4\int_o^{0.3396}\left(\frac{1+\text{Cos}\,2\theta}{2}\right)\left(\frac{1-\text{Cos}\,2\,\theta}{2}\right)d\theta$$

$$= 12886893.66 - \frac{1}{3}\times\left(\frac{105.08}{2}\right)^4\left|\theta + \frac{\text{Sin}\,2\theta}{2}\right|_o^{0.3396}$$

$$+\frac{1}{12}\times(105.08)^4\,\Big|_o^{0.3396}\,(1 - \text{Cos}^2\,2\theta)\,d\theta$$

$$= 12886893.66 - \frac{1}{6}\times(105.8)^4\left[0.3396 + \frac{\text{Sin}\,(2\times 0.3396)}{2}\right]]$$

$$+\frac{1}{12}\times(105.08)^4\int_o^{0.3396}\left[1 - \left(\frac{1+\text{Cos}\,4\,\theta}{2}\right)\right]d\theta$$

$$= 12886893.66 - 13283049.35 + \frac{1}{12}\times(105.08)^4\int_o^{0.3396}\left(\frac{1}{2} - \frac{1}{2}\text{Cos}\,4\,\theta\right)d\theta$$

$$= -396155.69 + \frac{1}{24}(105.08)^4\left|\theta - \frac{\text{Sin}\,4\theta}{4}\right|_o^{0.3396}$$

$$= -396155.69 + \frac{1}{24}(105.08)^4\left[0.3396 - \frac{\text{Sin}\,(4\times 0.3396)}{4}\right]_o^{0.3396}$$

$$= -396155.69 + 483712.6275$$

$$= 87556.9375$$

$$I_2 = \int_o^{20} 21.07\,(35 - X)\left[\sqrt{105.08^2 - X^2} - 99.08\right] dX$$

$$= \int_o^{20} (737.45 - 21.07X)\left[\sqrt{105.08^2 - X^2} - 99.08\right] dX$$

$$= \int_o^{20} \Big[737.45\sqrt{105.08^2 - X^2} - 73066.546$$

$$- 21.07X\sqrt{(105.08)^2 - X^2} + 2087.6162\Big]\, dX$$

Put $X = 105.08 \text{ Sin } \theta$

$dX = 105.08 \text{ Cos } \theta \, d\theta$

At $X = 0$ $\quad \theta = 0$

At $X = 20$ $\quad \theta = 0.1915$

$$I_2 = 737.45\int_o^{0.1915} (105.08)^2 \text{Cos}^2\theta \, d\theta + \frac{21.07}{2}\int_o^{20}\sqrt{105.08^2 - X^2}$$

$$(-2X)\, dX - 73066.546\int_o^{26} dX + 2087.616\int_o^{20} X dX$$

$$= 8.143 \times 10^6 \int_o^{0.1915}\left(\frac{1 + \text{Cos } 2\theta}{2}\right) d\theta + \frac{21.07}{2}\left|\left[\frac{(105.08)^2 - X^2}{3/2}\right]^{3/2}\right|_o^{20}$$

$$- 73066.546\,|\,X\,|_o^{26} + 2087.616\left|\frac{X^2}{2}\right|_o^{20}$$

$$= \frac{8.143 \times 10^6}{2}\left|\theta + \frac{\text{Sin } 2\theta}{2}\right|_o^{0.1915} + \frac{21.07}{3}\left[\{(105.08)^2 - (20)^2\}^{3/2} - (105.08)^{2 \times 3/2}\right]$$

$$- 73066.546\,(20) + \frac{2087.616}{2}(400)$$

$$= \frac{8.143 \times 10^6}{2}\left[0.1916 + \frac{\text{Sin }(2 \times 0.1915)}{2}\right] - 438772.215$$

$I_2 = 58247.385$

$$I_3 = \int_{20}^{35} (662.45 - 16.07X)\left[\sqrt{105.08^2 - X^2} - 99.08\right] dX$$

$$= \int_{20}^{35} \Big[662.45\sqrt{105.08^2 - X^2} - 65635.546$$

$$- 16.07 \times \sqrt{105.08^2 - X^2} + 1592.216X\Big]\, dX$$

$$= 662.45 \int_{0.1915}^{0.3396} 105.08^2 \cos^2\theta \, d\theta - 65635.546 \, |X|_{20}^{35}$$

$$+ 1592.216 \left|\frac{X^2}{2}\right|_{20}^{35} + \frac{16.07}{2} \int_{20}^{35} \sqrt{105.08^2 - X^2}\,(-2X)\, dX$$

$$= 662.45 \times 105.08^2 \int_{0.1915}^{0.3396} \left(\frac{1 + \cos 2\theta}{2}\right) d\theta - 65635.546 \times 15$$

$$+ \frac{1592.216}{2}(35^2 - 20^2) + \frac{16.07}{2} \left|\frac{(105.08^2 - X^2)^{3/2}}{3/2}\right|_{20}^{35}$$

$$= \frac{662.45 \times 105.08^2}{2} \left|\theta + \frac{\sin 2\theta}{2}\right|_{0.1915}^{0.3396} - 65635.546 \times 15$$

$$+ \frac{1592.216}{2}(35^2 - 20^2) + \frac{16.07}{3}[(105.08^2 - 35^2)^{3/2} - (105.08^2 - 20^2)^{3/2}]$$

$$= \frac{662.45 \times 105.08^2}{2}\left[0.3396 - 0.1915 + \frac{\sin(2 \times 0.3396)}{2} - \frac{\sin(2 \times 0.1915)}{2}\right]$$

$$- 65635.546 \times 15 + \frac{1592.216}{2}(35^2 - 20^2) + \frac{16.07}{3}[(105.08^2 - 35^2)^{3/2} - (105.08^2 - 20^2)^{3/2}]$$

I_3 = 8838.028 . Adding values of three integrals. We have

MydX = 87556.9375 + 58247.385 + 8838.028

= 154642.3505 . Now calculate $\int y^2 dX$

$$\int y^2\, dX = 2\int_o^{35} \left[\sqrt{105.08^2 - X^2} - 99.08\right]^2 dX$$

$$= 2\int_o^{35} \left[105.08^2 - X^2 + 99.08^2 - 2X99.08\sqrt{105.08^2 - X^2}\right] dX$$

$$= 2\int_o^{35} (20858.653 - X^2 - 198.16\sqrt{105.08^2 - X^2})\, dX$$

$$= 2 \times 20858.653 \, |X|_o^{35} - \frac{2}{3}|X^3|^{35} - 198.16 \times 2\int_o^{0.3396} 105.08^2 \cos^2\theta\, d\theta$$

$$= 2 \times 20858.653\,(35) - \frac{2}{3}(35^3) - 198.16 \times 2 \times 105.08^2 \int_o^{0.3396} \left(\frac{1 + \cos^2\theta}{2}\right) d\theta$$

$$= 2 \times 20858.653 \times 35 - \frac{2}{3} + 35^3 - \frac{198.16 \times 2 \times 105.08^2}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_o^{0.3396}$$

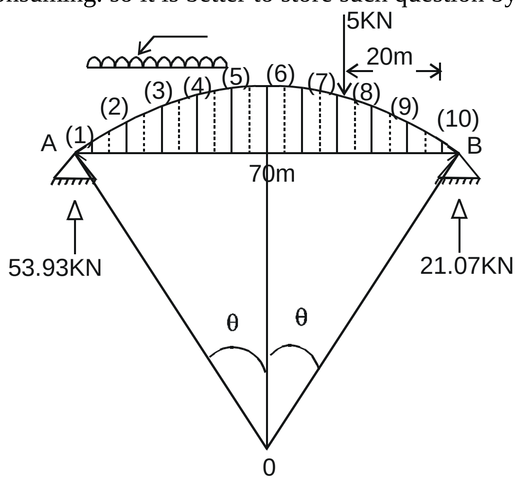
$\int y^2 d X = 1229.761$

$$H = \frac{\int MydX}{\int y^2 dX}$$

$$= \frac{154642.3505}{1239.761}$$

$$\boxed{H = 125.75 \text{ KN}}$$

D. CIRCULAR ARCH BY NUMERICAL INTEGRATION:- As you have seen algebraic integration is lengthy, laborious and time consuming. so it is better to store such question by numerical integration.



$$y = \sqrt{105.08^2 - X^2} - 99.08$$

$M_{ac} = 53.93 (35 - X) - (35 - X) \qquad 20 < X < 35$

$M_{bd} = 21.07 (35 - X) \qquad 0 < X < 20$

$M_{dc} = 21.07 (35 - X) - 5 (15 - X) \qquad 29 < X < 35$

Attempting in a tabular form

Section	X	Y	M	MY	Y^2
1	31.5	1.167	176.505	205.981	1.362
2	24.5	3.104	456.015	1415.47	9.635
3	17.5	0.533	637.525	2889.901	20.548
4	10.5	5.474	721.035	3946.446	29.965
5	3.5	5.942	760.545	4198.29	35.307
6	3.5	5.942	606.205	3602.07	35.307
7	10.5	5.474	493.715	2702.596	29.965
8	17.5	4.533	368.725	1671.430	20.548
9	24.5	3.104	221.235	686.713	9.635
10	31.5	1.167	73.745	86.060	1.362
				∑21405.157	∑193.634

$$S = 105.08\,(2 \times 0.3396) = 71.370 \text{ m}$$

$$dS = \frac{71.37}{10} = 7.137 \text{ m}$$

$$H = \frac{\sum Myds}{\sum y2ds} = \frac{21405.157 \times 7.137}{193.634 \times 7.137}$$

H = 110.54 KN , Accuracy can be increased by taking more segments.

For secant variation of inertia follow the same procedures established already in this Chapter.

Space for taking Notes: