CHAPTER ONE

STABILITY, DETERMINACY OF STRUCTURES AND CONSISTENT DEFORMATIONS METHOD

1.1. STABILITY OF STRUCTURES:

Before deciding the determinacy or indeterminacy of a structure we should first of all have a structure which is stable. The question of determinacy or indeterminacy comes next. We shall now discuss 2-D or single plane structures. (Defined and accommodated in a single plane).

1.1.1. STABLE STRUCTURE:

A stable structure is the one, which remains stable for any conceivable (imaginable) system of loads. Therefore, we do not consider the types of loads, their number and their points of application for deciding the stability or determinacy of the structure. Normally internal and external stability of a structure should be checked separately and if its overall stable then total degree of indeterminacy should be checked.

1.2. ARTICULATED STRUCTURES:

This may be defined as “A truss, or an articulated structure, composed of links or bars, assumed to be connected by frictionless pins at the joints, and arranged so that the area enclosed within the boundaries of the structure is subdivided by the bars into geometrical figures which are usually triangles.”

1.3. CONTINUOUS FRAME:

“ A continuous frame is a structure which is dependent, in part, for its stability and load carrying capacity upon the ability of one or more of its joints to resist moment.” In other words, one or more joints are more or less rigid.

1.4. DETERMINACY:

A statically indeterminate structure is the one in which all the reactive components plus the internal forces cannot be calculated only from the equations of equilibrium available for a given force system.These equations, of course, are

$$\sum H = 0,\ \sum V = 0 \text{ and } \sum M = 0$$

The degree of indeterminacy for a given structure is, in fact , the excess of total number of reactive components or excess of members over the equations of equilibrium available.

It is convenient to consider stability and determinacy as follows.

a) With respect to reactions, i.e. external stability and determinacy.

b) With respect to members, i.e. internal stability and determinacy.

c) A combination of external and internal conditions, i.e. total stability and determinacy.

1.4.1. EXTERNAL INDETERMINACY:

A stable structure should have at least three reactive components, (which may not always be sufficient) for external stability of a 2-D structure, which are non-concurrent and non-parallel.

Fig. 1.1. Stable & determinate.

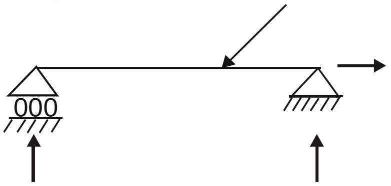


Fig. 1.2. Stable & determinate.

External indeterminacy is, in fact, the excess of total number of reactive components over the equations of equilibrium available.

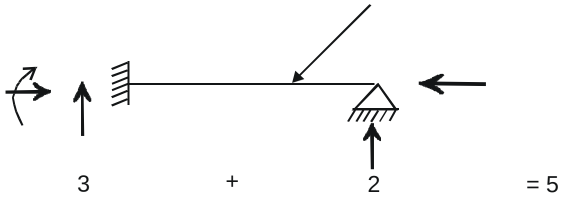


Fig. 1.3.

No. of reactions possible = 5
No. of Equations of equilibrium available = 3
Degree of External indeterminacy = 5 – 3 = 2

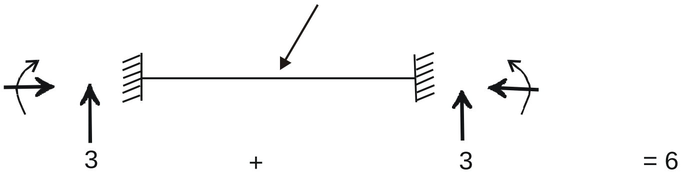


Fig. 1.4

Stable & Indeterminate to 2nd degree. (Fig. 1.3)

Fig. 1.4. Stable & externally indeterminate to 3rd degree.

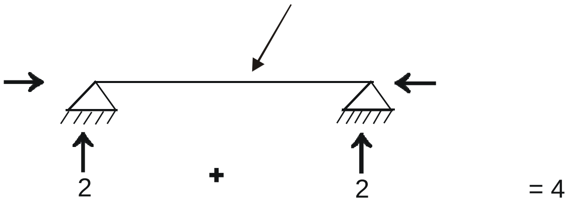


Fig. 1.5.

Stable & Indeterminate to Ist degree. (Fig. 1.5)

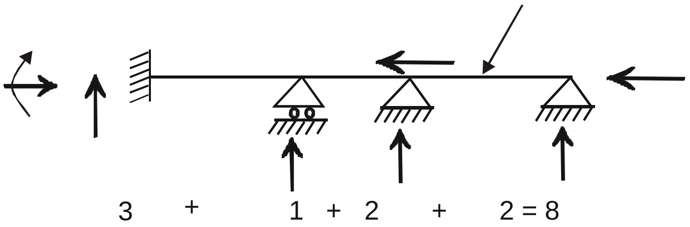


Fig. 1.6.

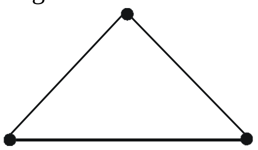
Stable & externally indeterminate to 5th degree. (Fig. 1.6)

Remove any five suitable redundant reactions to make it statically determinate.

1.4.2. INTERNAL INDETERMINACY:

This question can be decided only if the minimum number of reactive components necessary for external stability and determinacy are known and are acting on the structure. This type of indeterminacy is normally associated with articulated structures like trusses. We assume that the structure whose internal indeterminacy is being checked is under the action of minimum reactive components required for external stability at the supports.

The basic form of the truss is a triangle.



To make the truss, add two members and one joint and repeat.

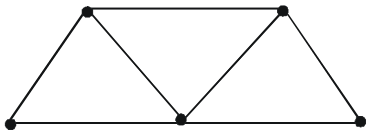


Fig 1.7

Let us assume that

j = Total number of joints.

b = Total number of bars.

r = Minimum number of reactive components required for external stability/determinacy.

$$\boxed{b + r} \quad = \quad \boxed{2j}$$

total number of unknowns.

total number of equations available (at joints).

1. If $b + r = 2j$ Stable & internally determinate. Check the arrangement of members also.

2. If $b + r > 2j$ Stable & internally indeterminate. (degree of indeterminacy would be decided by the difference of these two quantities).

3. If $b + r < 2j$ Unstable.

A structure is said to have determinacy or indeterminacy only if it is stable. Now we consider some examples.

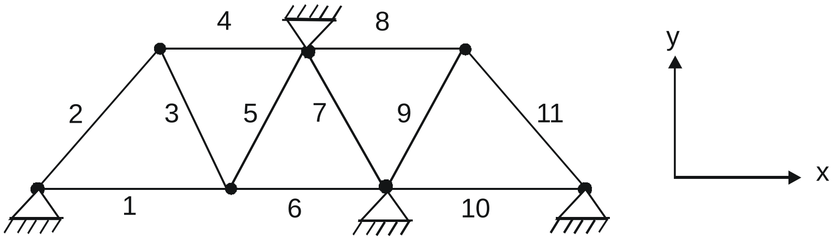


Fig. 1.8.

$b = 11$

$r = 3$ (Minimum external reactions required for external stability/determinacy)

$j = 7$

$b + r = 2j$

$11 + 3 = 2 \times 7$

$14 = 14$

This truss of fig. 1.8 is stable and internally determinate.

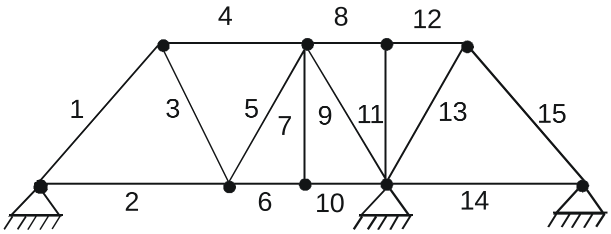


Fig. 1.9.

$b = 15$

$r = 3$

$j = 9$

$b + r = 2j$

$15 + 3 = 2 \times 9$

$18 = 18$

The truss of fig. 1.9 is stable and internally determinate.

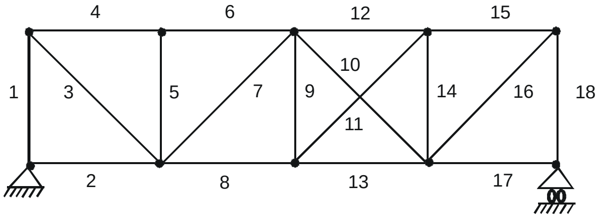


Fig. 1.10.

b = 18
r = 3
j = 10
b + r = 2 j
18 + 3 = 2 × 10
21 > 20

This truss of fig. 1.10 is stable & internally indeterminate to 1st degree.

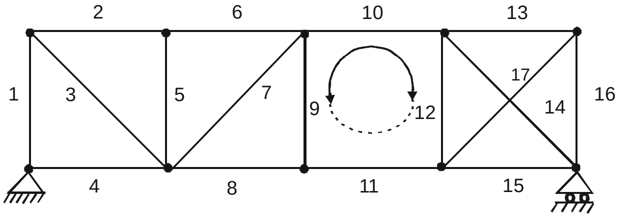


Fig. 1.11.

b = 16
r = 3
j = 10
b + r = 2 j
17 + 3 = 2 × 10
20 = 20

This truss is Unstable by inspection although the criterion equation is satisfied. The members in indicated square may get displaced and rotated due to gravity loads.

Always inspect member positions. Insert one member in the encircled box or manage prevention of sliding by external supports to make it stable.

NOTE:- The difference between the internal and the external indeterminacy is only in the definition of ‘r’

1.4.3. TOTAL INDETERMINACY

The question of total indeterminacy is of little interest and we have got different equations for different types of structures. For example, the previous equation, i.e., b + r = 2 j can be used to check the total degree of indeterminacy of an articulated structure like truss by slightly modifying the definition of “ r ” which should now be considered as the “total number of reactive components available”.

$b + r = 2j$

where b = Total number of bars.

r = Total number of reactive components available.

j = Total number of joints

Example No. 1: Determine the external and internal conditions of stability and determinateness for the following structures:-

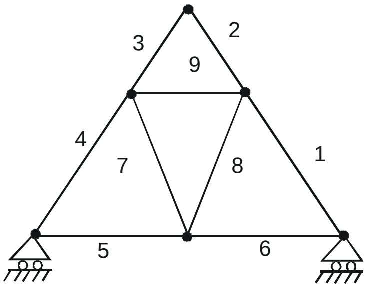


Fig. 1.12

(i) External Stability And Determinacy:-

Number of reactive components available = 2

Number of equations of equilibrium available = 3

∴ Unstable. (Visible also)

(ii) Internal Stability And Determinacy

$b = 9$
$r = 3$
$j = 6$
$b + r = 2j$
$9 + 3 = 2 \times 6$
$12 = 12$
Degree of Indeterminacy = $D = 12 - 12 = 0$

∴ Stable and Internally Determinate, if arrangement is improved to have $\Sigma = 3$.

Example No. 2:

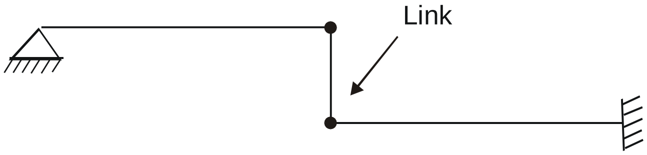


Fig. 1.13.

*In this case the presence of a pin at each end of the link makes one additional type of movement possible if reaction components are removed. Two condition equations are therefore provided by the link in terms of algebraic sum of moments equal to zero at the joints of link.

External Stability and Determinacy.

Number of reactive components = 5

Number of equations of equilibrium available = 3 + 2* = 5

Degree of indeterminacy = 5 – 5 = 0

∴ Stable and Externally Determinate. (Structure of fig. 1.13.)

Example No. 3:

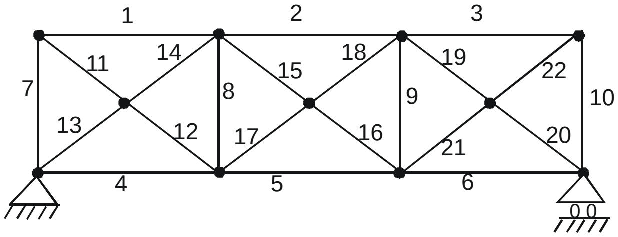


Fig. 1.14.

(i) External Stability and Determinacy:–

Number of reactions = 3

Number of equations = 3

D = 3 – 3 = 0

∴ Externally Stable and Determinate

(ii) Internal Stability and Determinacy:-

b = 22

r = 3

j = 11

b + r = 2 j

D = (b + r) – 2 j

= (22 + 3) – (2 × 11)

= 25 – 22

D = 3 where D = Degree of indeterminacy.

∴ Stable and indeterminate to 3rd degree.

Example No. 4:

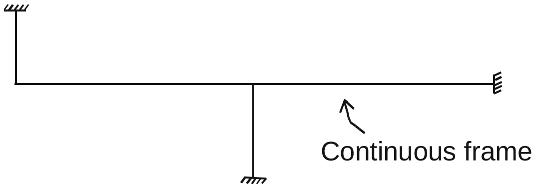


Fig. 1.15.

External Stability and Determinacy:-

Number of reactions = 9

Number of equations = 3

D = 9 – 3 = 6

∴ Stable and Indeterminate to 6th degree. (fig. 1.15).

Example No. 5:

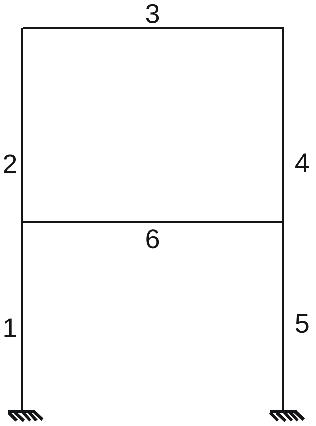


Fig 1.16

(i) External Stability And Determinacy :-

Number of reactions = 6
Number of equations = 3
Degree of indeterminacy = 6 – 3 = 3

∴ Stable and externally Indeterminate to 3rd degree.

(ii) Internal Stability and Determinacy :-

b = 6
r = 3, where r is the minimum reactive components required for external
j = 6 stability and determinacy.

Degree of indeterminacy of rigid jointed structure. (Fig. 1.16)

$D = (3b + r) - 3j$
$D = (3 \times 6 + 3) - (3 \times 6)$
$D = 21 - 18$
$D = 3$

∴ Stable and indeterminate to 3rd degree.

Example No. 6:

(i) External Stability and Determinacy :-

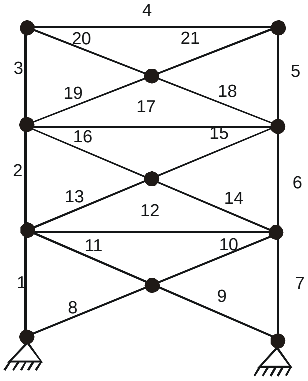


Fig. 1.17.

Number of reactions = 4
Number of equations = 3
D = 4 – 3 = 1
∴ Stable and indeterminate to Ist degree.

(ii) Internal Stability and Determinacy :-

b = 21
r = 3
j = 11
D = (b + r) – 2 j
= (21 + 3) – 2 × 11
D = 24 – 22 = 2
∴ Stable and indeterminate to 2nd degree.

Note: In case of a pin jointed structure, there is one unknown per member and in case of rigid jointed structure there are three unknowns at a joint.

Example No. 7:

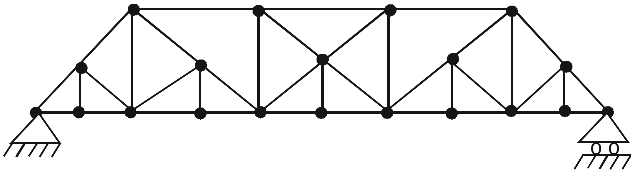


Fig. 1.18.

(i) External Stability and Determinacy :-

Number of reactions = 3
Number of equations = 3
D = 3 – 3 = 0
∴ Stable and Determinate.

(ii) Internal Stability and Determinacy :-

b = 38
r = 3
j = 20
D = (b + r) – 2 j
= (38 + 3) – 2 × 20
= 41 – 40
D = 1
∴ Stable and indeterminate to Ist degree. (Fig. 1.18)

Example No. 8:

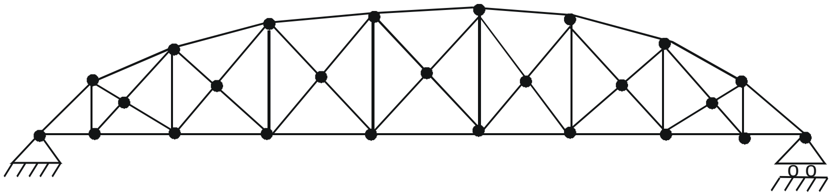


Fig. 1.19.

(i) External Stability and Determinacy :-

Number of reactions = 3

Number of equations = 3

D = 3 – 3 = 0

∴ Stable and Determinate.

(ii) Internal Stability and Determinacy :-

b = 54

r = 3

j = 25

b + r = 2 j

54 + 3 > 2 × 25

57 > 50

D = 57 – 50 = 7

∴ Stable and indeterminate to 9th degree. (Fig. 1.19)

Example No. 9:

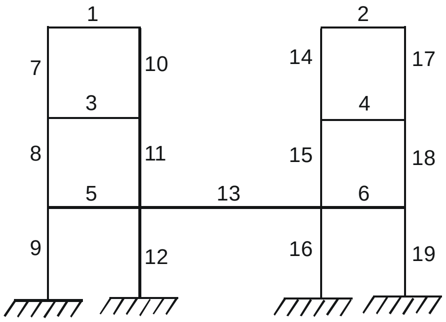


Fig. 1.20.

(i) External Stability and Determinacy :-

Number of reactions = 12

Number of equations = 3

D = 12 – 3 = 9

∴ Stable and indeterminate to 9th degree.

(ii) Internal Stability and Determinacy :-

b = 19

r = 3

j = 16

D = (3 b + r) = 3 j

= (3 × 19 + 3) = 3 × 16

= 60 > 48

D = 60 – 48 = 12

∴ Stable and Internally Indeterminate to twelfth degree. (Fig. 1.20)

Example No. 10:

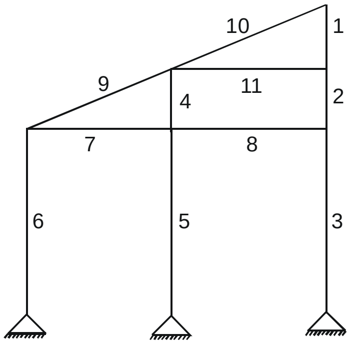


Fig. 1.21.

(i) External Stability and Determinacy :-

Number of reactions = 6
Number of equations = 3
D = 6 – 3 = 3
∴ Stable and Indeterminate to 3rd degree.

(ii) Internal Stability and Determinacy : -

b = 11
r = 3
j = 9

$$D = (3b + r) - 3j$$
$$= (3 \times 11 + 3) - 3 \times 9$$
$$= 36 - 27$$
$$D = 9$$

∴ Stable and indeterminate to 9th degree. (Fig. 1.21)

Example No. 11:

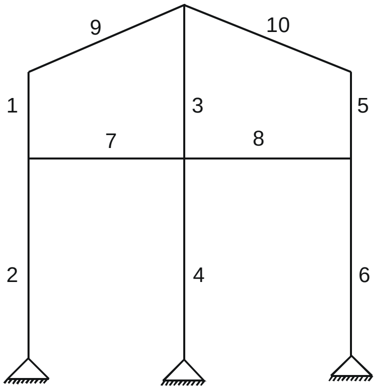


Fig. 1.22.

(i) External Stability and Determinacy :-

Number of reactions = 6

Number of equations = 3

D = 6 – 3 = 3

∴ Stable and indeterminate to 3rd degree.

(ii) Internal Stability and Determinacy :-

b = 10

r = 3

j = 9

D = (3 b + r) – 3 j

= (3 × 10 + 3) – 3 × 9

D = 33 – 27

D = 6

∴ Stable and indeterminate to 6th degree. (Fig. 1.22)

Example No. 12:

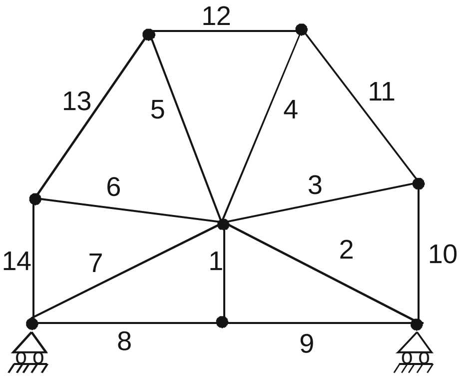


Fig. 1.23.

(i) External Stability and Determinacy :-

Number of reactions = 2

Number of equations = 3

∴ Unstable Externally. (Visible also)

(ii) Internal Stability and Determinacy :-

b = 14

r = 3

j = 8

D = (b + r) – 2 j

= (14 + 3) – 2 × 8

D = 1

∴ Stable and Internal Indeterminacy to Ist degree.

Example No. 13:

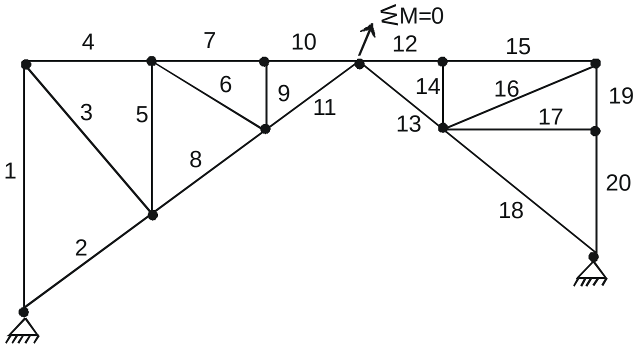


Fig. 1.24.

(i) **External Stability and Determinacy :-**

Number of reactions = 4

Number of equations = 3 + 1 = 4

D = 4 – 4 = 0

$\therefore$ Stable and Determinate.

(ii) **Internal Stability and Determinacy :-**

b = 20

r = 4 (Note this. A roller at either support will create instability)

j = 12

(b + r) = 2 j

(20 + 4) = 2 × 12

24 = 24

D = 24 – 24 = 0

(Here minimum r is 4 for internal stability and determinacy.)

$\therefore$ **Stable and determinate.**

Example No. 14:

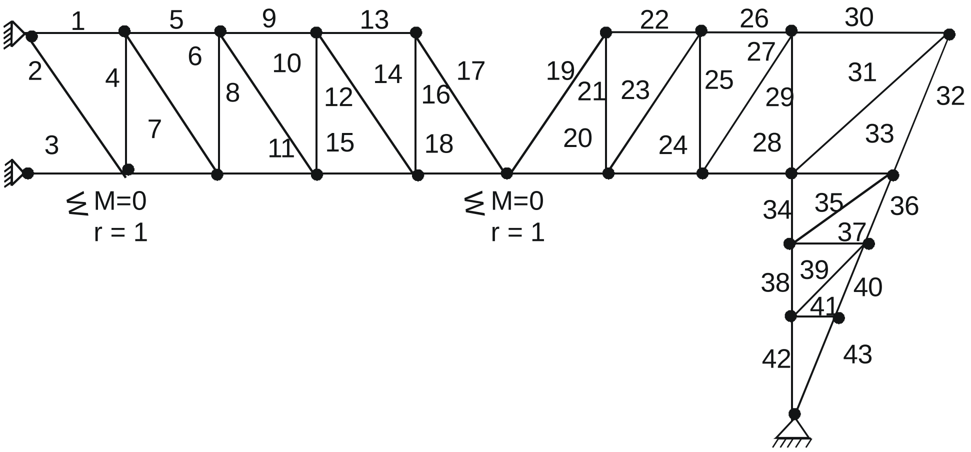


Fig. 1.25.

(i) External Stability and Determinacy :-

Number of reactions = 6
Number of equations = 3 + 2 = 5
D = 6 – 5 = 1
∴ Stable and Indeterminate to Ist degree.

(ii) Internal Stability and Determinacy :-

b = 43
r = 3 + 2 = 5 (take notice of it). Two pins where $\Sigma M = 0$
j = 24
b + r = 2 j
43 + 5 = 2 × 24
48 = 48
D = 48 – 48 = 0
∴ Stable and Determinate. (Fig. 1.25)

Example No. 15:

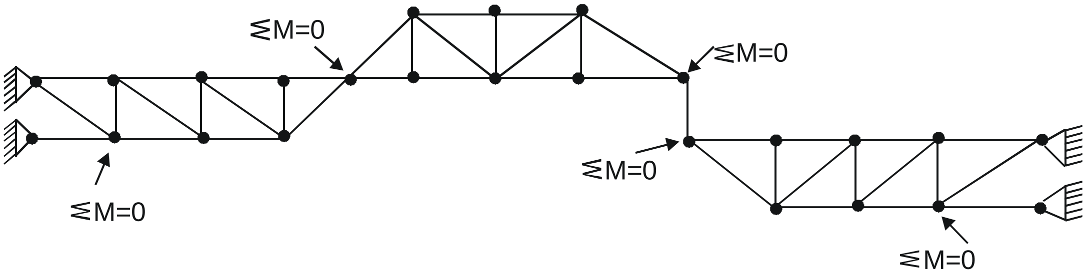


Fig. 1.26.

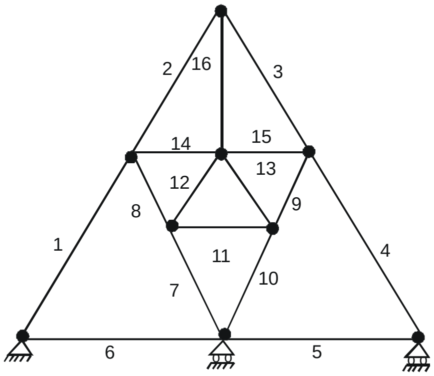
(i) External Stability and Determinacy :-

Number of reactions = 8
Number of equations = 8 = (3 + 5)
D = 8 – 8 = 0
∴ Stable and Determinate.

(ii) Internal Stability and Determinacy :-

b = 42
r = 3 + 5 = 8. There are 5 joints where $\Sigma M = 0$
j = 25
b + r = 2 j
42 + 8 = 2 × 25
50 = 50
D = 50 – 50 = 0
∴ Stable and Determinate.

Example No. 16:



(i) External Stability and Determinacy :-

Number of reactions = 4
Number of equations = 3
D = 4 – 3 = 1
∴ Stable and Indeterminate to Ist degree.

(ii) Internal Stability and Determinacy :-

b = 16
r = 3
j = 9
D = (b + r) – 2 j
= (16 + 3) – 2 × 9
= 19 – 18
D = 1
∴ Stable and Indeterminate to Ist degree.

In the analysis of statically determinate structures, all external as well as internal forces are completely known by the application of laws of statics.Member sizes do not come into the picture as no compatibility requirements are to be satisfied. However, in the analysis of indeterminate structures we should have member sizes, sectional and material properties before doing the analysis as member sizes would be involved in the determination of deflections or rotations which are to be put in compatibility equations afterwards. Now we discuss methods for finding deflection and rotations.

1.5. METHODS FOR FINDING DEFLECTION AND ROTATION;-

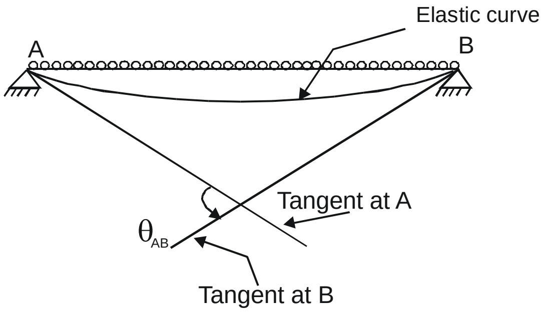
Usually following methods are used in this classical analysis of structures..

--- Unit - load method. (Strain energy method).
--- Moment - area method.
--- Conjugate beam method (a special case of moment - area method).

1.5.1. MOMENT AREA THEOREM (1) ;-

The change of slope between tangents drawn at any two points on the elastic curve of an originally straight beam is equal to the area of the B.M.D between these two points when multiplied by 1/EI (reciprocal of flexural stiffness),

$\theta_{AB} = \frac{1}{EI}$ (Area of B.M.D. between A & B)



$\theta_{AB} = \frac{1}{EI}\,(\text{AREA})_{AB}$

Signs of Change of Slope:-

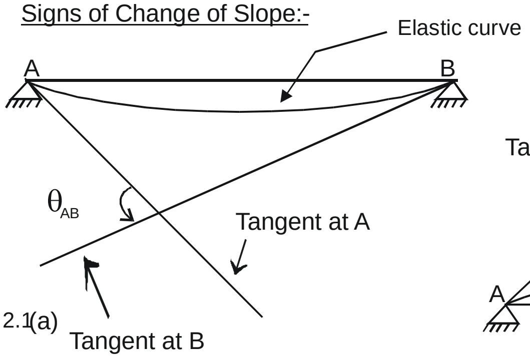


Fig 2.1(a)

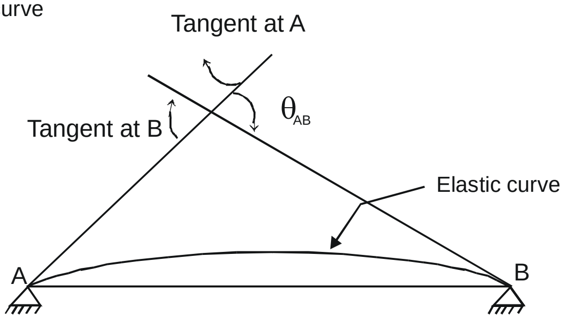


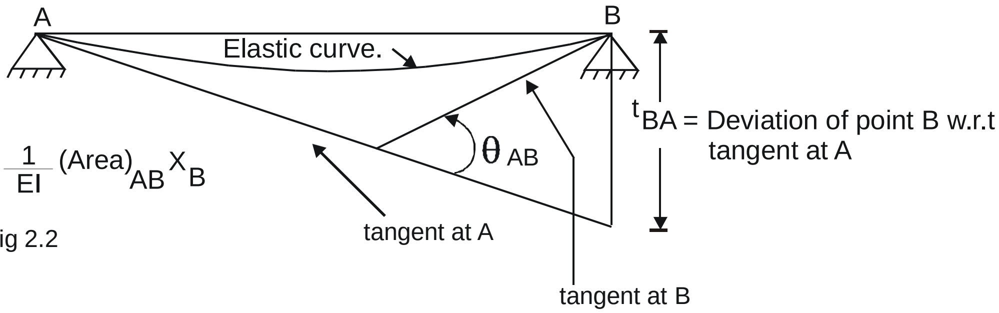
Fig 2.1(b)

(a) Positive change of slope, θ_{AB} is counterclockwise from the left tangent. (Fig. 2.1a)

(b) Negative change of slope, θ_{AB} is clockwise from the left tangent. (Fig. 2.1b)

1.5.2. MOMENT AREA THEOREM (2) :-

"The deviation of any point on elastic curve from the tangent drawn at some other point on the elastic curve is equal to $\frac{1}{EI}$ multiplied by the moment of the area of the bending moment diagram between these two points". The moment may generally be taken through a point where deviation is being measured.



$t_{BA} = \frac{1}{EI}\,(\text{Area})_{AB}\,X_B$

Fig 2.2

1.5.3. SIGN CONVENTION FOR DEVIATIONS:-

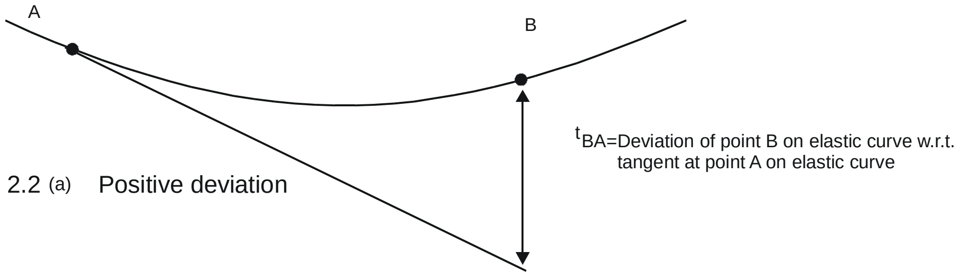


Fig 2.2 (a) Positive deviation

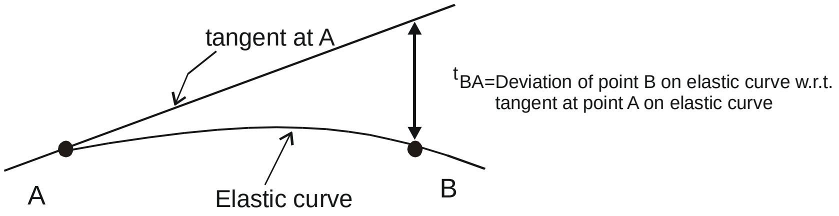


Fig 2.2 (b) Negative deviation

(a) Positive Deviation:- B located above the reference tangent. (Tangent at A; Fig. 2.2a)

(b) Negative Deviation:- B located below the reference tangent. (Tangent at A; Fig. 2.2b)

1.5.4. INEQUALITY OF t_{BA} AND t_{AB}

Depending upon loading, these two deviations t_{ab} and t_{ba} may not be equal if loading is unsymmetrical about mid span of the member.

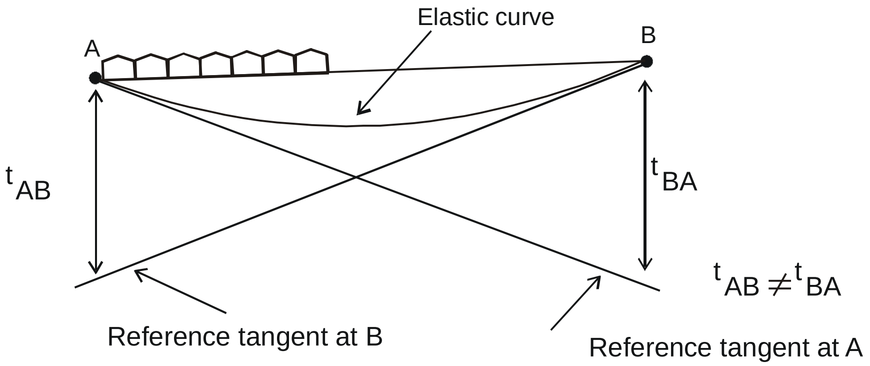


Fig. 2.3

1.6. BENDING MOMENT DIAGRAM BY PARTS:

In order to compute deviations and change of slope by moment area method, bending moment diagram may be drawn in parts i.e. one diagram for a particular load starting from left to right. Same sign convention would be followed for bending moment and shear force as have been followed in subjects done earlier. Bending moment would be positive if elastic curve resembles sagging i.e. compression at top fibers and tension at the bottom fibers while shear force would be positive at a section of a portion being considered as a free body when left resultant force acts upwards and right resultant force acts downwards. Negative bending moment and shear force would be just opposite to this.

1.6.1. SIGN CONVENTIONS FOR SHEAR FORCE AND BENDING MOMENT

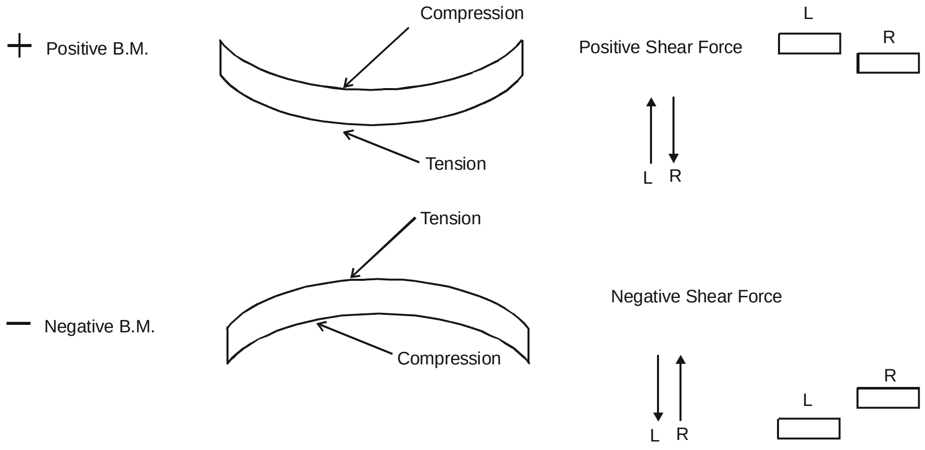
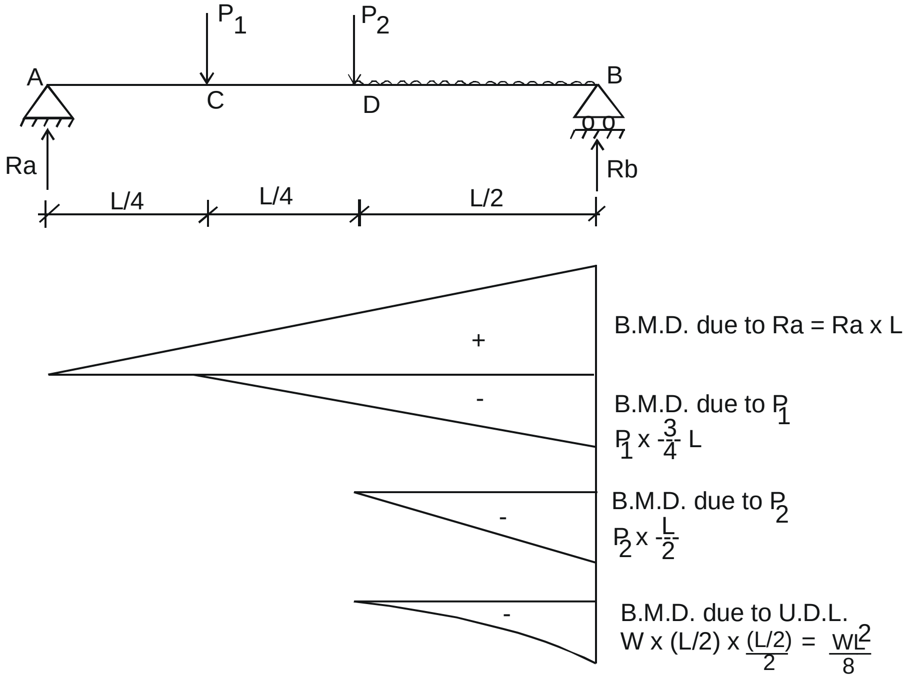


Fig 2.4

Consider the following loaded beam. Start from faces on LHS and move towards RHS. Construct BMS due to all forces encountered treating one force at a time only.



We observe that the moment effect of any single specified loading is always some variation of the general equation. Like

$$y = kX^n \qquad (1)$$

This Relationship has been plotted below. While drawing bending moment diagrams by parts and starting from left, for example, Ra is acting at A. Imagine that Ra is acting while support at A has been removed and beam is fixed adequately at B (just like a cantilever support), the deflected shape whether sagging or hogging will determine the sign of B.M.D. Similar procedure is adopted for other loads.

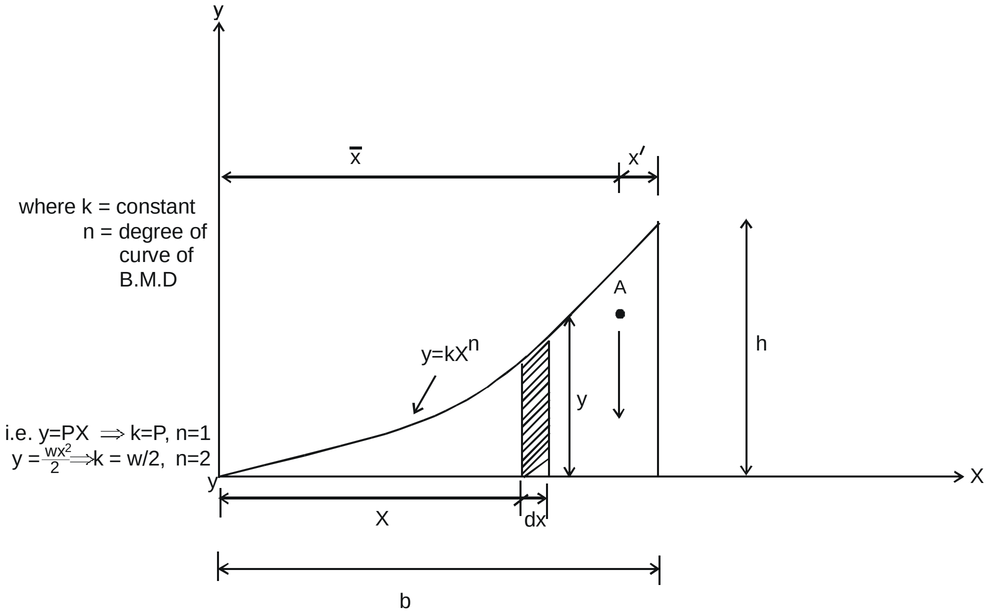


Fig. 2.6
Generalized variation of B.M. w.r.t. x

$$\text{In general} \qquad \bar{X} = \int \frac{Xd\,A}{A}$$

Area of the strip = $ydX = kX^n\,dX$ by putting value of y.

$$\text{Total area} = A = \int_o^b kX^n\,dX$$

$$A = \left| \frac{kX^{n+1}}{n+1} \right|_o^b$$

$$A = \frac{Kb^{(n+1)}}{(n+1)}$$

We want to find the total area under the curve in terms of ‘b’ and ‘h’ and for that the constant ‘k’ has to be evaluated from the given boundary conditions.

At $X = b$, $y = h$
Put this in (1) , $y = kX^n$
we get $h = kb^n$
or $k = \frac{h}{b^n}$ Put this in equation for A above.

$$A = \frac{h\,b^{n+1}}{b^n\,(n+1)}$$ Simplifying

$$= \frac{h\,b^n \,.\, b}{b^n\,(n+1)}$$

So $$\boxed{A = \frac{bh}{(n+1)}} \qquad (2)$$

Now its centroid would be determined with reference to fig. 2.6..

$$\bar{X} = \int \frac{X\,d\,A}{A}$$

$$= \int \frac{X\,(ydX)}{A}$$ Put $y = kX^n$

$$= \int \frac{X\,kX^n\,dX}{A}$$

$$= \int_o^b \frac{k\,X^{n+1}\,dX}{A}$$ Now put $k = \frac{h}{b^n}$ and $A = \frac{bh}{(n+1)}$ we have

$$= \int_o^b \frac{h/b^n\,(X)^{n+1}\,dX}{bh/(n+1)}$$

$$= \int_o^b \frac{h\,(X^{n+1})\,dX(n+1)}{hb^{n+1}}$$ simplifying step by step

$$= \frac{(n+1)}{b^{\,n+1}} \int_o^b X^{n+1}dX$$

$$= \frac{(n+1)}{b^{\,n+1}} \left| \frac{X^{n+2}}{(n+2)} \right|_o^b$$

$$= \frac{(n+1)}{b^{\,n+1}}\;\frac{b^{n+2}}{(n+2)}$$

$$= \frac{(n+1)}{b^{\,(n+1)}} \,.\, \frac{b^{n+1}\,.\,b}{(n+2)}$$

$$\boxed{\bar{X} = \frac{b\,(n+1)}{(n+2)}} \qquad (3)$$

$\bar{X}$ is the location of centroid from zero bending moment

From above figure 2.6, we have

$$\bar{X} + X^{/} = b$$

$$\therefore \quad X^{/} = b - \bar{X}$$

$$= b - \frac{b\,(n+1)}{(n+2)}$$ Simplify

$$= \frac{b\,(n + 2) - b\,(n + 1)}{(n + 2)}$$

$$= \frac{bn + 2b - bn - b}{(n + 2)}$$

$$\boxed{X^{/} = \frac{b}{(n+2)}} \qquad (4)$$

This gives us the location of centroid from the ordinate of B.M.D

$$\boxed{A= \frac{bh}{(n+1)}} \qquad (2)$$

Note:- While applying these two formulae to calculate the deflection and the rotation by moment area method and with diagrams by parts, it must be kept in mind that these two relationship assume zero slope of the B.M. Diagram at a suitable point. It may not be applied to calculate A & $\bar{X}$ within various segments of the B.M.D where this condition is not satisfied. Apply the above equations for area and centroid to the following example.

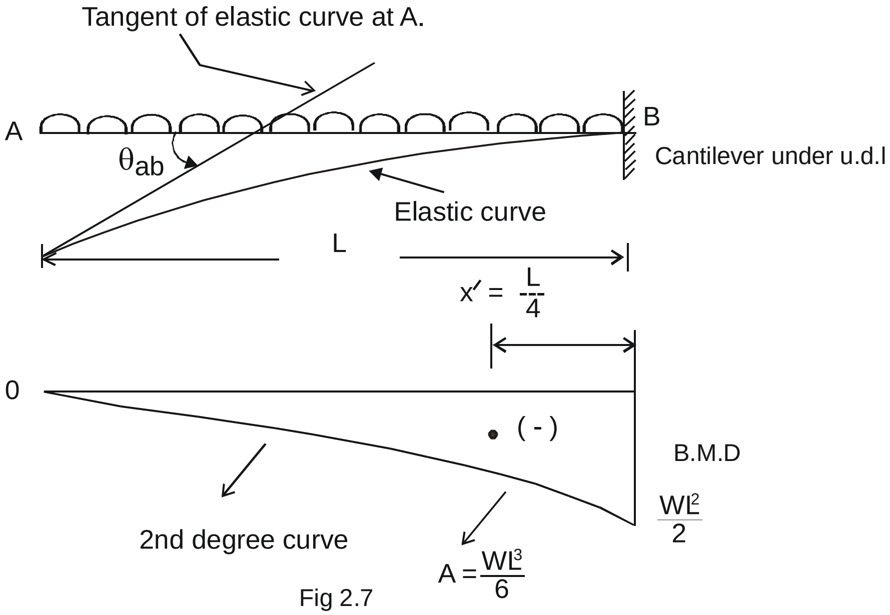


Fig 2.7

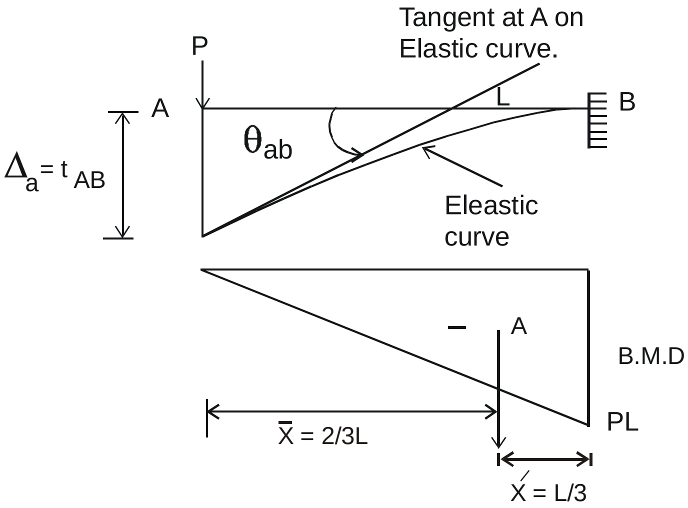


Fig. 2.8

(–ve) sign in the deflection of diagram below does not mean that area is (–ve) but ordinate of BMD is (–ve). For loads the fig. 2.7.

$$\Delta a = \frac{1}{EI}\left(A \times \frac{3L}{4}\right)$$

$$= \frac{1}{EI}\left[\frac{-WL^3}{6} \times \frac{3L}{4}\right]$$

$$= \frac{-WL^4}{8EI}$$

1.7. FIRST THEOREM OF CONJUGATE BEAM METHOD :–

In simple words the absolute slope at any point in the actual beam is equal to the shear force at the corresponding point on the conjugate beam which is loaded by $\frac{M}{EI}$ diagram due to loads on actual beam.

1.7.1. SECOND THEOREM OF CONJUGATE BEAM METHOD :-

The absolute deflection at any point in the actual beam is equal to the B.M at the corresponding point on the conjugate beam which is loaded by $\frac{M}{EI}$ diagram.

The reader is reminded to draw conjugate beams for actual beams under loads very carefully by giving due consideration to support conditions of actual beam. In general for a fixed and free end of actual beam, the corresponding supports would be free and fixed in conjugate beam respectively. Deflection Δ at any point on actual beam is associated with the bending moment at corresponding point on conjugate beam while rotation θ at any point on actual beam is associated with shear force at corresponding point on conjugate beam. At an actual hinge support Δ is equal to zero and θ is there indicating non development of moment at the support (Shear force present, bending moment zero). The corresponding support conditions in conjugate beam would be such where bending moment is zero and shear force may be there i.e., a hinge is indicated. See the following example.

EXAMPLE :- Calculate the central deflection by the conjugate beam method:

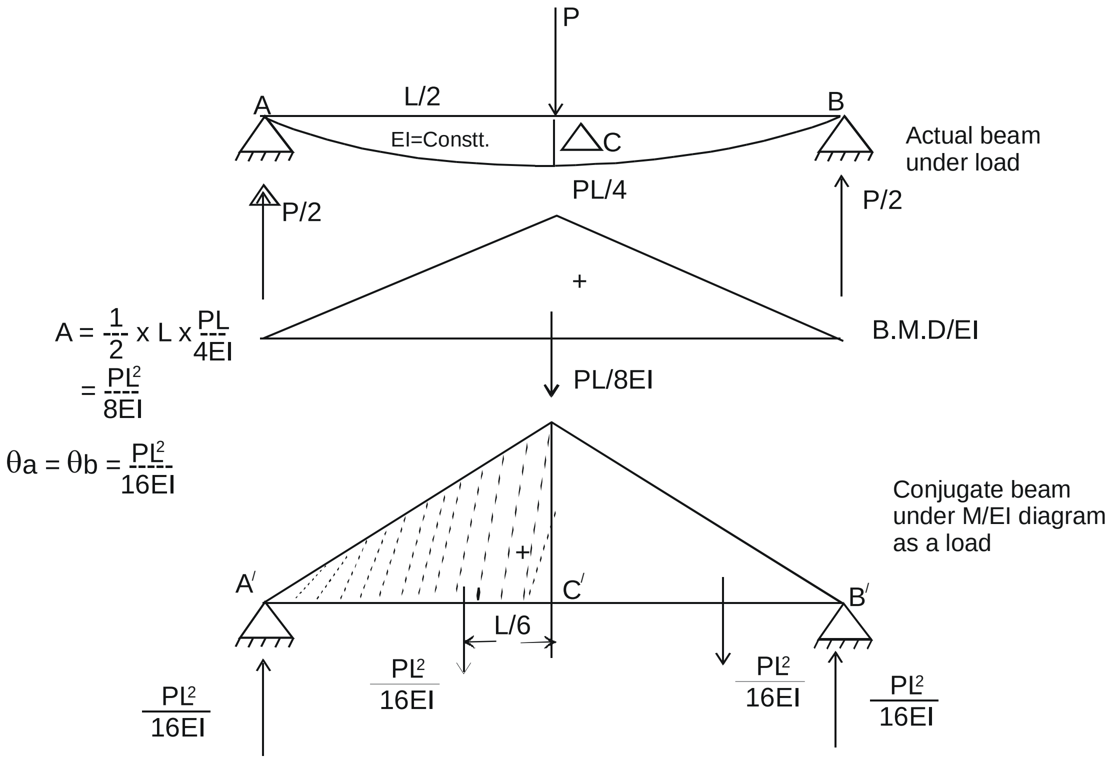


Fig. 2.9

$$\Delta C = Mc' = \frac{PL^2}{16EI} \times \frac{L}{2} - \frac{PL^2}{16EI} \times \frac{L}{6}$$

(considering forces on LHS of

$$= \frac{PL^3}{32EI} - \frac{PL^3}{96EI} = \frac{3PL^3 - PL^3}{96EI} = \frac{2EPL^3}{96EI}$$

point C of shaded area)

$$\Delta C = \frac{PL^3}{48EI}$$

1.8. STRAIN ENERGY :-

"The energy stored in a body when it undergoes any type of deformation (twisting, elongation, shortening & deflection etc.) under the action of any external force is called the strain energy." If this strain energy is stored in elastic range it is termed as elastic strain energy. All rules relating to strain energy apply. The units of strain energy are the same as that of the work i.e., joule (N – mm, N – m).

1.8.1. TYPES OF STRAIN ENERGY :-

1.8.1.1 STRAIN ENERGY DUE TO DIRECT FORCE :-

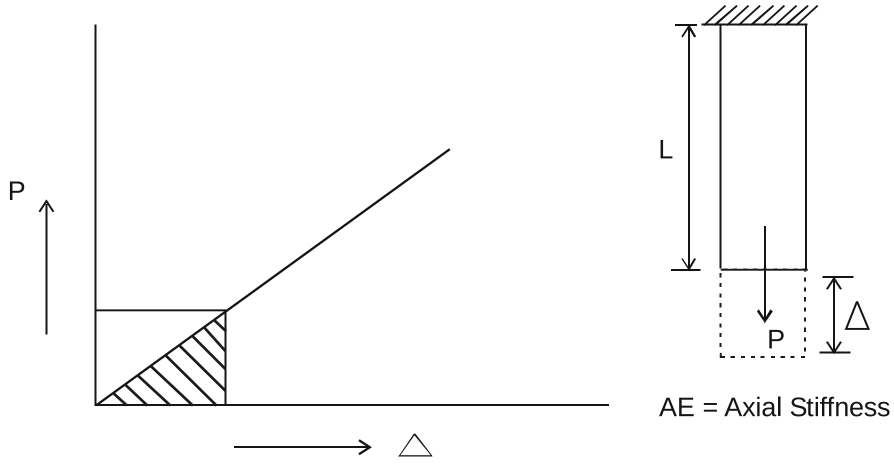


Fig. 2.10

Work done by a gradually increased force 'P' is equal to area of load – deflection diagram = P/2 Δ. (From graph)

$\ldots$ Stress $\propto$ Strain (Hooke's Law)

So $f \propto \in$

$f = \text{Constt} \,.\, \in$

$f = E \,.\, \in$

$$\frac{P}{A} = E \times \frac{\Delta}{L}$$

so $\Delta = \frac{PL}{AE}$ Strain energy will be $\frac{1}{2}$ PΔ from above. So putting it we have.

$\Rightarrow$ $U = \frac{P}{2}\left(\frac{PL}{AE}\right)$, where U is the internal strain energy stored.

$U = \frac{P^2L}{2AE}$ (for single member)

$U = \Sigma \frac{P^2L}{2AE}$ (for several members subjected to axial forces)

1.8.1.2. STRAIN ENERGY DUE TO BENDING, SHEAR FORCE AND TORSION :-

(1) $U = \int_O^L \frac{M^2 \, dX}{2 \, EI}$. This is elastic strain energy stored due to bending.

(2) Strain Energy Due to shear force:- $U = \int_O^L \frac{Q^2 \, ds}{2AG}$ where Q is shear force and G is shear modulus

(3) Strain Energy Due to Torsion:- $U = \int_{O}^{L} \frac{T^2\,ds}{2GJ}$ (Consult a book on strength of Materials). Where T is Torque and J is polar moment of inertia.

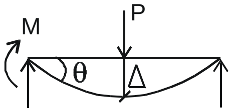
1.9. CASTIGLIANO'S THEOREM :-

In 1879, Castigliano published two theorems connecting the strain energy with the deformations and the applied loads.

1.9.1 CASTIGLIANO'S FIRST THEOREM :-

The partial derivative of the total strain energy stored with respect to a particular deformation gives the corresponding force acting at that point.

Mathematically



$$\frac{\partial U}{\partial \Delta} = P$$ Where U is strain energy stored in bending

and $$\frac{\partial U}{\partial \theta} = M\,.$$ Here Δ is connected with loads and θ with moment.

1.9.2. CASTIGLIANO'S SECOND THEOREM :-

The partial derivative of the total strain energy stored with respect to a particular force gives the corresponding deformation at that point.

Mathematically,

$$\frac{\partial U}{\partial P} = \Delta$$

and $$\frac{\partial U}{\partial M} = \theta$$ Here Δ is connected with loads and θ with moment.

1.10. CONSISTENT DEFORMATION METHOD :-

This method may be termed as redundant force method or simply a force method. In this method, the statically indeterminate structure is idealized as a basic determinate structure under the action of applied loads plus the same structure under the action of redundant forces considered one by one. The deformations produced at the points of redundancy are calculated in the above-mentioned basic determinate structures and then these calculated deformations are put into compatibility requirement for the structure. Normally these are satisfied at a joint.

Now for a given beam, various possible Basic determinate structures (BDS) would be given. A clever choice of BDS for a given structure can reduce the amount of time and labour.

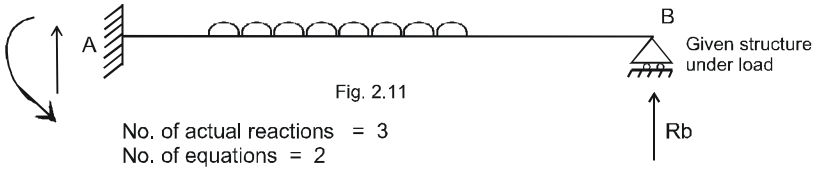


Fig. 2.11

No. of actual reactions = 3
No. of equations = 2
degree of indeterminacy = 3 - 2 = 1

1. First alternative

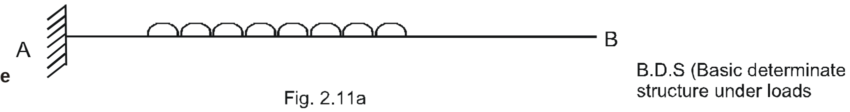


Fig. 2.11a

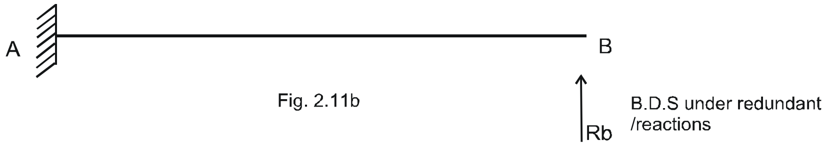


Fig. 2.11b

2. second alternative

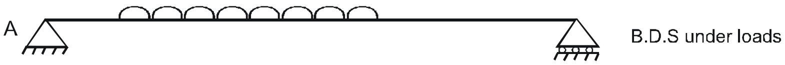


Fig. 2.12

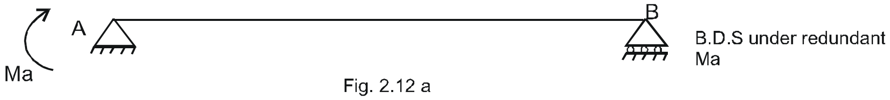


Fig. 2.12 a

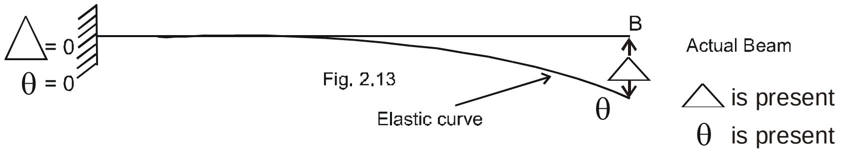


Fig. 2.13

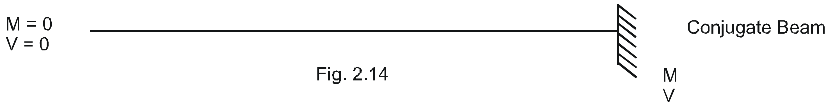


Fig. 2.14

An indeterminate structure can be made determinate in several ways and the corresponding quantities may be calculated very easily. However, we will notice that a clever choice of making a basic determinate structure will reduce the time of our computations tremendously. In Figs. 2.11 and 2.12 various options regarding choice of BDS are given while Figs. 2.13 and 2.14 illustrate how to make conjugate beam for a given beam using the guidelines stated earlier. Consider another loaded beam in Fig. 2.15.

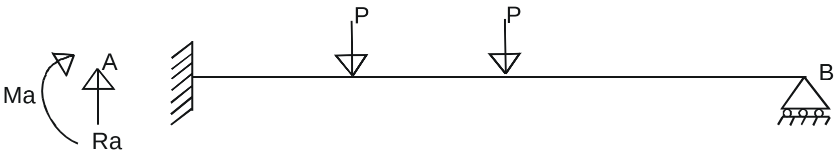
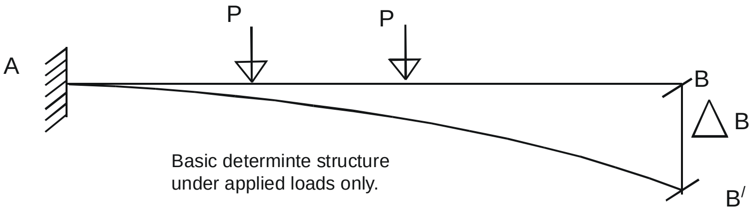


Fig. 2.15

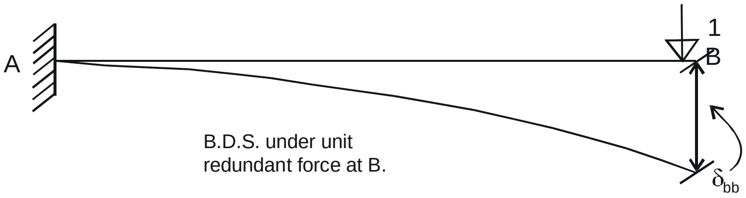


Basic determinte structure
under applied loads only.

Rb has been chosen
as redundant.

Fig. 2.15 a

where ΔB is the deflection at point B due to the applied loads.



B.D.S. under unit
redundant force at B.

Fig. 2.15 b

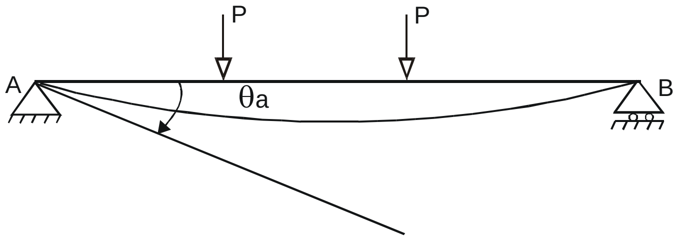
So compatibility of deformation at B requires that

$\Delta B + Rb \times \delta bb = 0$ (Deflection Produced by loads Plus that by redundant should be equal to zero at point B)

where ΔB = Deflection at B due to applied loads in a BDS.

δbb = deflection at B due to redundant at B in a BDS.

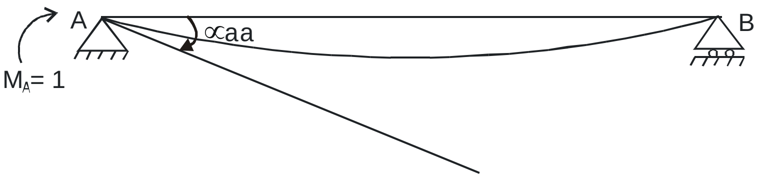
or $Rb = -\dfrac{\Delta B}{\delta bb}$ (sign is self-adjusting)



Ma has been
considered as
redundant force.

Fig. 2.16

θa = Slope at point. A due to applied loads only in a BDS.
The other option of a simple beam as BDS is shown in fig. 2.16.



B.D.S. under unit redundant moment at A.
where ∝aa = slope at A due to unit redundant moment at A.

Fig 2.16a

Compatibility equation $\theta a + Ma . \propto aa = 0$ (Slope created by loads + slope created by redundant moment should be zero)

or $$Ma = -\frac{\theta a}{\propto aa}$$

"In consistent deformation method (force method), there are always as many conditions of geometry as is the number of redundant forces."

1.11. Example No. 1:- Analyze the following beam by the force method. Draw S.F. & B.M. diagrams.

SOLUTION :-

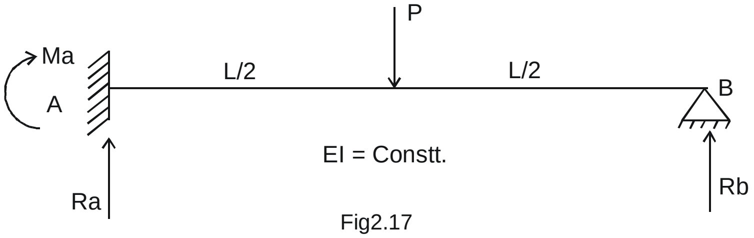


Fig2.17

Number of reactions = 3
Number of equations = 2
Degree of Indeterminacy = 3 – 2 = 1
Indeterminate to Ist degree.

SOLUTION: (1) Chose cantilever as a basic determinate structure.

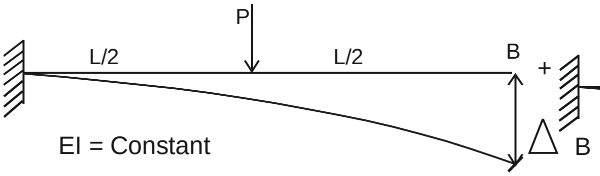


Fig 2.17a

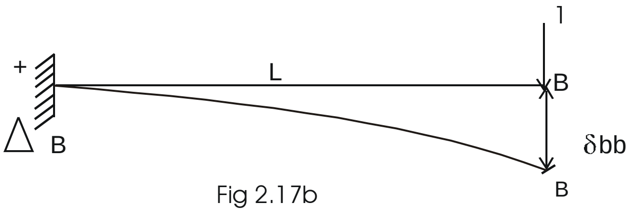


Fig 2.17b

B.D.S. under applied loads.

δbb=Deflection of point B due to unit load at B
B.D.S. under unit redundant force at B.

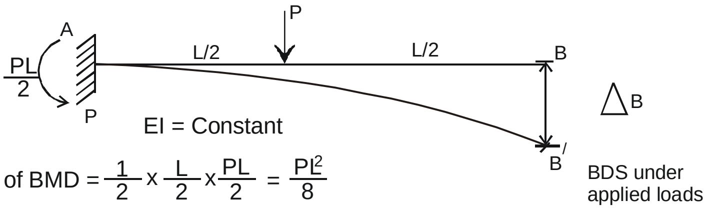
Therefore, now compatibility requirement is

$\Delta B + Rb \times \delta bb = 0$ (Deflection created by actual loads + deflection created by redundant Rb should be equal to zero at support B)

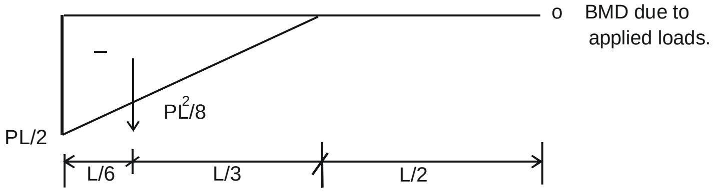
or $$Rb = -\frac{\Delta B}{\delta bb} \quad \rightarrow (1)$$

Therefore, determine these deflections ΔB and δbb in equation (1) either by moment area method or by unit load method.

1.11.1. DETERMINE ΔB AND δbb BY MOMENT - AREA METHOD :-



Area of BMD = $\frac{1}{2} \times \frac{L}{2} \times \frac{PL}{2} = \frac{PL^2}{8}$



$$\Delta B = \frac{I}{EI}\left[-\frac{PL^2}{8}\left(\frac{L}{2}+\frac{L}{3}\right)\right]$$
$$= \frac{I}{EI}\left[-\frac{PL^2}{8}\times\frac{5L}{6}\right]$$
$$\Delta B = -\frac{5PL^3}{48EI}$$

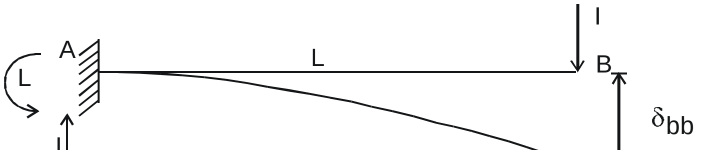
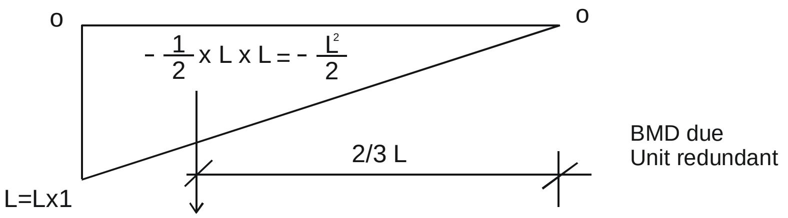


Fig 2.18 a



$$\delta bb = \frac{I}{EI}\left[-\frac{L^2}{2}\times\frac{2L}{3}\right]$$

$\delta bb = -\dfrac{L^3}{3EI}$, Putting ΔB and δbb in equation (1)

$Rb = -\left[-\dfrac{5PL^2}{48EI} \Big/ -\dfrac{L^3}{3EI}\right]$ By putting ΔB and δbb in compatibility equation

$$= -\frac{5PL^3}{48EI} \times \frac{3EI}{L^3} \qquad = -\frac{5P}{16}$$

The (– ve) sign with Rb indicates that the direction of application of redundant force is actually upwards and the magnitude of redundant force Rb is equal to $\dfrac{5P}{16}$. Apply evaluated redundant at point B.

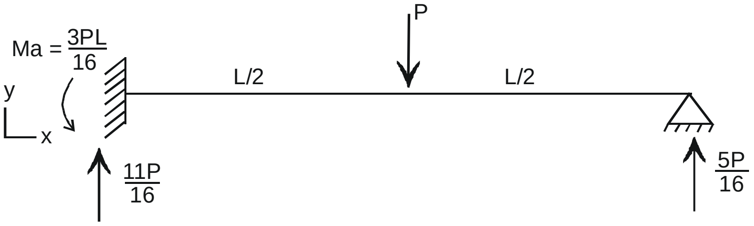


Fig. 2.19

$\sum fy = 0$

$Ra + Rb = P$

$Ra = P - Rb = P - \dfrac{5P}{16} = \dfrac{11P}{16}$. Now moment at A can be calculated.

Direction of applied moment at A $= \dfrac{5P}{16} \times L - P \cdot \dfrac{L}{2} = \dfrac{5PL}{16} - \dfrac{PL}{2}$

$$= \frac{5PL - 8PL}{16}$$

$$= -\frac{3\,PL}{16}$$

The (–ve) sign with $\dfrac{3\,PL}{16}$ indicates that the net applied moment about ‘A’ is clockwise. Therefore, the reactive moment at the support should be counterclockwise (giving tension at top). Apply loads and evaluate redundant on the given structure.

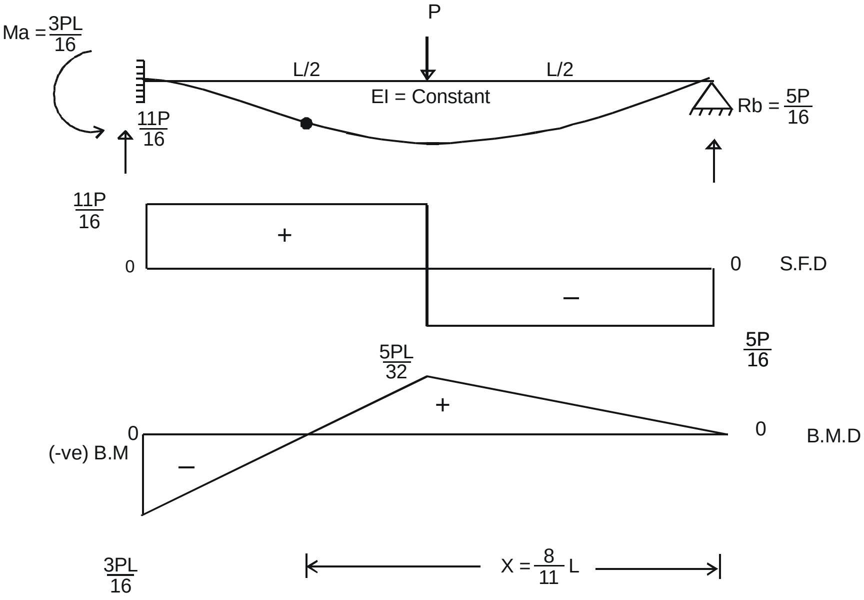


Fig. 2.20

LOCATION OF POINT OF CONTRAFLEXURE :-

$$MX = \frac{5\,PX}{16} - P\left(X - \frac{L}{2}\right) = 0$$

$$= \frac{5\,PX}{16} - PX + \frac{PL}{2} = 0$$

$$= -\frac{11PX}{16} + \frac{PL}{2} = 0$$

$$= \frac{PL}{2} = \frac{11PX}{16}$$

$$\boxed{X = \frac{8L}{11}}$$

***Note*:-** In case of cantilever, moment – area method is always preferred because slope is absolute everywhere.

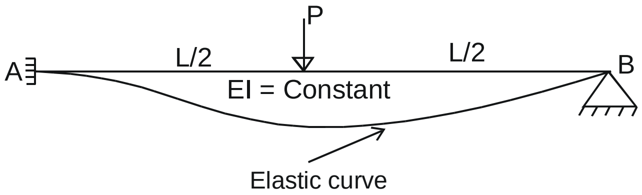


Fig. 2.21

Solution: (2) As a second alternative, Chose Simply Supported Beam as a basic determinate structure.

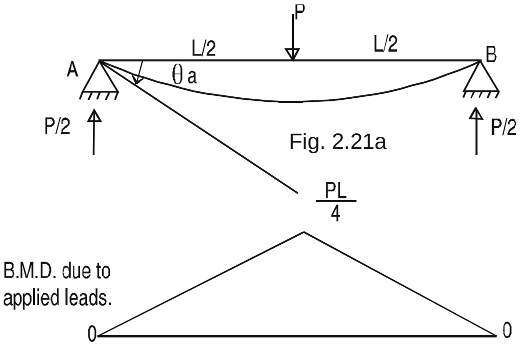


Fig. 2.21a

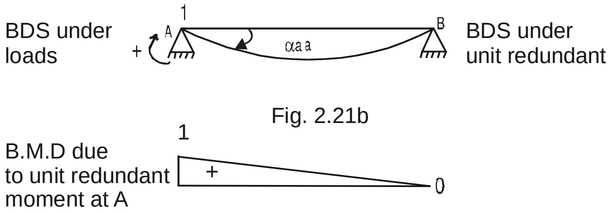


Fig. 2.21b

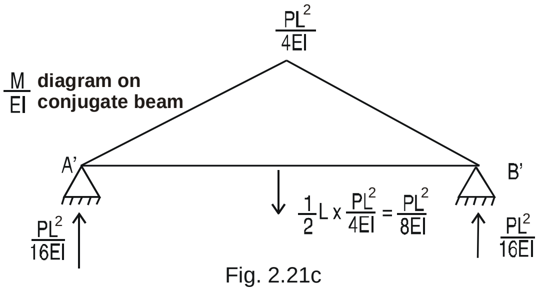


Fig. 2.21c

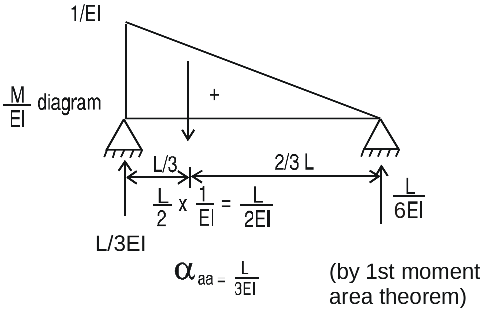


Fig 2.21d

$$\propto aa = \frac{L}{3EI}$$

$$\theta\ a = \frac{PL^2}{16EI} \quad \text{(by 1}^{st}\text{ moment area theorem)}$$

For fixed end, there is no rotation. Therefore compatibility equation becomes

$\theta\ a + Ma \propto aa = 0$ (slope at A created by loads + slope at A created by redundant should be zero).

So $Ma = -\dfrac{\theta\ a}{\propto aa}$

θ a & ∝ aa are the flexibility co–efficients. Putting these in compatibility equation

we have, $$Ma = -\frac{PL^2}{16EI} \times \frac{3EI}{L}$$

$$Ma = -\frac{3PL}{16}$$

The (–ve) sign with Ma indicates that the net redundant moment is in opposite direction to that assumed. Once M_a is known, R_a and R_b can be calculated.

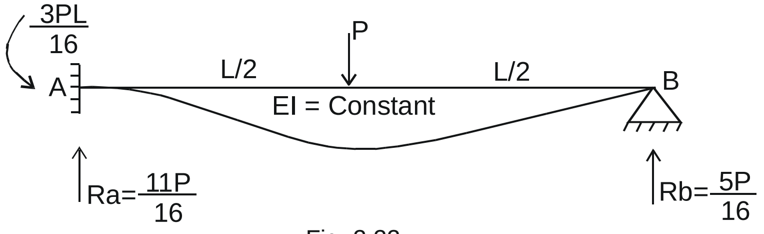


Fig. 2.22

To calculate R_b, $\sum M_a = 0$

$$R_b \times L - P \times \frac{L}{2} + \frac{3PL}{16} = 0$$

$$R_b \times L = \frac{PL}{2} - \frac{3PL}{16}$$

$$= \frac{8\,PL - 3\,PL}{16}$$

$$R_b \times L = \frac{5PL}{16}$$

$$\boxed{R_b = \frac{5P}{16}}$$

$\sum fy = 0$

$R_a + R_b = P$ so $R_a = P - R_b$

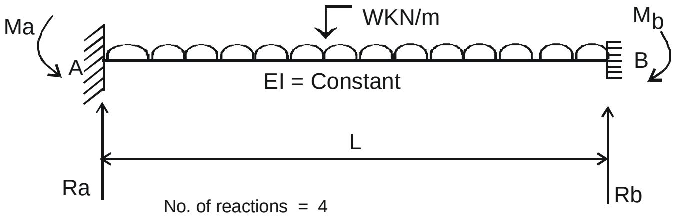
$$= P - \frac{5P}{16}$$

$$\boxed{R_a = \frac{11P}{16}}$$

***Note*:-** In case of simply supported beam, conjugate beam method is preferred for calculating slopes and deflections.

1.12. Example No. 2:- Analyze the following beam by the force method. Draw S.F. and B.M. diagrams.

SOLUTION :-



No. of reactions = 4
No. of equations = 2
Degree of Indeteminacy = 4 - 2 = 2
Indeterminate to 2nd degree.

Fig. 2.23

Choosing cantilever with support at A as BDS. Vertical reaction at B and moment at B will be redundants. To develop compatibility equations at B regarding translation and rotation at B, we imagine the BDS under applied loads and then under various redundants separately.

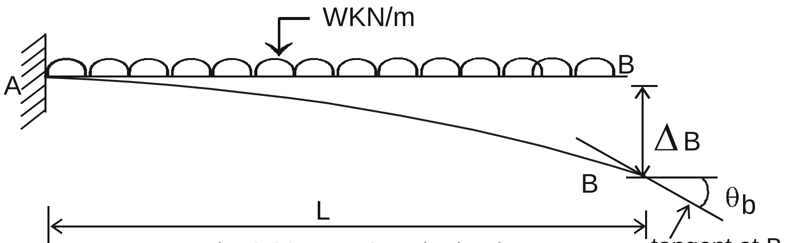


Fig. 2.23a B.D.S under loads

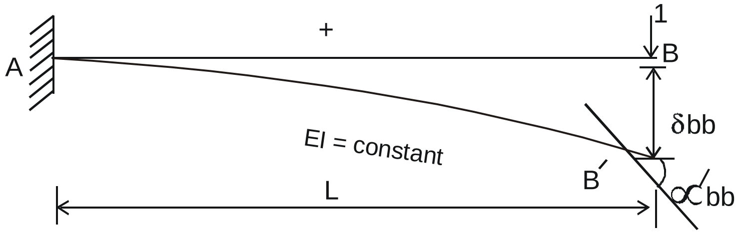


Fig. 2.23b B.D.S. under redundant unit vertical force at B

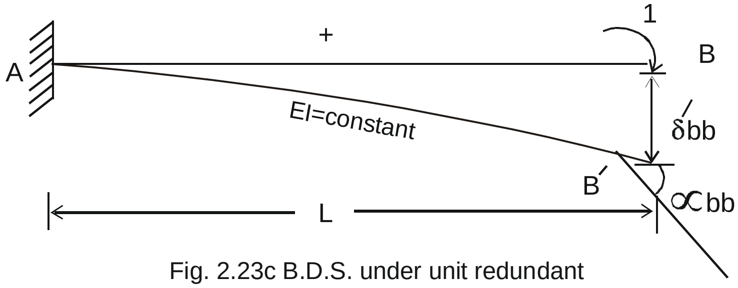


Fig. 2.23c B.D.S. under unit redundant moment at B

Compatibility Equations

$\Delta B + Vb \times \delta bb + Mb \times \delta' bb = 0$ → (1) For vertical displacement at B

$\theta B + Vb \times \propto' bb + Mb \times \propto bb = 0$ → (2) For redundant moment at B

Notice that rotation produced by Unit load at B (α'bb) and deflection produced by unit moment of B (δ'bb) are denoted by dash as superscript to identify them appropriately.

In matrix form

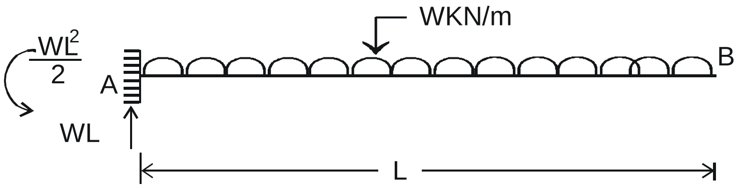
$$\underset{\uparrow}{\begin{bmatrix} \delta bb & \delta' bb \\ \propto' bb & \propto bb \end{bmatrix}} \underset{\uparrow}{\begin{bmatrix} Vb \\ Mb \end{bmatrix}} = \underset{\uparrow}{\begin{bmatrix} -\Delta B \\ -\theta B \end{bmatrix}}$$

Structure flexibility matrix. Column vector of redundants. Column vector of flexibility coefficients.

$$\begin{bmatrix} Vb \\ Mb \end{bmatrix} = \begin{bmatrix} \delta bb & \delta' bb \\ \propto' bb & \propto bb \end{bmatrix} \begin{bmatrix} -\Delta B \\ -\theta B \end{bmatrix}$$

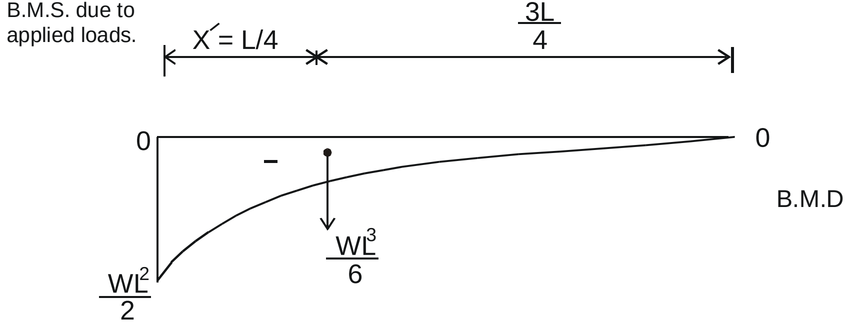
Now we evaluate ΔB, θb, δbb, $\alpha' bb$, $\delta' bb$ and $\propto bb$ with the help of moment area theorems separately, where Δ = Deflection at B in BDS due to applied loads

θb = Rotation at B in BDS due to applied loads.



B.D.S. under loads

Fig. 2.24a



Calculate area of BMD and fix its centroid

$$A = \frac{bh}{(n+1)} = \frac{L \times (-WL^2)}{(2+1)} = -\frac{WL^3}{6}$$

b = width of BMD.

h = ordinate of BMD.

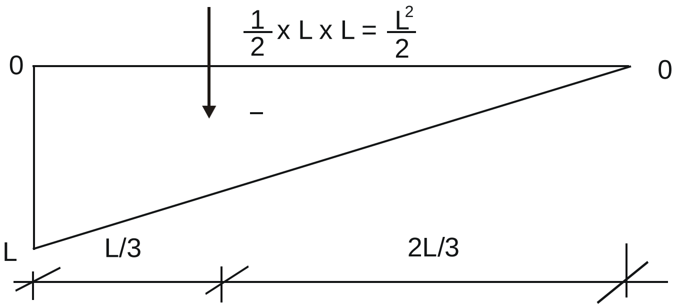
$$X' = \frac{b}{n+2} = \frac{L}{(2+2)} = \frac{L}{4}$$

By applying second theorem of moment area, we have

$$\Delta B = \frac{1}{EI}\left[-\frac{WL^3}{6} \times \frac{3}{4} L\right] = -\frac{WL^4}{8EI}$$

$$\theta b = \frac{1}{EI}\left[-\frac{WL^3}{6}\right] = -\frac{WL^3}{6EI}$$

Fig. 2.24b B.M.D. due to unit redundant force at B



B.D.S. under unit redundant force at B.

$$\delta bb = \frac{1}{EI}\left[-\frac{L^2}{2}\times\frac{2}{3}L\right] = -\frac{L^3}{3EI}$$; δbb = Deflection at B due to unit redundant at B

$$\alpha' bb = \frac{1}{EI}\left[-\frac{L^2}{2}\right] = -\frac{L^2}{2EI}$$; α′bb = Rotation at B due to unit redundant at B

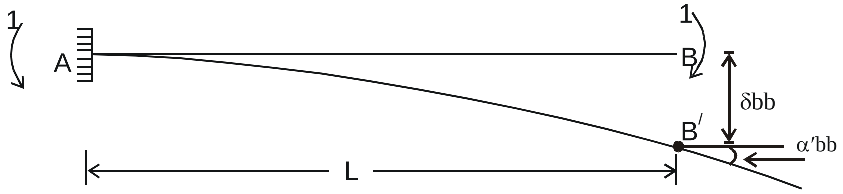
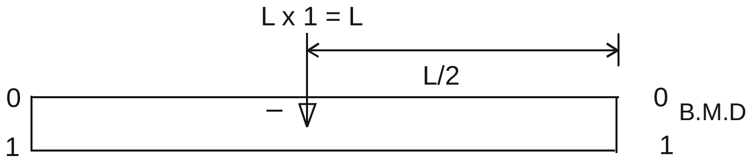


Fig. 2.24c B.D.S under unit redundant moment at B



$$\delta' bb = \frac{1}{EI}\left[-L\times\frac{L}{2}\right] = -\frac{L^2}{2EI}$$

$$\propto bb = \frac{1}{EI}\left[-L\right] = -\frac{L}{EI}$$

Normally BMD’s are plotted on the compression side of beam.

Putting values in first equation, we have

$$-\frac{WL^4}{8EI} - Vb \times \frac{L^3}{3EI} - \frac{L^2}{2EI} Mb = 0 \qquad (1)$$

multiply by 24 and simplify to get equation (3)

Putting values in second equation, we have (2)

multiply by 6 and simplify to get equation (4)

$$-\frac{WL^3}{6EI} - \frac{Vb \text{ x } L^2}{2EI} - \frac{L \text{ x } Mb}{EI} = 0$$

$$-3\,WL^4 - 8\,L^3 \times Vb - 12\,L^2 \times Mb = 0 \qquad (3)$$

or $$3\,WL^4 + 8\,L^3\,Vb + 12\,L^2\,Mb = 0 \qquad (3)$$

$$-WL^3 - 3\,L^2\,Vb - 6\,L\,Mb = 0 \qquad (4)$$

or $$WL^3 + 3\,L^2\,Vb + 6\,L\,Mb = 0 \qquad (4)$$

Multiply (4) by 2 L & subtract (4) from (3)

$$3\,WL^4 + 8\,L^3\,Vb + 12\,L^2\,Mb = 0 \qquad (3)$$

$$2\,WL^4 + 6\,L^3\,Vb + 12\,L^2\,Mb = 0 \qquad (4)$$

$$WL^4 + 2\,L^3\,Vb = 0$$

$$WL^4 = -2\,L^3\,Vb$$

$$Vb = -\frac{WL^4}{2L^3}$$

$$\boxed{Vb = -\frac{WL}{2}}$$

The (–ve) sign with Vb shows that the unit redundant load at B is in upward direction.(Opposite to that assumed and applied)

Putting the value of Vb in (3)

$$3\,WL^4 + 8\,L^3\left(-\frac{WL}{2}\right) + 12\,L^2\,Mb = 0$$

or $$3\,WL^4 - 4\,WL^4 + 12\,L^2\,Mb = 0$$

$$WL^4 = 12\,L^2\,Mb$$

$$Mb = \frac{WL^4}{12L^2}$$

$$\boxed{Mb = \frac{WL^2}{12}}$$

The (+ve) sign with Mb indicates that the assumed direction of the unit redundant moment at B is correct. Now apply the computed redundants at B and evaluate and apply reactions at A.

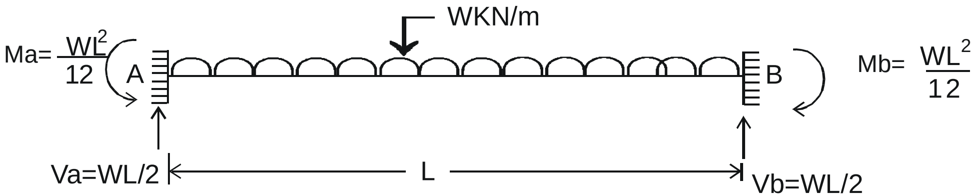
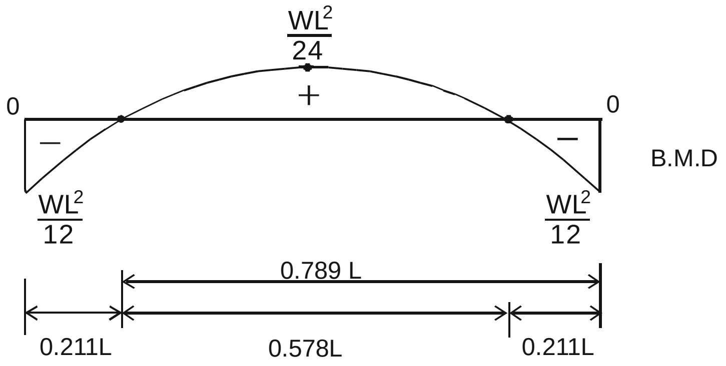


Fig. 2.25



Points of Contraflexure : -

B as origin :- write moment expression

$$Mx = \frac{WL}{2}X - \frac{WL^2}{12} - \frac{WX^2}{2} = 0$$

Multiply by $\frac{12}{W}$ and re-arrange.

$$6X^2 - 6LX + L^2 = 0$$

$$X = + \frac{6L \pm \sqrt{36L^2 - 4 \times 6 \times L^2}}{2 \times 6}$$

$$= \frac{6L \pm \sqrt{36L^2 - 24L^2}}{12}$$

$$= \frac{6L \pm \sqrt{12L^2}}{12}$$

$$= \frac{6L \pm 2\sqrt{3L^2}}{12}$$

$$= \frac{6L \pm 3.464L}{12}$$

$$= \frac{9.464\ L}{12} \quad , \quad \frac{2.536\ L}{12}$$

$$X = 0.789\ L \quad , \quad 0.211\ L$$ Location of point of contraflexure From both ends.

$$\boxed{X = 0.211\ L}$$

Same can be done by taking A as origin and writing moment expression : –

$$Mx' = \frac{WLX'}{2} - \frac{WL^2}{12} - \frac{WX'^2}{2} = 0$$

$$6\ WLX' - WL^2 - 6\ WX'^2 = 0$$ Simplify

$$LX' - \frac{L^2}{6} - X'^2 = 0$$

$$X'^2 - LX' + \frac{L^2}{6} = 0$$

$$X' = \frac{L \pm \sqrt{L^2 - 4 \times 1 \times \frac{L^2}{6}}}{2 \times 1}$$

$$= \frac{L \pm \sqrt{L^2 - \frac{2\ L^2}{3}}}{2}$$

$$= \frac{L \pm \sqrt{\frac{L^2}{3}}}{2}$$

$$= \frac{L \pm \sqrt{\frac{1}{3} . L^2}}{2}$$

$$X' = \frac{L \pm 0.577\ L}{2}$$

$$X' = 0.789\ L , \quad 0.211\ L$$ Location of points of contraflexure.

$$\boxed{X' = 0.211\ L}$$

We get the same answer as before.

This is a flexibility method and was written in matrix form earlier. The matrix inversion process is given now for reference and use.

1.13. MATRIX INVERSION : -

These co-efficients may also be evaluated by matrix Inversion so basic procedures are given.

$$\text{Inverse of matrix} = \frac{\text{Adjoint of matrix}}{\text{Determinant of matrix}}$$

Adjoint a matrix = Transpose (Interchanging rows & columns) of matrix of co-factors.

Co-factors of an element = $(-1)^{i+j}$ × minor of element.where i = Row number in which that element is located and j = Column number in which that element is located.

Minor of element = Value obtained by deleting the row & the column in which that particular element is located and evaluating remaining determinant.

Let us assume a matrix :

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 5 & 9 \\ 8 & 10 & 11 \end{bmatrix}$$

$$\begin{aligned} \text{Determinant of matrix } A &= 1\,(5 \times 11 - 10 \times 9\,) - 3\,(44 - 72) + 7\,(\,4 \times 10 - 8 \times 5\,) \\ &= -35 + 84 + 0 \\ &= 47 \end{aligned}$$

MINORS OF MATRIX :-

Find out the minors for all the elements of the matrix. Then establish matrix of co-factors.

$$\text{Matrix of Minors} = \begin{bmatrix} -35 & -28 & 0 \\ -37 & -45 & -14 \\ -8 & -19 & -7 \end{bmatrix}$$

$$\text{Matrix of co-factors} = \begin{bmatrix} -35 & 28 & 0 \\ 37 & -45 & 14 \\ -8 & 19 & -7 \end{bmatrix}$$

$$\text{Adjoint of matrix A} = \begin{bmatrix} -35 & 37 & -8 \\ 28 & -45 & 19 \\ 0 & 14 & -7 \end{bmatrix}$$

$$\text{Inverse of matrix} = \frac{1}{49}\begin{bmatrix} -35 & 37 & -8 \\ 28 & -45 & 19 \\ 0 & 14 & -7 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -0.71 & 0.755 & -0.163 \\ 0.571 & -0.918 & 0.387 \\ 0 & 0.286 & -0.143 \end{bmatrix}$$

$$A \text{ x } A^{-1} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Check for correct matrix inversion}$$

$Aij \text{ x } Bjk = Cik$

$$A\,A^{-1} = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 5 & 9 \\ 8 & 10 & 11 \end{bmatrix} \begin{bmatrix} -0.71 & 0.755 & -0.163 \\ 0.571 & -0.918 & 0.387 \\ 0 & 0.286 & -0.143 \end{bmatrix}$$

$$= \begin{bmatrix} -1\times0.71+3\times0.571+7\times0 & 1\times0.755-3\times0.918+7\times0.286 & -1\times0.163+3\times0.387\ -7\times0.143 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Proved.}$$

1.14. 2^{ND} DEGREE INDETERMINACY :-

Example No. 3:

Solve the following continuous beam by consistent deformation method.

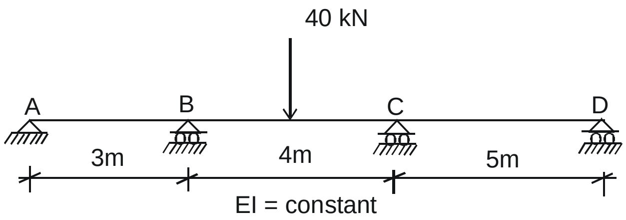


Fig. 2.26

In this case, we treat reaction at B and C as redundants and the basic determinate structure is a simply supported beam AD.

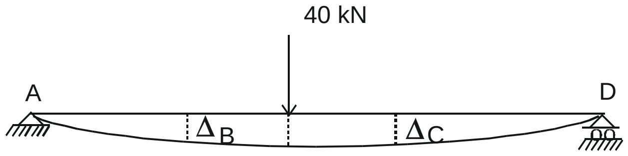


Fig. 2.26 a
Bending under applied loads

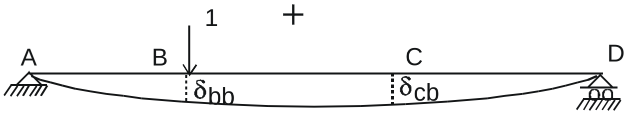


Fig. 2.26 b
Bending under unit redundant force at B

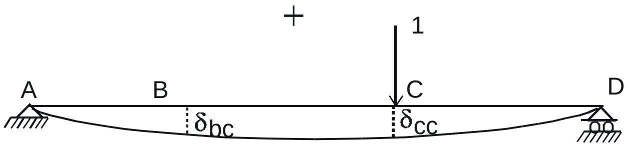


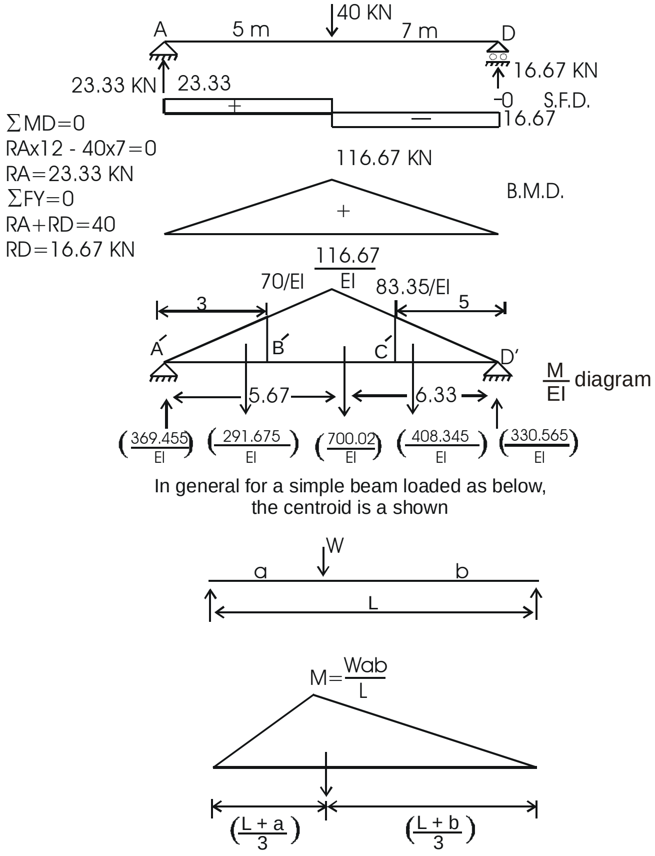
Fig. 2.26 c
Bending under unit redundant force at C

Compatibility equations are as follows:

$$\Delta B + \delta bb \times Rb + \delta bc \times Rc = 0 \rightarrow \text{(1) For compatibility at B}$$

$$\Delta C + \delta cb \times Rb + \delta cc \times Rc = 0 \rightarrow \text{(2) For compatibility at C}$$

Evaluate the flexibility co-efficients given in equation (1) and (2). Using Conjugate beam method.



In general for a simple beam loaded as below, the centroid is a shown

Fig. 2.27

$\Sigma MD' = 0$, Calculate $_{RA}$'

$$R_A' \times 12 = \frac{291.675}{EI}\left(7 + \frac{1}{3}5\right) + \frac{408.345}{EI}\left(\frac{2}{3} \text{ x } 7\right)$$

$$= \frac{2527.85}{EI} + \frac{1905.61}{EI}$$

$$R_A' = \frac{369.455}{EI}$$

$$\Sigma Fy = 0$$

$$R_A' + R_D' = \frac{369.455}{EI} + R_D' = \frac{700.02}{EI}$$

$$R_D' = \frac{700.02}{EI} - \frac{369.455}{EI}$$

$R_D' = \dfrac{330.565}{EI}$. Now ordinates of $\dfrac{M}{EI}$ diagram are determined by comparing Similar triangles.

$$\frac{116.67}{5\ EI} = \frac{Y}{3} \quad \Rightarrow Y = \frac{70}{EI}$$

Now by using conjugate beam method (theorem 2)

$$\Delta B = \frac{1}{EI}\left[369.455 \times 3 - \left(\frac{1}{2} \times 3 \times 70\right) \times \frac{3}{3}\right]$$

$$\Delta B = \frac{1003.365}{EI}\ KN - m^3$$

Determine

$$\frac{116.67}{7} = \frac{Y}{5}$$

$$Y = 83.34$$

$$\Delta C = \frac{1}{EI}\left[330.565 \times 5 - \left(\frac{1}{2} \times 5 \times 83.34\right) \times \frac{5}{3}\right]$$

$$\Delta C = \frac{1305.575}{EI}\ KN - m^3$$

Now apply unit redundant at B.

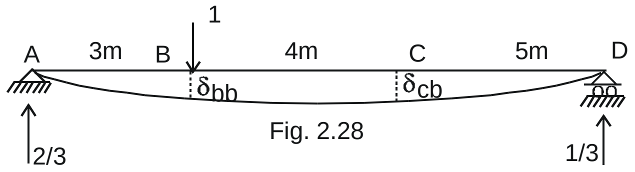
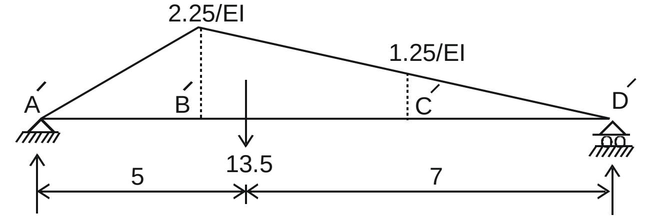


Fig. 2.28



Conjugate beam under M/EI

Computing Co-efficients by Conjugate beam method. (Theorem 2)

$$M_B' \;=\; \delta bb \;=\; \frac{1}{EI}\;[\;7.875 \times 3 \;-\; 3.375 \times 1\;]$$

$$\delta bb \;=\; \frac{20.25}{EI}\;\;KN - m^3$$

Determine ordinate $\frac{2.25}{9} = \frac{Y}{5}$

$$Y \;=\; \frac{1.25}{EI}$$

$$M_C' \;=\; \delta cb \;=\; \frac{1}{EI}\left[5.625 \times 5 - 3.125 \;\times \frac{5}{3}\right]$$

$$\delta cb \;=\; \frac{22.92}{EI}\;\;KN - m^3$$

Now apply unit redundant at C.

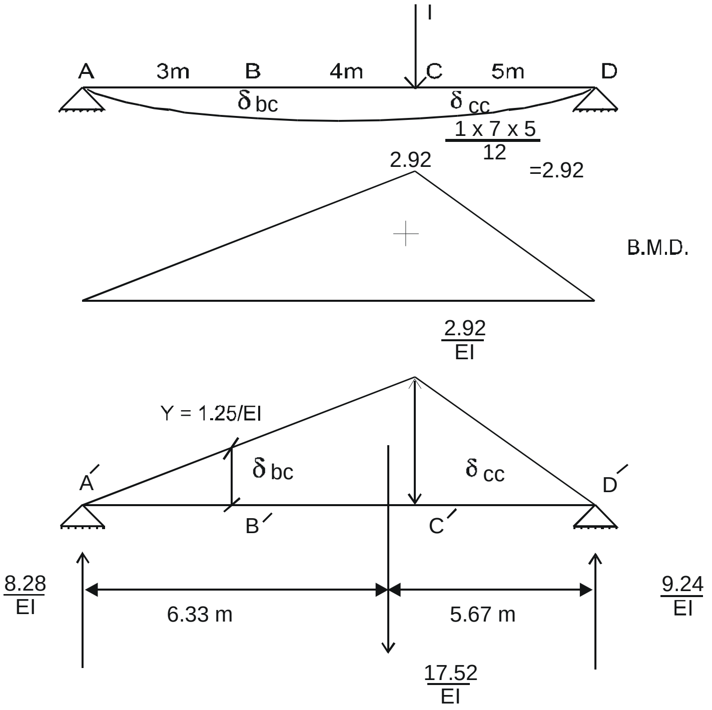


Fig. 2.29 Conjugate beam under M/EI

Moment at B’ in conjugate beam gives

$$M_B' \;=\; \delta bc \;=\; \frac{1}{EI}\left[8.28 \text{ x } 3 - \frac{1}{2} \text{ x } 1.25 \text{ x } 3 \text{ x } 1\right]$$

$$M_C' \;=\; \delta bc \;=\; \frac{22.965}{EI}\;\;KN - m^3 \qquad (\delta bc = \delta cb\;) \text{ PROVED.}$$

$$\delta cc \;=\; \frac{1}{EI}\left[9.24 \text{ x } 5 - \frac{1}{2} \text{ x } 2.92 \text{ x } 5 \text{ x } \frac{5}{3}\right]$$

$$\delta cc \;=\; \frac{34.03}{EI}\;\;KN - m^3.$$

Inserting evaluated Co-efficients in equation (1) and (2)

$$\frac{1003.365}{EI} + \frac{20.25}{EI} Rb + \frac{22.965}{EI} Rc = 0 \quad (1)$$

$$1003.365 + 20.25 \; Rb + 22.965 \; Rc = 0 \qquad \text{(3) Canceling 1/EI throughout}$$

$$\frac{1305.575}{EI} + \frac{22.92}{EI} Rb + \frac{34.03}{EI} Rc = 0 \quad \text{(4) Cancelling } \frac{1}{EI} \text{ throughout}$$

$$1305.575 + 22.92 \; Rb + 34.03 \, Rc = 0 \qquad (4)$$

Multiply (3) by 22.92 and (4) by 20.25 & subtract (4) from (3)

$$22997.1258 + 464.13 \; Rb + 526.357 \; Rc = 0 \qquad (3)$$

$$\underline{26437.8938 + 464.13 \; Rb + 689.1075 \, Rc = 0} \qquad (4)$$

$$- \; 3460.768 - 162.75 \; Rc = 0$$

$\boxed{Rc = -21.264 \text{ KN}}$ Putting this in equation (3)

$$1003.365 + 20.25 \quad Rb - 22.963 \times 21.264 = 0$$

$$\boxed{Rb = -25.434 \text{ KN}}$$

The (–ve) signs with the values of the redundants are suggestive of the fact that the directions of the actual redundants are in fact upwards. Now apply loads and evaluated redundants to original beam calculate remaining reaction.

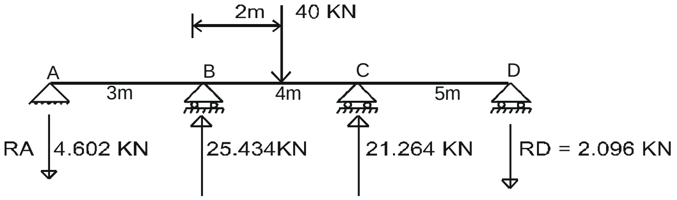


Fig. 2.30

$\sum Fy = 0$ Considering all upwards at this stage as Ra and Rd are unknown.

$$R_A + R_D + 25.434 + 21.264 - 40 = 0$$

$$R_A + R_D = -6.698 \qquad \rightarrow \quad (1)$$

$\sum MD = 0$ Considering all upward reactions

$$R_A \times 12 + 25.454 \times 9 - 40 \times 7 + 21.264 \times 5 = 0$$

$\boxed{R_A = -4.602 \text{ KN}}$. It actually acts downwards.

$$R_D = -R_A - 6.698$$

$$= 4.602 - 6.698$$

$\boxed{R_D = -2.096 \text{ KN}}$ All determined reactions are shown in figure 2.30 above sketch SFD and BMD.

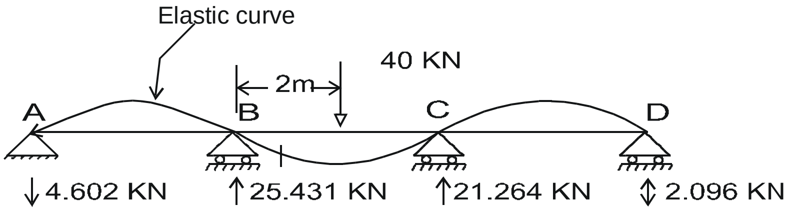
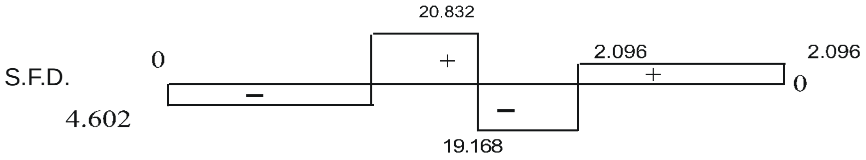
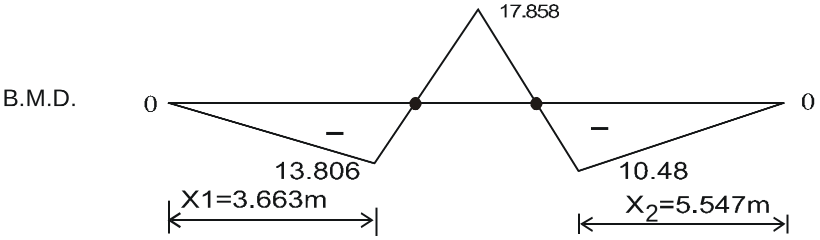


Fig. 2.31





LOCATION OF POINTS OF CONTRAFLEXURE :- These are in Span BC.

A as origin. Write moment expression and equate to zero.

$$MX_1 = -4.602\,X_1 + 25.434\,(X_1 - 3)$$

$$-4.602\,X_1 + 25.434\,X_1 - 76.302 = 0$$

$$X_1 = 3.663 \text{ m from A.}$$

D as origin. Write moment expression and equate to zero.

$$MX_2 = -2.096\,X_2 + 21.264\,(X_2 - 5) = 0$$

$$-2.096\,X_2 + 21.264\,X_2 - 106.32 = 0$$

$$19.168\,X_2 - 106.32 = 0$$

$$X_2 = 5.547 \text{ m.}$$

These locations are marked above in BMD.

1.15. 3RD DEGREE INDETERMINACY :-

Example No. 4:

Solve the frame shown below by consistent deformation method.

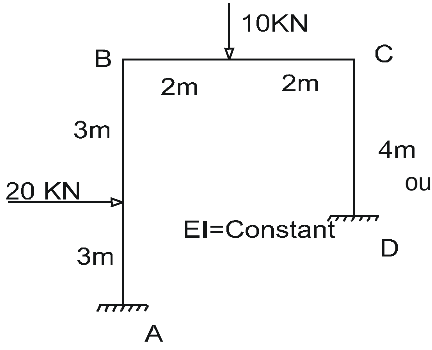
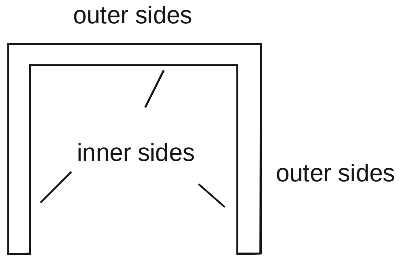


Fig. 2.32



B.M is +ve for
Tension on inner sides

1.15.1. SOLUTION:

Sign convention for S.F. and B.M. remains the same and are shown above as well. In this case, any force or moment which creates tension on the inner side of a frame would be considered as a (+ve) B.M. Removing right hand support to get BDS. The loads create three defermations as shown.

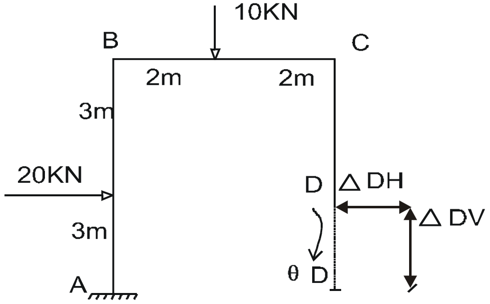


Fig. 2.33 (a) M - Diagram

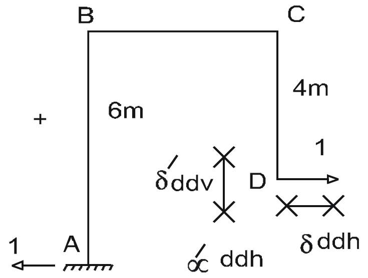


Fig. 2.33 (b) mH-Diagram

Note: ΔDH = Deflection of point D in horizontal direction due to applied loads on BDS.

ΔDV = Deflection of point D in vertical direction due to applied loads on BDS.

θ D = Rotation of point D due to applied loads on BDS.

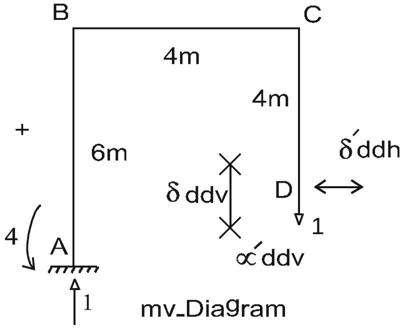


Fig. 2.33c B.D.S. under unit vertical redundant force at D

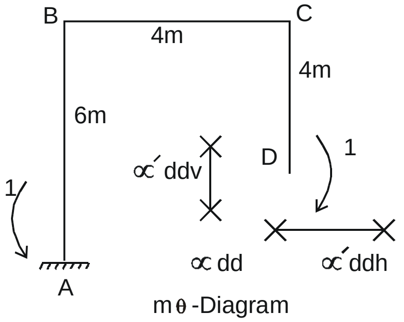


Fig. 2.33d B.D.S. under unit rotational redundant moment at D

Where (See mH diagram Fig. 2.33b)

δddh = Deflection of point D due to unit load at D in horizontal direction acting on BDS.

$\delta' ddv$ = Deflection of point D, in vertical direction due to unit load at D in horizontal direction.

$\alpha' ddh$= Rotation of point D, due to unit load in horizontal direction at D acting on BDS.

(See mV diagram Fig: 2.33c)

δddv = Deflection of point D due to unit load at D in vertical direction.

$\delta' ddh$ = Deflection of point D (in horizontal direction) due to unit vertical load at D.

$\alpha' ddv$ = Rotation of point D due to unit vertical load at D.

(See mθ diagram Fig: 2.33d))

$\alpha' ddh$ = Horizontal deflection of point D due to unit moment at D.

$\alpha' ddv$ = Vertical deflection of point D due to unit moment at D.

αdd = Rotation of point D due to unit moment at D.

Compatibility equations :-

$\Delta D_H + H_D \times \delta ddh + V_D \times \delta' ddv + M_D \times \propto' ddh = 0$ (1) Compatibility in horizontal direction at D.

$\Delta D_V + H_D \times \delta' ddh + V_D \times \delta dd_V + M_D \times \propto' dd_V = 0$ (2) Compatibility in vertical direction at D.

$\theta_D + H_D \times \propto' ddh + V_D \times \propto' ddv + M_D \times \propto dd = 0$ (3) Compatibility of rotation at D

Now evaluate flexibility co-efficients used in above three equations. We know that

$$\Delta \text{ or } \theta = \int \frac{1}{EI} \,(\, Mmdx \,)$$

There are 12 co-efficients to be evaluated in above three equations.

So $$\Delta D_H = \int \frac{M \times mH}{EI}\, dx \qquad (1)$$

$$\delta ddh = \int \frac{(mH)^2 \; dx}{EI} \qquad (2)$$

$$\delta' ddh = \int \frac{mH \;\, mv\, dx}{EI} \qquad (3)$$

$$\Delta Dv = \int \frac{M \times \,(mv\,)\;\, dx}{EI} \qquad (4)$$

$$\delta' ddv = \int \frac{(mH \times mv\,)\;\, dx}{EI} \qquad (5)$$

$$\delta\, ddv = \int \frac{(mv)^2\, dx}{EI} \qquad (6)$$

$$\propto' ddv = \int \frac{mv \times m\theta}{EI}\, dx \qquad (7)$$

$$\theta_D = \frac{1}{EI}\int (M)(m\,\theta)\ dx \qquad (8)$$

$$\propto' ddh = \frac{1}{EI}\int (mH)(m\,\theta)\ dx \qquad (9)$$

$$\propto' ddv = \frac{1}{EI}\int (mv)(m\theta)\ dx \qquad (10)$$

$$\propto dd = \frac{1}{EI}\int (m\theta)^2\ dx \qquad (11)$$

Multiplying the corresponding moment expressions in above equations, we can evaluate above deformations. Draw M-diagram.

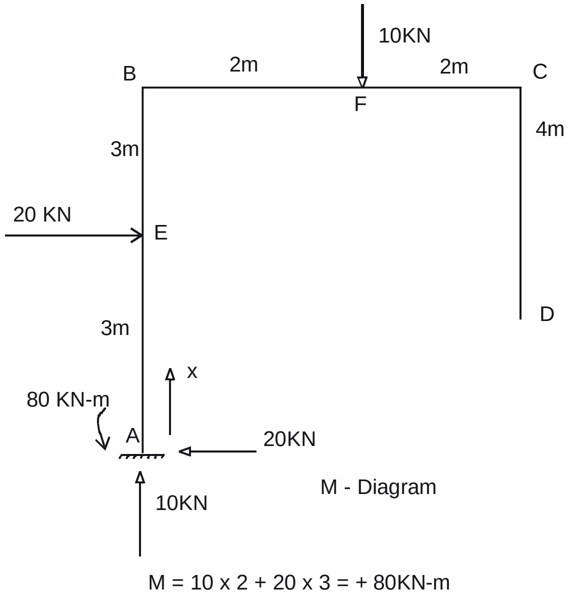
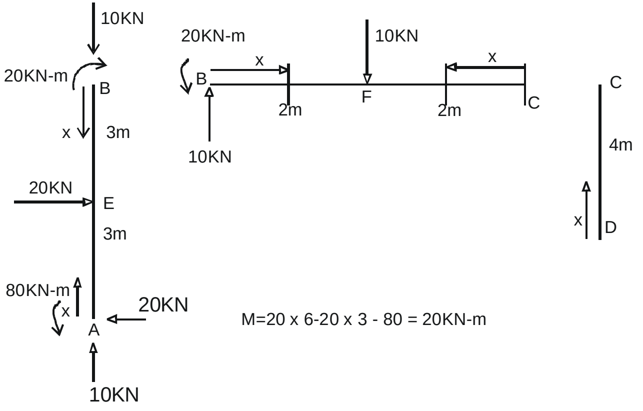
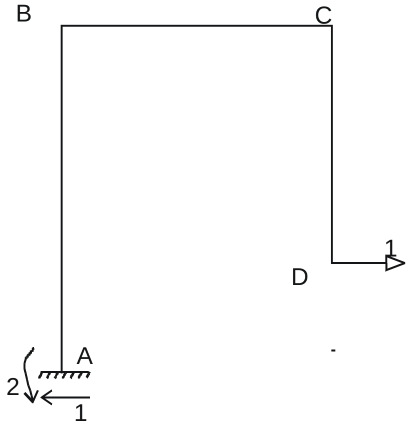
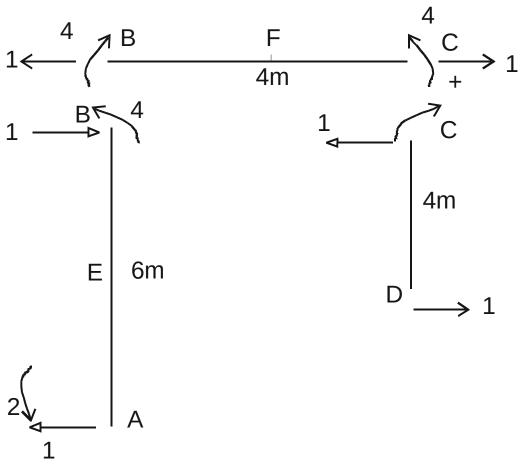


Fig. 2.34 B.D.S under applied loads

M – Diagram by parts







MH - Diagram

Fig. 2.34a

Fig 2.34b

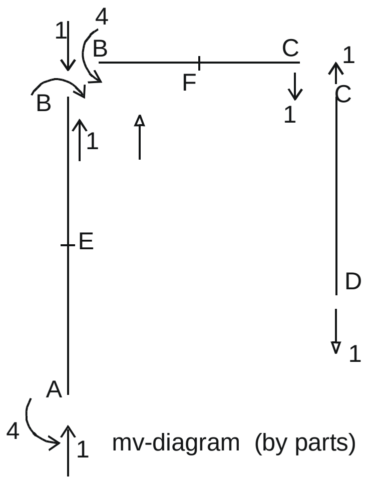


Fig 2.34c

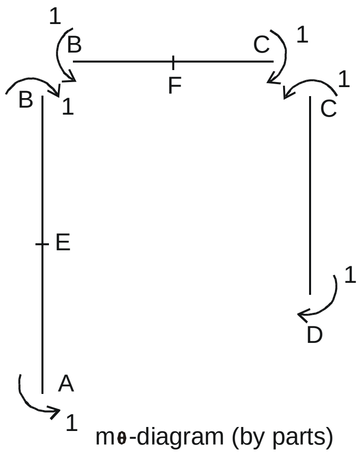


Fig 2.34d

Moments expressions in various members can now be written in a tabular form.

Portion	Origin	Limits	M	mH	mv	MO
AE	A	0 – 3	20X – 80	X – 2	– 4	–1
BE	B	0 – 3	– 20	– X + 4	– 4	–1
BF	B	0 – 2	10X – 20	4	X – 4	–1
CF	C	0 – 2	0	4	– X	–1
CD	D	0 – 4	0	X	0	–1

Put these moment expressions, integrate and evaluate co-efficients

$$\Delta D_H = \frac{1}{EI}\int M\,(\,mH\,)\;\;dX$$

$$\Delta D_H = \frac{1}{EI}\left[\int_o^3 (20X - 80)\,(X - 2)\,dX + \int_o^3 (-X+4)\,(-20)\,dX + \int_o^2 (10X - 20)\;4\,dX + 0 + 0\right]$$

$$= \frac{1}{EI}\left[\int_o^3 (20X^2 - 80X - 40X + 160\,) + \int_o^3 (20X - 80\,)dX + \int_o^2 (\,40X - 80\,)\,dX\right]$$

$$= \frac{1}{EI}\left[\left|\frac{20X^3}{3} - \frac{80X^2}{2} - \frac{4X^2}{2} + 160X\right|_o^3 + \frac{20X^2}{2} - 80X\,\Big|_o^3 + \left|\frac{40X^2}{2} - 80\,X\right|_o^4\right]$$

$$= \frac{1}{EI}\left[\left(\frac{20 \times 3^3}{3} - 40 \times 3^2 - 20 \times (3)^2 + 160 \times 3\right) + (10 \times 9 - 80 \times 3) + (20 \times 4 - 80 \times 2)\right]$$

$$\boldsymbol{\Delta\, D_H = -\frac{110}{EI}}$$

$$\delta\ ddh = \frac{1}{EI}\int (\,mH\,)^2\;\;dX$$

$$= \frac{1}{EI}\left[\int_o^3 (X - 2)^2\,dX + \int_o^3 (-X + 0)^2 dX + \int_o^2 16\,dX + \int_o^2 16\,dX + \int_o^4 X^2\;\;dX\right]$$

$$\delta ddh = \frac{1}{EI}\left[\int_o^3 (X - 4X + 4)\,dX + \int_o^3 (16 - 8X + X^2)\,dX + \int_o^2 16\,dX + \int_o^2 16\,dX + \int_o^4 X^2\;\;dX\right]$$

$$= \frac{1}{EI}\left[\left|\frac{X^3}{3} - \frac{4X^2}{2} + 4X\right|_o^3 + \left|16X - \frac{8X^2}{2} + \frac{X^3}{3}\right|_o^3 + \left|16X\right|_o^2 + \left|16X\right|_o^2 + \left|\frac{X^3}{3}\right|_o^4\right]$$

$$= \frac{1}{EI}\left[\left(\frac{3^3}{3} - 2\,(3)^2 + 4 \times 3\right) + \left(16 \times 3 - 4 \times 9 + \frac{3^3}{3}\right) + \left((16 \times 2) + (16 \times 2) + \Big|\,\frac{(4)^3}{3} - 0\right)\right]$$

$$\boxed{\delta ddh = \frac{109.33}{EI}}$$

$$\delta' ddV = \frac{1}{EI}\int (\,mH\,)\,(\,mv\,)\;\;dX$$

$$= \frac{1}{EI}\left[\int_o^3 (X - 2)\,(\,-4\,)\;\;dX + \int_o^3 (-X + 4\,)\,(-4\,)\,dX + \int_o^2 (\,4\,)\,(X-4\,)\,dX + \int_o^2 4\,(-X\,)\,dX + 0\right]$$

$$= \frac{1}{EI}\left[\int_o^3 (-\,4X+8\,)\,dX + \int_o^3 (4X - 16\,)\,dX + \int_o^2 (4X-16\,)\,dX + \int_o^2 -\,4XdX\right]$$

$$= \frac{1}{EI}\left[\left|-\frac{4X^2}{2} + 8X\right|_o^3 + \left|\frac{4X^2}{2} - 16X\right|_o^3 + \left|\frac{4X^2}{2} - 16X\right|_o^2 + \left|-\frac{4X^2}{2}\right|_o^2\right]$$

$$= \frac{1}{EI}\left[\Big| -2 \times (3)^2 + 8 \times 3 \Big| + (2 \times 3^2 - 16 \times 3) + (2 \times 2^2 - 16 \times 2) + (-2 \times 2^2)\right]$$

$$\boxed{\delta' ddV = -\frac{56}{EI}}$$

$$\propto' ddh = \frac{1}{EI}\int (mH)(m\theta)\, dX$$

$$= \frac{1}{EI}\left[\int_o^3 (-1)(X-2)\, dX + \int_o^3 (-1)(-x+4)\, dX + \int_o^2 -4\, dX + \int_o^2 -4\, dX + \int_o^4 -XdX\right]$$

$$= \frac{1}{EI}\left[\Big| -\frac{X^2}{2} + 2X \Big|_o^3 + \Big| \frac{X^2}{2} - 4X \Big|_o^3 + \Big| -4X \Big|_o^2 + \Big| -4X \Big|_o^2 + \Big| -\frac{X^2}{2} \Big|_o^4\right]$$

$$= \frac{1}{EI}\left[\left(-\frac{9}{2} + 2 \times 3\right) + \left(\frac{9}{2} - 4 \times 3\right) + (-4 \times 2) + (-4 \times 2) + \left(-\frac{4^2}{2} - 0\right)\right]$$

$$\boxed{\alpha' ddh = -\frac{30}{EI}}$$

$$\theta_D = \frac{1}{EI}\int M(m\theta)\, dX$$

$$= \frac{1}{EI}\left[\int_o^3 -(20X - 80)\, dX + \int_o^3 20\, dX + \int_o^2 (-10X + 20)\, dX + 0 + 0\right]$$

$$= \frac{1}{EI}\left[\Big| -\frac{20X^2}{2} + 80X \Big|_o^3 + \Big| 20X \Big|_o^3 + \Big| -\frac{10X^2}{2} + 20X \Big|_o^2\right]$$

$$= \frac{1}{EI}\left[(-10 \times 3^2 + 80 \times 3) + (20 \times 3) + (-5 \times 4 + 20 \times 2)\right]$$

$$\boxed{\theta_D = \frac{230}{EI}}$$

$$\Delta\, Dv = \frac{1}{EI}\int M(mv)\, dX$$

$$= \frac{1}{EI}\left[\int_o^3 (20X - 80)(-4)\, dX + \int_o^3 (-20)(-4)\, dX + \int_o^2 (10X - 20)(X - 4)\, dX + 0 + 0\right]$$

$$= \frac{1}{EI}\left[\int_o^3 (-80X + 320)\, dX + \int_o^3 80\, dX + \int_o^2 (10X^2 - 20X - 40X + 80)\, dX\right]$$

$$= \frac{1}{EI}\left[\Big| -80\frac{X^2}{2} + 320X \Big|_o^3 + \Big| 80X \Big|_o^3 + \Big| 10\frac{X^3}{3} - \frac{60X^2}{2} + 80X \Big|_o^2\right]$$

$$= \frac{1}{EI}\left[(-40 \times 9 + 320 \times 3) + (80 \times 3) + \left(\frac{10}{3} \times 8 - 30 \times 4 + 80 \times 2\right)\right]$$

$$\boxed{\Delta Dv = \frac{906.67}{EI}}$$

$$\delta' ddh = \frac{1}{EI}\int (\, mH \,)\,(\, mv \,)\, dX$$

$$= \frac{1}{EI}\left[\int_{o}^{3} (X-2)\,(-4)\, dX + \int_{o}^{3} (-X + 4)\,(-4)\, dX + \int_{o}^{2} 4\,(X-4)\, dX + \int_{o}^{2} - 4XdX + 0\right]$$

$$= \frac{1}{EI}\left[\int_{o}^{3} (-4X + 8)dX + \int_{o}^{3} (4X - 16)\, dX + \int_{o}^{2} (4X - 16)\, dX + \int_{o}^{2} - 4XdX\right]$$

$$= \frac{1}{EI}\left[\Big| -\frac{4X^2}{2} + 8X \Big|_{o}^{3} + \Big| \frac{4X^2}{2} - 16X \Big|_{o}^{3} + \Big| \frac{4X^2}{2} - 16X \Big|_{o}^{2} + \frac{4X}{2}\Big|_{o}^{2}\right]$$

$$= \frac{1}{EI}\Big[(-2 \times 9 + 8 \times 3) + (2 \times 9 - 16 \times 3) + (2 \times 4 - 16 \times 2) + (-2 \times 4)\Big]$$

$$\boxed{\delta' ddh = -\frac{56}{EI}}$$

$$\delta ddv = \frac{1}{EI}\int (\, mv^2 \,)\, dX$$

$$= \frac{1}{EI}\left[\int_{o}^{3} 16\, dX + \int_{o}^{3} 16\, dX + \int_{o}^{2} (X - 4)^2\, dX + \int_{o}^{2} (-X)^2 dX + 0\right]$$

$$= \frac{1}{EI}\left[\int_{o}^{3} 16\, dX + \int_{o}^{3} 16\, dX + \int_{o}^{2} (X^2 - 8X + 16)dX + \int_{o}^{2} + X^2\, dX\right]$$

$$= \frac{1}{EI}\left[\Big| 16X \Big|_{o}^{3} + \Big| 16X \Big|_{o}^{3} + \Big| \frac{X^3}{3} - \frac{8X^2}{2} + 16X \Big|_{o}^{2} + \Big| + \frac{X^3}{3}\Big|_{o}^{2}\right]$$

$$= \frac{1}{EI}\left[(16 \times 3) + (\, 16 \times 3 \,) + \left(\frac{8}{3} - 4 \times 4 + 16 \times 2\right) + \left(+\frac{8}{3}\right)\right]$$

$$\boxed{\delta ddv = \frac{117.33}{EI}}$$

$$\propto'ddv = \frac{1}{EI}\int mv \times m\theta \ dX$$

$$= \frac{1}{EI}\left[\int_o^3 + 4\ dX + \int_o^3 + 4\ dX + \int_o^2 (-X + 4)\ dX + \int_o^2 XdX\right]$$

$$= \frac{1}{EI}\left[\left|4X\right|_o^3 + \left|4X\right|_o^3 + \left| -\frac{X^2}{2} + 4X\right|_o^2 + \left|\frac{X^2}{2}\right|_o^2\right]$$

$$= \frac{1}{EI}\left[(4 \times 3) + (4 \times 3) + (-2 + 4 \times 2) + \left(\frac{2^2}{2}\right)\right]$$

$$\boxed{\propto'ddv = \frac{32}{EI}}$$

$$\propto dd = \frac{1}{EI}\int (\ m\theta\ )^2 \ \ dX$$

$$= \frac{1}{EI}\left[\int_o^3 (-1)^2\, dX + \int_o^3 (-1)^2\, dX + \int_o^2 (-1)^2\, dX + \int_o^2 (-1)^2\, dX + \int_o^4 (-1)^2 \ dX\right]$$

$$= \frac{1}{EI}\left[\ \left| X\right|_o^3 + \left| X\right|_o^3 + \left| X\right|_o^2 + \left| X\right|_o^2 + \left| X\right|_o^4\right]$$

$$= \frac{1}{EI}\left[\ 3 + 3 + 2 + 2 + 4\ \right]$$

$$\boxed{\propto dd = \frac{14}{EI}}$$

Putting all values of evaluated co-efficients, equations 1,2 and 3 become

$$-\frac{110}{EI} + \frac{109.33}{EI} \times H_D - \frac{56}{EI} \times V_D - \frac{30}{EI}\ M_D = 0 \qquad (1)$$

and $$\frac{906.67}{EI} - \frac{56}{EI} \times H_D + \frac{117.33}{EI} \times V_D + \frac{32}{EI}\ M_D = 0 \qquad (2)$$

and $$\frac{230}{EI} - \frac{30}{EI} \times H_D + \frac{32}{EI} \times V_D + \frac{14}{EI}\ M_D = 0$$ (3) Simplifying

$$-110 + 109.33\ H_D - 56\ V_D - 30\ M_D = 0 \qquad \rightarrow (1)$$
$$906.67 - 56\ H_D + 117.33\ V_D + 32\ M_D = 0 \qquad \rightarrow (2)$$
$$230 - 30\ H_D + 32\ V_D + 14\ M_D = 0 \qquad \rightarrow (3)$$

From Eq (1)

$$MD \quad = \frac{-110 + 109.33\ H_D - 56\ V_D}{30} = -3.67 + 3.64\ H_D - 1.86\ V_D \quad \rightarrow \quad (4)$$

Putting in Eq (2)

$$906.67 - 56\ H_D + 117.33\ V_D + 32\ (-3.67 + 3.64\ H_D - 1.86\ V_D) = 0$$
$$906.67 - 56\ H_D + 117.33\ V_D - 117.44 + 116.5\ H_D - 59.52\ V_D = 0$$
$$789.23 + 60.5\ H_D + 57.81\ V_D = 0$$
$$H_D = -13.045 - 0.95\ V_D \quad \rightarrow \quad (5)$$

Putting the value of H_D in Eq (4)

$$M_D = -3.67 + 3.64\ (-13.045 - 0.95\ V_D) - 1.86\ V_D$$
$$M_D = -51.15 - 5.32\ V_D \quad \rightarrow \quad (6)$$

Putting the values of M_D & H_D in Eq (3)

$$230 - 30\ (-13.045 - 0.95\ V_D) + 32\ V_D + 14\ (-51.15 - 5.32\ V_D) = 0$$
$$230 + 391.35 + 28.5\ V_D + 32\ V_D - 716.1 - 74.5\ V_D = 0$$
$$-14\ V_D - 94.75 = 0$$
$$V_D = -6.78\ KN$$

Putting in (5) & (6)

$$H_D = -6.61\ KN, \qquad M_D = -15.08\ KN\text{–}m$$

From any equation above. We get

$$\boxed{V_D = -\ 12.478\ \ KN}$$

Apply the evaluated structural actions in correct sense on the frame. The correctness of solution can be checked afterwards by equilibrium conditions.

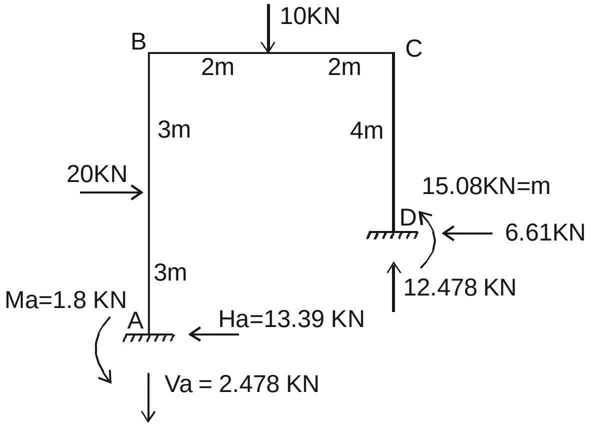


Fig. 2.35 shows all reactions after Evaluation

$\sum Fx = 0$

$20 - Ha - 6.61 = 0$

$$\boxed{Ha = 13.39 \text{ KN}}$$

$\sum Fy = 0$

$Va + 12.478 - 10 = 0$ (asuming Va upwards)

$$\boxed{Va = -2.478 \text{ KN}}$$

0

$Ma + 20 \times 3 + 10 \times 2 - 12.478 \times 4 - 6.61 \times 2 - 15.08 = 0$ (assuming Ma clockwise)

$$\boxed{Ma = -1.8 \text{ KN-m}}$$

$\Sigma Ma = 0$ $\quad 12.478 \times 4 + 15.08 + 6.61 \times 2 + 1.8 - 20 \times 3 - 10 \times 2 = 0$ Proved.

1.16. ANALYSIS OF STATICALLY EXTERNALLY INDETERMINATE TRUSSES :-

A truss may be statically indeterminate if all external reactive components and internal member forces may not be evaluated simply by the help of equations of equilibrium available. The indeterminacy of the trusses can be categorized as follows :-

(1) Trusses containing excessive external reactive components than those actually required for external stability requirements.

(2) Trusses containing excessive internal members than required for internal stability requirements giving lesser the number of equations of equilibrium obtained from various joints.

(3) A combination of both of the above categories i.e. excessive external reactions plus excessive internal members.

INTERNAL INDETERMINACY:-

$b + r = \lfloor 2j \rfloor$

There are two equations of equilibrium per joint where

b = number of bars or members.

r = minimum number of external reactive components required for external stability (usually 3).

j = number of joints.

The above formula can also be used to check the total indeterminacy of a truss if we define ‘r’ as the total number of reactive components which can be provided by a typical support system.

1.17. METHOD OF MOMENTS AND SHEARS :

A simple method is presented to evaluate axial member forces in parallel chord trusses. For other types of trusses method of joints, method of sections or Maxwell’s diagram may be used. For determining forces in members of trusses, this method has been used throughout this text. To develop the method, consider the truss loaded as shown below:

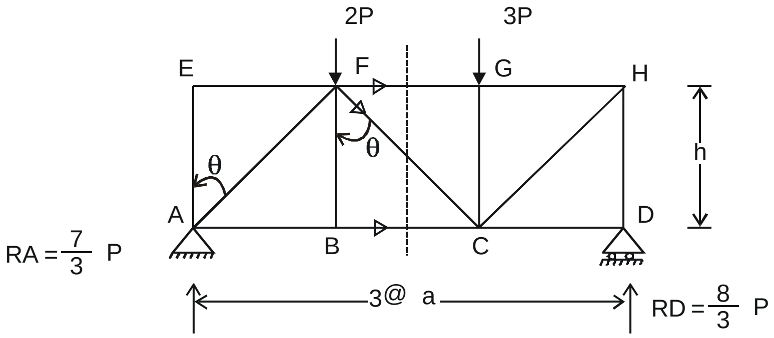


Fig. 2.36 A typical Truss under loads

Consider the equilibrium of L.H.S. of the section. Take ‘D’ as the moment centre: we find Ra

$$Ra \times 3a = 2P \times 2a + 3P \times a$$

$$Ra = \frac{7Pa}{3a} = \frac{7P}{3}$$

$\sum Mc = 0$ and assuming all internal member forces to be tensile initially, we have

$Ra \; x \; 2a - 2P \times a + S_{FG} \times h = 0$ (considering forces on LHS of section)

or $$S_{FG} = -\left(\frac{Ra \times 2a - 2\,Pa}{h}\right)$$

The (–ve) sign indicates a compressive force. Or

$$S_{FG} = \left(\frac{Ra \times 2a - 2\,Pa}{h}\right) = \frac{Mc}{h}$$ where numerator is Mc. Therefore

The force in any chord member is a function of bending moment.

“To find out the axial force in any chord member, the moment centre will be that point where other two members completing the same triangle meet and the force will be obtained by taking moments about that point and dividing it by the height of truss. The signs of the chord members are established in the very beginning by using an analogy that the truss behaves as a deep beam. Under downward loads, all upper chord members are in compression while all lower chord members are in tension.

Similarly, $S_{BC} = \frac{M_F}{h}$ (using the guide line given in the above para)

Consider the equilibrium of left hand side of the section and

$\sum Fy = 0$

$$Ra - 2P - S_{FC} \cos\theta = 0$$

$$S_{FC} = \left(\frac{Ra - 2P}{\cos\theta}\right)$$ where Ra – 2P is equal to shear force V due to applied loads at the section. So in general the force in any inclined member is a function of shear force.

$$S_{FC} = \frac{V}{\cos\theta}$$

The general formula is :

$$S = \frac{\pm (V)}{\pm (\cos\theta)}$$

Where V is the S.F. at the section passing through the middle of inclined member and ‘θ‘ is the angle measured from “the inclined member to the vertical” at one of its ends. Use (+ve) sign as a pre-multiplier with the Cosθ if this angle is clockwise and (–ve) sign if θ is anticlockwise. Take appropriate sign with the S.F also. This treatment is only valid for parallel chord trusses.

The force in the vertical members is determined by inspection or by considering the equilibrium of forces acting at the relevant joints. To illustrate the method follow the example below.

1.17.1: EXAMPLE :– Analyze the following truss by the method of moment & shear.
SOLUTION:- Determine reactions and Draw SFD and BMD.

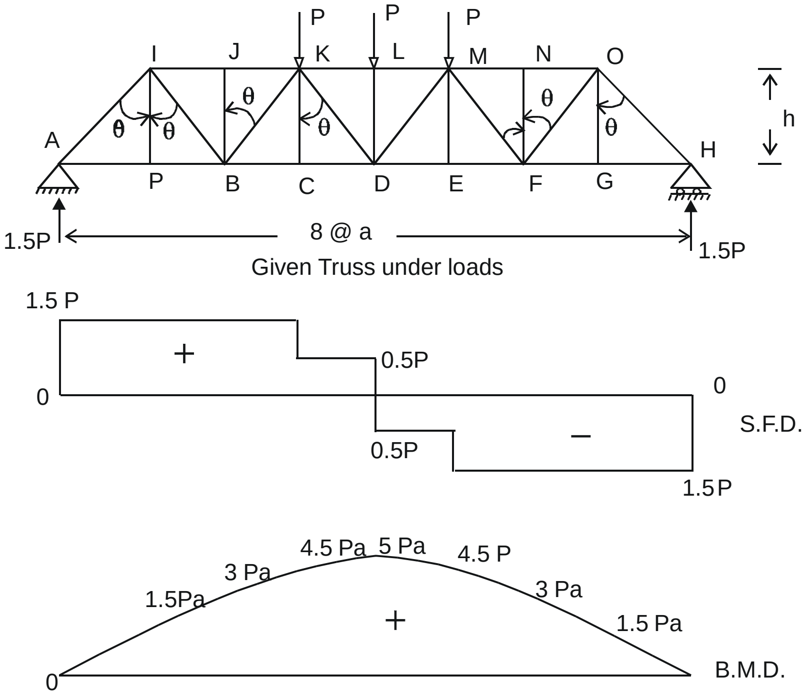


Fig. 2.37

TOP CHORD MEMBERS.

Considering the beam analogy of truss, all top chord members are in compression. Picking bending moment, at appropriate moment centers, from BMD and dividing by height of Truss.

$$\text{Sij} = -\frac{3\text{ Pa}}{\text{h}}$$

$$\text{Sjk} = -\frac{3\text{ Pa}}{\text{h}}$$

$$\text{Stl} = -\frac{5\text{ Pa}}{\text{h}}$$

$$\text{Slm} = -\frac{5\text{ Pa}}{\text{h}}$$

$$\text{Smn} = -\frac{3\text{ Pa}}{\text{h}}$$

$$\text{Sno} = -\frac{3\text{ Pa}}{\text{h}}$$

Negative sign means compression.

BOTTOM CHORD MEMBERS.

All are in tension. Taking appropriate moment point and dividing by height of Truss.

$$Sap = Spb = +\frac{1.5\ Pa}{h}$$

$$Sbc = Scd = +\frac{4.5\ Pa}{h}$$

$$Sde = Sef = +\frac{4.5\ Pa}{h}$$

$$Sfg = Sgh = +\frac{1.5\ Pa}{h}$$

INCLINED MEMBERS.

The force in these members has been computed by the formula. $\frac{\pm V}{\pm(\text{Cos}\theta)}$. Follow the guidelines.

$$Sai = \frac{1.5\ P}{-\text{Cos}\theta}$$

$$Sib = \frac{1.5\ P}{+\text{Cos}\theta}$$

Length AI = $\sqrt{a^2 + h^2}$

(if a and h are given, length and Cos θ will have also late values)

$$Sbk = \frac{1.5\ P}{-\text{Cos}\theta}$$

$$\text{Cos}\ \theta = \frac{h}{\sqrt{a^2 + h^2}}$$

$$Skd = \frac{0.5\ P}{+\text{Cos}\theta}$$

$$Sdm = \frac{-0.5\ P}{-\text{Cos}\theta} = \frac{0.5\ P}{\text{Cos}\theta}$$

$$Smf = \frac{-1.5\ P}{+\text{Cos}\theta}$$

$$Sfo = \frac{-1.5\ P}{-\text{Cos}\theta} = \frac{1.5\ P}{\text{Cos}\theta}$$

$$Soh = \frac{-1.5\ P}{+\text{Cos}\theta}$$

VERTICAL MEMBERS.

For all vertical members of trusses in this book, member forces have been determined by Inspection or by Equilibrium of joints. So

$Sip = Sbj = Sck = Sem = Sfn = Sgo = 0$

$Sld = -P$ (If a and h values are given, all forces can be numerically evaluated)

1.18. EXTERNALLY REDUNDANT TRUSSES – FIRST DEGREE

EXAMPLE 5 :- Analyze the following truss by the force method. (consistent deformation method). The following data is given.

$E = 200 \times 10^6$ KN/m^2

A=$5\text{x}10^{-3}$m^2 for inclineds and verticals,

A=$4\text{x}10^{-3}$m^2 for top chord members,

A=$6\text{x}10^{-3}$m^2 for bottom chord members

SOLUTION:-

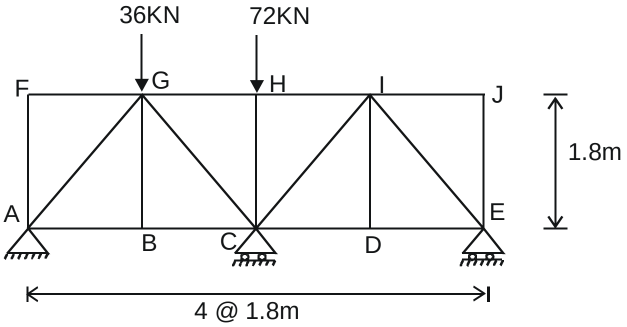


Fig. 2.38 Given Truss under loads

TOTAL INDETERMINACY :-

$b + r = 2j$ where r = total reactions which the supports are capable of providing.

$17 + 4 \neq 2 \times 10$

$21 \neq 20$

$D = 21 - 20 = 1$

Indeterminate to Ist degree.

Apply check for Internal Indeterminacy :-

$b + r = 2j$ where r = Minimum number of external reactions required for stability.

$17 + 3 = 2 \times 10$

$20 = 20$

This truss is internally determinate and externally indeterminate to 1st degree, therefore, we select reaction at point "C" as the redundant force. Remove support at C, the Compatibility equation is :

$\Delta C + \delta cc \times Rc = 0$ (Deflection at C due to loads plus due to redundant should be zero.)

or $Rc = -\dfrac{\Delta c}{\delta cc}$. Now we have to calculate Δc and δcc to get Rc.

where $\Delta c = \sum \dfrac{F' UL}{AE}$ where F' = Force induced in members due to applied loads acting on BDS.

$\delta cc = \sum \dfrac{U^2 L}{AE}$ U = Forces in members due to Unit load applied in direction of applied loads, at external redundant support in BDS.

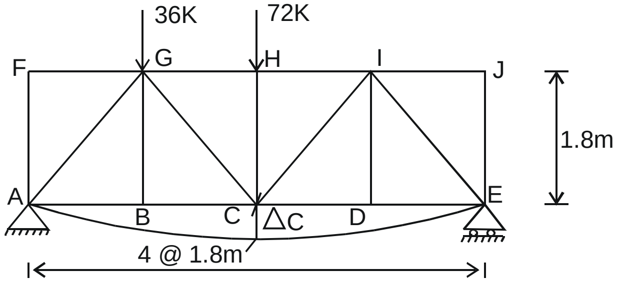


Fig 2.39a B.D.S under applied Loads
(F-Diagram)

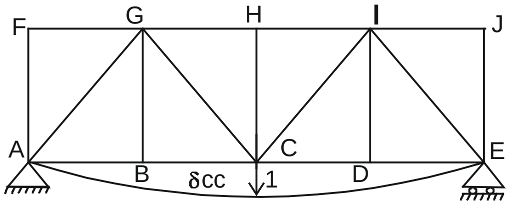
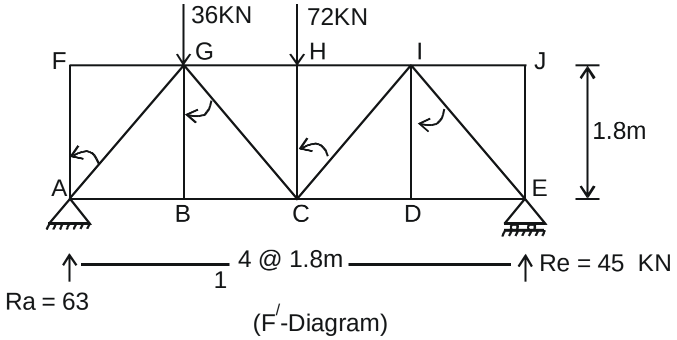
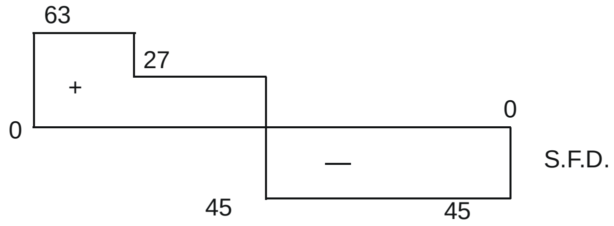


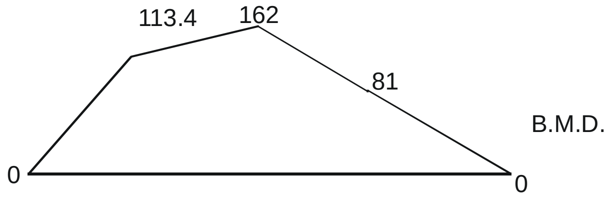
Fig 2.39b B.D.S under unit Vertical Redundant at C
(U-Diagram)

Analyze the given truss by the method of moments and shears as explained already for F' and U forces in members.



(F'-Diagram)
Fig 2.40 B.D.S under Loads





Determine forces in all members of trusses loaded as shown in this question and enter the results in a tabular form. (using method of moments and shears, F' and U values for members have been obtained).

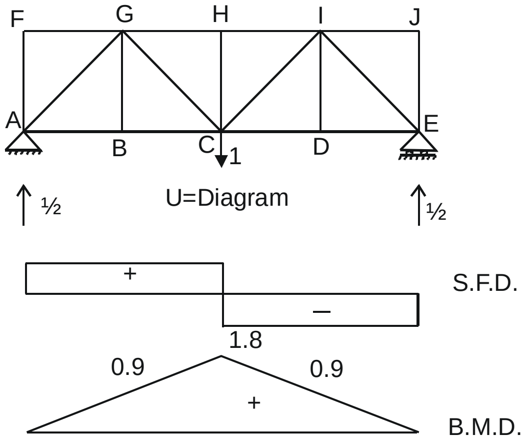


Fig 2.41 B.D.S under Unit redundant force at C

Member	F′ (KN)	U	Ax 10^{-3} $(m)^2$	L (m)	$\frac{F'UL}{AE} \times 10^{-3}$ (m)	$\frac{U^2L}{AE} \times 10^{-3}$ (m)	Fi = Fi′ – Rc × U_1 (KN)
FG	0	0	4	1.8	0	0	0
GH	– 90	– 1	″	″	0.2025	2.25×10^{-3}	+ 2.5
H I	– 90	– 1	″	″	0.2025	2.25×10^{-3}	+ 2.5
I J	0	0	4	″	0	0	0
AB	+63	+0.5	6	1.8	0.04725	0.375×10^{-3}	+16.75
BC	+63	+0.5	″	″	0.04725	0.375×10^{-3}	+16.75
CD	+45	+0.5	″	″	0.03375	0.375×10^{-3}	– 1.25
DE	+45	+0.5	″	″	0.03375	0.375×10^{-3}	– 1.25
AG	– 89.1	–0.707	″	2.55	0.16063	1.275×10^{-3}	– 23.7
GC	+38.2	GC	5	″	0.06887	1.275×10^{-3}	– 27.2
C I	+63.64	+0.707	″	″	0.11473	1.275×10^{-3}	– 1.76
I E	–63.64	–0.707	″	″	0.11473	1.275×10^{-3}	+1.76
AF	0	0	″	1.8	0	0	0
BG	0	0	″	″	0	0	0
HC	– 72	0	″	″	0	0	– 72
I D	0	0	″	″	0	0	0
J E	0	0	″	″	0	0	0
					$\sum \frac{F'UL}{AE} = 1.02596 \times 10^{-3}$	$\sum \frac{U^2L}{AE} = 11.1 \times 10^{-6}$	

$$\Delta\,C = \sum\frac{F'\,UL}{AE} \;=\; 1.02596\times10^{-3} \;= 1025.96\times10^{-6}\text{ m}$$

$$\delta\text{ cc} \;=\; \sum\frac{U^2\,L}{AE} \;=\; 11.1\times10^{-6}\;\;\text{m}$$. Putting these two in original compatibility equation

$$\text{Rc} \;=\; -\;\frac{\Delta\,C}{\delta cc} \;=\; \frac{-1025.96\times10^{-6}}{11.1\times10^{-6}}$$

$$\text{Rc} \;=\; -\;92.5\text{ KN.}$$

The (–ve) sign with Rc shows that the assumed direction of redundant is incorrect and Rc acts upward.

If Fi is net internal force due to applied loading and the redundants, acting together, then member forces an calculated from

$$\text{Fi} \;=\; \text{Fi}' \;-\; \text{Rc} \;\times\; \text{Ui}$$

The final axial force in any particular member can be obtained by applying the principle of superposition and is equal to the force in that particular member due to applied loading (±) the force induced in the same member due to the redundant with actual signs.

Apply the principle of superposition and insert the magnitude of redundant Rc with its sign which has been obtained by applying the compatibility condition to calculate member forces.

1.19. SOLUTION OF 2ND DEGREE EXTERNALLY INDETERMINATE TRUSSES:-

Example-6 : Solve the following truss by consistent deformation method use previous member properties.

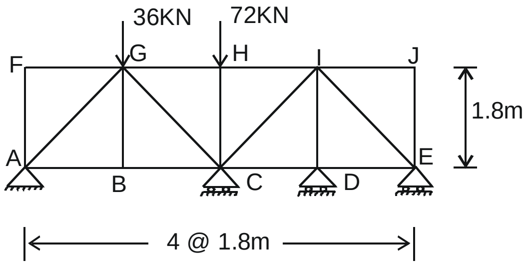


Fig 2.42 Given Truss

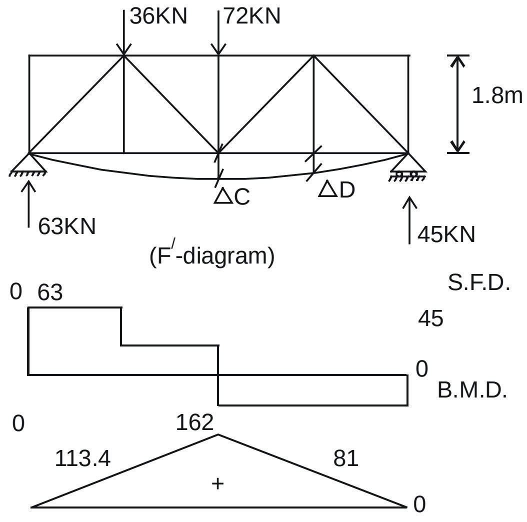


Fig 2.42a B.D.S under loads

+

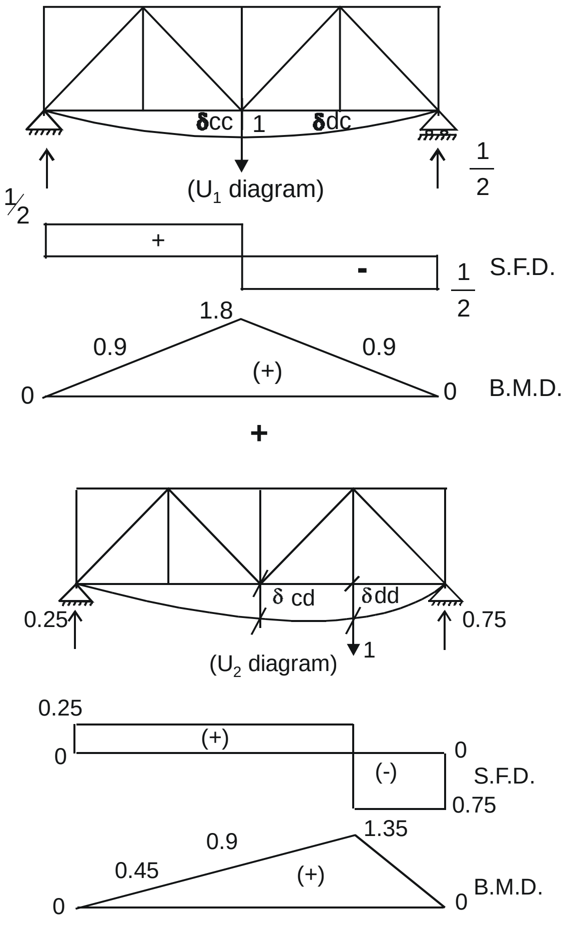


Fig 2.42 c B.D.S under unit redundant at D

Compatibility equations are:

$\Delta C + Rc.\ \delta cc + Rd \times \delta cd = 0$ (1) Compatibility of deformations at C

$\Delta D + Rc\ .\ \delta dc + Rd\ .\ \delta dd = 0$ (2) Compatibility of deformations at D

$\delta cd = \delta dc$ by the law of reciprocal deflection.

δcc = deflection of point C due to unit load at C.

δdc = deflection of point D due to unit load at C.

δdd = deflection of point D due to unit load at D.

δcd = deflection of point C due to unit load at D.

Flexibility coefficients of above two equations are evaluated in tabular form (Consult the attached table)

$$\Delta C = \sum \frac{F'U_1L}{AE} = 1026.2 \times 10^{-6} \text{ m}$$

$$\Delta D = \sum \frac{F'U_2L}{AE} = 579.82 \times 10^{-6} \text{ m}$$

$$\delta cc = \sum \frac{U_1^2L}{AE} = 11.1 \times 10^{-6} \text{ m}$$

$$\delta dd = \sum \frac{U_2^2 L}{AE} = 9.3565 \times 10^{-6} \text{ m}$$

$$\delta cd = \sum \frac{U_1 U_2 L}{AE} = 6.291 \times 10^{-6} \text{ m}$$

$$\delta dc = \sum \frac{U_1 U_2 L}{AE} = 6.291 \times 10^{-6} \text{ m}$$ Put these in equations 1 and 2

$$1026.2 \times 10^{-6} + 11.1 \times 10^{-6} \text{ Rc} + 6.291 \times 10^{-6} \text{ Rd} = 0 \quad \rightarrow \quad (1)$$

$$579.82 \times 10^{-6} + 6.291 \times 10^{-6} \text{ Rc} + 9.3565 \times 10^{-6} \text{ Rd} = 0 \quad \rightarrow \quad (2)$$

Simplify

$$1026.2 + 11.1 \text{ Rc} + 6.291 \text{ Rd} = 0 \quad \rightarrow \quad (3)$$

$$579.82 + 6.291 \text{ Rc} + 9.3565 \text{ Rd} = 0 \quad \rightarrow \quad (4)$$

From (3)

$$\text{Rc} = \left(\frac{-1026.2 - 6.291 \text{ Rd}}{11.1}\right) \quad \rightarrow \quad (5)$$

Put Rc in (4) & solve for Rd

$$579.82 + 6.291 \left(\frac{-1026.2 - 6.291 \text{ Rd}}{11.1}\right) + 9.3565 \text{ Rd} = 0$$

$$-1.786 + 5.791 \text{ Rd} = 0$$

$$\boxed{\text{Rd} = +0.308 \text{ KN}}$$

So, from (5), $\Rightarrow$ $\text{Rc} = \left(\frac{-1026.2 - 6.291 \times 0.308}{11.1}\right)$

$$\boxed{\text{Rc} = -92.625 \text{ KN}}$$

$\therefore$ Rc = – 92.625 KN
Rd = + 0.308 KN

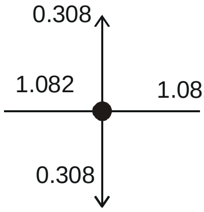
These signs indicate that reaction at C is upwards and reaction at D is downwards.

By superposition, the member forces will be calculated as follows

$\text{Fi} = \text{Fi} + \text{Rc} \times U_1 + \text{Rd} \times U_2$ which becomes.

$\text{Fi} = \text{Fi} - \text{Rc} \times U_1 + \text{Rd} \times U_2$. It takes care of (–ve) sign with Rc.

Equilibrium checks:–



Joint D

$\sum \text{Fx} = 0$

$\sum \text{Fy} = 0$

Equilibrium is satisfied. Only check at one joint has been applied. In fact this check should be satisfied at all joints.

Table 79−A

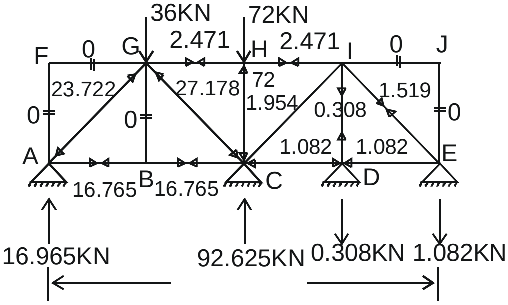


Fig 2.43 Result of analyzed Truss

Now find remaining reactions Ra and Re.

$\sum Fy = 0$

$Ra + Re + 92.625 - 0.308 - 36 - 72 = 0$

$Ra + Re = 15.683 \qquad \rightarrow \quad (1)$

$\sum M_A = 0$

$Re \times \Delta \times 1.8 - 0.308 \times 3 \times 1.8 + 92.625 \times 2 \times 1.8 - 72 \times 2 \times 1.8 - 36 \times 1.8 = 0$

$$\boxed{Re = -1.082 \text{ KN}}$$

As $\quad Ra + Re = 15.863$

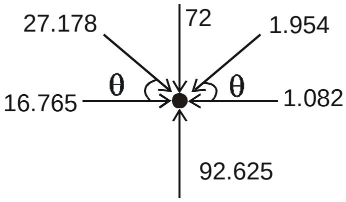
So $\quad Ra = 15.863 + 1.082$

$$\boxed{Ra = 16.945 \text{ KN}}$$

Now truss is determinate. Calculate member forces and apply checks in them.

Joint (C)

$\sum Fx = 0$



$- 1.082 - 16.765 - 1.954 \times 0.707 + 27.178 \times 0.707 = 0$

$- 0.0136 = 0$

$0 \cong 0$ equilibrium is satisfied.

$\sum Fy = 0$

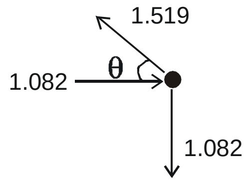
$- 72 + 92.625 - 1.954 \times 0.707 - 27.178 \times 0.707 = 0$

$0.0286 = 0$

$0 \cong 0$ equilibrium is satisfied

Joint (E)

$\sum Fy = 0$



$1.519 \times 0.707 - 1.087 = 0$

$0 = 0$

$\sum Fx = 0$

$082 - 1.519 \times 0.707 = 0$

$0 = 0$ equilibrium is satisfied.

1.20. Example–7:- SOLUTION OF 3^{RD} DEGREE EXTERNALLY INDETERMINATE TRUSSES:-

Now we solve the following truss by consistent deformation method. Choosing reaction of B, C and D as redundant.

SOLUTION:-

First step. Choose BDS Draw BDS under loads and subsequently under applied unit loads at points of redundancy also.

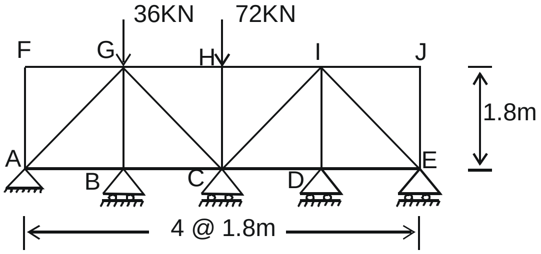


Fig 2.44 Given 3rd degree externally indeterminate truss under loads

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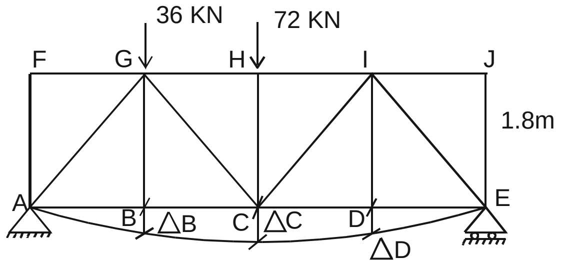


Fig 2.44(a) B.D.S under loads

+

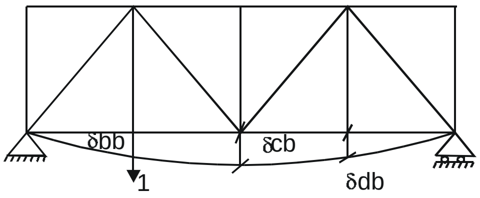


Fig 2.44(b) B.D.S under redundant unit load at B
(U1 diagram)

+

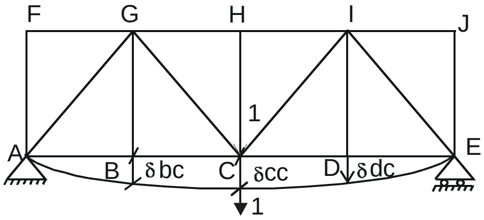


Fig 2.44(c) B.D.S under redundant unit load at C
(U2 diagram)

+

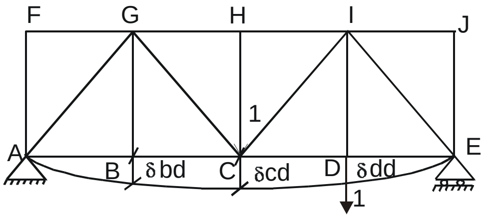


Fig 2.44(d) B.D.S under redundant unit load at D
(U3 diagram)

Step No.2: Compatibility equations are:

$\Delta B + Rb.\delta bb + Rc.\delta bc + Rd \times \delta bd = 0$ For joint B → (1)

$\Delta C + Rb.\delta cb + Rc.\delta cc + Rd \times \delta cd = 0$ For joint C → (2)

$\Delta D + Rb.\delta db + Rc.\delta dc + Rd \times \delta dd = 0$ For joint D → (3)

Step No.3: Evaluation of Flexibility co-efficients

$\Delta B = \sum \frac{F'U_1L}{AE}$ $\Delta C = \sum \frac{F'U_2L}{AE}$ $\Delta D = \sum \frac{F'U_3L}{AE}$

$\delta bb = \sum \frac{U_1^2L}{AE}$ $\delta bc = \sum \frac{U_1U_2L}{AE}$ $\delta bd = \sum \frac{U_1U_3L}{AE}$

$\delta cb = \sum \frac{U_1U_2L}{AE}$ $\delta cc = \sum \frac{U_2^2L}{AE}$ $\delta cd = \sum \frac{U_2U_3L}{AE}$

$\delta db = \sum \frac{U_1U_3L}{AE}$ $\delta dc = \sum \frac{U_2U_3L}{AE}$ $\delta dd = \sum \frac{U_3^2L}{AE}$

By law of reciprocal deflections :-

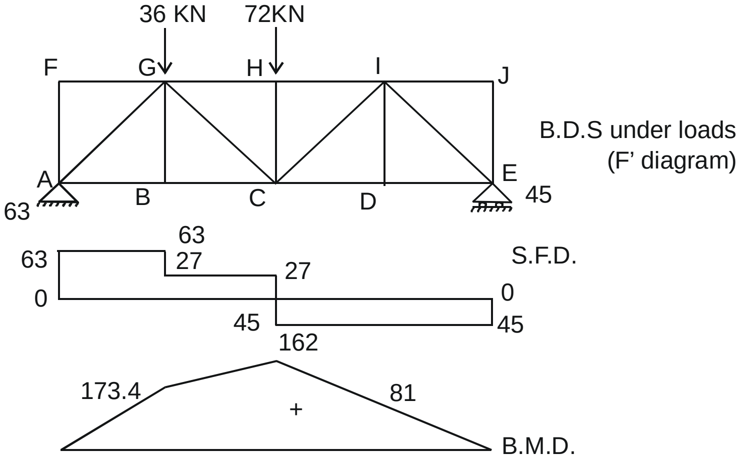
We know that

$\delta bc = \delta cb$

$\delta bd = \delta db$

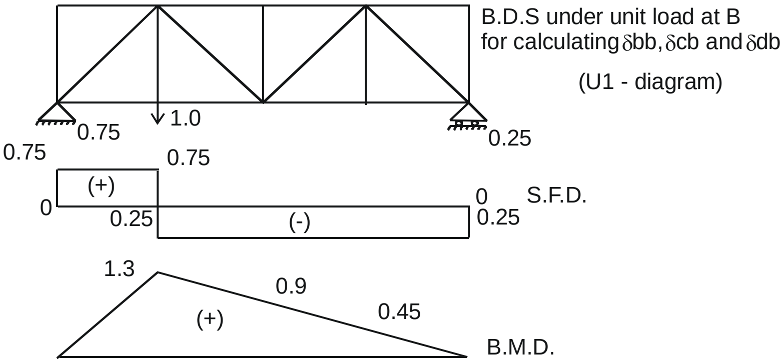
$\delta cd = \delta dc$

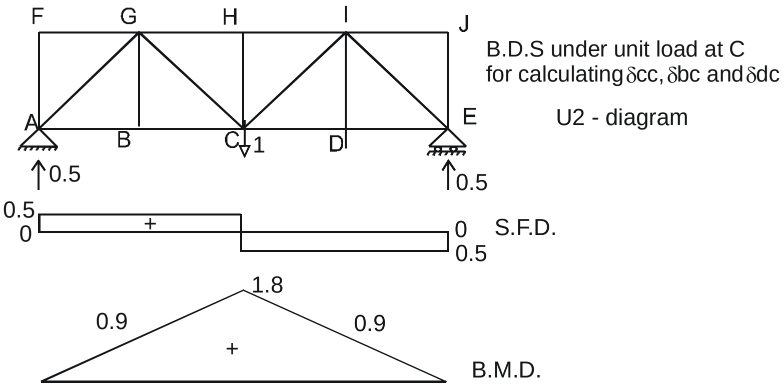
In order to find member forces due to applied forces in BDS, consider.



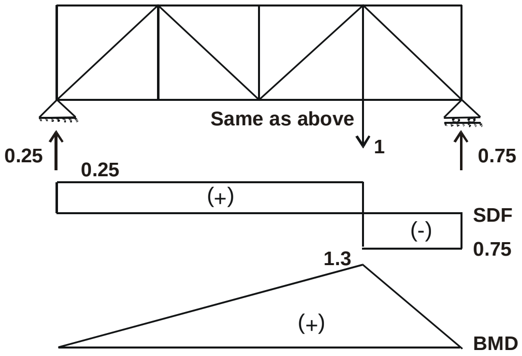
The above SFD and BMD are used to calculate member forces by method of moments and shears. Finally ΔB, ΔC and ΔD due to applied loads on BDS are calculated in a tabular form as given below:

Table 84–A





U3 diagram for δbd, δcd and δdd



From the previous table we have the values of all flexibility co-efficients as given below:

$\Delta B = 391.65 \times 10^{-6}$ m

$\Delta C = 1026.2 \times 10^{-6}$ m

$\Delta D = 692.42 \times 10^{-6}$ m

$\delta bb = 9.3616 \times 10^{-6}$ m, and $\delta cc = 11.1 \times 10^{-6}$ m, $\delta dd = 9.3565 \times 10^{-6}$ m

$\delta bc = \delta cb = 6.417 \times 10^{-6}$ m

$\delta bd = \delta db = 3.517 \times 10^{-6}$ m

$\delta cd = \delta dc = 6.291 \times 10^{-6}$ m

Putting the values of flexibility co-efficients into compatibility equations we have.

$$391.65 \times 10^{-6} + 9.3616 \times 10^{-6}\,Rb + 6.292 \times 10^{-6}\,Rc + 3.517 \times 10^{-6}\,Rd = 0 \quad \rightarrow (1)$$

$$1026.2 \times 10^{-6} + 6.292 \times 10^{-6}\,Rb + 11.1 \times 10^{-6}\,Rc + 6.291 \times 10^{-6}\,Rd = 0 \quad \rightarrow (2)$$

$$579.82 \times 10^{-6} + 3.517 \times 10^{-6}\,Rb + 6.291 \times 10^{-6}\,Rc + 9.3565 \times 10^{-6}\,Rd = 0 \quad \rightarrow (3)$$

Step No. 4

Simplify equation (1), (2) and (3), we have

$$391.65 + 9.3620\,Rb + 6.292\,Rc + 3.517\,Rd = 0 \quad \rightarrow (4)$$

$$1026.2 + 6.292\,Rb + 11.1\,Rc + 6.291\,Rd = 0 \quad \rightarrow (5)$$

$$579.82 + 3.517\,Rb + 6.291\,Rc + 9.357\,Rd = 0 \quad \rightarrow (6)$$

Multiply (4) by 6.291 & (5) by 3.517 & subtract (5) from (4)

$$391.65 \times 6.291 + 9.362 \times 6.291 Rb + 6.292 \times 6.291\,Rc + 3.517 \times 6.291 Rd = 0$$

$$1026.2 \times 3.517 + 6.292 \times 3.517\,Rb + 11.1 \times 3.517\,Rc + 3.517 \times 6.291 Rd = 0$$

$$-1145.275 + 36.767\,Rb + 0.544\,Rc = 0 \quad \rightarrow (7)$$

Multiply (5) by 9.357 & (6) by 6.291 & subtract (6) from (5) :-

$$1026.2 \times 9.357 + 6.292 \times 9.357\,Rb + 11.1 \times 9.357\,Rc + 6.291 \times 9.357 Rd = 0$$

$$579.82 \times 6.291 + 3.517 \times 6.291 Rb + 6.291 \times 6.291\,Rc + 6.291 \times 9.357 Rd = 0$$

$$5954.506 + 36.749\,Rb + 64.286\,Rc = 0 \quad \rightarrow (8)$$

From (7), $Rb = \left(\dfrac{1145.275 - 0.544\,Rc}{36.767}\right)$

Put Rb in (8) & solve for Rc

$$5954.506 + 36.749\left(\frac{1145.275 - 0.544\,Rc}{36.767}\right) + 64.286\,Rc = 0$$

$$5954.506 + 1144.71 - 0.544\,Rc + 64.286\,Rc = 0$$

$$7099.22 + 63.742\,Rc = 0$$

$$\boxed{Rc = -111.374\text{ KN}}$$

Put this value in equation (7) and solve for Rb

$$Rb = \left(\frac{1145.275 - 0.544 \times 111.374}{36.767}\right)$$

$$\boxed{Rb = +32.797\text{ KN}}$$

Put Rb and Rc values in equation (4) to get Rd.

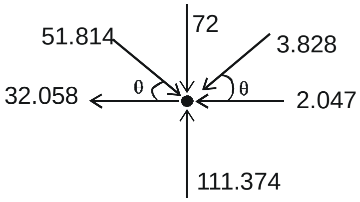
$$391.65 + 9.362 \times 32.797 + 6.292 \times (111.374) + 3.517\,Rd = 0$$

$$\boxed{Rd = +0.588\text{ KN}}$$

After reactions have been calculated, truss is statically determinate and member forces can be easily calculated by $Fi = Fi^{/} + RbU_1 + RcU_2 + RdU_3$ as given in table. Apply checks on calculated member forces.

Step No. 5: Equilibrium checks.

Joint (C)



$\sum Fx = 0$

$-2.047 - 32.058 - 3.828 \times 0.707 + 51.814 \times 0.707 = 0$

$-0.179 \cong 0$

$0 = 0$

$\sum Fy = 0$

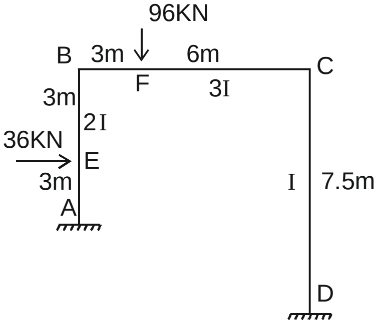
$111.374 - 72 - 3.828 \times 0.707 - 51.814 \times 0.707 = 0$

$0.035 \cong 0$

$0 = 0$ (satisfied) Solution is alright.

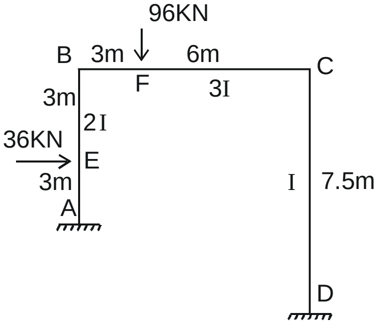
1.21: ANALYSIS OF 3-DEGREE REDUNDANT FRAMES

Example No. 8: Analyze the following frame by consistent deformation method.



SOLUTION :-

The given frame is statically indeterminate to the 3rd degree. So that three redundants have to be removed at support D or A. Consider H_D, V_D & M_D as the redundants



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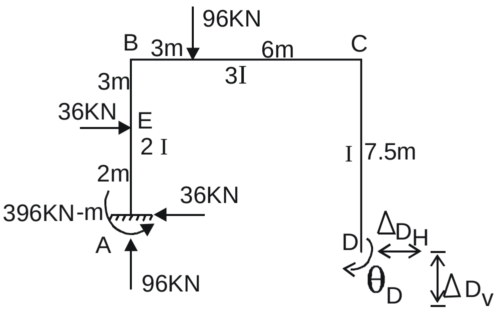
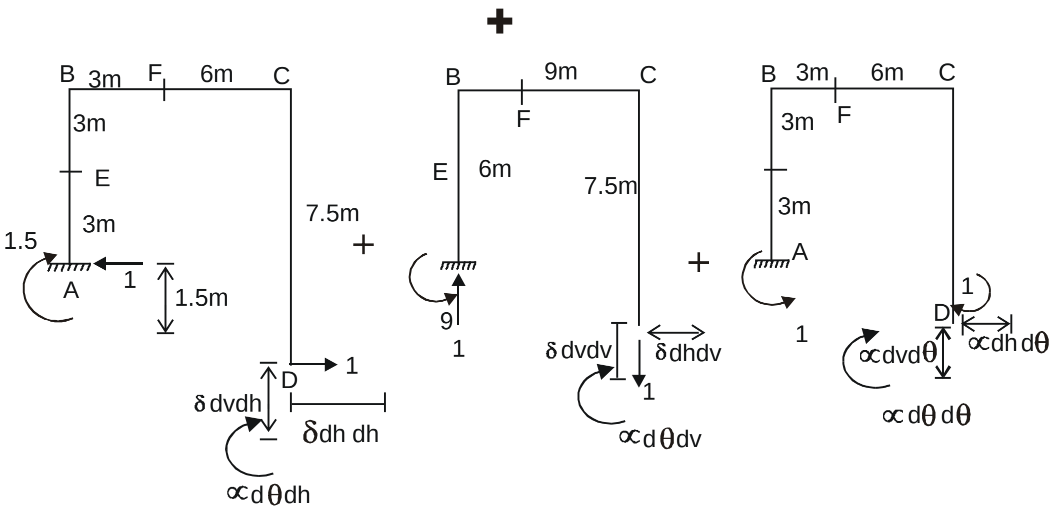


Fig. 2.45 B.D.S under loads



mH-Diagram mV-Diagram mθ-diagram

(BDS under redundants)

Compatibility Equations:-

$\Delta D_H + H_D \times \delta dh.dh + V_D \times \delta dhdv + M_D \times \alpha dhd\theta = 0$ (1) compatibility in horizontal direction at D.

$\Delta D_V + H_D \times \delta dv.dh + V_D \times \delta dvdv + M_D \times \alpha dvd\theta = 0$ (2) compatibility in vertical direction at D.

$\theta_D + H_D \times \alpha d\theta.dh + V_D \times \alpha d\theta dv + M_D \times \alpha d\theta d\theta = 0$ (3) rotational compatibity at D.

We have to determine the following flexibility co-efficients.

ΔD_H = Horizontal deflection of point D due to applied loads.

ΔD_V = Vertical deflection of point D due to applied loads.

θ_D = Rotation of point D due to applied loads.

$\delta dhdh$ = Horizontal deflection of point D due to unit horizontal redundant force at D

$\delta dhdv$ = Horizontal deflection of point D due to unit vertical redundant force at D

$\alpha d\theta dh$ = angular deflection of point D due to unit angular redundant force at D

$\delta dvdh$ = Vertical deflection of point D due to unit horizontal redundant force at D

$\delta dvdv$ = Vertical deflection of point D due to unit vertical redundant force at D

$\alpha d\theta dv$ = Rotation deflection of point D due to unit vertical redundant force at D

$\alpha dhd\theta$ = Horizontal rotation of point D due to unit rotation at pt D

$\alpha dvd\theta$ = Vertical rotation of point D due to unit rotation at pt D

$\alpha d\theta d\theta$ = Rotation rotation of point D due to unit rotation at pt D

$\delta dvdh = \delta dhdv$ (reciprocal deformations)

$\alpha d\theta dh = \alpha dhd\theta$ (reciprocal deformations)

$\alpha d\theta dv = \alpha dvd\theta$ (reciprocal deformations)

Now these flexibility co-efficients can be evaluated by following formulae.

$$\Delta D_H = \int \frac{M \times mH}{EI}\, dX$$

$$\Delta D_V = \int \frac{M \times mV}{EI}\, dX$$

$$\theta_D = \int \frac{M \text{ x } m\theta}{EI}\, dX$$

$$\delta dhdh = \int \frac{(mH)^2\, dX}{EI}$$

$$\delta dvdv = \int \frac{(mv)^2\, dX}{EI}$$

$$\left.\begin{aligned} \alpha d\theta dh &= \alpha dhd\theta = \int \left(\frac{mH \times m\theta}{EI}\right) dX \\ \delta dhdv &= \delta dvdh = \int \left(\frac{mv \times mH}{EI}\right) dX \\ \alpha d\theta dv &= \alpha dvd\theta = \int \left(\frac{mv \times m\theta}{EI}\right) dX \end{aligned}\right] \text{ from law of reciprocals deformations}$$

$$\alpha d\theta d\theta = \int \frac{m^2\theta}{EI}\, dX$$

ESTABLISH MOMENT EXPRESSIONS BY FREE BODY DIAGRAMS:

Note: Moments giving compression on outside and tension on inside of frame (sagging) will be positive.

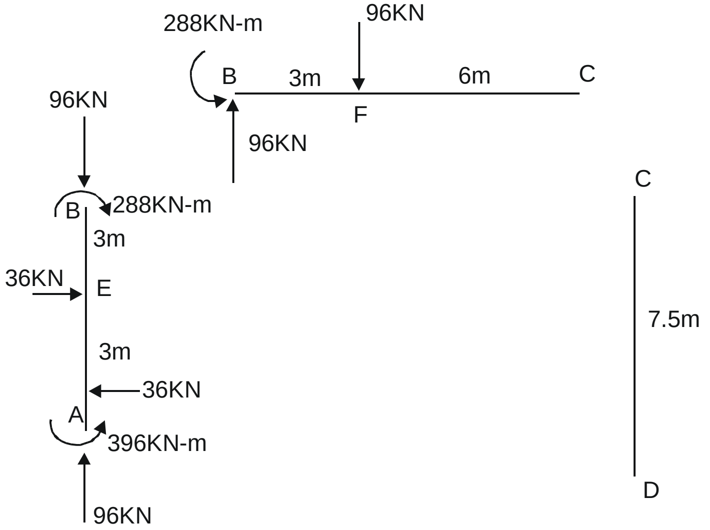


Fig 2.46 B.D.S under loads (M-diagram)

$\Sigma M_b = 0$

$$M_b + 36 \times 6 - 396 - 36 \times 3 = 0$$

$$M_b = +288 \text{ KN} - \text{m.}$$

$\Sigma M_c = 0$

$$M_c + 96 \times 9 - 288 - 96 \times 6 = 0$$

$$M_c + 0 = 0$$

$$M_c = 0$$

Free body m – Diagrams

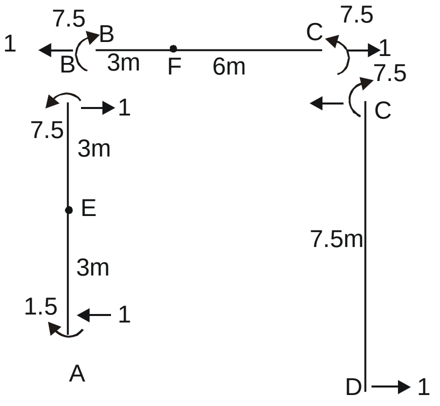


Fig. 2.46a mH-Diagam

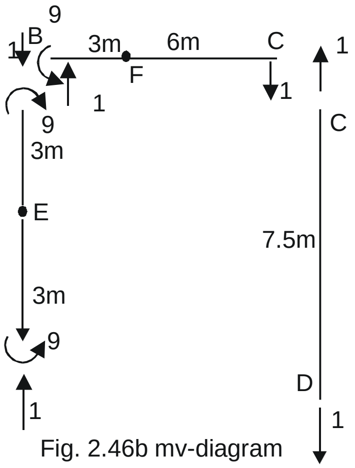


Fig. 2.46b mv-diagram

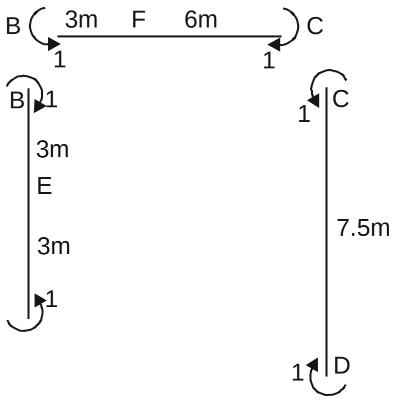


Fig. 2.46 mθ diagram

Write moment expressions alongwith limits in a tabular form

Portion	**Origin**	**Limits**	**M**	**MH**	**Mv**	**Mθ**	**I**
AE	A	0 – 3	36X–396	X + 1.5	– 9	– 1	2I
BE	B	0 – 3	– 288	–X + 7.5	– 9	– 1	2I
BF	B	0 – 3	96X–288	+ 7.5	+ X – 9	– 1	3I
CF	C	0 – 6	0	+ 7.5	– X	– 1	3I
CD	D	0 –7.5	0	+ X	0	– 1	I

It may be done in a tabular form or may be directly evaluated.

CALCULATIONS OF FLEXIBILITY CO-EFFICIENTS:-

$$\Delta DH = \frac{1}{EI}\int M \times mH\, dX$$

$$= \frac{1}{2EI}\int_o^3 (36X - 396)(X+1.5\,)dX + \frac{1}{2EI}\int_o^3 (-288)(-X+7.5)\, dX + \frac{1}{3EI}\int_o^3 (96X - 288)(7.5)dX + \int_o^6 0 + \int_o^{7.5} 0$$

$$= \frac{1}{2EI}\int_o^3 (36X^2+54X - 396X - 594)\, dX + \frac{1}{2EI}\int_o^3 (288X - 2160)\, dX + \frac{1}{3EI}\int_o^3 (720X - 2160)\, dX$$

$$= \frac{1}{2EI}\int_o^3 (36X^2 - 54X - 2754)\, dX + \frac{1}{3EI}\int_o^3 (720X - 2160)dX \text{ , (First two integrals have been combined)}$$

$$= \frac{1}{2EI}\left|\frac{36X^3}{3} - \frac{54X^2}{2} - 2754\, X\right|_o^3 + \frac{1}{3EI}\left|\frac{720X^2}{2} - 2160X\right|_o^3$$

$$= \frac{1}{2EI}\left(12 \times 3^3 - \frac{54}{2} \times 3^2 - 2754 \times 3\right) + \frac{1}{3EI}\left(\frac{720}{2} \times 3^2 - 2160 \times 3\right) - \frac{4090.5}{EI} - \frac{1080}{EI}$$

$$\boxed{\Delta D_H = \frac{51.705}{EI}}$$

$$\delta dhdh = \frac{1}{EI}\int mH^2\, dX$$

$$= \frac{1}{2EI}\int_o^3 (X + 1.5)^2 dX + \frac{1}{2EI}\int_o^3 (-X+7.5)^2 dX + \frac{1}{3EI}\int_o^3 (7.5)^2 dX + \frac{1}{3EI}\int_o^6 (7.5)^2 dX + \frac{1}{EI}\int_o^{7.5} X^2 dX$$

$$= \frac{1}{2EI}\int_o^3 (X^2+3X+2.25)dX + \frac{1}{2EI}\int_o^3 (X^2-15X+56.25)dX + \frac{1}{3EI}\int_o^3 56.25\, dX + \frac{1}{3EI}\int_o^6 56.25\, dX + \frac{1}{EI}\int_o^{7.5} X^2\, dX$$

$$= \frac{1}{2EI}\left|\frac{X^3}{3} + \frac{3X^2}{2} + 2.25X\right|_o^3 + \frac{1}{2EI}\left|\frac{X^3}{3} - \frac{15X^2}{2} + 56.25X\right|_o^3 + \frac{1}{3EI}\Big|56.25X\Big|_o^3 + \frac{1}{3EI}\Big|56.25X\Big|_o^6 + \frac{1}{EI}\left|\frac{X^3}{3}\right|_o^{7.5}$$

$$= \frac{1}{2EI}\left(\frac{3^3}{3}\times\frac{3}{2}\times 3^2 + 2.25\times 3\right) + \frac{1}{2EI}\left(\frac{3^3}{3} - \frac{15}{2}\times 3^2 + 56.25\times 3\right) + \frac{1}{3EI}(56.25\times 3) + \frac{1}{3EI}(56.25\times 6) + \frac{1}{3EI}\left(\frac{7.5^3}{3}\right)$$

$$= \frac{14.625}{EI} + \frac{55.125}{EI} + \frac{56.25}{EI} + \frac{112.5}{EI} + \frac{140.625}{EI}$$

$$\boxed{\delta dhdh = +\frac{379.125}{EI}}$$

$$\alpha dhd\theta = \frac{1}{EI}\int (mH \times m\theta)\, dX$$

$$\alpha dhd\theta = \frac{1}{2EI}\int_o^3 (X+1.5)(-1)dX + \frac{1}{2EI}\int_o^3 (-X+7.5)(-1)dX + \frac{1}{3EI}\int_o^3 (7.5)(-1)dX + \frac{1}{3EI}\int_o^6 (7.5)(-1)dX + \frac{1}{EI}\int_o^{7.5} (X)(-1)dX$$

$$= \frac{1}{2EI}\int_o^3 (-X-1.5)dX + \frac{1}{2EI}\int_o^3 (X-7.5)dX + \frac{1}{3EI}\int_o^3 (-7.5)dX + \frac{1}{3EI}\int_o^6 (-7.5)dX + \frac{1}{EI}\int_o^{7.5} (-X)$$

$$= \frac{1}{2EI}\int_o^3 (-9)dX + \frac{1}{2EI}\int_o^3 (-7.5)dX + \frac{1}{3EI}\int_o^6 (-7.5)dX + \frac{1}{EI}\int_o^{7.5} (-X)dX$$

$$= \frac{1}{2EI}\Big|-9X\Big|_o^3 + \frac{1}{3EI}\Big|-7.5X\Big|_o^3 + \frac{1}{3EI}\Big|-7.5X\Big|_o^6 + \frac{1}{EI}\left|-\frac{X^2}{2}\right|_o^{7.5}$$

$$= \frac{1}{2EI}(-9\times 3) + \frac{1}{3EI}(-7.5\times 3) + \frac{1}{3EI}(-7.5\times 6) + \frac{1}{EI}\left(-\frac{(7.5)^2}{2}\right)$$

$$\boxed{\alpha dhd\theta = -\frac{64.125}{EI}}$$

$$\Delta Dv = \frac{1}{EI}\int (M \times mv)\, dX$$

$$\Delta Dv = \frac{1}{2EI}\int_o^3 (36X - 396)(-9)\, dX + \frac{1}{2EI}\int_o^3 (-288)(-9)\, dX + \frac{1}{3EI}\int_o^3 (96X - 288)(X-9)dX + 0 + 0$$

$$= \frac{1}{2EI}\int_o^3 (-324X+3564)\,dX + \frac{1}{2EI}\int_o^3 2592\,dX + \frac{1}{3EI}\int_o^3 (96X^2-864X-288X+2592)\,dX$$

$$= \frac{1}{2EI}\int_o^3 (-324X+6156)\,dX + \frac{1}{3EI}\int_o^3 (96X^2-1152X+2592)\,dX$$

$$= \frac{1}{2EI}\left|\frac{-324X^2}{2}+6156X\right|_o^3 + \frac{1}{3EI}\left|\frac{96X^3}{3}-\frac{1152X^2}{2}+2592X\right|_o^3$$

$$= \frac{1}{2EI}(-162\times 3^2+6156\times 3) + \frac{1}{3EI}(32\times 3^3-576\times 3^2+2592\times 3)$$

$$= \frac{8505}{EI} + \frac{1152}{EI}$$

$$\boxed{\Delta Dv = \frac{9657}{EI}}$$

$$\delta dvdv = \frac{1}{EI}\int (mv)^2\,dX$$

$$= \frac{1}{2EI}\int_o^3 (-9)^2\,dX + \frac{1}{2EI}\int_o^3 (-9)^2\,dX + \frac{1}{3EI}\int_o^3 (X-9)^2\,dX + \frac{1}{3EI}\int_o^6 (-X)^2\,dX + \frac{1}{EI}\int_o^{7.5} (0)\,dX$$

$$= \frac{1}{2EI}\int_o^3 162\,dX + \frac{1}{3EI}\int_o^3 (X^2-18X+81)\,dX + \frac{1}{3EI}\int_o^6 X^2\,dX$$

$$= \frac{162}{2EI}\left|X\right|_o^3 + \frac{1}{3EI}\left|\frac{X^2}{3}-\frac{18X^2}{2}+81X\right|_o^3 + \frac{1}{3EI}\left|\frac{X^3}{3}\right|_o^6$$

$$= \frac{81(3)}{EI} + \frac{1}{3EI}\left(\frac{3^3}{3}-9\times 3^2+81\times 3\right) + \frac{1}{3EI}\left(\frac{6^3}{3}\right)$$

$$\boxed{\delta dvdv = +\frac{324}{EI}}$$

$$\alpha dvd\theta = \frac{1}{EI}\int (mv\times m\theta)\,dX$$

$$\alpha dvd\theta = \frac{1}{2EI}\int_o^3 9\,dX + \frac{1}{2EI}\int_o^3 9\,dX + \frac{1}{3EI}\int_o^3 (-X+9)\,dX + \frac{1}{3EI}\int_o^6 \times\,dX + 0$$

$$= \frac{1}{2EI}\left|9X\right|_o^3 + \frac{1}{2EI}\left|9X\right|_o^3 + \frac{1}{3EI}\left|-\frac{X^2}{2}+9X\right|_o^3 + \frac{1}{3EI}\left|\frac{X^2}{2}\right|_o^6$$

$$= \frac{1}{2EI}(9\times 3) + \frac{1}{2EI}(9\times 3) + \frac{1}{3EI}\left(\frac{-9}{2}+9\times 3\right) + \frac{1}{3EI}\left(\frac{36}{2}\right)$$

$$\boxed{\alpha dvd\theta = +\frac{40.5}{EI}}$$

$$\alpha d\theta d\theta = \frac{1}{EI}\int (m\theta)^2 \, dX$$

$$\alpha d\theta d\theta = \frac{1}{2EI}\int_o^3 1dX + \frac{1}{2EI}\int_o^3 1dX + \frac{1}{3EI}\int_o^3 1dX + \frac{1}{3EI}\int_o^6 1dX + \frac{1}{EI}\int_o^{7.5} 1dX$$

$$= \frac{1}{2EI}\Big| X \Big|_o^3 + \frac{1}{3EI}\Big| X \Big|_o^3 + \frac{1}{3EI}\Big| X \Big|_o^6 + \frac{1}{EI}\Big| X \Big|_o^{7.5}$$

$$= \frac{1}{EI}(3) + \frac{1}{3EI}(3) + \frac{1}{3EI}(6) + \frac{1}{EI}(7.5)$$

$$\boxed{\alpha d\theta d\theta = +\frac{13.5}{EI}}$$

$$\theta_D = \frac{1}{EI}\int (M \text{ x } m\theta) \, dX$$

$$= \frac{1}{2EI}\int_o^3 (-36X + 396)\, dX + \frac{1}{2EI}\int_o^3 288\, dX + \frac{1}{3EI}\int_o^3 (-96X + 288)\, dX$$

$$= \frac{1}{2EI}\int_o^3 (-36X + 684)\, dX + \frac{1}{3EI}\int_o^3 (-96X + 288)\, dX$$

$$= \frac{1}{2EI}\left| -36\frac{X^2}{2} + 684X \right|_o^3 + \frac{1}{3EI}\left| -96\frac{X^2}{2} + 288X \right|_o^3$$

$$= \frac{1}{2EI}(-18 \times 9 + 684 \times 3) + \frac{1}{3EI}(-48 \times 9 + 288 \times 3)$$

$$\boxed{\theta_D = +\frac{1089}{EI}}$$

$$\delta dhdv = \frac{1}{EI}\int (m_H \times m_v)\, dX$$

$$\delta dhdv = \frac{1}{2EI}\int_o^3 (-9X - 13.5)dX + \frac{1}{2EI}\int_o^3 (+9X - 67.5)dX + \frac{1}{3EI}\int_o^3 (7.5x - 67.5)dX + \frac{1}{3EI}\int_o^6 (-7.5X)\, dX + 0$$

$$= \frac{1}{2EI}\int_o^3 (-81)dX + \frac{1}{3EI}\int_o^3 (7.5X - 67.5)\, dx + \frac{1}{3EI}\int_o^6 (-7.5X)\, dX$$

$$= \frac{1}{2EI}\left|-81X\right|_o^3 + \frac{1}{3EI}\left|\frac{7.5X^2}{2} - 67.5X\right|_o^3 + \frac{1}{3EI}\left|-\frac{7.5X^2}{2}\right|_o^6$$

$$= \frac{1}{2EI}(-81 \times 3) + \frac{1}{3EI}\left(\frac{7.5}{2} \times 9 - 67.5 \times 3\right) + \frac{1}{3EI}\left(-\frac{-7.5}{2} \times 36\right)$$

$$\boxed{\delta dhdv = -\frac{222.75}{EI}}$$

Putting above evaluated flexibility co–efficients in compatibility equations , we have.

$$(1) \Rightarrow -5170.5 + 379.125\, H_D - 222.75\, V_D - 64.125\, M_D = 0 \quad \rightarrow (4)$$

$$(2) \Rightarrow +9657 - 222.75\, H_D + 324\, V_D + 40.5\, M_D = 0 \quad \rightarrow (5)$$

$$(3) \Rightarrow +1089 - 64.125\, H_D + 40.5\, V_D + 13.5\, M_D = 0 \quad \rightarrow (6)$$

Multiply (4) by 222.75 & (5) by 379.125 Then add (4) & (5) to eliminate H_D

$$-(5170.5 \times 222.75) + (379.125 \times 222.75)H_D - (222.75)^2 V_D - (64.125 \times 222.75)M_D = 0$$

$$+(9657 \times 379.125) - (379.125 \times 222.75)H_D + (324 \times 379.125)V_D + (40.5 \times 379.125)\, M_D = 0$$

$$2509481.25 + 73218.9375\, V_D + 1070.72\, M_D = 0 \quad \rightarrow (7)$$

Multiply (5) by 64.125 & (6) by 222.75 & subtract (6) from (5) to eliminate H_D again

$$619255.125 - 14283.84\, H_D + 20776.5\, V_D + 2597.06\, M_D = 0$$

$$-242574.75 - 14283.84\, H_D + 9021.375\, V_D + 3007.125\, M_D = 0$$

$$376680.375 + 11755.125\, V_D - 410.065\, M_D = 0 \quad \rightarrow (8)$$

Now equation (7) and (8) are in terms of V_D and M_D

$$\text{From (7), } V_D = \left(\frac{-1070.72\, M_D - 2509481.25}{73218.9375}\right) \quad \rightarrow (9)$$

Put V_D in (8) to get M_D

$$376680.375 + 11755.125\left(\frac{-1070.72\, M_D - 2509481.25}{73218.9375}\right) - 410.065 M_D = 0$$

$$376680.375 - 171.90\, M_D - 402891.20 - 410.065\, M_D = 0$$

$$-26210.83 - 581.965\, M_D = 0$$

M_D = – 45.04 KN–m, put this in (9) to get V_D

$$V_D = \left[\frac{-1070.72 \times (45.04) - 2509481.25}{73218.9375}\right]$$

V_D = – 33.62 KN. Now put values of V_D and M_D in (4) to get H_D

$$-5170.5+379.125 \times H_D+222.75 \times 33.62 + 64.125 \times 45.04 = 0$$

$$379.125\ H_D + 5205.44 = 0$$

H_D = – 13.73 KN

$$\boxed{\begin{aligned} H_D &= -13.73 \text{ KN} \\ V_D &= -\ 33.62 \text{ KN} \\ M_D &= -\ 45.64 \text{ KN – m} \end{aligned}}$$

These reactions are applied to frame which becomes statically determinate now and shear force and moment diagram can be sketched (by parts) now.

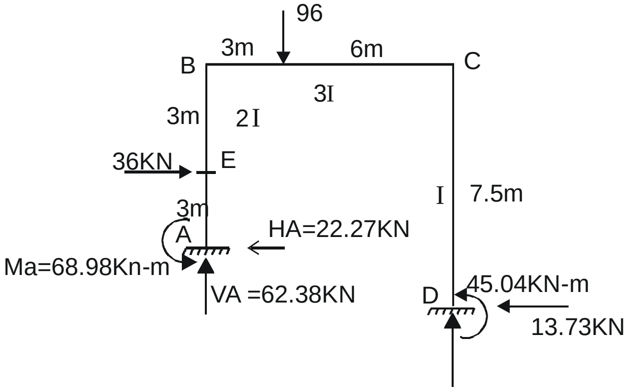


Fig. 2.47

Applying condition of equilibrium at A, reactions can be obtained.

$\sum$ FX = 0

$36 - H_A - 13.73 = 0$

$\boxed{H_A = 22.27 \text{ KN}}$

$\sum$Fy = 0

$V_A + 33.62 - 96 = 0$

$\boxed{V_A = 62.38 \text{ KN}}$

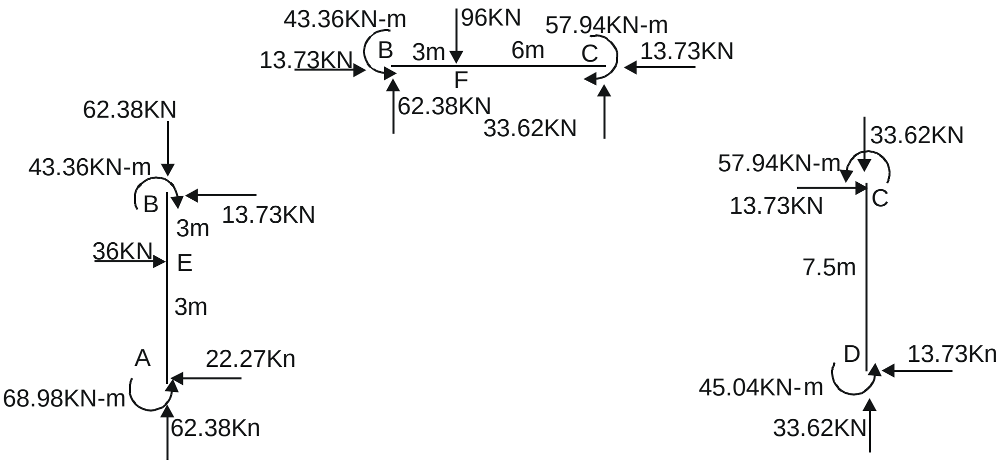
$\sum M = 0$

$M_A + 45.04 - 13.73 \times 1.5 + 33.62 \times 9 - 96 \times 3 - 36 \times 3 = 0$

$M_A - 68.98 = 0$

$\boxed{M_A = 68.98 \text{ KN-m}}$

Applying these reactions to frame, various free-body diagrams can be drawn and moments expressions can be set-up for determining combined deflections of any point due to applied loads and reactions (at supports) acting simultaneously.

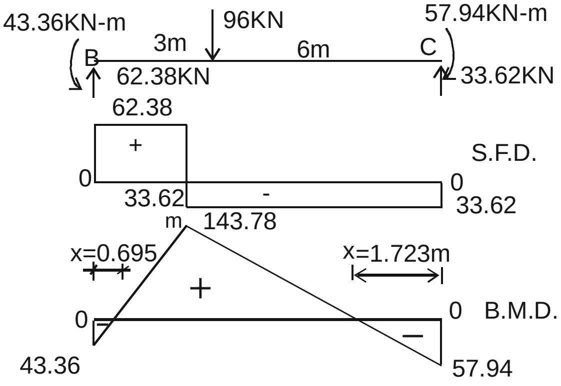


$\Sigma M_b = 0$, Mb+22.27 x 6-68.98-36 x 3 = 0
$M_b = 43.36$ KN-m

$\Sigma M_C=0$, M+62.38 x 9-43.36-96x 6=0
M_C=57.94 KN-m (for beam)

BENDING MOMENT AND SHEAR FORCE DIAGRAMS :–

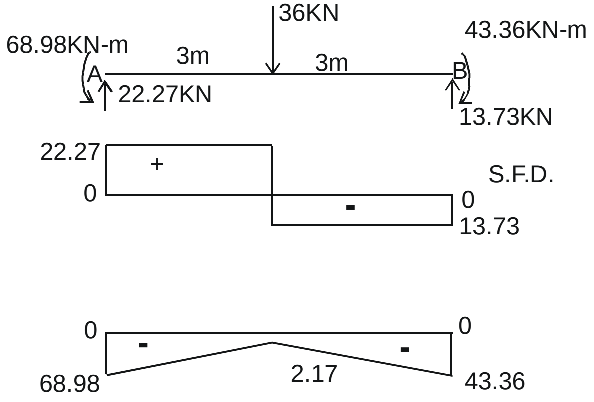
For beam BC



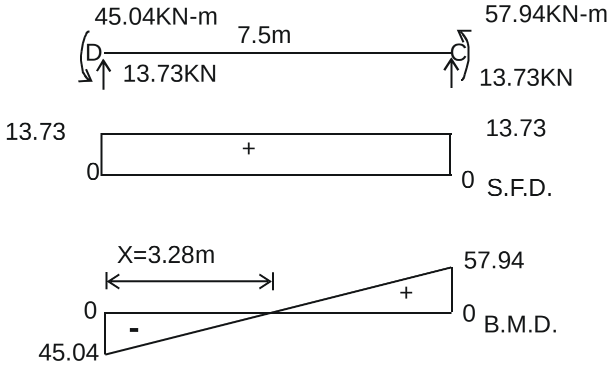
Mx = –45.04 + 13.73x = 0

x = 3.28 m

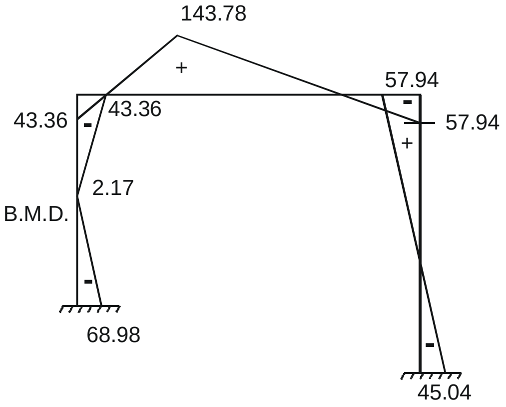
FOR COLUMN AB
(Seen rotated at 90°)

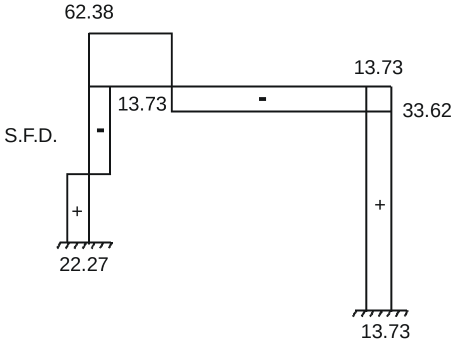


FOR COLUMN DC
(Seen rotated at 90°)



Mx=-45.04+13.73x = 0
x = 3.28m

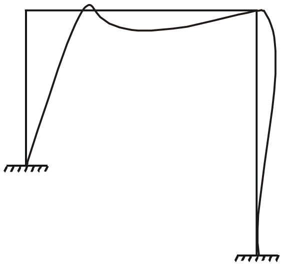




Composite S.F.D. for analysed frame

Fig. 2.48

Elastic Curve:-

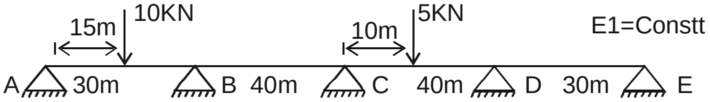


1.22: Analysis of Continuous Beams

Example No. 9:

Analyze the following beam by consistent deformation method. Check the results by the method of least work.

SOLUTION:-

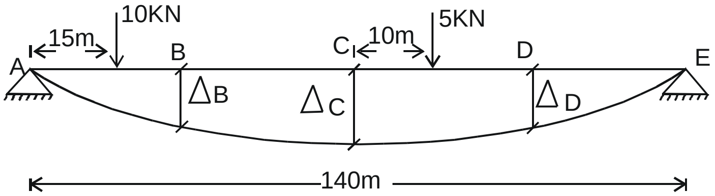


Number of reactions=5
number of equations=2

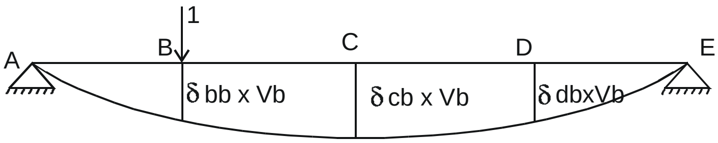
Fig. 2.56

Step No.1:

In this structure, we treat reactions at B, C & D as redundants and the B.D.S. is a simply supported beam AE.



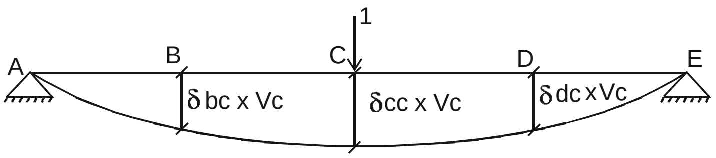
B.D.S. Under applied loads.
Fig. 2.56a



B.D.S. Under Unit redundant load at B.

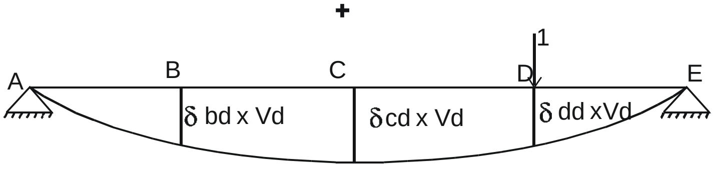
Fig. 2.56 b

+



B.D.S. under Unit redundant load at C.

Fig. 2.56c



B.D.S. under Unit redundant load at D.

Fig. 2.56d

Step No.2: Compatibility Equations.

$\Delta B + Vb \times \delta bb + Vc \times \delta bc + Vd \times \delta bd = 0 \rightarrow (1)$ Compatibility of deformations at B

$\Delta C + Vb \times \delta cb + Vc \times \delta cc + Vd \times \delta cd = 0 \rightarrow (2)$ Compatibility of deformations at C

$\Delta D + Vb \times \delta db + Vc \times \delta dc + Vd \times \delta dd = 0 \rightarrow (3)$ Compatibility of deformation at D

Sketch BDS, Draw SFD, and $\frac{M}{EI}$ diagram for use in conjugate beam method.

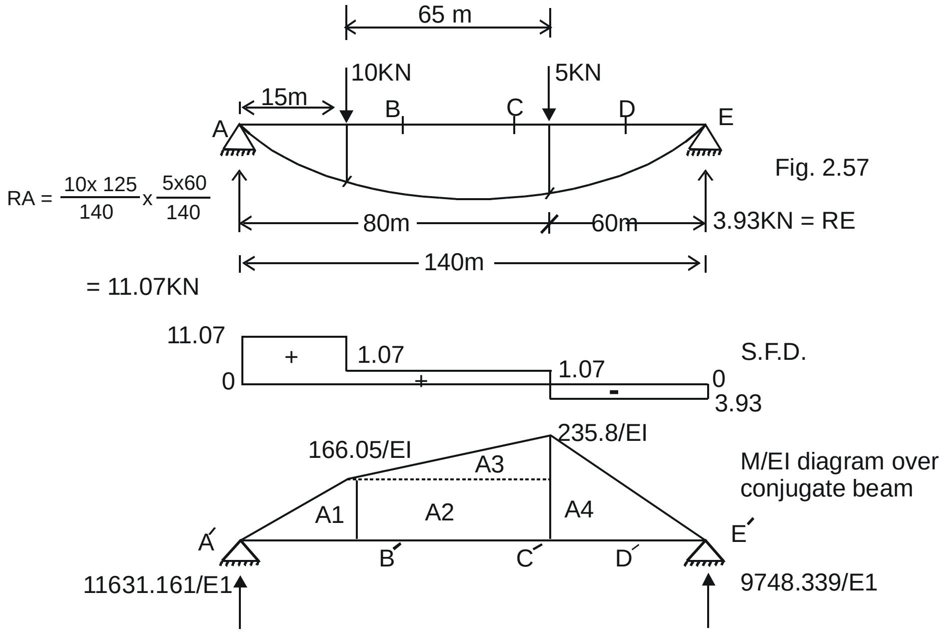


Fig. 2.57

Splitting above $\frac{M}{EI}$ in 4 parts as shown, calculate areas of these portions.

$$A_1 = \frac{1}{2} \times 15 \times \frac{166.05}{EI} = \frac{1245.375}{EI}$$

$$A_2 = \frac{166.05}{EI} \times 65 = \frac{10793.25}{EI}$$

$$A_3 = \frac{1}{2} \times \frac{69.75}{EI} \times 65 = \frac{2266.875}{EI}$$

$$A_4 = \frac{1}{2} \times 235.8 \times 60 = \frac{7074}{EI}$$

$$A1+A2+A3+A4 = \frac{21379.5}{EI}$$

$$\sum M'_E = 0$$

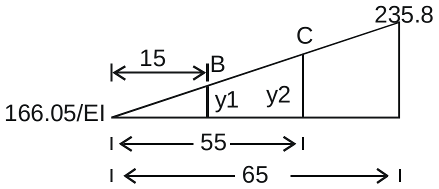
$$R_A' \text{ x } 140 = \frac{1}{EI}\left[1245.375\left(125+\frac{15}{3}\right)+10793.25\left(60+\frac{65}{2}\right)+2266.875\left(60+\frac{65}{3}\right)+7074\left(\frac{2}{3}\times 60\right)\right]$$

$$R_A' = \frac{11631.161}{EI}$$

$$R_E' = \frac{21379.5}{EI} - \frac{11631.161}{EI}$$

$$R_E' = \frac{9748.339}{EI}$$

Isolating the upper part of $\frac{M}{EI}$ diagram between two loads.



$$\frac{y_2}{55} = \frac{235.8}{65}$$

By conjugate beam method, ΔB would be moment at B' of conjugate beam loaded with $\frac{M}{EI}$ diagram.

$y_2 = 199.52$

$y_1 = 54.4$

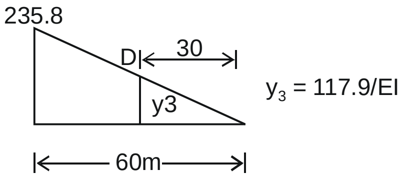
$$\Delta B = \frac{1}{EI}\left[11631.161\times 30 - 1245.375\left(15+\frac{15}{3}\right) - (166.05\times 15)\times 7.5 - \left(54.42\times\frac{15}{2}\right)\times\left(\frac{15}{3}\right)\right]$$

$$= \frac{303080.955}{EI} \quad \text{KN–m}^3$$

Moment at C' of conjugate beam

$$\Delta C = \frac{1}{EI}\left[11631.161\times 70-(1245.375)\left(\frac{15}{3}+55\right)-(166.05\times 55)\left(\frac{55}{2}\right)-\left(\frac{1}{2}\times 100.52\times 5.5\right)\times\left(\frac{1}{3}\times 55\right)\right]$$

$$= \frac{387716.812}{EI} \quad \text{KN–m}^3$$



Isolating the portion of $\frac{M}{EI}$ diagram between right support and 5 KN load.

Moment at D' of conjugate beam

$$\Delta D = \frac{1}{EI}\left[9748.339 \times 30 - \left(\frac{1}{2} \times 117.9 \times 30\right) \times \frac{30}{3}\right]$$

$$\Delta D = \frac{274765.17}{EI} \qquad \text{KN–m}^3$$

If we construct $\frac{M}{EI}$ diagram for above figures 2.56b, 2.56c and 2.56d and place them over conjugate beam, we have δcb= 34501.88, δcc= 57166.66, δcd= 34501.88 on similar lines as above. From conjugate beam for fig: 2.56b, you will have

$$\delta bb = \frac{1}{EI}\left[982.086 \times 30 - (353.565)\left(\frac{30}{30}\right)\right] = \frac{25926.93}{EI}$$

$$\delta cb = \frac{1}{EI}\left[667.884 \times 70 - \left(\frac{1}{2} \times 15 \times 70\right)\left(\frac{70}{3}\right)\right] = \frac{34501.88}{EI}$$

$$\delta db = \frac{1}{EI}\left[667.884 \times 30 - \left(\frac{1}{2} \times 6.423 \times 30\right)\left(\frac{30}{3}\right)\right] = \frac{19073.07}{EI}$$

We already know from law of reciprocal deflections that

δcb = δbc

δbd = δdb

δcd = δdc

From conjugate beam for fig: 2.5d, you will have

$$\delta cd = \frac{1}{EI}\left[667.884 \times 70 - \left(\frac{15 \times 70}{2}\right)\left(\frac{70}{3}\right)\right] = \frac{34501.88}{EI}$$

$$\delta dd = \frac{1}{EI}\left[982.086 \times 30 - \left(\frac{1}{2} \times 23.571 \times 30\right)\left(\frac{30}{EI}\right)\right] = \frac{25926.93}{EI}$$

Putting above flexibility co-efficients in compatibility equations, we have

303080.955 + 25926.93 Vb + 34500 Vc + 19073.07 Vd = 0 → (1)

387716.812 + 34501.88 Vb + 57166.67 Vc + 34501.88 Vd = 0 → (2)

274765.17 + 1907307 Vb + 34500 Vc + 25926.93 Vd = 0 → (3)

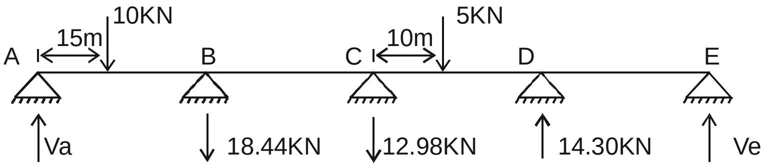
Solving above three linear – simultaneous equations, we have

$$\boxed{\text{Vd} = -14.30 \text{ KN}}$$

$$\boxed{\text{Vc} = 12.98 \text{ KN}}$$

$$\boxed{\text{Vb} = 18.44 \text{ KN}}$$

Now the continuous beam has become determinate. Apply loads and redundants reactions, other support reactions can be determined.



$\sum M_E = 0$

$Va \times 140 - 10 \times 125 - 18.44 \times 110 - 12.98 \times 70 - 5 \times 60 + 14.3 \times 30 = 0$

$$\boxed{Va = 28.9 \text{ KN}}$$

$\sum Fy = 0$

gives $Ve = 3.22$ KN upwards

Now shear force and BMD can be plotted as the beam is statically determinate now.