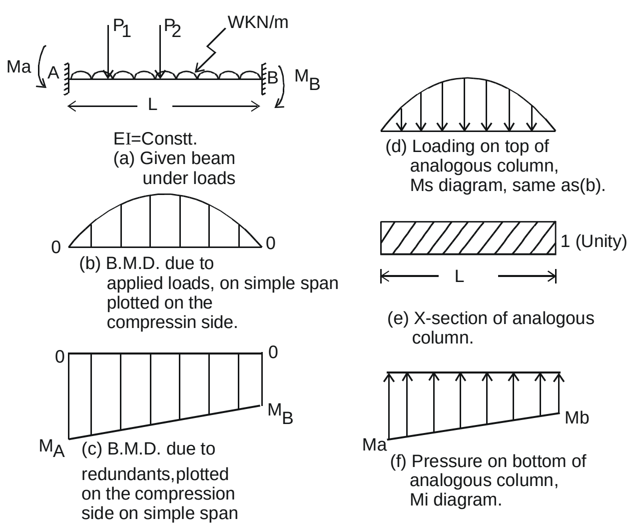
## CHAPTER SEVEN

# 7. INTRODUCTION TO COLUMN ANALOGY METHOD

The column analogy method was also proposed by Prof. Hardy Cross and is a powerful technique to analyze the beams with fixed supports, fixed ended gable frames, closed frames & fixed arches etc., These members may be of uniform or variable moment of inertia throughout their lengths but the method is ideally suited to the calculation of the stiffness factor and the carryover factor for the members having variable moment of inertia. The method is strictly applicable to a maximum of 3rd degree of indeterminacy. This method is essentially an indirect application of the consistent deformation method.

The method is based on a mathematical similarity (i.e. analogy) between the stresses developed on a column section subjected to eccentric load and the moments imposed on a member due to fixity of its supports. *(We have already used an analogy in the form of method of moment and shear in which it was assumed that parallel chord trusses behave as a deep beam). In the analysis of actual engineering structures of modern times, so many analogies are used like slab analogy, and shell analogy etc. In all these methods, calculations are not made directly on the actual structure but, in fact it is always assumed that the actual structure has been replaced by its mathematical model and the calculations are made on the model. The final results are related to the actual structure through same logical engineering interpretation.

In the method of column analogy, the actual structure is considered under the action of applied loads and the redundants acting simultaneously on a BDS. The load on the top of the analogous column is usually the B.M.D. due to applied loads on simple spans and therefore the reaction to this applied load is the B.M.D. due to redundants on simple spans considers the following fixed ended loaded beam.



The resultant of B.M.D's due to applied loads does not fall on the mid point of analogous column section which is eccentrically loaded.

Msdiagram = BDS moment diagram due to applied loads.
Mi diagram = Indeterminate moment diagram due to redundants.

If we plot (+ve) B.M.D. above the zero line and (−ve) B.M.D below the zero line (both on compression sides due to two sets of loads) then we can say that these diagrams have been plotted on the compression side.

(The conditions from which $M_A$ & $M_B$ can be determined, when the method of consistent deformation is used, are as follows). From the Geometry requirements, we know that

(1) The change of slope between points A & B = 0; or sum of area of moment diagrams between A & B = 0 (note that EI = Constt:), or area of moment diagrams of fig.b = area of moment diagram of fig..c.

(2) The deviation of point B from tangent at A = 0; or sum of moment of moment diagrams between A & B about B = 0, or Moment of moment diagram of fig.(b) about B = moment of moment diagram of fig.(c) about B. Above two requirements can be stated as follows.

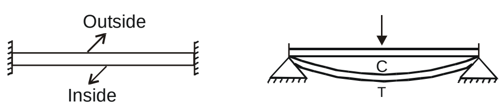
(1) Total load on the top is equal to the total pressure at the bottom and;

(2) Moment of load about B is equal to the moment of pressure about B), indicates that the analogous column is on equilibrium under the action of applied loads and the redundants.

## 7.1. SIGN CONVENTIONS:−

It is necessary to establish a sign convention regarding the nature of the applied load (Ms − diagram) and the pressures acting at the base of the analogous column (Mi−diagram.)

1. Load ( P) on top of the analogous column is downward if Ms/EI diagram is (+ve) which means that it causes compression on the outside or (sagging) in BDS vice-versa. If EI is constant, it can be taken equal to units.




2. Upward pressure on bottom of the analogous column ( Mi − diagram) is considered as (+ve).

3. Moment (M) at any point of the given indeterminate structure ( maximum to $3^{rd}$ degree) is given by the formula.

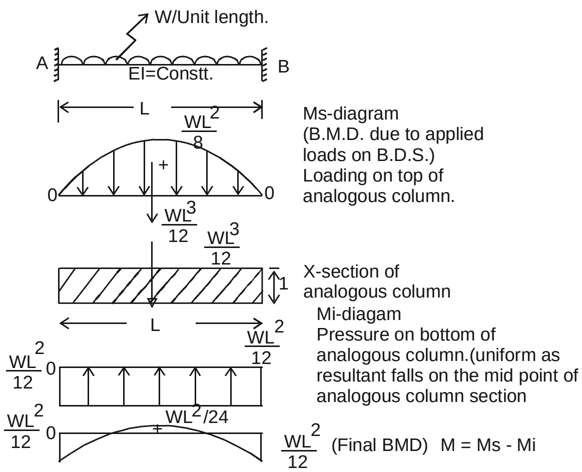
$$M = Ms - Mi,$$

which is (+ve) if it causes compression on the outside of members.

**EXAMPLE NO. 1:**− Determine the fixed−ended moments for the beam shown below by the method of column analogy.

**SOLUTION:**

Choosing BDS as a simple beam. Draw Ms diagram. Please it on analogous column.




Pressure at the base of the column = $\frac{P}{A}$

A = L × I (area of analogous column section).

$$= \frac{WL^3}{12(Lx1)}$$

$$Mi = \frac{WL^2}{12}$$ . In this case, it will be uniform as resultant of Ms diagram falls on centroid of analogous column)

(MS)a = 0 , (Ms at point A to be picked up for M-s diagram)

$$Ma = (Ms - Mi)_a$$ , (net moment at point A)

$$= 0 - \frac{WL^2}{12}$$

$$Ma = -\frac{WL^2}{12}$$

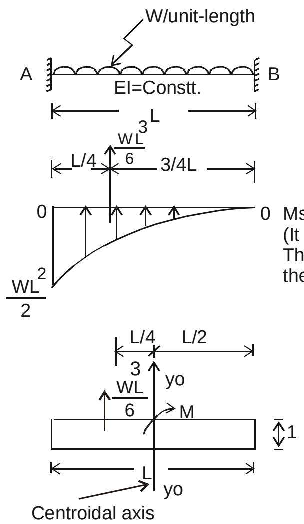
$$Mb = (Ms{-}Mi)b = \left(0 - \frac{WL^2}{12}\right) = \frac{-WL^2}{12}$$

$$Mc = (Ms - Mi)c = \frac{WL^2}{8} - \frac{WL^2}{12}$$

$$Mc = \frac{3\,WL^2 - 2\,WL^2}{24} = \frac{WL^2}{24}$$ . Plot these values to get M = Ms − Mi diagram.

The beam has been analyzed.

**EXAMPLE NO. 2:- SOLVING THE PREVIOUS EXAMPLE, IF B.D.S. IS A CANTILEVER SUPPORTED AT 'A'**.




Ms-diagram
(It creates hagging so load acts upwards)
The resultant of Ms diagram does not fall on the centroid of analogous column.

X-section of analogous column. Carrying eccentric load of $WL^3/6$

Eccentric load $wL^3/6$ acts on centre of analogous column x-section with an associated moment as well
(Eccentric load = Concentric load plus accomprying moment)

Area of Ms diagram $\quad A = \dfrac{bh}{(n+1)} = \dfrac{L \times WL^2}{2(2+1)} = \dfrac{WL^3}{6}$

$$X' = \frac{b}{(n+2)} = \frac{L}{(2+2)} = \frac{L}{4} \qquad \text{(from nearest and)}$$

Alternatively centroid can be located by using the following formula)

$$\bar{X} = \frac{\int MXdX}{\int MdX}$$

$$\int MdX = \int_o^L \left(-\frac{WX^2}{2}\right) dX = -\frac{W}{2} \left|\frac{X^3}{3}\right|_o^L = -\frac{WL^3}{6} \text{ ( Same as above)}$$

$$\int MXdX = \int_o^L \left(-\frac{WX^2}{2}\right) XdX = \int_o^L -\frac{WX^3}{2} dx$$

$$= -\frac{W}{2} \left| \frac{X^4}{4} \right|_o^L = -\frac{WL^4}{8}$$

$$\bar{X} = \frac{\int MXdX}{\int MdX}$$

$$\bar{X} = \frac{-WL^4}{8} \times \frac{6}{(-WL^3)} = \frac{3}{4} L.$$ (from the origin of moment expression or from farthest end)

**NOTE :** Moment expression is always independent of the variation of inertia.

**Properties of Analogous Column X−section :−**

1. Area of analogous column section, $A = L \times 1 = L$
2. Moment of inertia, $I\, y_o\, y_o = \frac{L^3}{12}$
3. Location of centroidal column axis, $C = \frac{L}{2}$

$A\, e' = M = \left(\frac{WL^3}{6}\right)\left(\frac{L}{4}\right) = \frac{WL^4}{24}$ , ( $\frac{L}{4}$ is distance between axis yo− yo and the centroid of Ms diagram where the load equal to area of Ms diagram acts.)

$$(Mi)a = \frac{P}{A} \pm \frac{Mc}{I}$$ (P is the area of Ms diagram and is acting upwards so negative $C = \frac{L}{2}$ and $I = \frac{L^3}{12}$ )

$$= \frac{-WL^3}{6 . L} - \frac{WL^4 . L . 12}{24 . 2 . L^3}$$ (Load P on analogous column is negative)

$$= \frac{-WL^2}{6} - \frac{WL^2}{4}$$ ( Reaction due to MC/I would be having the same direction at A as that due to P while at B these two would be opposite)

$$= \frac{-2WL^2 - 3\,WL^2}{12}$$

$$= \frac{-5}{12} WL^2$$

$$(Ms)a = \frac{-WL^2}{2}$$

$$Ma = (Ms - Mi)a$$

$$= \frac{-WL^2}{2} + \frac{5}{12} WL^2$$

$$= \frac{-6\,WL^2 + 5\,WL^2}{12}$$

$$Ma = \frac{-WL^2}{12}$$

$$Mb = (Ms - Mi)b$$

$$(Mi)b = \frac{P}{A} \pm \frac{Mc}{I}$$

$$= \frac{-WL^3}{6 \times L} + \frac{WL^4 \times L \times 12}{24 \times 2 \times L^3}$$

$$= \frac{-WL^2}{6} + \frac{WL^2}{4}$$

$$= \frac{-2WL^2 + 3WL^2}{12}$$

$$= \frac{WL^2}{12}$$

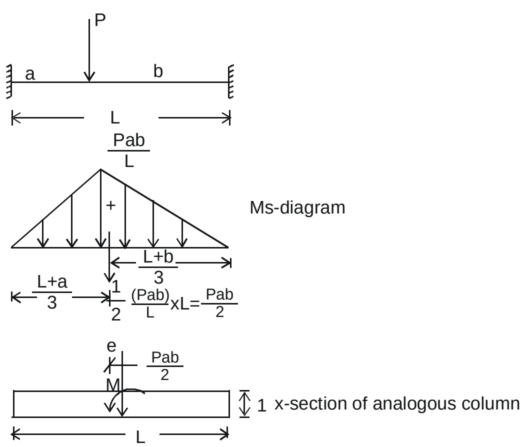
$$(Ms)b = 0$$

$$Mb = (Ms - Mi)b = 0 - \frac{WL^2}{12} = -\frac{WL^2}{12}$$

Same results have been obtained but effort / time involved is more for this BDS).

**EXAMPLE NO. 3:**– Determine the F.E.Ms. by the method of column analogy for the following loaded beam.

**3.1 SOLUTION:– CASE 1 ( WHEN BDS IS A SIMPLE BEAM )**




$$e = \frac{L}{2} - \left(\frac{L + a}{3}\right) = \frac{3L - 2L - 2a}{6} = \left(\frac{L - 2a}{6}\right)$$

( The eccentricity of load w.r.t mid point of analogous column)

$$M = \left(\frac{Pab}{2}\right)\left(\frac{L - 2a}{6}\right) = \frac{Pab}{12}(L - 2a)$$

**Properties of Analogous Column X – section**

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1. $A = L \times 1 = L$

2. $I = \frac{L^3}{12}$

3. $C = \frac{L}{2}$

$$(Mi)a = \frac{P}{A} \pm \frac{Mc}{I}$$

$$= \frac{Pab}{2L} + \frac{Pab}{\cancel{12}}(L - 2a) \times \frac{L \times \cancel{12}}{2 \times L^3}$$

$$= \frac{Pab}{2L} + \frac{Pab}{2L^2}(L - 2a)$$

$$= \frac{PabL + PabL - 2Pa^2b}{2L^2}$$

$$= \frac{2PabL - 2Pa^2b}{2L^2}$$

$$(Mi)a = \frac{PabL - Pa^2b}{L^2}$$

$$= \frac{Pab(L-a)}{L^2} \qquad \therefore a + b = L$$

$$b = L - a$$

$$= \frac{Pab \cdot b}{L^2}$$

$$(Mi)a = \frac{Pab^2}{L^2}$$

$$(Ms)\,a = 0$$

Net moment at A = Ma = (Ms – Mi) a

$$= 0 - \frac{Pab^2}{L^2}$$

$$Ma = -\frac{Pab^2}{L^2}$$

**The (–ve) sign means that it gives us tension at the top when applied at A.**

$$(Mi)b = \frac{P}{A} \pm \frac{MC}{I}$$

$$= \frac{Pab}{2L} - \frac{Pab}{12L^2}(L - 2a) \times \frac{L \times 12}{2 \times L^3}$$

$$= \frac{Pab}{2L} - \frac{Pab}{2L^2}(L - 2a)$$

$$= \frac{PabL - PabL + 2Pa^2b}{2L^2}$$

$$= \frac{2Pa^2b}{2L^2}$$

$$(Mi)b = \frac{Pa^2b}{L^2}$$

$$(Ms)b = 0$$

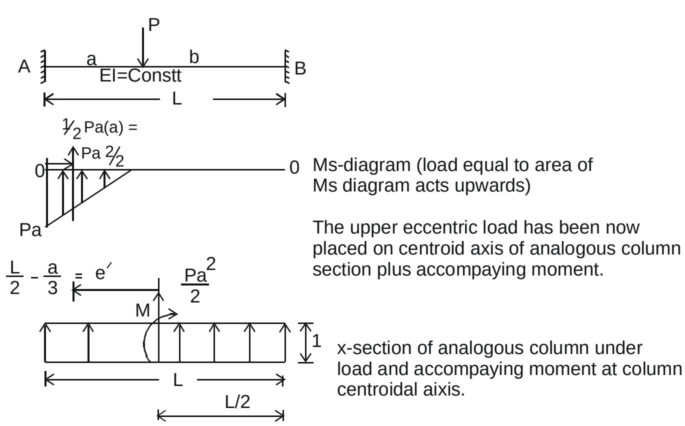
$$Mb = (Ms - Mi)a = 0 - \frac{Pa^2b}{L^2}$$

$$Mb = \frac{-Pa^2b}{L^2}$$

**The minus sign means that it gives us tension at the top.**

**EXERCISE 3.2:- If B.D.S. is a cantilever supported at A:–**

We solve the same exercise 3.1 but with a different BDS.



$$e = \frac{L}{2} - \frac{a}{3} = \left(\frac{3L - 2a}{6}\right)$$

$$Pe = M = \frac{Pa^2}{2}\left(\frac{3L - 2a}{6}\right) = \frac{Pa^2\,(3L - 2a)}{12}$$

**Properties of Analogous Column section** :– $A = L$, $I = \frac{L^3}{12}$, $C = \frac{L}{2}$

$$(Mi)a = \frac{P}{A} \pm \frac{MC}{I}$$

$$= \frac{-Pa^2}{2L} - \frac{Pa^2\,(3L - 2a)\,.\,L\,.\,\cancel{12}}{\cancel{12}\,.\,2\,.\,L^3}$$

(Due to upward $P = Pa^2/2$, reaction at A and B is downwards while due to moment, reaction at B is upwards while at A it is downwards. Similar directions will have the same sign to be additive or vice–versa)

$$= \frac{-Pa^2}{2L} - \frac{Pa^2\,(3L - 2a)}{2L^2}$$

$$= \frac{-Pa^2L - 3Pa^2L + 2Pa^3}{2L^2}$$

$$= \frac{-4\,Pa^2L + 2Pa^3}{2L^2}$$

$$= \frac{-2Pa^2L + Pa^3}{L^2}$$

$$= \frac{Pa^2\,(a - 2L)}{L^2}$$

$$= \frac{-Pa^2\,(2L - a)}{L^2}$$ , We can write $2L - a = L + L - a = L + b$

$$(Mi)a = \frac{-Pa^2\,(L + b)}{L^2}$$

$$(Ms)a = -Pa$$

$$Ma = (Ms - Mi)a$$

$$= -Pa + \frac{Pa^2(L + b)}{L^2}$$

$$= \frac{-PaL^2 + Pa^2\,L + Pa^2b}{L^2}$$

$$= \frac{-\,PaL\,(L-a) + Pa^2\,b}{L^2}$$

$$= \frac{-\,PabL + Pa^2\,b}{L^2}$$

$$= \frac{-\,Pab\,(L-a)}{L^2}$$

$$= \frac{-\,Pab\,.\,b}{L^2}$$

$$Ma = \frac{-\,Pab^2}{L^2}$$ ( Same result as was obtained with a different BDS)

$$(Mi)b = \frac{P}{A} \pm \frac{MC}{I}$$

$$= \frac{-\,Pa^2}{2L} + \frac{Pa^2\,(3L-2a)}{2L^2}$$

$$= \frac{-\,Pa^2\,L + 3Pa^2L - 2Pa^3}{2L^2}$$

$$= \frac{2\,Pa^2\,L - 2Pa^3}{2L^2}$$

$$= \frac{Pa^2\,L - Pa^3}{L^2}$$

$$= \frac{+Pa^2\,(L-a\,)}{L^2}$$

$$(Mi)b = \frac{Pa^2\,b}{L^2}$$

$$(Ms)b = 0$$

$$Mb = (Ms - Mi)b$$

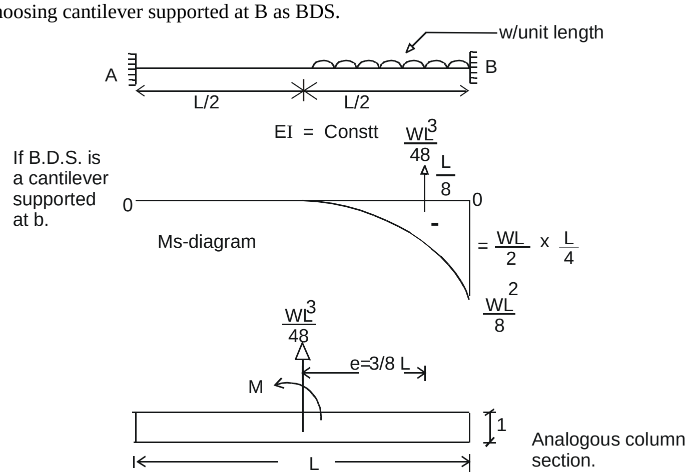
$$= 0 - \frac{Pa^2\,b}{L^2}$$

$$Mb = \frac{-\,Pa^2\,b}{L^2}$$ ( Same result as obtained with a different BDS)

**EXAMPLE NO.4:–** Determine the F.F.Ms. by the method of column analogy for the following loaded beam.

**SOLUTION:–**

Choosing cantilever supported at B as BDS.




Eccentricity = $e = \frac{L}{2} - \frac{L}{8} = \frac{4L - L}{8} = \frac{3L}{8}$

Moment = $Pe = M = \frac{WL^3}{48} \times \frac{3L}{8} = \frac{WL^4}{128}$ Where P = Area of Ms diagram $= \frac{WL^3}{48} = \left(\frac{bh}{n+1}\right)$

### Properties of Analogous column section.

$A = L$, $I = \frac{L^3}{12}$ and $C = \frac{L}{2}$

**Step 1:** Apply P= Area Of BMD(Ms diagram ) due to applied loads in a BDS at the center of analogous column section i.e. at L/2 from either side.

**Step 2:** The accompanying moment Pe, where e is the eccentricity between mid point of analogous column section and the point of application of area of Ms diagram, is also applied at the same point along with P.

**Step 3:** Imagine reactions due to P and M=Pe. At points A and B, use appropriate signs.

$(Mi)a = \frac{P}{A} \pm \frac{MC}{I}$ ( Subtractive reaction at A due to P)

$= \frac{-WL^3}{48.L} + \frac{WL^4 \times L \times 12}{128 \times 2 \times L^3}$ ( P is upwards, so negative. Reactions due to this P at A and B will be downwards and those due to moment term will be upward at A and downward at B. Use opposite signs now for A)

$= \frac{-WL^2}{48} + \frac{3WL^2}{64}$

$= \frac{-4WL^2 + 9WL^2}{192}$

$= \frac{+5WL^2}{192}$

$(Ms)a = 0$ ( Inspect BMD drawn on simple determinate span)

$Ma = (Ms - Mi)a$

$= 0 - \dfrac{5WL^2}{192}$

$Ma = -\dfrac{5WL^2}{192}$

$(Mi)b = \dfrac{P}{A} \pm \dfrac{MC}{I}$ ( Additive reactions at B as use negative sign with $\dfrac{Mc}{I}$ term)

$= -\dfrac{4WL^2 - 9WL^2}{192}$

$= -\dfrac{13\,WL^2}{192}$

$(Ms)b = -\dfrac{WL^2}{8}$

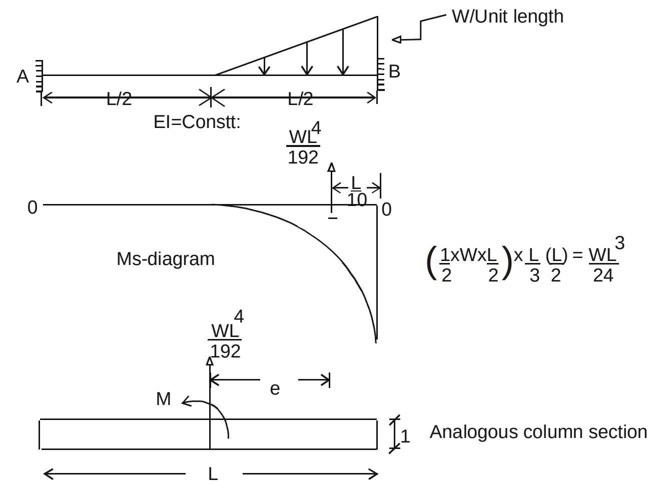
$Mb = (Ms - Mi)b$

$= -\dfrac{WL^2}{8} + \dfrac{13\,WL^2}{192} = \dfrac{-24\,WL^2 + 13\,WL^2}{192}$

$Mb = \dfrac{-11}{192}WL^2$

**The beam is now statically determinate etc.**

**EXAMPLE NO. 5:–** Determine the F.E. M's by the method of column analogy for the following loaded beam.
**SOLUTION:–**




$A = \dfrac{bh}{n+1}$

$A = \dfrac{\frac{L}{2} \times \frac{WL^3}{24}}{4} = \dfrac{WL^4}{192}$

$\left(\frac{1}{2} \times W \times \frac{L}{2}\right) \times \frac{L}{3}\left(\frac{L}{2}\right) = \frac{WL^3}{24}$

$\acute{X} = \dfrac{b}{n+2}$

$= \dfrac{L}{2(3+2)}$

$\acute{X} = \dfrac{L}{10}$

$e = \dfrac{L}{2} - \dfrac{L}{10} = \dfrac{5L - L}{10} = \dfrac{4L}{10} = \dfrac{2}{5}L$

$M = \left(\dfrac{WL^4}{192}\right) \times \left(\dfrac{2\,L}{5}\right) = \dfrac{WL^5}{480}$

Comment [A1]:

**Properties of Analogous column section.**

$A = L\,, \quad I = \frac{L^3}{12}, \quad C = \frac{L}{2}$

$(Mi)a \quad = \frac{P}{A} \pm \frac{MC}{I}$

$(Mi)a \quad = \frac{-WL^4}{192L} + \frac{WL^5 \times L \times 12}{480 \times 2 \times L^3}$ (Downward reaction at A due to P and upward reaction at A due to M)

$= \frac{-WL^3}{192} + \frac{WL^3}{80}$

$= \frac{-80WL^3 + 192\,WL^3}{15360}$

$= \frac{112\,WL^3}{15360}$ ( Divide by 16)

$(Mi)a = \frac{7\,WL^3}{960}$

$(Ms)a = 0$

$Ma = (Ms - Mi)a$

$Ma = 0 - \frac{7}{960}\,WL^3 = \frac{-7}{960}\,WL^3$

$(Mi)b = \frac{P}{A} \pm \frac{MC}{I}$

$= -\frac{WL^3}{192} - \frac{WL^3}{80}$

$= \frac{-80\,WL^3 - 192\,WL^3}{15360}$

$= \frac{-272\,WL^3}{15360}$

$= \frac{-17\,WL^3}{960}$

$(Ms)b = \frac{-WL^3}{24}$

$Mb = (Ms - Mi)\,b$

$$= \frac{-WL^3}{24} + \frac{17}{960} WL^3$$

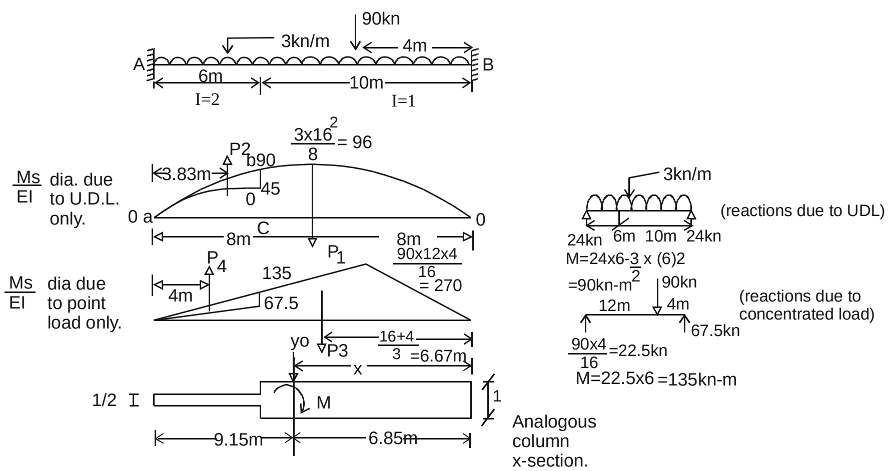
$$= \frac{-40\ WL^3 + 17\ WL^3}{960}$$

$$Mb = \frac{-23\ WL^3}{960}$$

***Note :*** After these redundant end moments have been determined, the beam is statically determinate and reactions , S.F, B.M, rotations and deflections anywhere can be found.

## 7.2. STRAIGHT MEMBERS WITH VARIABLE CROSS – SECTION.

**EXAMPLE NO. 6:–** Determine the fixed–end moments for the beam shown by the method of column analogy
**SOLUTION:–** BDS is a simple beam.




The above two $\frac{Ms}{EI}$ diagrams will be taken full first and then load corresponding to areas of these diagrams on left 6m distance will be subtracted. ($P_2$ and $P_4$ will be subtracted from $P_1$ and $P_3$ respectively).

In this solution, two basic determinate structures are possible.

(1) a simply supported beam.

(2) a cantilever beam.

This problem is different from the previous one in the following respects.

(a) Ms – diagram has to be divided by a given value of I for various portions of span.

(b) The thickness of the analogous column X – section will also vary with the variation of inertia. Normally, the width 1/EI can be set equal to unity as was the case in previous problem, when EI was set equal to unity.

(c) As the dimension of the analogous column X – section also varies in this case, we will have to locate the centroidal axis of the column and determine its moment of inertia about it.

## (1) SOLUTION:- By choosing a simple beam as a B.D.S.

$P_1 = \frac{2}{3} \times 16 \times 96 = 1024$ KN ( Load corresponding to area of entire BMD due to UDL)

$\int MdX = \int_o^6 (24X - 1.5X^2)\, dX$ (Simply supported beam moment due to UDL of left 6/ portion)

$= \left| 12X^2 - 0.5X^3 \right|_o^6 = 12 \times 36 - 0.5 \times 216 = 432 - 108 = 324$

area of abc = 324

$\int MXdX = \int_o^6 (24X - 1.5X^2)\, XdX$

$= \int_o^6 (24X^2 - 1.5X^3)\, dX$

$= \left| \frac{24}{3} X^3 - \frac{1.5}{4} X^4 \right|_o^6 = 8 \times 6^3 - \frac{1.5}{4} \times 6^4$

$= 1242$

$\bar{X} = \frac{\int MxdX}{\int MdX} = \frac{1242}{342} = 3.83$ m from A. (of left 6/ portion of BMD)

$P_2 = \frac{1}{2}$( area abc) $= \frac{324}{2} = 162$ KN( To be subtracted from Ms diagram )

$P_3 = \frac{1}{2} \times 16 \times 270 = 2160$ KN ( Area of BMD due to concentrated Load)

$P_4 = \frac{1}{2} \times 6 \times 67.5 = 202.5$ KN ( To be subtracted from Ms diagram )

**Properties of Analogous column x – section**.

Area $= A = 1 \times 10 + \frac{1}{2} \times 6 = 13 \text{ m}^2$

$\bar{X} = \frac{\int XdA}{A} = \frac{(1 \times 10)\,5 + (1/2 \times 6 \times 13)}{13}$ from R.H.S.

= 6.85 m ( From point B) . It is the location of centroidal axis Yo–Yo.

$Iy_0\,y_0 = \frac{1 \times 10^3}{12} + 10(1.85)^2 + \frac{0.5 \times 6^3}{12} + (0.5 \times 6) \times (6.15)^2 = 240 \text{ m}^4$

by neglecting the contribution of left portion about its own centroidal axis.

Total load to be applied at the centroid of analogous column x – section.

$= P_1 + P_3 - P_2 - P_4$

$= 1024 + 2160 - 162 - 202.5$

$= 2819.5$ KN

Applied Moment about centroidal axis = M = + 1024 (1.15) – 2160 (0.18) – 162 (5.32) – 202.5 (5.15)
= – 1116 KN–m , clockwise (Note: distance 5.32 = 9.15 – 3.83 (and 5.15 = 9.15 – 4)

**The (–ve) sign indicates that the net applied moment is clockwise**.

$(Mi)a = \frac{P}{A} \pm \frac{MC}{I}$ ( subtractive reactions at A)

$= \frac{2819.5}{13} - \frac{1116 \times 9.15}{240}$ , (Preserve at A due to $\frac{Mc}{I}$ is downwards so negative).

= + 174.34 KN–m

(Ms)a = 0

Ma = (Ms – Mi)a = 0 – 174.34
= – 174.34 KN–m

$(Mi)b = \frac{2819.5}{13} + \frac{1116 \times 6.85}{240}$ , ( Note the difference in the values of C for points A and B.)

= + 248.74 KN–m

(Ms)b = 0

Mb = (Ms – Mi)b

= 0 – 248.74

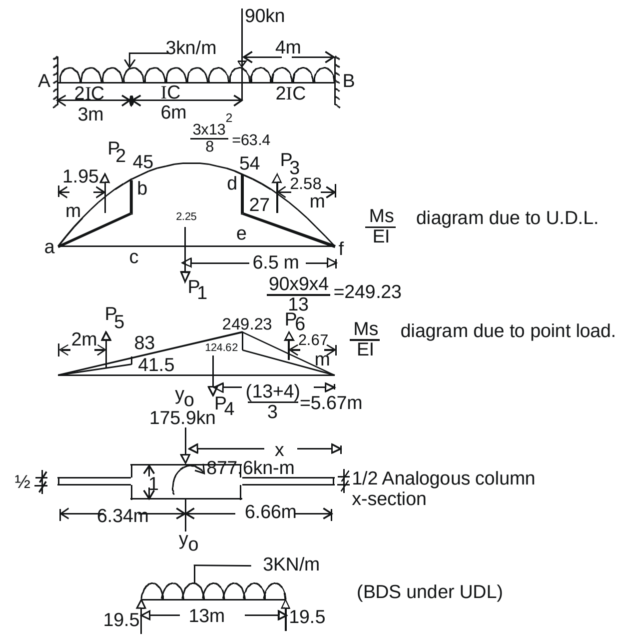
= – 248.74 KN–m

The –ve sign with Ma & Mb indicates that these cause compression on the inside when applied of these points.

**EXAMPLE NO.7:–** Determine the F.E.Ms. by the method of column analogy.

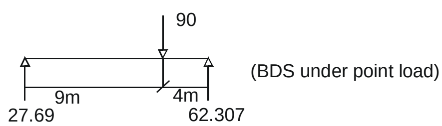
**SOLUTION:–**

1. Choosing a simple beam as a B.D.S.




$(M_3)L \quad = 19.5 \times 3 - 1.5(3)^2 \; = 45$ KN–m ( 3m from A )

$(M_4)R \quad = 19.5 \times 4 - \frac{3}{2}\,(4)^2 \; = 54$ KN–m ( 4m from B)




$(M_3)L \;=\; 27.69 \times 3 \;=\; 83$ KN–m( 3m from A)

$(M_4)R \;=\; 62.307 \times 4 = \; 249.22$ (4m from B)

$$\int M dX \;=\; \text{area abc} \;=\; \int_o^3 (19.5\,X - 1.5\,X^2)\, dX$$

$$= \left| \frac{19.5}{2} X^2 - \frac{1.5}{3} X^3 \right|_o^3 = 74.25$$

$$\int MXdX = \int_o^3 (19.5\, X^2 - 1.5\, X^3)\, dX = \left| \frac{19.5}{3} X^3 - \frac{1.5}{4} X^4 \right|_o^3$$

$$= 145.12$$

$$\bar{X} = \frac{145.12}{74.25} = 1.95.m \text{ ( From point A as shown )}$$

$$\text{Area def} = \int MdX = \int_o^4 (19.5X - 1.5\, x^2)\, dX = 124$$

$$\int MXdX = \int_o^4 (19.5\, X^2 - 1.5\, x^3)\, dX$$

$$= \left| \frac{19.5X^3}{3} - \frac{1.5}{4} X^4 \right|_o^4$$

$$= 320$$

$$\bar{X} = \frac{320}{124} = 2.58 \text{ m ( From point B )}$$

$$P_1 = \frac{2}{3} \times 63.4 \times 13 = 549.5 \text{ KN( Due to entire BMD due to UDL )}$$

$$P_2 = \frac{1}{2} \text{ (area abc)} = \frac{1}{2} (74.25) = 37.125 \text{ KN ( To be subtracted )}$$

$$P_3 = \frac{1}{2} \text{ (area def)} = \frac{1}{2} (124) = 62 \text{ KN ( To be subtracted )}$$

$$P_4 = \frac{1}{2} \times 249.23 \times 13 = 1620 \text{ KN ( Entire area of BMD due to point load)}$$

$$P_5 = \frac{1}{2} \times 41.5 \times 3 = 62.25 \text{ KN ( To be subtracted )}$$

$$P_6 = \frac{1}{2} \times 4 \times 124.62 = 249.23 \text{ KN ( To be subtracted )}$$

**Properties of Analogous column x – section**.

$$A = \frac{1}{2} \times 4 + 1 \times 6 + \frac{1}{2} \times 3 = 9.5m^2$$

$$\bar{X} = \frac{(0.5 \times 4) \times 2 + (1 \times 6) \times 7 + (0.5 \times 3) \times (11.5)}{9.5}$$

$\bar{X}$ = 6.66 ( From point B) meters

$$Iy_oy_o = \frac{0.5 \times 4^3}{12} + (0.5 \times 4)(4.68)^2 + \frac{1 \times 6^3}{12} + (1 \times 6)(0.34)^2$$

$$+ \frac{0.5 \times 3^2}{12} + (1.5)(4.84)^2$$

$$= 101.05$$

Total concentric load on analogous column x – section to be applied at centroidal column axis )

$P = P_1 - P_2 - P_3 + P_4 - P_5 - P_6 = 549.5 - 37.125 - 62 + 1620 - 62.25 - 249.23$
$= 1759$ KN

Total applied moment at centroid of analogous column due to above six loads is

= 549.5 (0.16) + 37.125 (4.39) – 62(4.08) + 1620 (0.99) + 62.25 (4.34) – 249.2 (3.99)
= + 877.6 clockwise.

$$(Mi)a = \frac{P}{A} \pm \frac{MC}{I}$$ ( Reactions due to P and M are subtractive at A)

$$= \frac{1759}{9.5} - \frac{877.6 \times 6.34}{101.05}$$

= + 130 KN–m

(Ms)a = 0
Ma = (Ms – Mi)a = 0 – 130 = – 130 KN–m

$$(Mi)b = \frac{P}{A} \pm \frac{MC}{I}$$

$$= \frac{1759}{9.5} + \frac{877.6 \times 6 \times 6.66}{101.05}$$ ( Reactions due to P and M are additive at B)

= + 243 KN–m

(Ms)b = 0

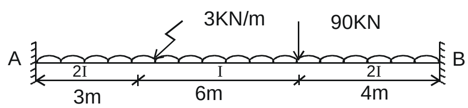
Mb = (Ms – Mi)b

= 0 – 243

Mb = – 243 KN–m

**Now the beam has become determinate.**

**EXAMPLE NO. 7:-** (2) Choosing cantilever supported at B as a B.D.S. Let us solve the loaded beam shown below again.




$A = \frac{bh}{n+1}$

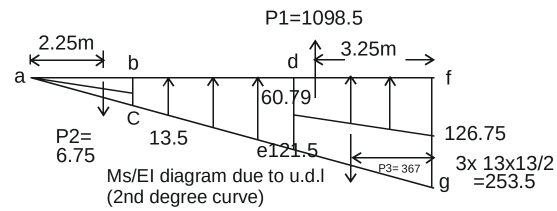
$P_1 = \frac{13x253.5}{3}$

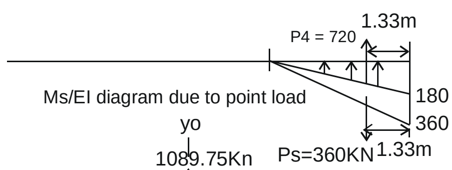
$=1098.5$

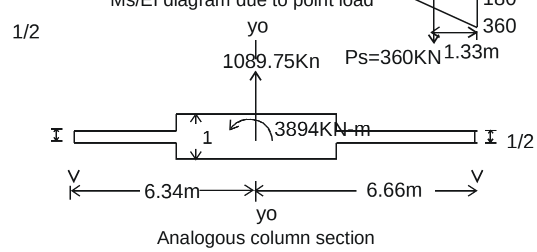
$X' = \frac{b}{n+2} = \frac{13}{4} = 3.25$

$P_4 = A = \frac{bh}{n+1} = \frac{4x360}{2} = 720$

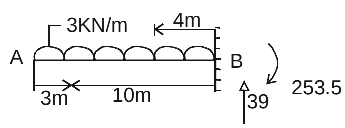
$X' = \frac{b}{n+2} = \frac{4}{3} = 1.33$




BDS under UDL




$$\text{Area abc} = \int MdX = \int_o^3 \left(-\frac{3}{2}X^2\right)dX$$

$$= \left| -1.5\frac{X^3}{3} \right|_o^3 = 0.5 \times 3^3 = -13.5 \quad \text{( Upwards to be subtracted)}$$

$$\int MXdX = \int_o^3 (1.5X^3) \,|\, dX = \left| -\frac{1.5X^4}{4} \right|_o^3$$

$$= -30.375$$

Location of centroidal axis from B: ( 1/2 × 3 + 1 × 6+1/2 × 4)X′ =(1/2 × 4 × 2+1 × 6 × 7+1/2 × 3 × 11.5)
9.5X'= 63.25 0r X' = 6.66m from B or 6.34 m from A. (already done also)

location of centroid of area abc = X $= \frac{-30.375}{-13.5} = 2.25$ m ( From A)

$$\text{Area defg} = \int MdX = \int_o^4 (39X - 253.5 - 1.5X^2)dX$$

Moment expression taken from B considering BDS under UDL.

$$= \left| 39\frac{X^2}{2} - 253.5\,X - \frac{1.5}{3}\,X^3 \right|_o^4$$

$$= -734 \qquad \text{(Area is always positive).}$$

$$\int MXdX = \int_o^4 (39X^2 - 253.5X - 1.5X^3)dX$$

$$= \left| \frac{39X^3}{3} - \frac{253.5X^2}{2} - \frac{1.5X^4}{4} \right|_o^4$$

$$= -1292$$

$$\bar{X} = \frac{-1292}{-734}$$

$\bar{X}$ = + 1.76 m From B (Centroid of area defg)

$P_1$ = 1098.5 KN ( Area of entire BMD due to UDL )

$P_2 = \frac{1}{2}$(area abc) $= \frac{1}{2}(13.5) = 6.75$ K( To be subtracted)

$P_3 = \frac{1}{2}$( area defg) $= \frac{1}{2}(734) = 367$ KN( To be subtracted )

$P_4$ = 720 KN( Area of entire BMD due to point Load )

$P_5 = \frac{1}{2} \times 180 \times 4 = 360$ KN

Total concentric load on analogous column X – section is

$P = P_1 + P_2 + P_3 - P_4 + P_5$

$= -1098.5 + 6.75 + 367 - 720 + 360$

$= -1084.75$ KN( It is upward so reactions due to this will be downward)

Total applied moment at centroid of column

$= -6.75\,(6.34 - 2.25) + 1098.5\,(6.66 - 3.25)$

$- 367\,(6.66 - 1.76) + 720\,(6.66 - 1.33) - 360\,(6.66 - 1.33)$

$= 3894$ KN–m (anticlockwise)

## Properties of Analogous column X – section.

$A = \frac{1}{2} \times 4 + 1 \times 6 + \frac{1}{2} \times 3 = 9.5$

$\bar{X} = 6.66$ meters From B as in previous problem.

$I_{y_o y_o} = 101.05\ m^4$ as in previous problem.

$(Mi)a = \frac{P}{A} \pm \frac{MC}{I}$ ( Reactions are subtractive at A)

$= \frac{-1084.75}{9.5} + \frac{3894 \times 6.34}{101.05}$

$(Mi)a = +130$ KN–m ( Same answer as in previous problem )

$(Ms)a = 0$

$Ma = (Ms - Mi)a$

$Ma = (0 - 130) = -130$ KN–m

$(Mi)b = \frac{P}{A} \pm \frac{MC}{I}$ ( Reactions are additive at B )

$= \frac{-1084.75}{9.5} - \frac{3894 \times 6.66}{101.05}$

$= -370.83$ KN–m

$(Ms)b = -253.5 - 360 = -613.5$ KN–m

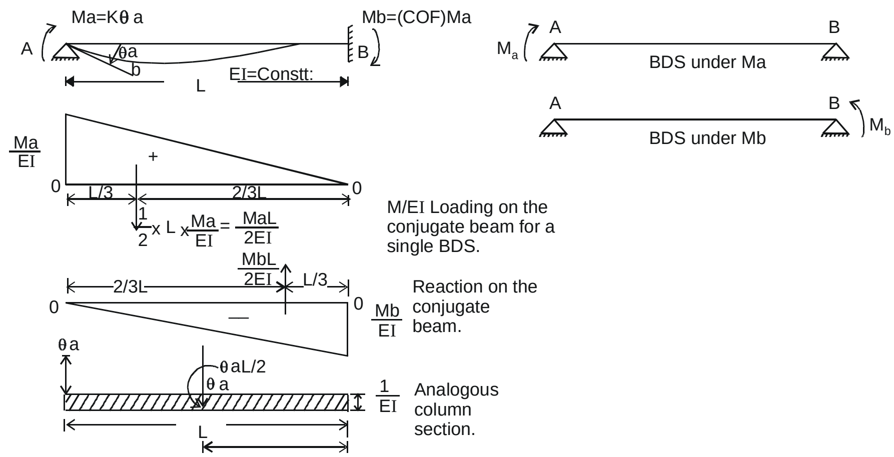
$Mb = (Ms - Mi)b = -613.5 + 370.83$

$Mb = -243$ KN–m

Now beam is determinate. Please note that the final values of redundant moments at supports remain the same for two BDS. However, amount of effort is different.

## 7.3. STIFFNESS AND CARRYOVER FACTORS FOR STRAIGHT MEMBERS WITH CONSTANT SECTION:–

For the given beam, choose a simple beam as BDS under Ma and Mb




By choosing a B.D.S. as simple beam under the action of Ma and Mb, we can verify by the use of conjugate beam method that θb = 0. In this case, we are required to find that how much rotation at end A is required to produce the required moment Ma. In other words, θa (which is in terms of Ma and Mb can be considered as an applied load on the analogous column section). The moments computed by using the formula $\frac{P}{A} \pm \frac{MC}{I}$ will give us the end moments directly because in this case Ms diagram will be zero.

So, M = Ms – Mi = 0 – Mi = – Mi.

### Properties of analogous column section:–

$$A = \frac{L}{EI}\ , \qquad I = \frac{1}{EI}\ \frac{L^3}{12} = \frac{L^3}{12EI}$$

factor

Downward load on analogous column = θa at A.

Accompanying moment = $\theta a \times \frac{L}{2}$ ( About centroidal column axis )

and $C = \frac{L}{2}$ for use in above formula.

$$Ma = \frac{P}{A} + \frac{MC}{I}$$

$$= \frac{\theta a\ EI}{L} + \frac{\theta a \times L \times L \times 12EI}{2 \times 2 \times L^3}$$ ( Reactions are additive at A and are upwards)

$$= \frac{\theta a\ EI}{L} + \frac{3\theta a\ EI}{L}$$

$$Ma = \frac{4\ EI}{L}\theta a$$

Where $\frac{4\ EI}{L} = Ka$

Where Ka = stiffness factor at A.

$$Mb = \frac{P}{A} \pm \frac{MC}{I}$$ ( Reactions are subtractive at B)

$$= \frac{\theta a\ EI}{L} - \frac{3\theta a\ EI}{L}$$

$$= \frac{-2\theta a\ EI}{L}$$

$$= \frac{-2EI}{L}\ .\ \theta a$$

The (−ve) sign with Mb indicates that it is a (−ve) moment which gives us tension at the top or compression at the bottom.

$(COF)_{a \to b}$ Carry−over factor from A to B $= \frac{Mb}{Ma} = \frac{2}{4} = +\frac{1}{2}$

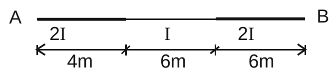
"**BY PUTING θA EQUAL TO UNITY , MA & MB WILL BE THE STIFFNESS FACTORS AT THE CORRESPONDING JOINTS". STIFFNESS FACTOR IS THE MOMENT REQUIRED TO PRODUCE UNIT ROTATION.**

In the onward problems of members having variable X-section, we will consider θa = θb = 1 radians and will apply them on points A & B on the top of the analogous column section. The resulting moments by using the above set of formulas will give us stiffness factor and COF directly.

**EXAMPLE NO. 8:–** Determine the stiffness factors at A & at B and the carry-over factors from A to B and from B to A for the straight members with variable X-sections shown in the figure below..

**SOLUTION:–**

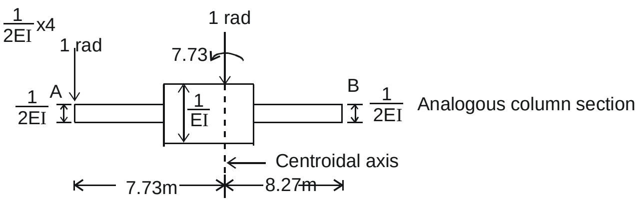
Draw analogous column section and determine its properties.




$$A = \frac{1}{2EI} \times 6 + \frac{1}{EI} \times 6 + \frac{1}{2EI} \times 4$$

$$= \frac{3}{EI} + \frac{6}{EI} + \frac{2}{EI}$$

$$= \frac{11}{EI}$$




Taking moments of areas about point B.

$$\bar{X} = \frac{(0.5 \times 6) \times 3 + (6 \times 1) \times 9 + (4 \times 0.5) \times 14}{11}$$

$$\bar{X} = 8.27 \text{ meters from B.}$$

$$I = \frac{0.5 \times 6^3}{12} + (0.5 \times 6) \times (5.27)^2 + \frac{1 \times 6^3}{12} + (1 \times 6) \times$$

$$(0.73)^2 + \frac{0.5 \times 4^3}{12} + (0.5 \times 4) \times (5.73)^2$$

$$I = \frac{181.85}{EI}$$

Consider loads acting at centroid of analogous column and determine indeterminate moments at A and B.

$$Ma = \frac{P}{A} \pm \frac{MC}{I}$$

$$= \frac{P}{A} + \frac{MC}{I} = \frac{1 \times EI}{11} + \frac{7.73 \times 7.73 \times EI}{181.85}$$

$$Ma = 0.419\,EI = 0.419 \times 16\,\frac{EI}{L}$$ , (by multiplying and dividing RHS by L)

$$Ma = 6.71\,\frac{EI}{L}$$

$$\boxed{Ka = 6.71}$$

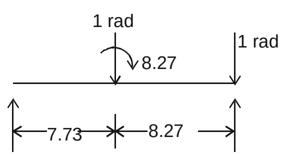
$$Mb = \frac{EI}{11} - \frac{7.73 \times 8.27 \times EI}{181.85} \times \frac{16}{L} \qquad \text{(by multiplying and dividing by L)}$$

$$= -4.17\frac{EI}{L}$$

$$(COF)_{A\to B} = \frac{Mb}{Ma} = \frac{4.17}{6.71} = 0.62$$

$$\boxed{(COF)A\to B = 0.62}$$

Now applying unit radian load at B. This eccentric load can be replaced by a concentric load Plus accompanying moment.




Considering eccentric 1 rad load to be acting at centroid of section alongwith moment.

$$Ma = \left[\frac{EI}{11} - \frac{(8.27 \times 7.73 \times EI)}{181.85}\right]\frac{16}{L}, \qquad \text{(multiplying and dividing by L)}$$

$$Ma = -4.17\frac{EI}{L}$$

$$Mb = \left[\frac{EI}{11} + \frac{(8.27 \times 8.27 \times EI)}{181.85}\right]\frac{16}{L} \qquad \text{(multiplying and dividing by L)}$$

$$Mb = 7.47\frac{EI}{L}$$

$$\boxed{Kb = 7.47}$$

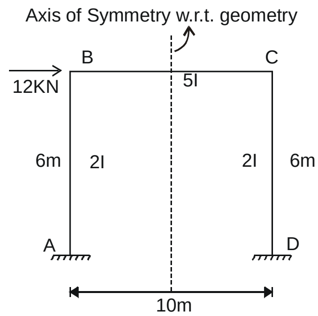
$$(COF)_{b\to a} \text{ Carry–over factor from B to A} = \frac{Ma}{Mb} = \frac{4.17}{7.47}$$

$$\boxed{(COF)_{b\to a} = 0.56}$$

## 7.4. APPLICATION TO FRAMES WITH ONE AXIS OF SYMMETRY:–

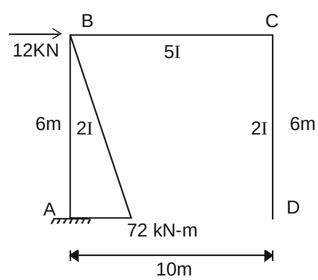
**EXAMPLE NO. 9:-** Analyze the quadrangular frame shown below by the method of column analogy. Check the solution by using a different B.D.S.

**SOLUTION:–**

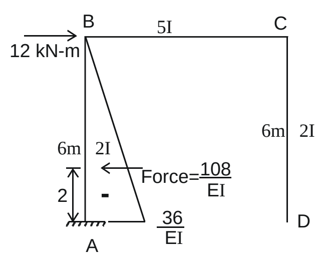



The term "axis of symmetry" implies that the shown frame is geometrically symmetrical (M.O.I. and support conditions etc., are symmetrical) w.r.t. one axis as shown in the diagram. The term does not include the loading symmetry (the loading can be and is unsymmetrical).

Choosing the B.D.S. as a cantilever supported. at A.

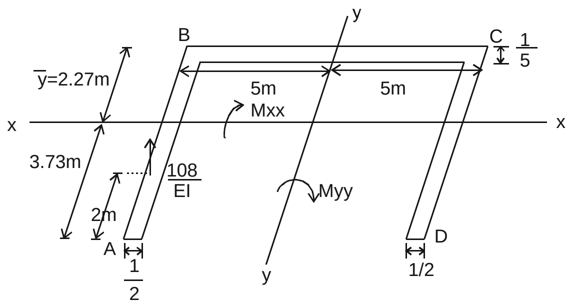



Ms-diagram




$\frac{Ms}{EI}$ - Diagram

According to our sign convention for column analogy, the loading arising out of negative $\frac{Ms}{EI}$ giving tension on outside will act upwards on the analogous column section. Sketch analogous column section and place load.




**(1) Properties of Analogous Column Section:–**

$$A = \left(\frac{1}{2}\times 6\right)\times 2 + \frac{1}{5} \times 10 = \frac{8}{EI}$$

$$\bar{y} = \left[\frac{\left(\frac{1}{5} \times 10\right)\times\frac{1}{10} + 2\left[\frac{1}{2} \times 6 \times 3\right]\frac{1}{EI}}{\frac{8}{EI}}\right] = 2.27 \text{ m about line BC. (see diagram)}$$

$$Ixx = 2\left[\frac{0.5 \times 6^3}{12} + \left(\frac{1}{2}\times 6\right) x\ (0.73)^2\right] + \frac{10 \times (1/5)^3}{12} + (0.2 \times 10) \times (2.27)^2$$

$$= \frac{31.51}{EI} m^4$$

$$Iyy = \frac{0.2 \times 10^3}{12} + 2\left[\frac{6 \times 0.5^3}{12} + (6 \times 0.5) \times (5)^2\right]$$

$$= \frac{167}{EI} m^4$$

$$Mxx = 108 \times 1.73 = \frac{187}{EI} \text{ clockwise.}$$

$$Myy = 108 \times 5 = \frac{540}{EI} \text{ clockwise.}$$

Applying the formulae in a tabular form for all points. Imagine the direction of reactions at exterior frame points due to loads and moments.

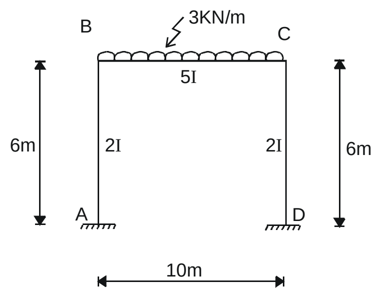
Ma = ( Ms– Mi)a

$$( Mi)a = \frac{P}{A} \pm \frac{M_x\, y}{Ix} \pm \frac{M_y\, X}{Iy}$$

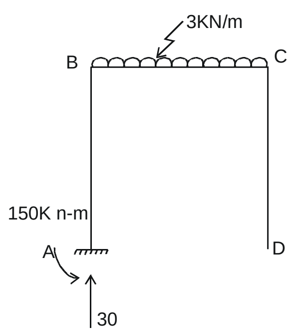
| POINT | Ms | P/A | $\frac{Mx\ y}{Ix}$ | $\frac{My\ X}{Iy}$ | Mi | M = Ms–Mi |
|---|---|---|---|---|---|---|
| A | – 72 | – 13.5 | – 22.14 | – 16.17 | – 51.81 | – 20.19 |
| B | 0 | – 13.5 | + 13.47 | – 16.17 | – 16.20 | + 16.20 |
| C | 0 | – 13.5 | + 13.47 | + 16.17 | + 16.14 | – 16.14 |
| D | 0 | – 13.5 | – 22.14 | + 16.17 | – 19.47 | + 19.47 |

**Note:** Imagine the direction of reaction due to P, Mx and My at all points A, B, C and P. Use appropriate signs. Repeat the analysis by choosing a different BDS yourself.

**EXAMPLE NO. 10:–** Analyze the quadrangular frame shown by the method of column analogy.

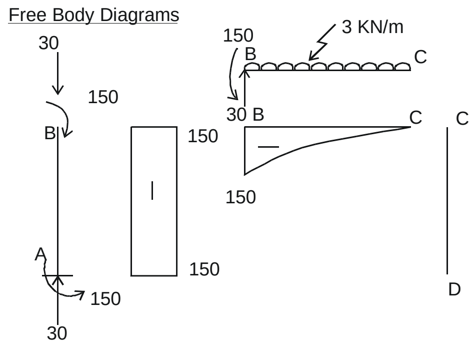


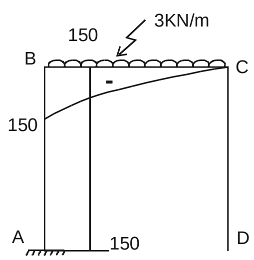

**Choosing B.D.S. as a cantilever supported at A.**



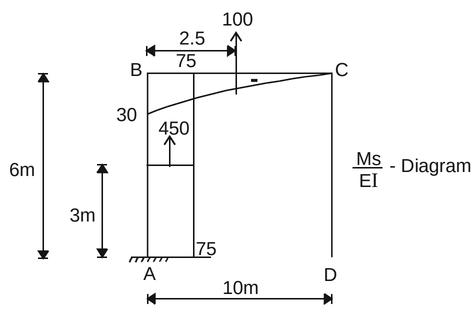

BDS under loads

Draw Ms−diagram by parts and then superimpose for convenience and clarity.



Ms-Diagram

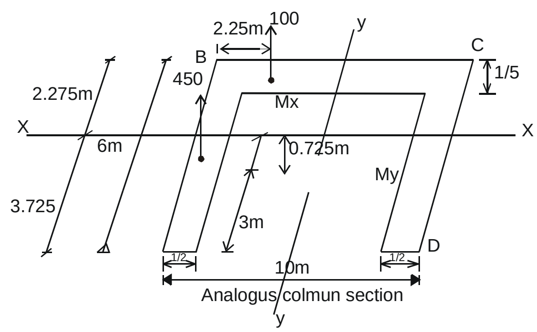



For Portion BC

$$\text{Area} = \frac{bb}{n+1} = \frac{10 \times 30}{2+1} = \frac{300}{3} = 100$$

$$X' = \frac{b}{n+2} = \frac{10}{2+2} = \frac{10}{4} = 2.5 \text{ from B.}$$

**Note:** As BMD on portions BC and AB are negative the loads equal to their areas will act upwards. Now sketch analogous column section carrying loads arising from $\frac{M}{EI}$ contributions.




**Properties of analogous column section:–**

$$A = 2\left[\frac{1}{2}\times 6\right] + \frac{1}{5}\times 10 = \frac{8}{EI} \text{ (as before)}$$

$$\bar{y} = \frac{\left(\frac{1}{5}\times 10\right)\times\frac{1}{10} + 2\left[\left(6\times\frac{1}{2}\right)\times 3\right]}{8} = 2.275 \text{ about line BC (as before)}$$

$$Ix = 2\left[\frac{1}{2}\times 6^3 + \left(\frac{1}{2}\times 6\right)\times(0.725)^2\right] + \left[10\times\left(\frac{1}{5}\right)^3 + \left(10\times\frac{1}{5}\right)\times(2.275)^2\right]$$

$$= \frac{31.51}{EI}\,m^4 \quad \text{(as before)}$$

$$Iy = 2\left[\frac{6\times 0.5^3}{12} + (6\times 0.5)\times 5^2\right] + \frac{0.2\times 10^3}{12}$$

$$= \frac{166.79}{EI}\,m^4 \quad \text{(as before)}$$

Mx = 450 × 0.725 – 100 × 2.275 = 95.75 KN–m Clockwise

My = 450 × 5 + 100 × 2.75 = 2525 KN–m clockwise.

P = 100 + 450 = 550 KN

Now this eccentric load P and $M_X$ and My are placed on column centroid.

Applying the formulae in a tabular form.

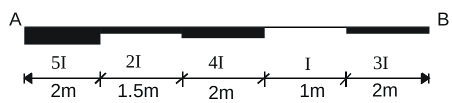
Ma = ( Ms– Mi)a

and $(Mi)a = \frac{P}{A} \pm \frac{M_x\,y}{Ix} \pm \frac{M_y\,x}{Iy}$

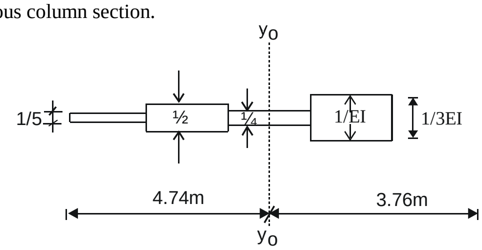
| POINT | Ms | P/A | $\frac{Mx.y}{Ix}$ | $\frac{My.x}{Iy}$ | Mi | M = Ms–Mi |
|---|---|---|---|---|---|---|
| A | – 150 | – 68.75 | – 11.32 | – 75.69 | – 155.76 | 5.76 |
| B | – 150 | –68.75 | + 6.91 | – 75.69 | – 137.53 | –12.47 |
| C | 0 | –68.75 | + 6.91 | + 75.69 | 13.85 | –13.85 |
| D | 0 | –68.75 | –11.32 | + 75.69 | –4.38 | 4.38 |

**EXAMPLE NO. 4:–** Determine stiffness factors corresponding to each end and carry-over factors in both directions of the following beam.

**SOLUTION:–**




Sketch analogous column section.




## Properties of Analogous Column Section :–

$$A = \frac{1}{5}\times 2 + \frac{1}{2}\times 1.5 + \frac{1}{4}\times 2 + 1\times 1 + \frac{1}{3}\times 2$$

$$A = \frac{3.32}{EI}$$

Taking moment about B of various segments of column section.

$$\bar{X} = \frac{\frac{1}{3}\times 2\times 1 + 1\times 1\times 2.5 + \frac{1}{4}\times 2\times 4 + \frac{1}{2}\times 1.5\times 5.75 + \frac{1}{5}\times 2\times 7.5}{3.32}$$

$$X = \frac{12.4725}{3.32}$$

$\bar{X} = 3.76$ m from B.

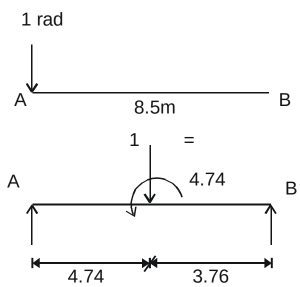
$$Iy_oy_o = \frac{1}{3} \times \frac{2^3}{12} + \left(\frac{1}{3} \times 2\right) \times (2.76)^2 + \frac{1 \times 1^3}{12} + (1 \times 1)(2.26)^2$$

$$+ \frac{\left(\frac{1}{4}\right) \times (2)^3}{12} + \left(\frac{1}{4} \times 2\right)(0.24)^2 + \frac{\left(\frac{1}{2}\right) \times (1.5)^3}{12}$$

$$+ \left(\frac{1}{2} \times 1.5\right)(1.99)^2 + \frac{\left(\frac{1}{5}\right) \times (2)^3}{12} + \left(\frac{1}{5} \times 2\right)(3.74)^2$$

$$= \frac{19.53}{EI}$$

1. Determination of stiffness factor at A (ka) and carry-over factor from A to B. Apply unit load at A and then shift it along with moment to centroidal axis of column as shown below:




$$Ma = \frac{P}{A} \pm \frac{MC}{I}$$

$$= 1 \times \frac{EI}{3.32} + 4.74 \times 4.74 \frac{EI}{19.53}$$

$= 1.45\ EI$ , multiply and divide by L

$$Ma = 1.45 \times 8.5 \times \frac{EI}{L} = 12.33 \frac{EI}{L}$$

$$\boxed{Ka = 12.33}$$

$$Mb = \frac{EI}{3.32} - \frac{4.74 \times 3.26 \times EI}{19.53}$$

$$= -0.61\ EI = -0.61 \times 8.5 \times \frac{EI}{L} = -5.19 \frac{EI}{L}$$ (multiply and divide by L)

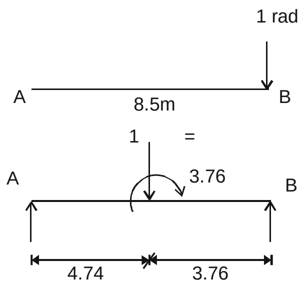
$$Mb = -5.19 \frac{EI}{L}$$

$$(COF)a \rightarrow b = \frac{Mb}{Ma} = \frac{5.19}{12.33} = 0.42$$

$$\boxed{(COF)a \rightarrow b = 0.42}$$

2. Determination of stiffness factor at B (Kb) and carry-over from B to A. Apply a unit load at B and them shift it along with moment to centroidal axis of column as shown below:

Ma $= \frac{P}{A} \pm \frac{Mc}{I}$




Ma $= \frac{EI}{3.32} - \frac{3.76 \times 4.74 \times EI}{19.53}$

$= -0.61EI$ , multiply and divide by L.

$= -0.61 \times 8.5 \times \frac{EI}{L} = -5.19 \frac{EI}{L}$

Mb $= \frac{P}{A} \pm \frac{Mc}{I}$

$= \frac{EI}{3.22} + \frac{3.76 \times 3.76 \times EI}{19.53}$

$=1..03 \text{ EI} = 1.03 \times \frac{EI}{L} \times 8.5$ , multiply and dividing by L.

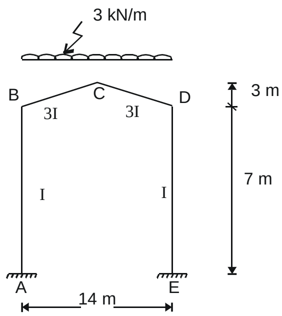
Mb $= 8.76 \frac{EI}{L}$

Kb = 8.76

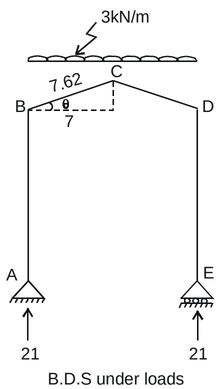
(COF)b $\rightarrow$ a $= \frac{Ma}{Mb} = \frac{5.19}{8.76} = 0.6$

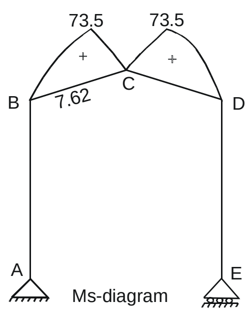
(COF) b $\rightarrow$ a = 0.6

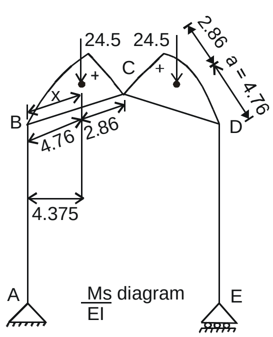
**EXAMPLE NO.12:–** Analyze the following gable frame by column analogy method.
**SOLUTION :–**




Choosing a simple frame as BDS



Taking the B.D.S. as a simply supported beam.

$M_X = 21X - 1.5X^2$ , taking X horizontally.

$M_X = Mc$ at $X = 7m$

$Mc = 21 \times 7 - 1.5 \times 7^2$

$= 73.5$ KN–m

$\sin\theta = \frac{3}{7.62} = 0.394$

$\cos\theta = \frac{7}{7.62} = 0.919$

$P_1 = P_2 = \frac{2}{3} \times 24.5 \times 7.62 = 124.46$

$P = P_1 + P_2 = 248.92$

$$\int M_X\, dX = \int_o^7 (21X - 1.5X^2)\, dX = \left|\frac{21}{2}X^2 - \frac{1.5}{3}X^3\right|_o^7 = 343$$

$$\int (M_X)X\, dX = \int_o^7 (21X^2 - 1.5X^3)dX = \left|\frac{21}{3}X^3 - \frac{1.5}{4}X^4\right|_o^7$$

$$= 7 \times 7^3 - \frac{1.5}{4} \times 7^4 = 1500.625$$

$$\bar{X} = \frac{\int (M_X)\, X\, dX}{\int M_X\, dX} = \frac{1500.625}{343}$$

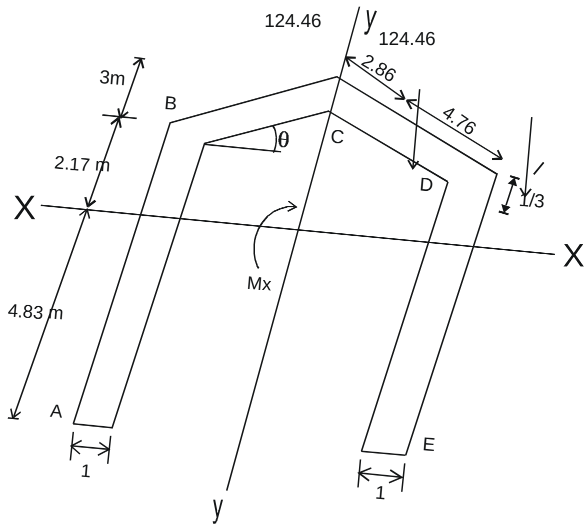
$\bar{X} = 4.375$ Horizontally from D or B. Shift it on the inclined surface.

$\cos\theta = \frac{4.375}{a}$

$a = \frac{4.375}{\cos\theta} = \frac{4.375}{0.919}$

$a = 4.76$

Now draw analogous column section and place loads on top of it.




## PROPERTIES OF ANALOGOUS COLUMN SECTION

$$A = 2\,(1 \times 7) + 2\left(\frac{1}{3} \times 7.62\right) = 19.08 \text{ m}^2$$

$$\bar{Y} = \frac{2[(1 \times 7) \times 3.5] + 2\left[\left(\frac{1}{3} \times 7.62\right) \times 8.5\right]}{19.08} = \frac{49 + 43 - 18}{19.08}$$

$\bar{Y}$ = 4.83 m from A or E

$$Ix = 2\left[\frac{1 \times 7^3}{12} + (1 \times 7)\,(4.83 - 3.5)^2\right]$$

$$+ 2\left[\frac{\left(\frac{1}{3}\right) \times (7.62)^3}{12} \times (0.394)^2 + \frac{1}{3}(7.62)\,(1.5 + 2.17)^2\right],$$

the first term in second square bracket is $\frac{bL^3}{12}\text{Sin}^2\theta$

= 154.17

So Ix $\cong$ 154 $m^4$

Now $Iy = 2\left[\frac{7 \times 1^3}{12} + (7 \times 1) \times 7^2\right]$

$$+ 2\left[\frac{\frac{1}{3} \times (7.62)^3}{12} \times (0.919)^2 + \left(\frac{1}{3} \times 7.62\right) \times (3.5)^2\right],$$

the first term in second square bracket is $\frac{bL^3}{12}\text{Cos}^2\theta$

=770.16

So Iy $\cong$ 770 $m^4$

**Total load on centroid of analogous column**

$P = P_1 + P_2 = 124.46 + 124.46 = 248.92$ KN

Mx = 2 × [124.46 × 4.05 ] , 4.05 = 2.17 + 4.76 Sinθ = 2.17 + 4.76 × 0.394.

Mx = 1007 (clockwise).

My = 0 (because moments due to two loads cancel out)

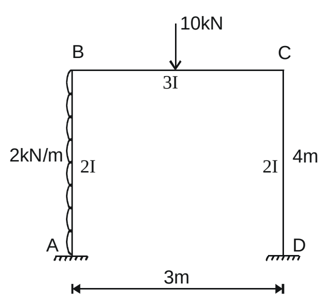
Applying the general formulae in a tabular form for all points of frame.

Ma = ( Ms– Mi)a

$(Mi)a = \frac{P}{A} \pm \frac{M_x\, y}{Ix} \pm \frac{M_y\, X}{Iy}$

| Point | Ms (A) | P/A (1) | $\frac{Mx.Y}{Ix}$ (2) | $\frac{My.X}{Iy}$ (3) | (B)=Mi (1)+(2) +(3) | M = Col (A)–(B) |
|---|---|---|---|---|---|---|
| A | 0 | + 13.05 | – 31.58 | 0 | – 18.53 | + 18.53 |
| B | 0 | + 13.05 | + 14.19 | 0 | + 27.24 | – 27.24 |
| C | + 73.5 | + 13.05 | + 33.81 | 0 | + 46.86 | +26.64 |
| D | 0 | + 13.05 | + 14.19 | 0 | + 27.24 | – 27.24 |
| E | 0 | + 13.05 | – 31.58 | 0 | – 18.53 | + 18.53 |

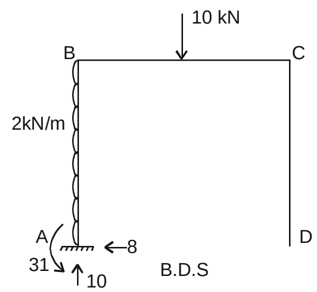
**EXAMPLE NO. 13:-** Analyze the frame shown in fig below by Column Analogy Method.



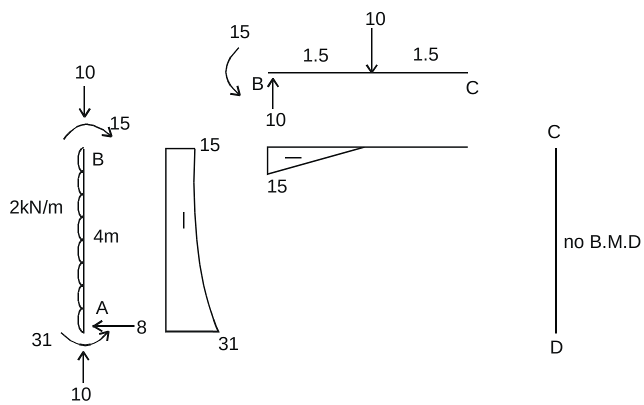

Choosing the B.D.S. as a cantilever supported at A.

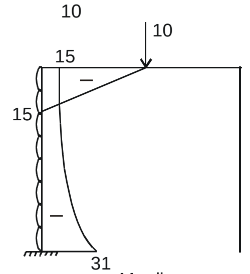
$M_A = 10 \text{ x } 1.5 + 2 \text{ x } 4 \text{ x } \frac{4}{2}$

$M_A = 31$ KN–m

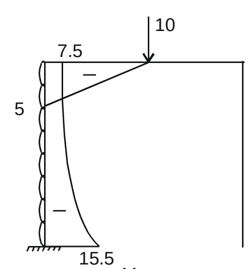



Draw Free Body Diagrams and sketch composite BMD:–



Ms-diagram



$\frac{Ms}{EI}$ diagram

**Properties Of Analogous Column Section :–**

Sketch analogous column section and show loads on it. BMD along column AB is split into a rectangle and other second degree curve.

$$A = \left(\frac{1}{2}\times 4\right)\times 2 + \left(\frac{1}{3}\times 3\right) = 5\ m^2$$

$$\bar{Y} = \frac{\left(3\times\frac{1}{3}\right)\times\left(\frac{1}{6}\right) + 2\left[\left(\frac{1}{2}\times 4\right)\times 2\right]}{5}$$

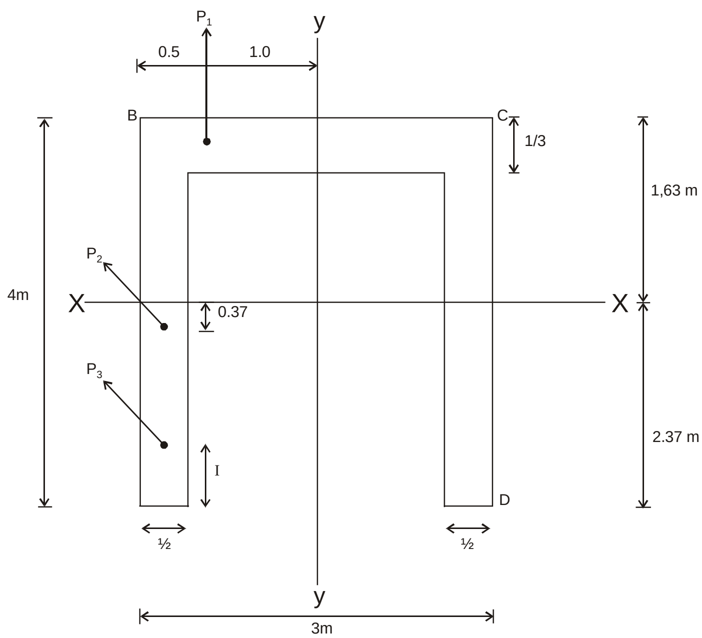
$\bar{Y}$ = 1.63 m From line BC

$$Ix = \frac{3\times\left(\frac{1}{3}\right)^3}{12} + \left(\frac{1}{3}\times 3\right)\times(1.63)^2 + 2\left[\frac{\frac{0.5+4^3}{12} + (0.5\times 4)\times(0.37)^2}{12}\right]$$

$$= 8.55\ m^4$$

$$Iy = \left(\frac{1}{3}\right)\times(3)^3 + 2\left[\frac{4\times 0.5^3}{12} + (4\times 0.5)\times(1.5)^2\right]$$

$$= 9.83\ m^4$$



Total load on top of analogous column section acting at the centroid.

P = 3.75 + 30 + 10.67 = 44.42 KN upward.

$P1 = \frac{1}{2} \times 1.5 \times 5 = 3.75, \quad P2 = 7.5 \times 4 = 30, \quad P3 = \frac{4 \times 7.5}{2 + 1} = 10$

$X' = \frac{4}{4} = 1$ meters for A.

$M_X = -3.75 \times 1.63 + 30 \times 0.37 + 10.67 \times 1.37$

= 19.61 KN-m clockwise.

My = 10.67 × 1.5 + 30 × 1.5 + 3.75 × 1 = 64.76 clockwise.

Applying the general formulae in a tabular form for all points of frame.

Ma = ( Ms– Mi)a

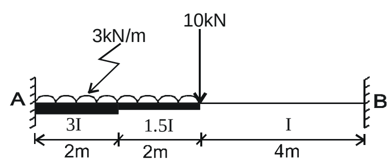
$(Mi)a = \frac{P}{A} \pm \frac{M_x\, y}{Ix} \pm \frac{M_y\, X}{Iy}$

| Point | Ms | P/A (1) | $\frac{Mx}{Ix} \cdot y$ (2) | $\frac{My}{Iy} \cdot X$ (3) | Mi (1)+(2) + (3) | M Ms – Mi |
|---|---|---|---|---|---|---|
| A | – 31 | –8.88 | – 5.44 | – 9.88 | – 24.2 | – 6.8 |
| B | – 15 | – 8.88 | + 3.74 | – 9.88 | –15.02 | + 0.02 |
| C | 0 | – 8.88 | + 3.74 | + 9.88 | + 4.74 | – 4.74 |
| D | 0 | – 8.88 | – 5.44 | + 9.88 | – 4.44 | + 4.44 |

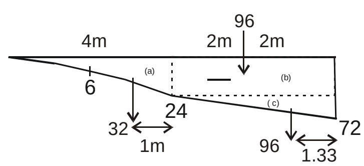
EXAMPLE NO. 14:- Analyze the following beam by column analogy method.

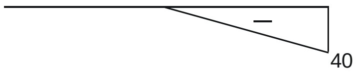
**SOLUTION :–**

Choosing B.D.S as cantilever supported at B



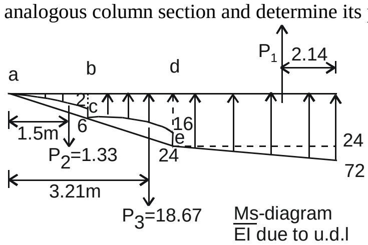

Ms-diagram
due to u.d.l. only



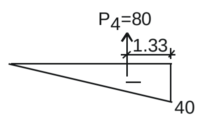



Ms diagram due to concentrated load only

Slectch analogous column section and determine its proteins

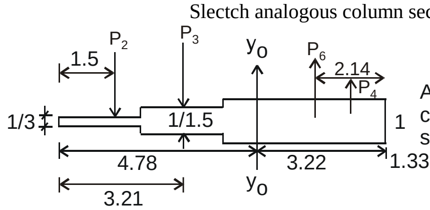



$\frac{Ms}{EI}$-diagram due to u.d.l




$\frac{MS}{EI}$ diagram due to point load.

Slectch analogous column section and determine its properties.




Analogous column section

$$P1 = \frac{24 \times 4}{3} + \frac{48 \times 4}{2} + 24 \times 4 = 224 \text{ KN}.$$ Corresponding to full Ms diagram, due to u.d.l.

**Location of P1 from B**

$224 \times X = 96 \times 1.33 + 96 \times 2 + 32 \times 5$

$X = 2.14$ meters

$P4 = \frac{1}{2} \times 4 \times 40 = 80$ KN, Corresponding to full Ms diagram due to point load.

**Note:** Area of 32 and its location of Ms diagram due to u.d.l. has been calculate d by formula e used in moment – area Theorems.

$$\text{area (abc)} = \int M_X dX = \int_o^2 -1.5X^2\, dX = \left|\frac{-1.5\,X^3}{3}\right|_o^2 = -4$$

$$\int (M_X)\, X\, dX = \int_o^2 -1.5X^3 dX = \left|\frac{-1.5\,X^4}{4}\right|_o^2 = -6$$

$$\bar{X} = \frac{-6}{-4} = 1.5\text{m from A}$$

$$\text{area (bcde)} = \int (M_X)\, dX = \int_o^4 -1.5X^2 dX - \int_o^2 -1.5\,X^2\, dX$$

$$= \left| -1.5\frac{X^3}{3} \right|_o^4 - \left| -1.5\frac{X^3}{3} \right|_o^2 = -28$$

$$\int (M_X)X\,dX = \int_o^4 -1.5\,X^3 dX - \int_o^2 -1.5\,X^3 dX = -90$$

$$X = \frac{-90}{-28}$$

$$= 3.21 \text{ meters from A (centroid of area bcde)}$$

$$P_3 = \frac{1}{1.5}(\text{area bcde}) = \frac{1}{1.5}(28) = 18.67 \text{ KN} \quad , \quad P_2 = \frac{1}{3}\text{area abc} = \frac{1}{3} \times 4 = 1.33$$

$$P_4 = 80 \text{ KN}$$

**Total concentric load on analogous column section**.

$$P = -P_1 + P_2 + P_3 - P_4$$

$$= -224 + 1.33 + 18.67 - 80$$

$$= -284 \text{ KN (upward)}$$

Total applied moment = M = – 224 × 1.68– 80 × 1.89 – 18.67 × 1.57 – 1.33 × 33 × 3.28

= – 426.79 KN-m (It means counter clockwise)

This total load P and M will now act at centroid of analogous column section.

**Properties of Analogous Column Section**.

$$A = \frac{1}{3} \times 2 + \frac{1}{1.5} \times 2 + 1 \times 4 = 6$$

$$\bar{X} = \frac{(1 \times 4) \times 2 + \left(2 \times \frac{1}{1.5}\right) \times 5 + \left(\frac{1}{3} \times 2\right) \times 7}{6}$$

$$= 3.22 \quad \text{from B.}$$

$$Iyoyo = \frac{1 \times 4^3}{12} + (1 \times 4)(1.22)^2 + \frac{\left(\frac{1}{1.5}\right) \times 2^3}{12} + \left(\frac{1}{1.5} \times 2\right)(1.78)^2$$

$$+\frac{\left(\frac{1}{3}\times 2^3\right)}{12}+\left(\frac{1}{3}\times 2\right)(3.78)^2$$

12

$= 25.70\ m^4$

$$(Mi)a = \frac{P}{A} \pm \frac{Mc}{I}$$

$$= \frac{-284}{6} + \frac{426.79 \times 4.78}{25.7}$$

= + 32.05 KN-m

(Ms)a = 0

Ma = (Ms – Mi)a = 0 – 32.05

**Ma = – 32.05 KN–m**

$$(Mi)b = \frac{P}{A} - \frac{Mc}{I}$$

$$= \frac{-284}{6} - \frac{426.79 \times 3.22}{25.7}$$

= – 100.81

(Ms)b = – 72 – 40 = – 112

Mb = (Ms – Mi)b

= – 112 + 100.81

**Mb = – 11.19 KN–m**

The beam has been analyzed. It is now statically determinate.