# CHAPTER-8

## Numerical Solution of Saint Venant's Equations

# The Saint-Venant equations

**The Saint-Venant equations for distributed routing are not amenable to analytical solution except in a few special simple cases. They are partial differential equations that, in general, must be solved using Numerical Methods.**

# Numerical Methods

**Methods for solving partial differential equations may be classified as;**

**1- Direct Numerical Method**

**2- Characteristics Methods**

## (1) Direct Numerical Method

**In direct methods, finite-difference equations are formulated from the original partial differential equations for continuity and momentum. Solutions for the flow rate and water surface elevation are then obtained for incremental times and distances along the stream or river.**

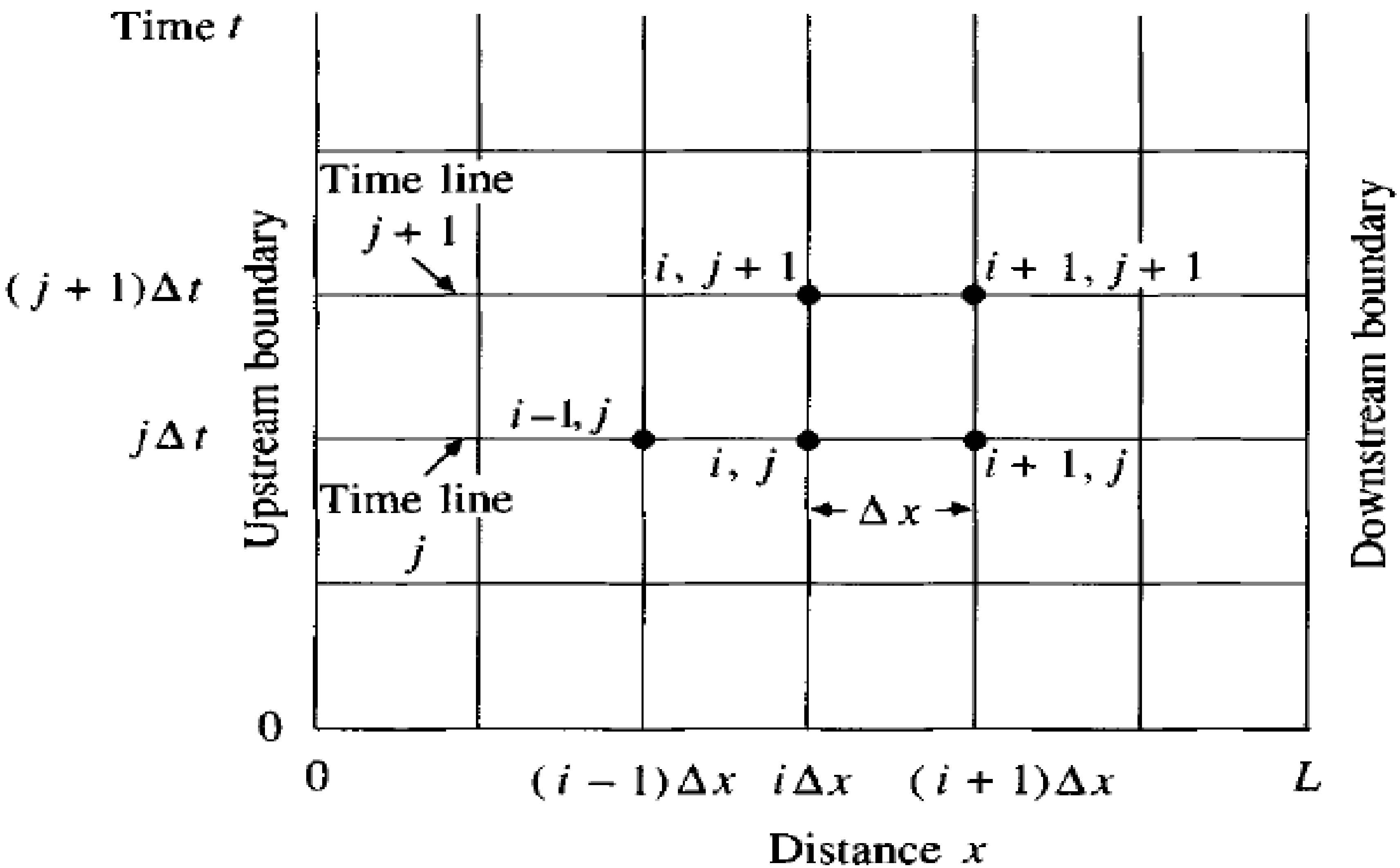
## (2) Characteristics Methods

**In characteristic methods, the partial differential equations are first transformed to a characteristic form, and the characteristic equations are solved analytically.**

# ❑ Numerical Methods:

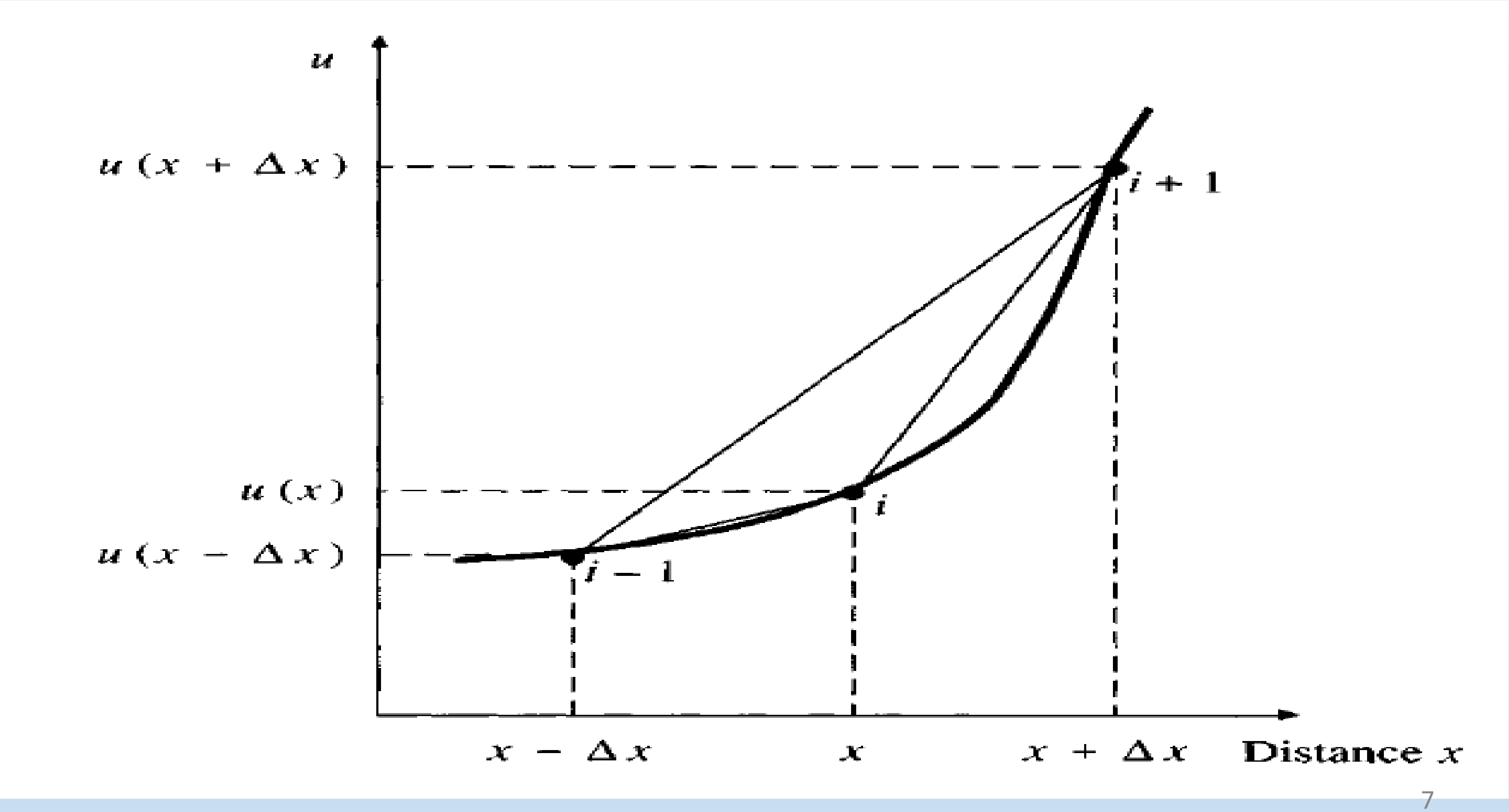
- In numerical methods for solving partial differential equations, the calculations are performed on a grid placed over the x-t plane.
- The x-t grid is a network of points defined by taking distance increments of length $\Delta x$, and time increments of duration $\Delta t$.
- The distance points are denoted by index i & time points by index j .
- A time line is a line parallel to the x axis as shown in Figure.

## The grid on the *x-t* plane used for numerical solution of the Saint-Venant equations by finite differences.



# Finite Differences

Finite-difference approximations can be derived for a function *u(x)* as shown in Fig.



# Finite Differences:

- A Taylor series expansion of u(x) at (x + Δx) produces;

$$u(x+\Delta x) = u(x) + \Delta x\, u'(x) + \frac{1}{2}\Delta x^2 u''(x) + \frac{1}{6}\Delta x^3 u'''(x) + \ldots \qquad (1)$$

**Where**

$$u'(x) = \partial u / \partial x$$

$$u''(x) = \partial^2 u / \partial x^2$$

# Finite Differences

**The Taylor series expansion at $x - \Delta x$ is**

$$u(x - \Delta x) = u(x) - \Delta x\, u'(x) + \frac{1}{2}\Delta x^2 u''(x) - \frac{1}{6}\Delta x^3 u'''(x) + \ldots \quad (2)$$

**A *central-difference* approximation uses the difference defined by subtracting eq (2) from (1)**

$$u(x + \Delta x) - u(x - \Delta x) = 2\,\Delta x\, u'(x) + 0(\Delta x^3)$$

$$u'(x) \approx \frac{u(x + \Delta x) - u(x - \Delta x)}{2\,\Delta x} \quad (3)$$

# Finite Differences:

Eq.(3) has an error of approximation of order $\Delta x^2$ . This approximation error, due to dropping the higher order terms, is also referred to as a **truncation error.**

A forward difference approximation is defined by subtracting u(x) from eq.(1)

$$u(x+\Delta x) - u(x) = \Delta x\, u'(x) + 0(\Delta x^2)$$

$$u'(x) \approx \frac{u(x+\Delta x) - u(x)}{\Delta x} \qquad (4)$$

which has an error of approximation of order $\Delta x$.

# Finite Differences:

- The backward difference approximation uses the difference defined by subtracting eq.(2) from u(x)

$$u(x) - u(x - \Delta x) = \Delta x\, u'(x) + 0(\Delta x^2)$$

$$u'(x) \approx \frac{u(x) - u(x - \Delta x)}{\Delta x}$$

Error of approximation is of the order Δx

# Finite Difference Schemes

There are Two schemes

**1- Explicit Scheme**

**2- Implicit Scheme**

## Explicit Scheme

- The unknown values are solved *sequentially* along a time line from one distance point to the next.
- Simpler but can be unstable.
  (smaller $\Delta x$ and $\Delta t$ are required)
- Convenient as results are given at grid points.
- Can treat slightly varying channel geometry from section to section.
- Less efficient
- so not suitable for routing the flood flows over a long period of time.
- It uses forward diff. app. for time scale and central diff. app. for space scales.

## Implicit Scheme

- The unknown values on a given time line are all determined *simultaneously.*
- More complicated, but with the use of computers this is not a serious problem once the method is programmed.
- The method is stable for large computation steps.
- Can treat significantly variations in channel sections.
- Much faster,
- Little loss of accuracy.
- Both for space and time, forward diff. app. are used.

# ➢Explicit Scheme:

- A forward-difference scheme is used for the time derivative and a central difference scheme is used for the space scale.

$$\frac{\partial u_i^{j+1}}{\partial t} = \frac{u_i^{j+1} - u_i^j}{\Delta t}$$

( For time derivatives, forward diff.)

$$\frac{\partial u_i^j}{\partial x} = \frac{u_{i+1}^j - u_{i-1}^j}{2\,\Delta x}$$

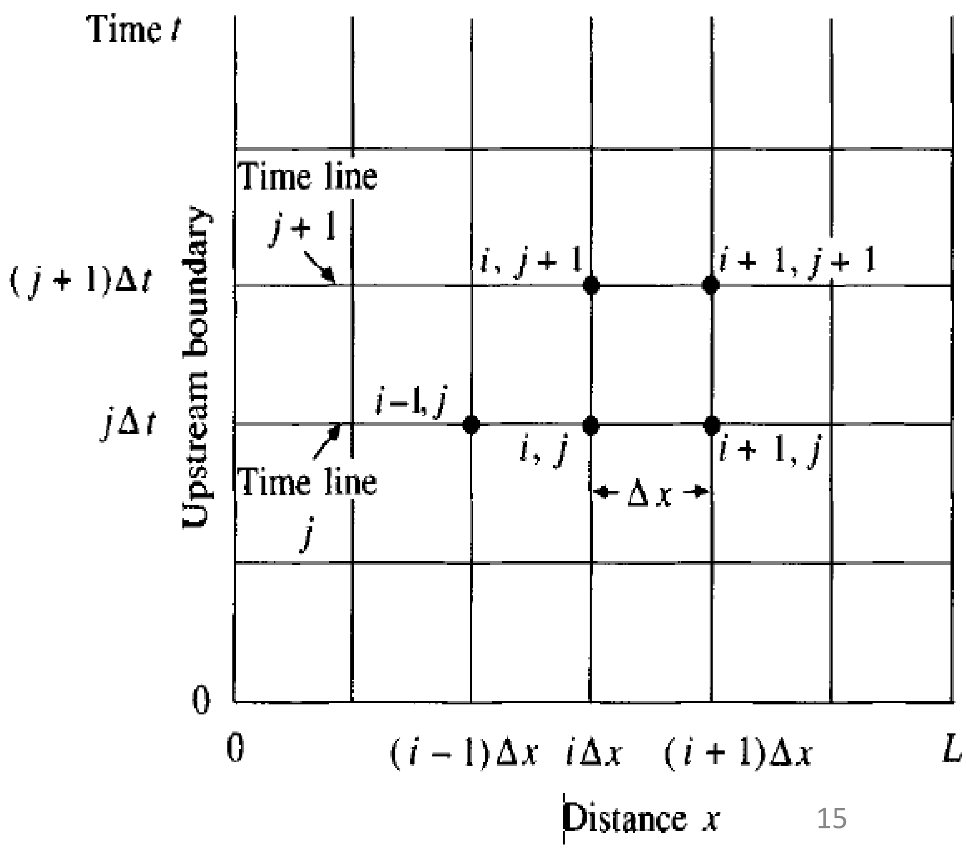
( For space scale derivatives, central diff.)

# Implicit Scheme

**Both the temporal and spatial derivative in terms of the dependent variable on the unknown time line. (j+1), forward diff.**

$$\frac{\partial u_{i+1}^{j+1}}{\partial x} = \frac{u_{i+1}^{j+1} - u_{i}^{j+1}}{\Delta x}$$

$$\frac{\partial u_{i+1}^{j+1}}{\partial t} = \frac{u_{i+1}^{j+1} - u_{i+1}^{j}}{\Delta t}$$



# Numerical Solution of The Kinematic Wave Routing Eqns.

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q$$ → **Continuity Equation (1)**

$$S_o = S_f$$ → **Momentum Equation (2)**

## ❑ Numerical Solution Of The Kinematic Wave Routing Eqns.

**Continues . . . . . .**

We know that ;

$$Q = A * V \quad \&$$

$$Q = A * \frac{1.49}{n} * R^{\frac{2}{3}} * S_o^{\frac{1}{2}}$$

**Let R =** $\frac{A}{P}$

$$Q = \frac{1.49}{n} * \left(\frac{A}{P}\right)^{\frac{2}{3}} * A * S_o^{\frac{1}{2}}$$

## ❑ Numerical Solution Of The Kinematic Wave Routing Eqns.

**Continues . . . . . .**

$$A = \left( \frac{nP^{2/3}}{1.49\sqrt{S_o}} \right)^{3/5} Q^{3/5} \longrightarrow \quad (3)$$

## ❑ Numerical Solution Of The Kinematic Wave Routing Eqns.

**Continues . . . . . .**

$$A = \alpha Q^{\beta} \qquad (4)$$

**Equation # 4 has two variables A & Q**

**Where**

$$\propto = \left( \frac{nP^{2/3}}{1.49\sqrt{S_o}} \right)^{3/5}$$

And $\beta =$

## ❑ Numerical Solution Of The Kinematic Wave Routing Eqns.

**Continues . . . . . .**

*Variable A* can be eliminated by differentiating eqn # 4 .

$$\frac{\partial A}{\partial t} = \alpha \beta Q^{\beta - 1}\left(\frac{\partial Q}{\partial t}\right) \longrightarrow \quad (5)$$

Substituting the value of $\frac{\partial A}{\partial t}$ in eqn # 1.

$$\frac{\partial Q}{\partial x} + \alpha \beta Q^{\beta - 1}\left(\frac{\partial Q}{\partial t}\right) = q \longrightarrow \quad (6)$$

## ❑ Numerical Solution Of The Kinematic Wave Routing Eqns.

**Continues . . . . . .**

**Objective :**

To solve eqn (6) for *Q(x, t)* at each point on the *x-t* grid.

**Unknown value =** $Q_{i+1}^{j+1}$

**Solutions are possible by ;**

1) **Linear Scheme**
2) **Non Linear Scheme**

# ❖ Linear Scheme;

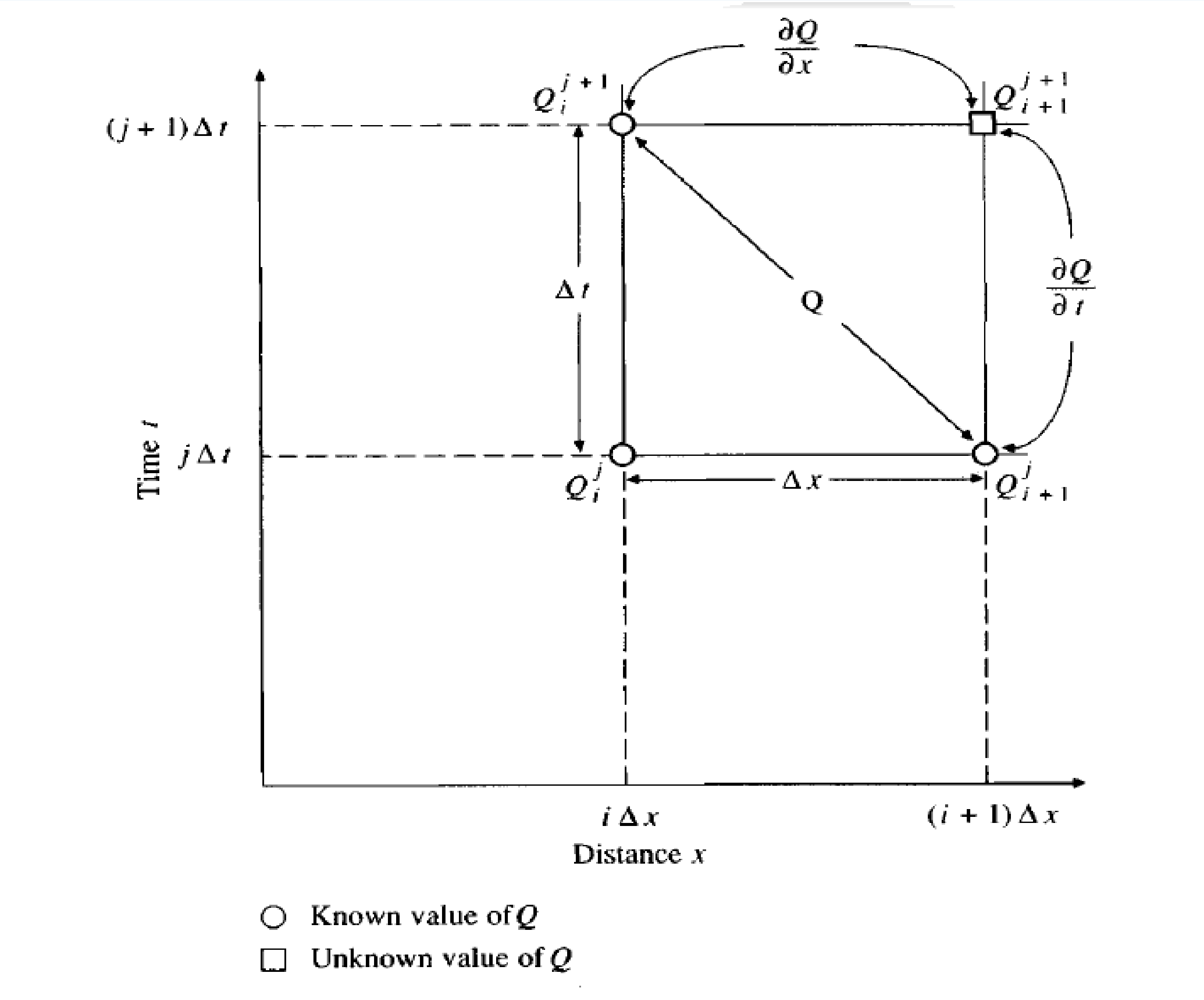



**FIGURE**
**Finite difference box for solution of the linear kinematic wave equation showing the finite difference equations.**

# ❖ Linear Scheme

- **The backward-difference method is used to set up the finite-difference equations.**

$$\frac{\partial Q}{\partial x} \approx \frac{Q_{i+1}^{j+1} - Q_{i}^{j+1}}{\Delta x} \qquad (7)$$

$$\frac{\partial Q}{\partial t} \approx \frac{Q_{i+1}^{j+1} - Q_{i+1}^{j}}{\Delta t} \qquad (8)$$

$$Q \approx \frac{Q_{i}^{j+1} + Q_{i+1}^{j}}{2} \qquad (9)$$

## ❖ Linear Scheme

**Average lateral flow;**

$$q \approx \frac{q_{i+1}^{j+1} + q_{i+1}^{j}}{2} \qquad \textbf{(10)}$$

**Sbstituting eqn (7)-(10) in eqn (6) ;**

$$\frac{Q_{i+1}^{j+1} - Q_{i}^{j+1}}{\Delta x} + \alpha\beta\left(\frac{Q_{i+1}^{j} + Q_{i}^{j+1}}{2}\right)^{\beta-1}\left(\frac{Q_{i+1}^{j+1} - Q_{i+1}^{j}}{\Delta t}\right) = \frac{q_{i+1}^{j+1} + q_{i+1}^{j}}{2} \qquad \textbf{(11)}$$

# ❖ Linear Scheme

**Equation # 11 can be solved for unknown** $Q_{i+1}^{j+1}$

Finally we get;

$$Q_{i+1}^{j+1} = \frac{\left[\dfrac{\Delta t}{\Delta x} Q_i^{j+1} + \alpha\beta Q_{i+1}^{j} \left(\dfrac{Q_{i+1}^{j} + Q_i^{j+1}}{2}\right)^{\beta-1} + \Delta t \left(\dfrac{q_{i+1}^{j+1} + q_{i+1}^{j}}{2}\right)\right]}{\left[\dfrac{\Delta t}{\Delta x} + \alpha\beta \left(\dfrac{Q_{i+1}^{j} + Q_i^{j+1}}{2}\right)^{\beta-1}\right]} \qquad (12)$$

# ❖ Non Linear Kinematic wave equation :

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q$$

→ **(1)** It is unconditionally stable.

**Where**

$$A = \alpha Q^{\beta}$$

# ❖ Non Linear Kinematic wave equation continued . . . .

- For Implicit solution, space & time derivation should be forward difference.
- The finite difference form of the equation (1) ;

$$\frac{Q_{i+1}^{j+1} - Q_{i}^{j+1}}{\Delta x} + \frac{A_{i+1}^{j+1} - A_{i+1}^{j}}{\Delta t} = \frac{q_{i+1}^{j+1} + q_{i+1}^{j}}{2} \longrightarrow \quad (2)$$

## ❖ Non Linear Kinematic wave equation continued . . . .

In eqn (2) ;

$$A_{i+1}^{j+1} = \alpha\left(Q_{i+1}^{j+1}\right)^{\beta} \longrightarrow \text{3A}$$

$$A_{i+1}^{j} = \alpha\left(Q_{i+1}^{j}\right)^{\beta} \longrightarrow \text{3B}$$

## ❖ Non Linear Kinematic wave equation continued . . . .

Substituting the values in eqn (2), we have;

$$\frac{\Delta t}{\Delta x} Q_{i+1}^{j+1} + \alpha \left(Q_{i+1}^{j+1}\right)^{\beta} = \frac{\Delta t}{\Delta x} Q_{i}^{j+1} + \alpha \left(Q_{i+1}^{j}\right)^{\beta} + \Delta t \left( \frac{q_{i+1}^{j+1} + q_{i+1}^{j}}{2} \right) \quad (4)$$

This eqn is a nonlinear eqn for $Q_{i+1}^{j+1}$ unknown. So a numerical solution scheme such as Newton's method will be required…

# Thanks