Plain & Reinforced Concrete-1

Sixth Term
Civil Engineering

CE-314

Lecture # 13

Flexural Analysis and Design of Beams
(Ultimate Strength Design of Beams)

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Doubly Reinforced Beams

"Beams having both tension and compression reinforcement to allow the depth of beam to be lesser than minimum depth for singly reinforced beam but still having tension-controlled behavior"

- By using lesser depth the lever arm reduces and to develop the same force more area of tension steel (besides extra compression steel) is required, so solution is costly.
- Ductility will be increased by providing compression steel.
- Hanger bars can also be used as compression steel reducing the cost up to certain cost.
- For high rise buildings the extra cost of the shallow deep beams is offset by saving due to less story height.

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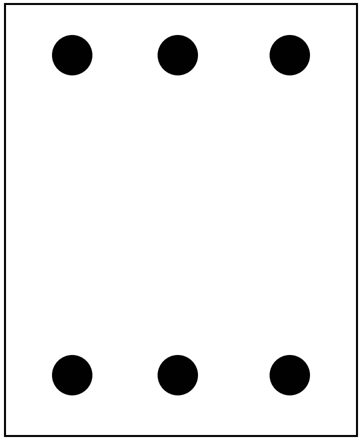
Doubly Reinforced Beams (contd…)

- Compression steel may reduce creep and shrinkage of concrete and thus reducing long term deflection.



Doubly Reinforced Beam

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Behavior of Doubly Reinforced Beams

Tension steel always yields in D.R.B.

There are two possible cases:

1. Case-I Compression steel is yielding at ultimate condition.
2. Case-II Compression steel is NOT yielding at ultimate condition.

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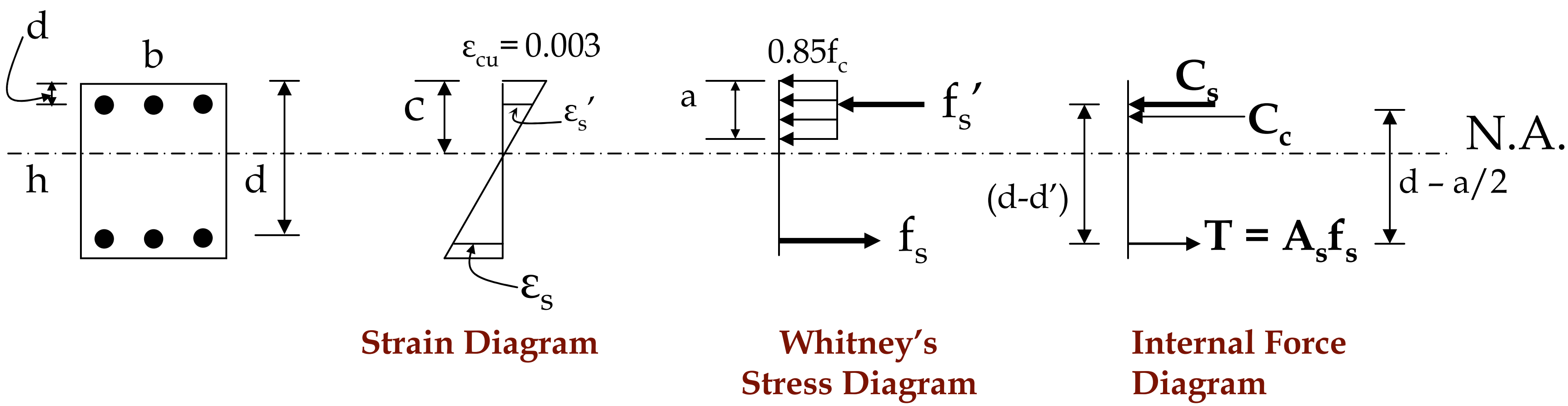
Behavior Doubly Reinforced Beams



$f_s = E_s\varepsilon_s$

$f_s' = E_s\varepsilon_s'$

$$T = A_sf_s$$

$$C_s = A_s'f_s'$$

$$C_c = 0.85f_c'ba$$

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Behavior Doubly Reinforced Beams (contd…)

Case-I Both Tension & Compression steel are yielding at ultimate condition

$f_s = f_y$ and $f_s' = f_y$

Location of N.A.

Consider equilibrium of forces in longitudinal direction

$$T = C_c + C_s$$

$$A_s f_y = 0.85 f_c' b a + A_s' f_y$$

$$\boxed{a = \frac{(A_s - A_s')f_y}{0.85 f_c' b}} \quad \text{and} \quad \boxed{c = \frac{a}{\beta_1}}$$

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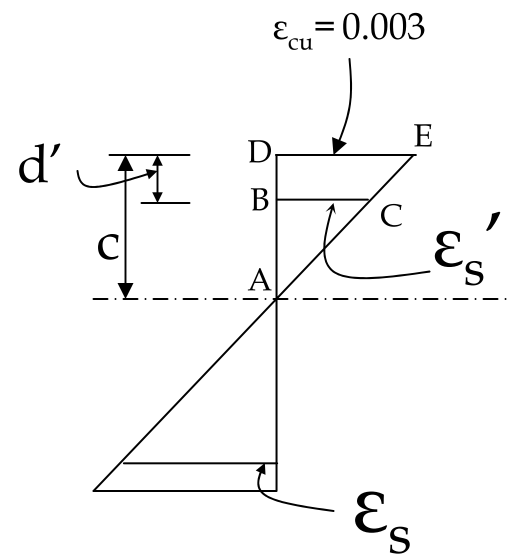
Δ ABC & ADE

$$\frac{\varepsilon_s'}{0.003} = \frac{c - d'}{c}$$

$$\varepsilon_s' = 0.003\left(\frac{c - d'}{c}\right)$$

$$\varepsilon_s' = 0.003\left(\frac{c - d'}{c}\right) \times \frac{\beta_1}{\beta_1}$$

$$\varepsilon_s' = 0.003\left(\frac{a - \beta_1 d'}{a}\right) \qquad (1)$$



Strain Diagram

If $\varepsilon_s' \geq \varepsilon_y$ compression steel is yielding.

If $\varepsilon_s' < \varepsilon_y$ compression steel is **NOT** yielding.

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T = total tensile force in the steel

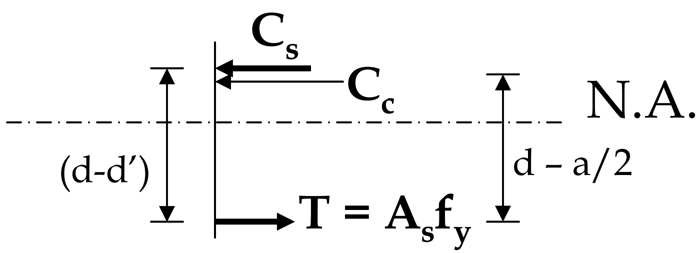
$$T = T_1 + T_2$$

T_1 is balanced by C_s

T_2 is balanced by C_c

$$T_1 = C_s$$

$$T_2 = C_c$$



Internal Force Diagram

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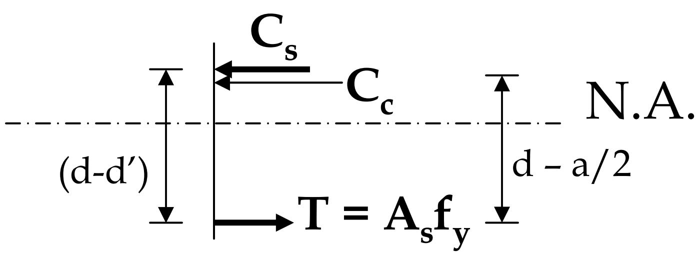
Moment Capacity by Compression Steel

$$M_{n_1} = C_s(d-d') = A_s' f_y (d-d')$$
$$= T_1(d-d')$$



Internal Force Diagram

Moment Capacity by Concrete

$$M_{n_2} = C_c\left(d-\frac{a}{2}\right) = T_2\left(d-\frac{a}{2}\right)$$
$$= (T - T_1)\left(d - \frac{a}{2}\right)$$
$$= (A_s - A_s')f_y\left(d - \frac{a}{2}\right)$$

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Total Moment Capacity

$$M_n = M_{n_1} + M_{n_2}$$

$$\boxed{M_n = A_s' f_y (d - d') + (A_s - A_s') f_y \left(d - \frac{a}{2} \right)}$$

Where

$$\boxed{a = \frac{(A_s - A_s') f_y}{0.85 f_c' b}}$$

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Case-II Compression steel is not yielding at ultimate condition. Similarly, even the tension steel may not be yielding

$f_s = f_y$ or not and $f_s' < f_y$

$$f_s' = E \times \varepsilon_s'$$

$$f_s' = 600\,\frac{a - \beta_1 d'}{a} \qquad f_s = 600\,\frac{\beta_1 d - a}{a}$$

Location of N.A. $\boxed{a = \dfrac{A_s f_s - A_s' f_s'}{0.85 f_c' b}}$ and $\boxed{c = \dfrac{a}{\beta_1}}$

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$$M_n = A_s' f_s' (d - d') + 0.85\, f_c' b\, a\, (d - a/2)$$

$$\text{or } A_s' f_s' (d - d') + (A_s - A_s')\, f_s\, (d - a/2)$$

BALANCED STEEL RATIO FOR DOUBLY REINFORCED SECTIONS

The following symbols may be used in the development of the expression for the balanced steel ratio in case of doubly reinforced sections.

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$\bar{\rho}_b$ = total balanced tension steel ratio in case of doubly reinforced beams, limiting steel ratio dividing the under-reinforced and the over-reinforced behavior

ρ_b = balanced tension steel ratio in case of corresponding singly reinforced beams

ρ' = compression steel ratio $= \dfrac{A'_s}{b\,d}$

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Total force in tension steel	=	Force in tension steel to balance concrete compression	+	Force in tension steel to balance steel compression

$$T = T_1 + T_2$$

$$\text{or} \quad T = C_c + C_s$$

$$A_s f_y = 0.85 f_c' \, b \, a + A_s' f_y$$

In the above equation, the compression steel is assumed to be yielding. If it is not actually yielding, adjustment is made later on.

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Divide the above equation throughout by bdf_y.

$$\frac{A_s}{b\,d} = 0.85\,\frac{f_c'}{f_y}\,\frac{a}{d} + \frac{A_s'}{b\,d}$$

$$\bar{\rho}_b = 0.85\,\frac{f_c'}{f_y}\,\frac{a}{d} + \rho'$$

From the strain diagram for the balanced failure,

$$\frac{a}{d} = \beta_1\,\frac{0.003\,E_s}{f_y + 0.003\,E_s}$$

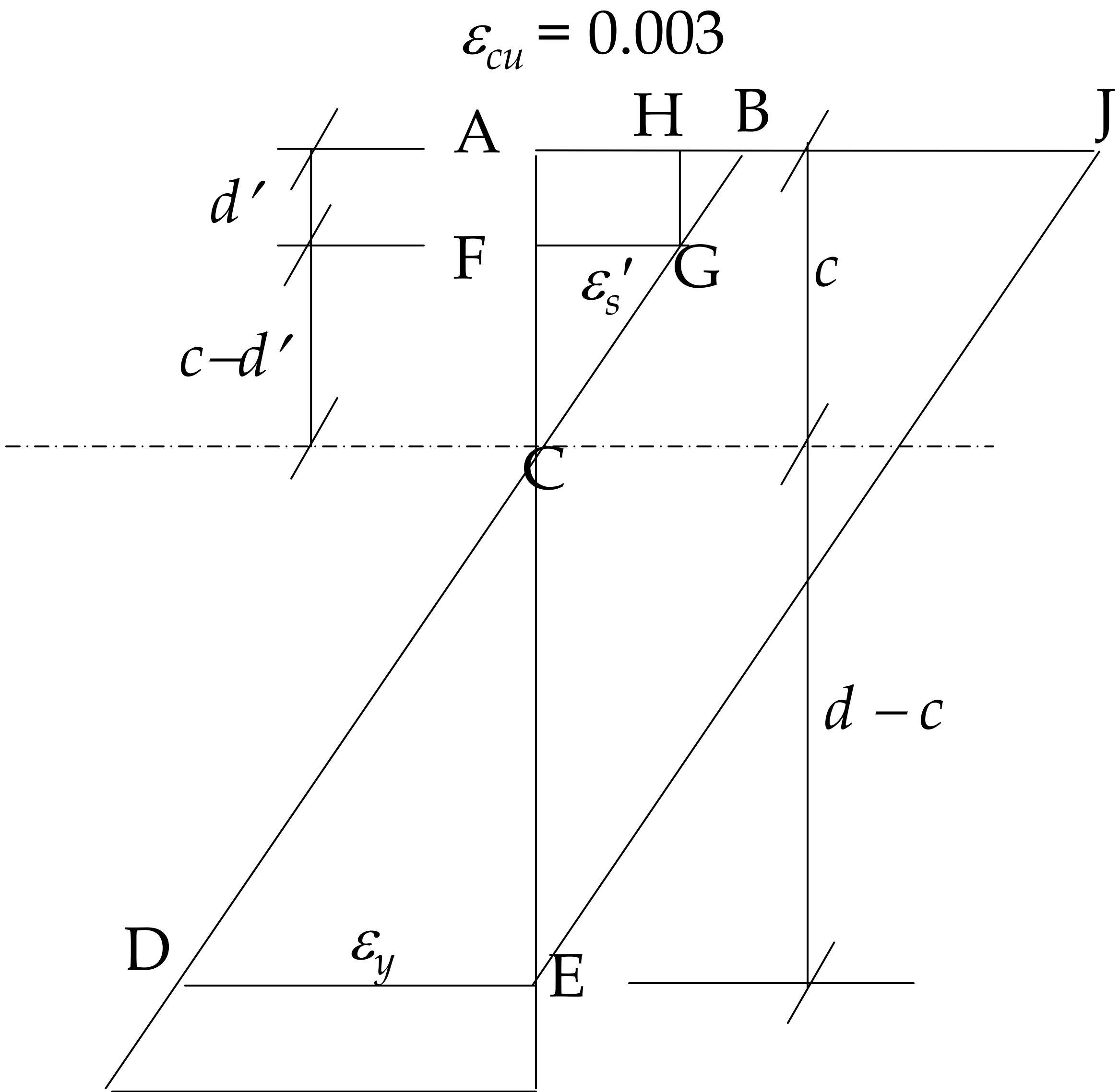
$$\bar{\rho}_b = 0.85\,\beta_1 \frac{f_c'}{f_y}\frac{600}{f_y + 600} + \rho'$$

$$\bar{\rho}_b = \rho_b + \rho'$$

The above formula is only valid if both the steels are yielding.

Tension steel is always yielding at the balanced condition by the definition.

However, the compression steel may or may not be yielding. If the compression steel has not yielded, the balanced steel ratio may be determined using Fig. 4.5 as follows:


$\varepsilon_{cu} = 0.003$
A
H
B
J
d'
F
G
ε_s'
c
$c-d'$
C
$d-c$
D
ε_y
E

From the similarity of triangles HBG and AJE, we get,

$$\frac{HB}{HG} = \frac{AJ}{AE} \quad \text{or} \quad \frac{HB}{d'} = \frac{0.003+\varepsilon_y}{d}$$

$$HB = \frac{0.003+\varepsilon_y}{d} \times d'$$

Now, $\varepsilon_s' = 0.003 - HB = 0.003 - \dfrac{d'}{d} \times (0.003 + \varepsilon_y)$

$$f_s' = E_s \left[\, 0.003 - \frac{d'}{d} \times (0.003 + \varepsilon_y) \,\right]$$

$$= 600 - \frac{d'}{d} \times (600 + f_y) \leq f_y$$

The effectiveness of the compression steel of magnitude A_s' is reduced in the ratio f_s' / f_y when the compression steel is not yielding. The balanced steel ratio, for the cases where compression steel is not yielding, is written as follows:

$$\bar{\rho}_b = \rho_b + \rho' \times \frac{f_s'}{f_y}$$

From the strain diagram for the tension-controlled failure, the following may be written:

$$\frac{a}{d} = \beta_1 \times \frac{3}{8} d$$

Putting this value of *a*/*d* in equation for $\bar{\rho}_b$ replaced with $\bar{\rho}_{\max}$, the following is obtained:

$$\bar{\rho}_{\max} = 0.85\beta_1 \times \frac{3}{8} d + \rho'$$

$\bar{\rho}_{\max}$ = ρ_{max} for singly reinforced sections with any extreme strain + ρ'

If the compression steel is not yielding, the formula may be written as follows:

$$\bar{\rho}_{mam} = \rho_{\max} + \rho' \times \frac{f_s'}{f_y}$$

MINIMUM TENSION STEEL RATIO FOR COMPRESSION STEEL YIELDING

From the strain diagram, $\dfrac{c}{c-d'} = \dfrac{0.003}{\varepsilon_y}$

$$c\,\varepsilon_y = 0.003\,c - 0.003\,d'$$

$$c\,(0.003 - \varepsilon_y) = 0.003\,d'$$

$$c = \frac{0.003}{0.003-\varepsilon_y}\,d' \qquad a = \beta_1 \frac{0.003}{0.003-\varepsilon_y}\,d'$$

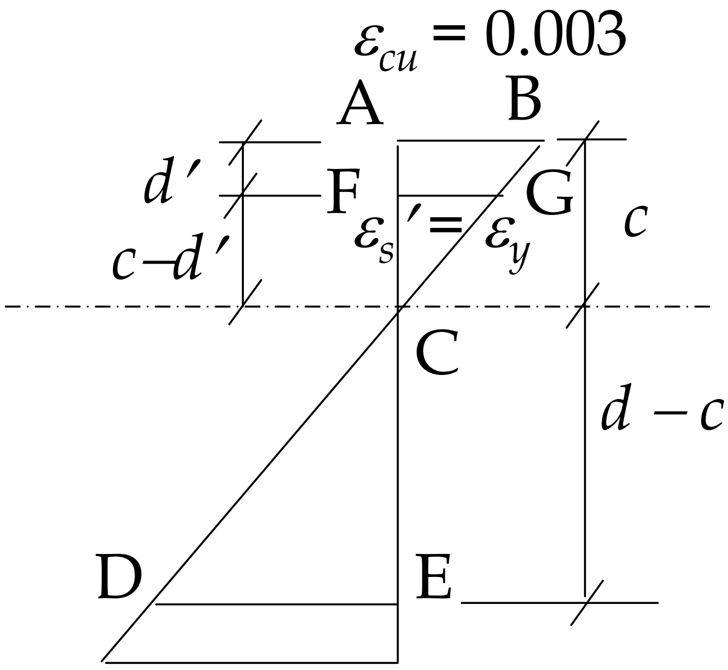


Fig.4.7. Ultimate Strain Diagram To Determine Minimum Steel Ratio For Compression Steel Yielding.

Summing the forces in the horizontal direction:

$$A_s f_y = 0.85 f_c' b a + A_s' f_y$$

Divide the above equation throughout by bdf_y.

$$\frac{A_s}{bd} = 0.85 \frac{f_c'}{f_y} \frac{a}{d} + \frac{A_s'}{bd}$$

$$\bar{\rho}_{cy} = 0.85 \frac{f_c'}{f_y} \frac{d'}{d} \beta_1 \left(\frac{0.003}{0.003 - \varepsilon_y} \right) + \rho'$$

$$= 0.85 \frac{f_c'}{f_y} \frac{d'}{d} \beta_1 \left(\frac{600}{600 - f_y} \right) + \rho'$$

ANOTHER CHECK FOR YIELDING OF COMPRESSION STEEL

The condition for yielding of the compression steel when maximum tensile steel ratio is to be used may be evaluated for different grades of steel as follows:

- For f_y = 280 MPa, $d' / d \leq 0.2$
- For f_y = 420 MPa, $d' / d \leq 0.1125$

ANALYSIS OF DOUBLY REINFORCED SECTIONS

Data: i) Dimensions like b, d, d' and L

ii) f_c', f_y and E_s

iii) Areas of tension and compression steels, A_s and A_s'

Required: Moment capacity, $\phi_b M_n$

1. Check the beam as a singly reinforced section to see whether the provided compression steel will be used for strength or not. For this purpose, ρ may be checked and it must be more than ρ_{max} for singly reinforced sections.

$$\rho_{\max} = 0.85\beta_1 \frac{3}{8}\frac{f_c'}{f_y}$$

for singly reinforced sections

2. Assume both tension and compression steels to be yielding and calculate the depth of N.A. (*c*) and depth of equivalent rectangular stress block (*a*).

$$a = \frac{(A_s - A_s')f_y}{0.85\, f_c' b} \qquad \text{and } c = a / \beta_1$$

3. Check for yielding of both the steels.

$$\varepsilon_y = f_y / E_s$$

$$\varepsilon_s = 0.003\,\frac{d-c}{c} \qquad \varepsilon_s' = 0.003\,\frac{c-d'}{c}$$

If $\varepsilon_s \geq \varepsilon_y \Rightarrow f_s = f_y$

If $\varepsilon_s' \geq \varepsilon_y \Rightarrow f_s' = f_y$

Alternatively calculate $\bar{\rho}_{cy}$ and make sure that $\rho \geq \bar{\rho}_{cy}$ for yielding of the compression steel. Similarly d'/d ratio may be evaluated for checking of the compression steel, as explained earlier.

4. If the tension and/or the compression steels are not yielding, write he expressions for f_s and/or f_s' in terms of unknown a and solve to get its new value.

$$f_s' = 600\,\frac{a - \beta_1 d'}{a} \qquad\qquad f_s = 600\,\frac{\beta_1 d - a}{a}$$

$T = C_s + C_c \Rightarrow$ tension failure

$$A_s\, 600\,\frac{\beta_1 d - a}{a} = 600\,\frac{a - \beta_1 d'}{a} A_s' + 0.85\, f_c' b\, a$$

$$\left(\frac{0.85\, f_c' b}{600}\right) a^2 + (A_s + A_s')\, a - \beta_1 (A_s d + A_s' d') = 0$$

Calculate *a* from this equation and then calculate f_s' and f_s from the earlier two equations.

5. Calculate the strength reduction factor (ϕ_b) depending on the value of the depth of N.A. and the extreme tensile strain.
6. Calculate the flexural strength depending upon the whether the steels are yielding or not.

Both Steels Are Yielding

$$f_s = f_s' = f_y$$

$$M_n = (A_s - A_s')\, f_y\, (d - a/2) + A_s'\, f_y\, (d - d')$$

$$M_u = \phi_b\, M_n$$

Any Steel Is Not Yielding

$$f_s' = 600\,\frac{a - \beta_1 d'}{a}$$

$$f_s = 600\,\frac{\beta_1 d - a}{a}$$

$$M_n = (A_s f_s - A_s' f_s')\,(d - a/2) + A_s' f_s'\,(d - d')$$

$$M_u = \phi_b M_n$$

Example 4.1

A doubly reinforced rectangular section has the following section properties:

- $A_s' = 568 \text{ mm}^2$ [2 – #19(US)]
- $A_s = 3060 \text{ mm}^2$ [6 – #25(US)]
- $b = 300$ mm
- $d' = 60$ mm
- $f_y = 300$ MPa

Calculate the design flexural strength for the following two conditions:

i) C – 20 concrete and $d = 525$ mm.

ii) C – 35 concrete and $d = 225$ mm.

Solution:
Case (i)

- $A_s' = 570\ \text{mm}^2$
- $A_s = 3060\ \text{mm}^2$
- $b = 300\ \text{mm}$
- $d' = 60\ \text{mm}$
- $d = 525\ \text{mm}$
- $f_y = 300\ \text{MPa}$
- $f_c' = 20\ \text{MPa}$
- $E_s = 200{,}000\ \text{MPa}$
- $\phi_b M_n = ?$

- $\rho = \dfrac{(3060)}{(300)(525)} = 0.0194$
- $\rho' = \dfrac{(568)}{(300)(525)} = 0.0036$
- $\rho_{max} = 0.85\beta_1 \dfrac{3}{8}\dfrac{f_c'}{f_y}$ singly reinforced sections

  $= 0.85^2 \times \dfrac{3}{8} \times \dfrac{20}{300} = 0.0181$
- $\overline{\rho}_{\max} = \rho max + \rho'$

  assuming the compression steel to be yielding

  $= 0.0181 + 0.0036 = 0.0217$

$$\bar{\rho}_{cy} = 0.85\,\frac{f_c'}{f_y}\,\frac{d'}{d}\,\beta_1\left(\frac{600}{600 - f_y}\right) + \rho'$$

$$= 0.85^2 \times \frac{20}{300} \times \frac{60}{525} \times \left(\frac{600}{600 - 300}\right) + 0.0036$$

$= 0.0146$

- $\rho > \rho_{max}$ for singly reinforced section, analyze as doubly reinforced section.
- $\rho > \bar{\rho}_{cy}$, compression steel is expected to be yielding.
- $\rho \leq \bar{\rho}_{max}$, tension steel is expected to be yielding and $\phi_b = 0.90$.

- $a = \dfrac{(A_s - A'_s) f_y}{0.85 f'_c b} = \dfrac{(3060 - 568) \times 300}{0.85 \times 20 \times 300} = 146.6$ mm

- $c = a / \beta_1 = 146.6 / 0.85 = 172$ mm
- $\varepsilon_y = f_y / E_s = 300 / 200{,}000 = 0.0015$

- $\varepsilon_s = 0.003 \dfrac{d - c}{c} = 0.003 \dfrac{525 - 172}{172} = 0.00616$

  $>$ ε_y and 0.005, tension steel is yielding and $\phi_b = 0.90$.

- $\varepsilon_s' = 0.003 \dfrac{c - d'}{c} = 0.003 \dfrac{172 - 60}{172} = 0.00195$

  $> \varepsilon_y$, the compression steel is yielding.

- $M_n = (A_s - A_s') f_y (d - a / 2) + A_s' f_y (d - d')$

  $= [(3060 - 568) \times 300 \times (525 - 146.6 / 2)$

  $+ 568 \times 300 \times (525 - 60)] / 10^6$

  $= 416.9$ kN-m

- $M_u = \phi_b M_n$

  $= 0.9 \times 416.9 = 375.2$ kN-m

Case (ii)

- $A_s' = 568 \text{ mm}^2$
- $A_s = 3060 \text{ mm}^2$
- $b = 300 \text{ mm}$
- $d' = 60 \text{ mm}$
- $d = 225 \text{ mm}$
- $f_y = 300 \text{ MPa}$
- $f_c' = 35 \text{ MPa}$
- $E_s = 200{,}000 \text{ MPa}$
- $\phi_b M_n = ?$

- $\rho = \dfrac{(3060)}{(300)(225)} = 0.0453$
- $\rho' = \dfrac{(568)}{(300)(225)} = 0.00841$
- For $f_c' > 28$ MPa, $\beta_1 = 1.05 - 0.00714\, f_c' \leq 0.85 = 0.8$
- $\rho_{max} = 0.85\beta_1 \dfrac{3}{8}\dfrac{f_c'}{f_y}$ singly reinforced sections

$$= 0.85 \times 0.8 \times \frac{3}{8} \times \frac{35}{300} = 0.0298$$

- $\bar{\rho}_{\max} = \rho_{max} + \rho'$

  assuming the compression steel to be yielding

  $= 0.0298 + 0.00841 = 0.03821$

- $$\bar{\rho}_b = 0.85\,\beta_1 \frac{f_c'}{f_y}\left(\frac{600}{600+f_y}\right) + \rho'$$

$$= 0.85 \times 0.8 \times \frac{35}{300} \times \left(\frac{600}{600+300}\right) + 0.00841$$

$$= 0.0613$$

$$\bar{\rho}_{cy} = 0.85\,\frac{f'_c}{f_y}\,\frac{d'}{d}\,\beta_1\left(\frac{600}{600-f_y}\right)+\rho'$$

$$= 0.85\times0.8\times\frac{35}{300}\times\frac{60}{225}\times\left(\frac{600}{600-300}\right)+0.00841$$

= 0.0507

- ρ > $\rho_{\max}$ for singly reinforced section, analyze as doubly reinforced section.
- ρ < $\bar{\rho}_{cy}$, compression steel is not yielding.
- ρ > $\bar{\rho}_{\max}$, extreme tensile strain is not greater than 0.005.
- ρ $\leq$ $\bar{\rho}_b$, tension steel is yielding.

Confirmation of above results (Are not always needed)

- Considering both the tension and the compression steels to be yielding, calculate depth of equivalent rectangular stress block (a) and the depth of N.A. (c).

- $a = \dfrac{(A_s - A'_s) f_y}{0.85 f'_c b} = \dfrac{(3060 - 568) \times 300}{0.85 \times 35 \times 300} = 83.8$ mm

- $c = a / \beta_1 = 83.8 / 0.80 = 104.8$ mm
- $\varepsilon_y = f_y / E_s = 300 / 200{,}000 = 0.0015$

$$\varepsilon_s = 0.003\,\frac{d-c}{c} = 0.003\,\frac{225-104.8}{104.8} = 0.00344$$

$> \varepsilon_y$, tension steel is yielding

$$\varepsilon_s' = 0.003\,\frac{c-d'}{c} = 0.003\,\frac{104.8-60}{104.8} = 0.00128$$

$< \varepsilon_y$, the compression steel is not yielding.

Calculation of revised value of a:

- $f_s' = 600\,\dfrac{a-\beta_1 d'}{a} = 600\,\dfrac{a-0.8\times 60}{a}$

- $T = C_s + C_c \Rightarrow$ tension failure

  $A_s f_y = A_s' f_s' + 0.85 f_c' b a$

$3060 \times 300 = 568 \times A_s' + 0.85 \times 35 \times 300 \times a$

$918{,}000\, a = 340{,}800\, a - 16{,}358{,}400 + 8925\, a2$

  $a^2 - 64.67\, a - 1832.9 = 0$

  $a = 86$ mm

  $c = a / \beta_1 = 86 / 0.80 = 107.5$ mm

- $\varepsilon_s = 0.003\dfrac{225-107.5}{107.5} = 0.00328$

  $> \varepsilon_y = 0.0015$, tension steel is still yielding

- $\phi_b = 0.65 + \dfrac{0.25}{0.005-\varepsilon_y}(\varepsilon_t - \varepsilon_y) = 0.78$
- $f_s' = 600\dfrac{a-\beta_1 d'}{a} = 600\dfrac{86-48}{86} = 265.12$ MPa
- $M_n = (A_s f_y - A_s' f_s')(d - a/2) + A_s' f_s' (d - d')$

  $= [(3060 \times 300 - 568 \times 265.12) \times (225 - 86/2)$

  $+ 568 \times 265.12 \times (225 - 60)] / 106$

  $= 164.5$ kN-m
- $M_u = \phi_b M_n = 0.78 \times 164.5 = 128.3$ kN-m

Continued on # 14