Structural Mechanics (CE- 312)
Unsymmetrical Bending

Prof. Dr. Zahid Ahmad SIDDIQI

Dr. Nauman KHURRAM

Department of Civil Engineering

UNIVERSITY OF ENGINEERING & TECHNOLOGY LAHORE

INTRODUCTION

- Introduction of subject.
- Introduction of instructors.
- Introduction of books & Reference Material
- Division of the Course Content
- Objectives of the Course

Teachers:

Prof. Dr. Zahid Ahmad

Dr. Nauman Khurram

Engr. Aamina Rajput

EVALUATION METHODOLOGY

THEORY PART

Quiz & Class Participation :	10 %
(Assignments, Presentations and Attendance):	
Mid-Semester Exam:	30 %
Final Semester Exam:	60 %

Final grades are assigned according to the approved policy.

PRACTICAL PART

Lab report and Vive Voce:	30 %
Lab Quiz:	30 %
External/Neutral Viva Voce Exam:	40 %

Attendance Requirement:

Attendance less than 75%, both in theory and lab part will attribute to the WF grade.

Objectives of Taking This Course

- Not interested, want to just pass.
- Not interested, want to get good grades.
- Interested want to work in this field.
- For what grade knowledge, you will study this course?
- How the teachers may help to achieve your target?
- Want to be an inspector, check teachers, check facilities, check neatness, check overall standard etc.

CHAPTER OUTLINE

- **Transformation of Stresses, Strains and Moment of Inertia:**

  Analysis of Stress and Strain at a point due to combined effect of axial force, shear force, bending and twisting moment. Mohr's circle for stresses and strains, relationships between elastic constants.

- **Experimental Stress Analysis:**

  strain rosette solution.

- **Introduction to Theory of Elasticity:**

  Stress tensor, plane stress and plane strain problems and formulation of stress function.

- **Theories of Yielding/Failure:**

  for ductile and brittle materials.

- **Unsymmetrical (Biaxial) Bending:**

  Symmetrical and unsymmetrical sections,

- **Shear Center:**

  Shear stress distribution in thin walled open sections and shear center.

- **Cylinders:**

  Thin, Thick and Compound Cylinders.

- **Columns:**

  Stability of columns, conditions of equilibrium, eccentrically loaded columns, initially imperfect columns.

UNSYMMETRICAL BENDING

Review of Flexure Theory

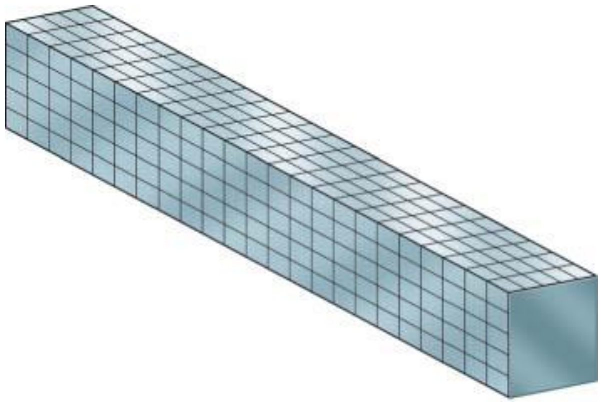
- In simple bending the Flexure (Bending) Theory was restricted to loads lying in a plane that contains an axis of symmetry of the cross section.

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$$

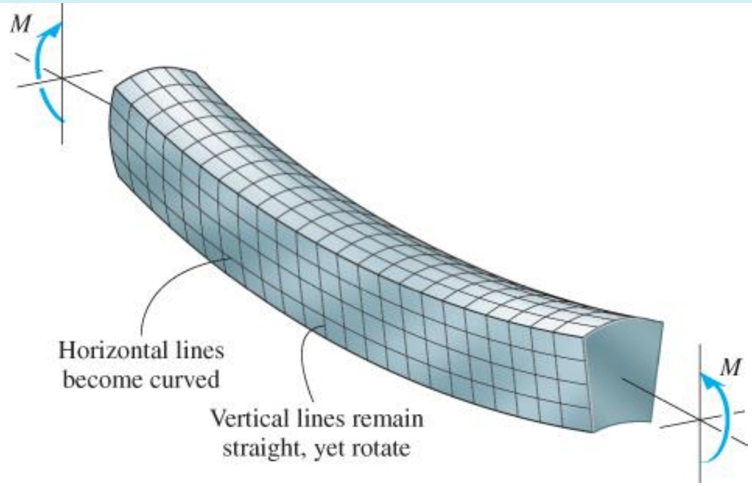
- The derivation of the equations that govern symmetrical bending and lead to the normal stress distribution is based on the following assumptions
  - Plane cross sections remain plane
  - Hooke's law is applicable (*i.e.* all the strains are within the elastic range

BENDING DEFORMATION OF A STRAIGHT MEMBER

When a bending moment is applied to a straight prismatic beam, the longitudinal lines become curved and vertical transverse lines remain straight and yet undergo a rotation.



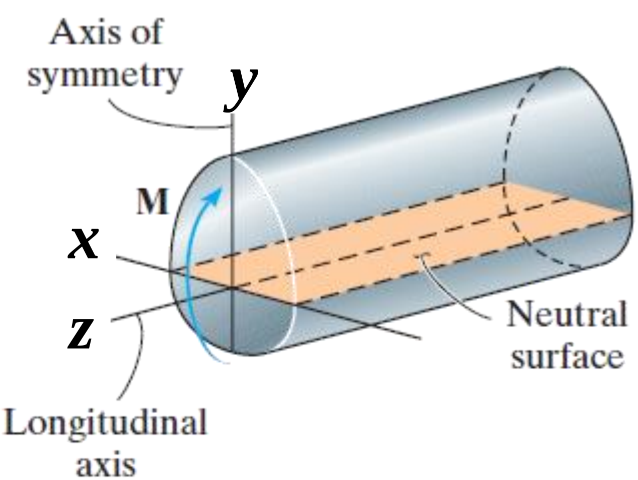
Before deformation



After deformation

Neutral Surface

A surface in a beam containing fibers that does not undergo any extension or compression thus not subjected to any tension or compression.

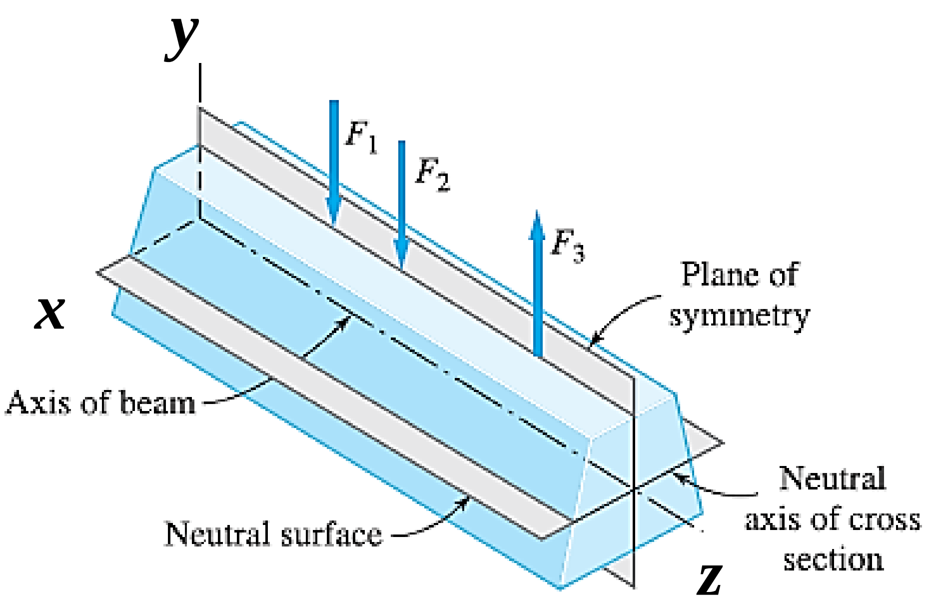


Neutral Axes

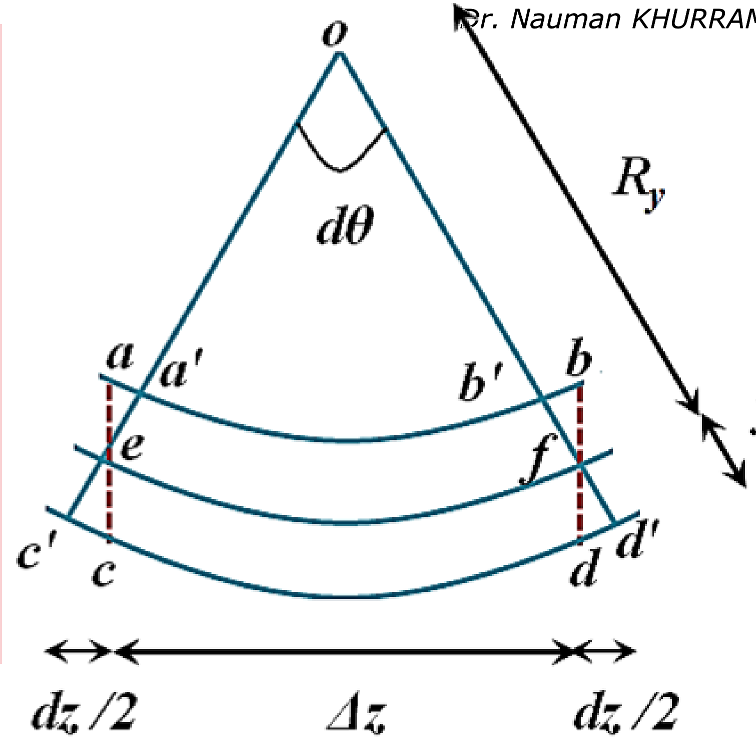
The intersection of neutral surface with any cross-section of the beam perpendicular to its longitudinal axes. All fibers on one side of the N.A are in the state of tension, which those on the opposite sides are in compression.

Flexure Formula

The beam has an axial plane of symmetry, which we take to be the ***zy***-plane. The applied loads (such as $\boldsymbol{F_1}$, $\boldsymbol{F_2}$, and $\boldsymbol{F_3}$ in Fig) lie in the plane of symmetry and are perpendicular to the axis of the beam (the z-axis).



Let ***ac*** and ***bd*** are the cross-sectional plane before bending having a differential distance ***Δz***. ***dθ*** is the angle subtended by the plane ***a'c'*** and ***b'd'*** after bending and ***ef*** is the neutral axes.
The strain at bottom may be calculated as following



$$\varepsilon_z = \frac{c'd' - cd}{cd} = \frac{c'd' - ef}{ef} \qquad (1)$$

$$\therefore \quad l = r.\theta$$

$$c'd' = (R+y)d\theta$$

$$ef = R.d\theta$$

$$(1) \Rightarrow \quad \varepsilon_z = \frac{(R+y)d\theta - R.d\theta}{R.d\theta}$$

$$\varepsilon_z = \frac{y d\theta}{R.d\theta} = \frac{y}{R}$$

$$\therefore \quad \sigma_z = \varepsilon_z.E$$

$$\frac{\sigma_z}{E} = \frac{y}{R} \qquad (2)$$

Applying $(\Sigma F)_z = 0$:

$$\int_A dP = \int_A (\sigma.dA) = 0$$

$$\frac{E}{R}\int_A y.dA = 0 \qquad (3)$$

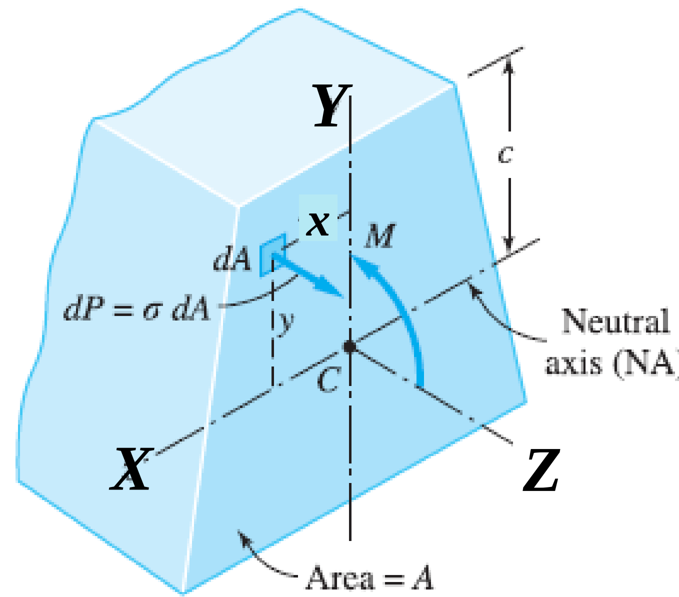
Since ***E*** and ***R*** cannot be zero thus the normal force will only be zero if ***y = 0***
i.e., along the neutral axis that coincides with the centroidal axis of the cross section.

Applying $(\Sigma M)_y = 0$:

$$\int_A dP.x = \int_A (\sigma.dA)x = 0 \qquad \therefore \sigma = \frac{E}{R}y$$

$$\frac{E}{R}\int_A yxdA = \frac{E}{R}.I_{xy} = 0 \qquad (4)$$

$$\therefore \quad I_{xy} = \int_A xy.dA$$



This shows that eqn. (***4***) is only valid if ***y-axes*** is the axes of symmetry (*i.e.* Product moment of inertia is equal to zero) and no moment is acting about the ***y-axes***.

Applying $(\Sigma M)_z = 0$:

$$\int_A dP.y = \int_A (\sigma.dA)y = M$$

$$\frac{E}{R}\int_A y^2 dA = \frac{E}{R}.I_x = M$$

$$\therefore \quad I_x = \int_A y^2 dA$$

$$\frac{E}{R} = \frac{M}{I_x} \qquad (5)$$

By Eqn. (2) and (5)

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R} \qquad (6)$$

Simple bending theory applies when bending takes place about an axes which is perpendicular to the plane of symmetry.

If such an axes drawn through the centroid and another mutually perpendicular to it through the centroid, then these axes are called the principal axes.

- ***The planes that are parallel to the principal axes and pass through the shear center are called the principal planes of bending***

Axes of Symmetry

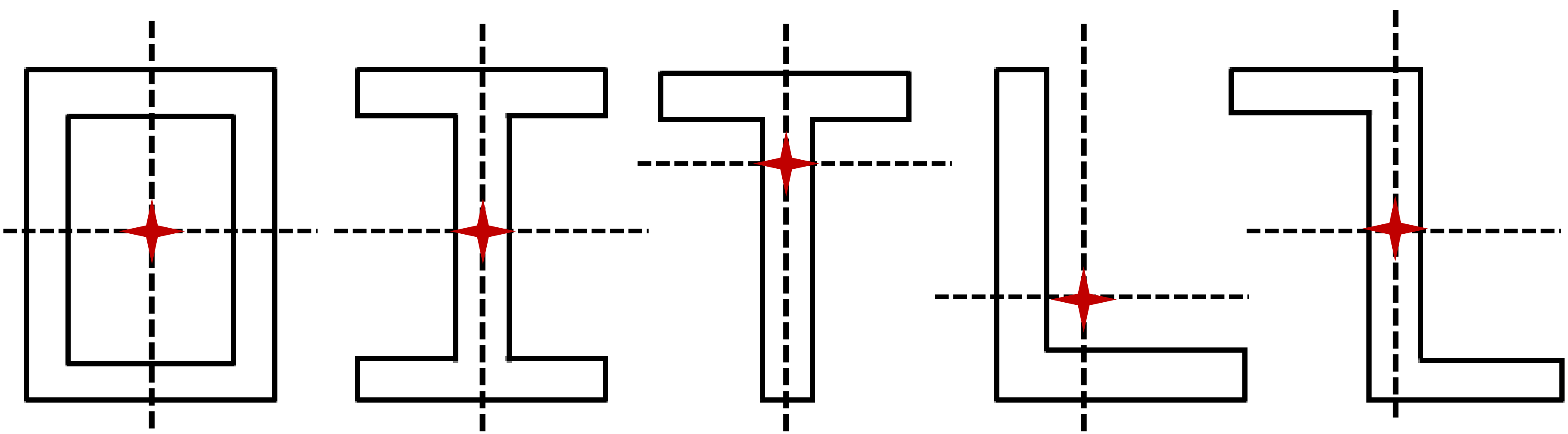
Axes of symmetry divides the section in such a fashion that one part is the mirror image of the other part.

Symmetrical sections

Sections which are having at-least one axis of symmetry are called the symmetrical sections

Unsymmetrical sections

Sections which are not having any axis of symmetry are called the unsymmetrical sections



Symmetric Sections **Unsymmetrical Sections**

Principal Axes

The axes about which the product moment of area (I_{yx} or I_{xy}) is found to be zero and second moment of area (I_x & I_y) are found to be minimum and maximum

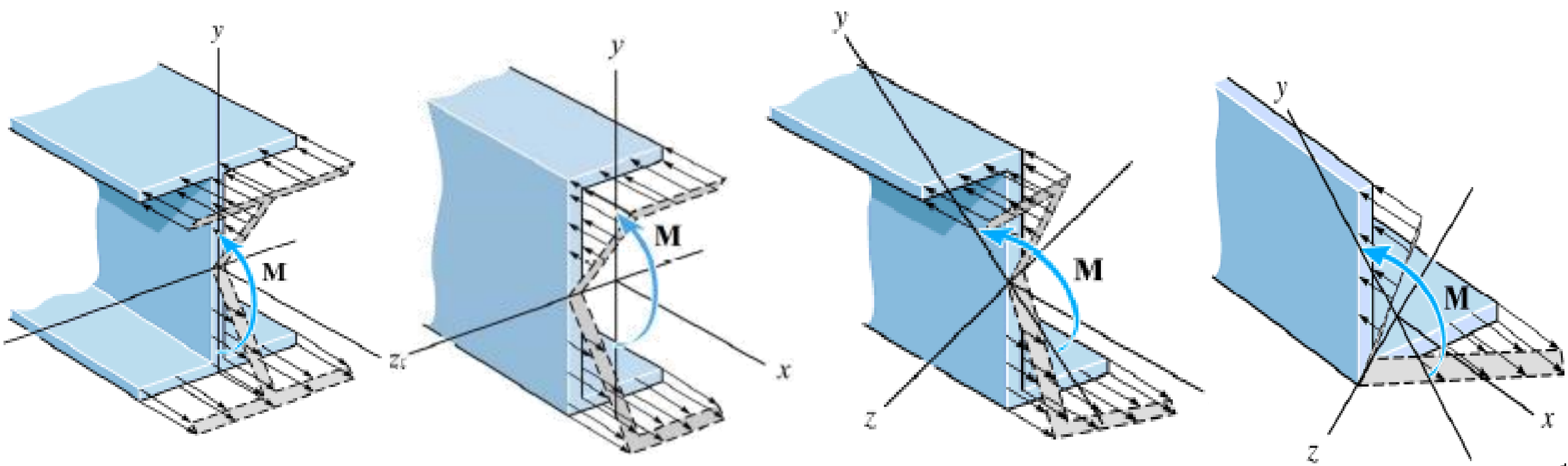
- A plane of symmetry in a section is automatically a principal plane
- All the plane sections whether they have an axes of symmetry or not have two perpendicular axes about which product moment of area is zero
- Simple (Symmetric) bending is the bending which takes place about a principal axis. ***i.e.*** moment is applied in a plane parallel to that axes or load is acting perpendicular to that axes
- Mainly unsymmetrical bending occurs if moments or loadings are not applied about the principal axes.
- In case of symmetric section principal axes always coincide with the centroidal axes

SYMMETRICAL BENDING

If the loading is perpendicular (or parallel) to the one of the Principal Axes the bending will be only in the direction of the loading, such bending is called the Symmetrical Bending.

For a symmetrical section to have symmetrical bending, the plane of loading must be parallel to or contain a central axes which is also a centroidal axes.

Unsymmetrical section may also be subjected to symmetrical bending if plane of loading contains a principal axes.



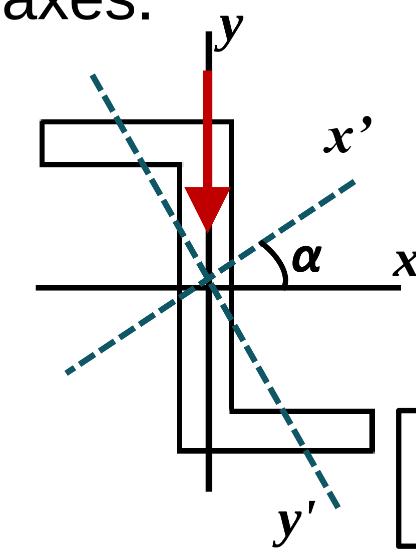
UNSYMMETRICAL BENDING

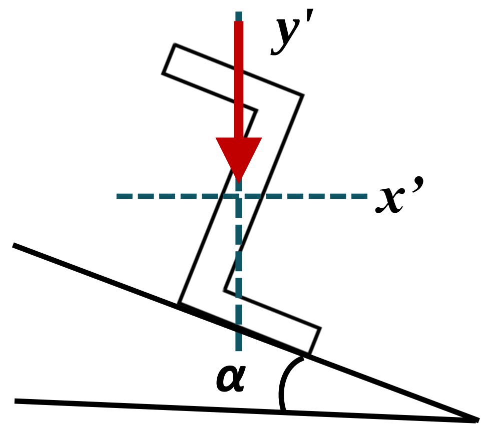
Unsymmetrical bending occurs if loading is not acting parallel or along one of the principal axes. Bending takes place out of the plane of the loading and as well in the plane.

Unsymmetrical bending can takes place both in the symmetric and unsymmetrical sections

Since loads are normally applied along or parallel to the centroidal axes, unsymmetrical bending is evident in the unsymmetrical sections whose principal axes do not coincide with the principal axes.

Inclination of roof is kept equal to the orientation of the principal axes from the plane of the loading to produce the symmetrical bending





Symmetrical bending in Z-section purlin

Unsymmetrical Bending of Symmetrical Sections

In symmetrical section unsymmetrical bending occurs when load is acting at an inclination to the axes of symmetry (centroidal axes or principal axes).

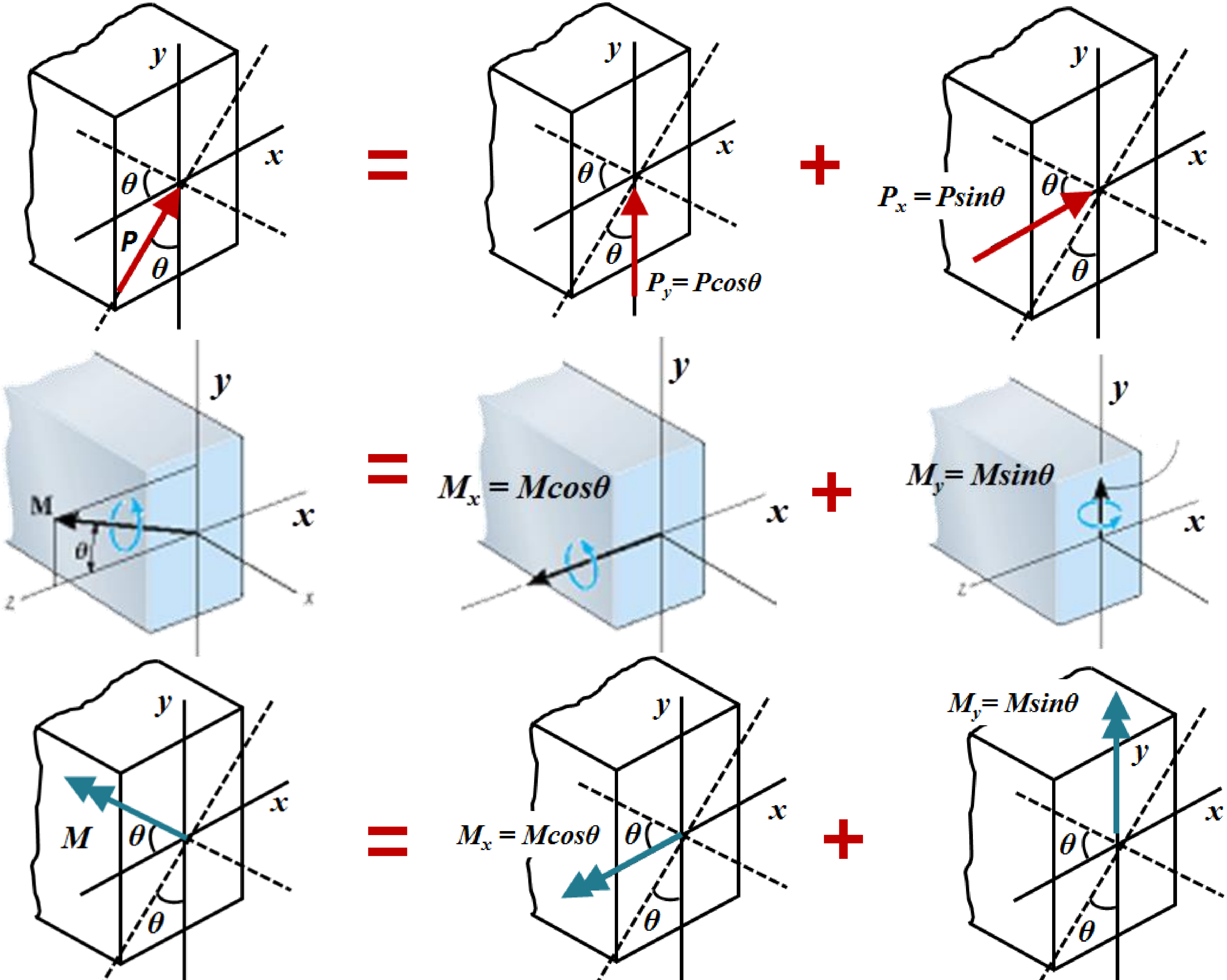
- In unsymmetrical bending the neutral axis of the x-section does not coincides with the axis of loading

Procedure to Solve:

1. Determine the inclination (***θ***) of the applied loading or resultant moment.
2. Resolved the applied loading or resultant moment into components directed along the principal axes.
3. The double-headed arrow are used to represent the bending moment as a vector direction (clock/counter clock wise) of which may determined by the right hand rule.

- Use flexure formula to determine normal stress caused by each moment component
- Use ***principle of superposition*** to determine resultant normal stress at any point on the section.
- In the addition of the stress components use the sign convention with respect to the tension or compression produced by the some particular component of the moment at any specified point. i.e., Consider tensile stress as positive and compressive stress as negative.
- For a section subjected to any arbitrary moment the stress at any specified pint can be determined by the following equation

$$\sigma_z = \frac{M_y . x}{I_y} + \frac{M_x . y}{I_x} \qquad (7)$$



y
x
θ
P
θ
=
y
x
θ
θ
P_y = Pcosθ
+
y
x
P_x = Psinθ
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M_x = Mcosθ
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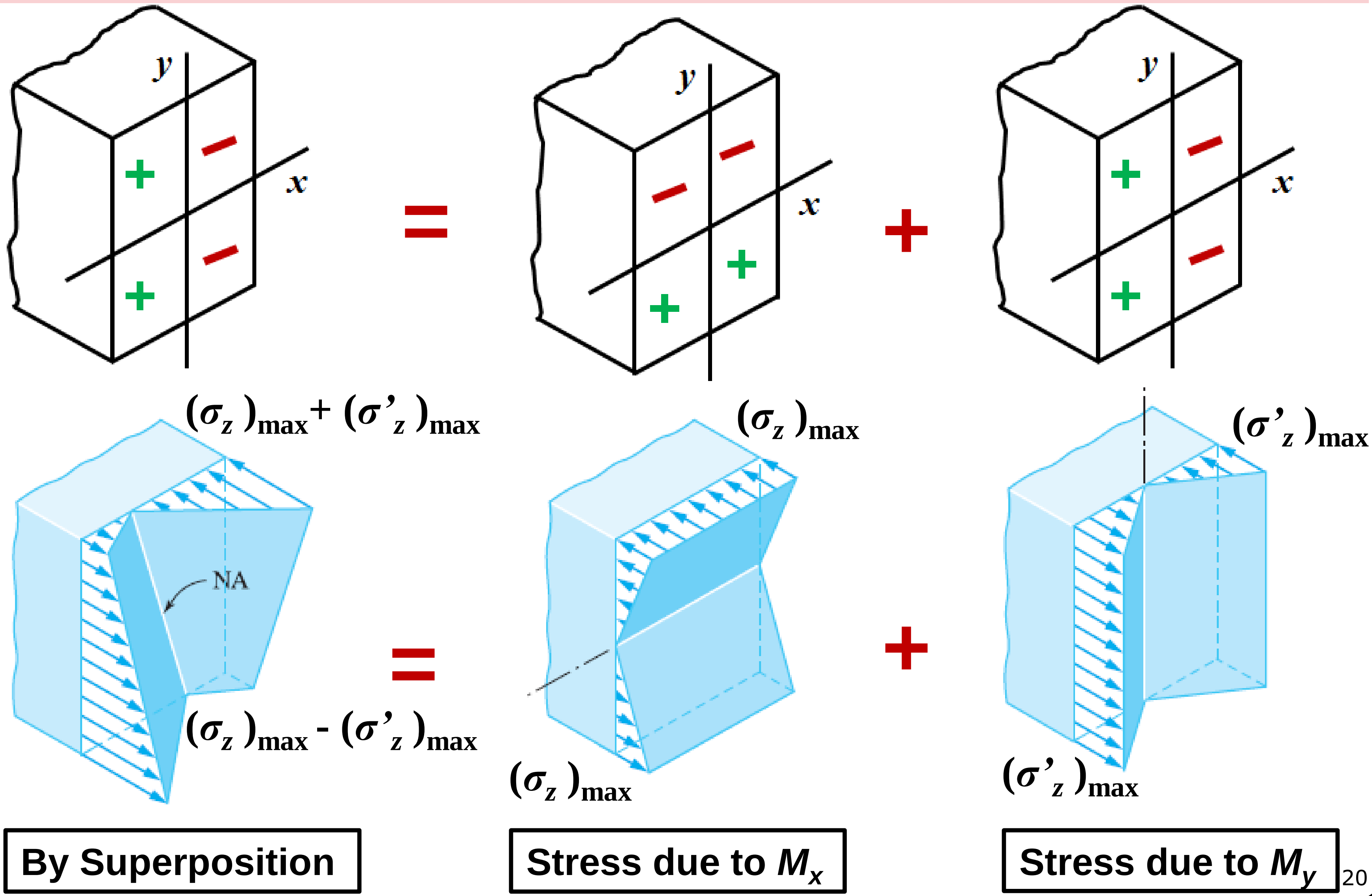
Note: The resultant stress after superposition depends upon the magnitude of the tensile and compressive stresses to be added at any specified point.



Inclination of the Neutral Axes (N.A.)

In general, the neutral axis for unsymmetrical bending is not parallel to the bending moment M. Because the neutral axis is the line where the bending stress is zero, its equation can be determined by setting $\boldsymbol{\sigma_z = 0}$ in the eqn. **(7)**, which yields

$$0 = \frac{M_y . x}{I_y} + \frac{M_x . y}{I_x} \qquad (7)$$

$$\frac{M_y . x}{I_y} = -\frac{M_x . y}{I_x}$$

$$\frac{y}{x} = -\frac{M \sin\theta}{M \cos\theta} . \frac{I_x}{I_y}$$

$$\therefore \quad M_x = M \cos\theta$$

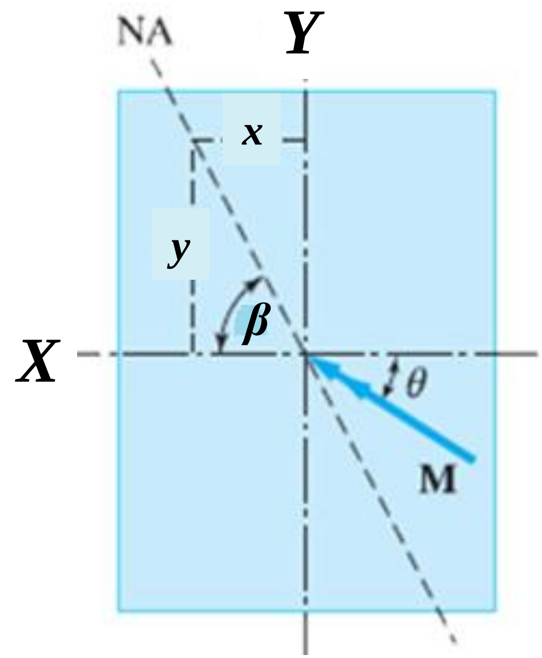
$$\therefore \quad M_y = M \sin\theta$$

$$\therefore \quad \frac{y}{x} = \tan\beta$$

$$\frac{y}{x} = -\frac{I_x}{I_y} . \tan\theta$$

$$\tan\beta = -\frac{I_x}{I_y} . \tan\theta$$

$$\beta = -\tan^{-1}\left[\frac{I_x}{I_y} . \tan\theta\right] \qquad (8)$$



- Inclination of N.A. ***(β)*** and load ***(θ)*** are from the same axes and ranging between **0 – 90^0**.

If $I_x > I_y$ then $\beta > \theta$

If $I_x < I_y$ then $\beta < \theta$

If $I_x = I_y$ then $\beta = \theta$

$$\beta = -\tan^{-1}\left[\frac{I_x}{I_y}.\tan\theta\right]$$

- The negative sign indicates the angle is in clock-wise
- Neutral axes always lies between couple vector, ***M*** (Resultant moment) and principal axes corresponding to the I_{min} (i.e. I_y)

Deflection

The deflections of symmetrical and unsymmetrical members in the directions of the principal axes may always be determined by application of the standard deflection formulae.

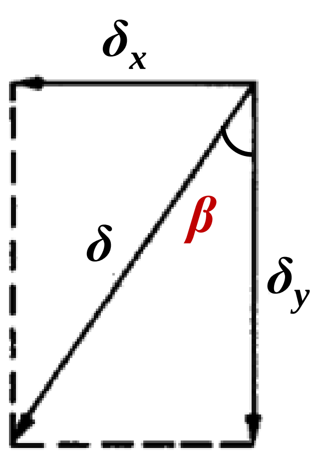
For example, the deflection at the free end of a cantilever carrying an end-point-load is ***$PL^3/3EI$***. With the appropriate value of ***I*** and the correct component of the load perpendicular to the principal axis used, the required deflection is obtained.

- The total resultant deflection is then given by combining the above values vectorally as shown Eqn. (10).
- The direction of the deflection will always be about the **N.A.**

$$\delta_y = \frac{P_y L^3}{3EI_x} \quad and \quad \delta_x = \frac{P_x L^3}{3EI_y} \qquad (9)$$

$$\delta = \sqrt{\delta_y{}^2 + \delta_x{}^2} \qquad (10)$$

$$\therefore \quad \tan\beta = \frac{\delta_x}{\delta_y}$$



$$\frac{\delta_x}{\delta_y} = \frac{P_x L^3}{3EI_y} \Big/ \frac{P_y L^3}{3EI_x}$$

$$\frac{\delta_x}{\delta_y} = \frac{P_x}{P_y} \cdot \frac{I_x}{I_y} = \frac{P\sin\theta}{P\cos\theta} \cdot \frac{I_x}{I_y}$$

$$\beta = \tan^{-1} \cdot \left[\frac{I_x}{I_y} . \tan\theta \right]$$

Alternatively

since bending always occurs about the N.A., the deflection equation can be written in the form

$$\delta = \frac{PL^3}{3EI_{N.A}} \qquad \textit{for cantilever beam at free end}$$

where $I_{N.A.}$ is the second moment of area about the N.A. and W' is the component of the load perpendicular to the N.A. The value of IN,A. may be found either graphically using

$$I_{N.A} = \frac{1}{2}\left[\left(I_x + I_y\right) + \left(I_x - I_y\right)\cos 2\beta\right] - I_{xy}\sin 2\beta$$

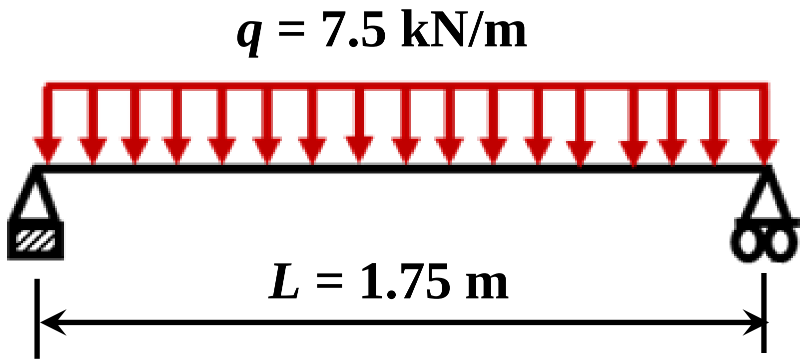
$$OR \qquad I_{N.A} = \frac{1}{2}\left[\left(I_x + I_y\right) + \left(I_x - I_y\right)\sec 2\beta\right]$$

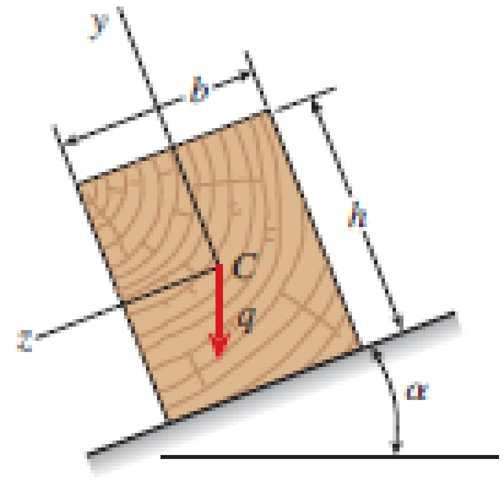
where β, is the angle between the N.A. and the principal x axis.

The Equation mentioned above will be derived in next sections

EXAMPLE PROBLEM

A wood beam of rectangular cross section is simply supported on a span of length L =1.75 m. The longitudinal axis of the beam is horizontal, and the cross-section is tilted at an angle of 22.5^0. The load on the beam is a vertical uniform load of intensity q = 7.5 kN/m acting through the centroid C. Determine the orientation of the neutral axis and calculate the maximum tensile stress σ_{max}. if ***b*** = 80 mm, ***h*** =140 mm. Also determine the maximum deflection



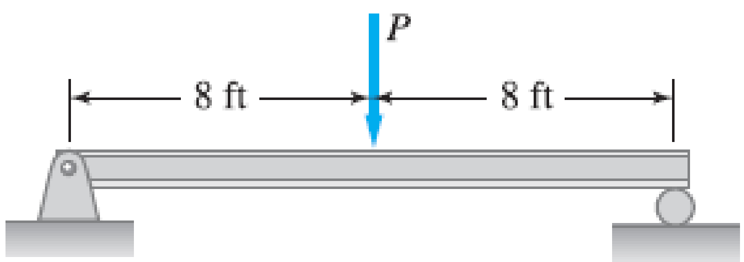


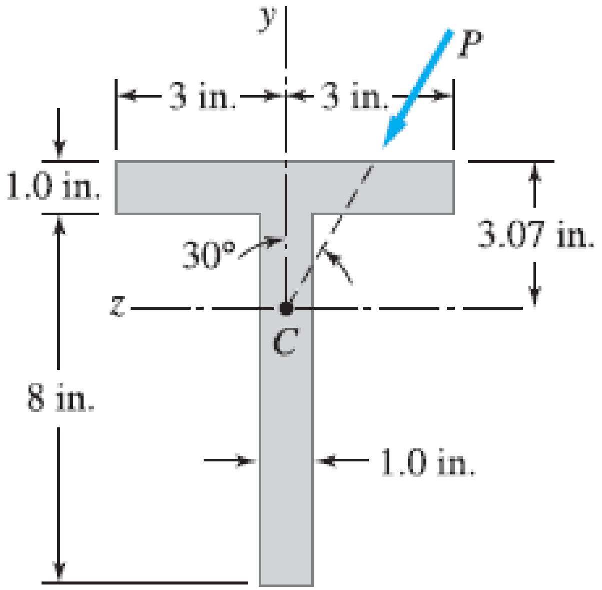
Problem 11.24: (Book by Andrew Pytel)

The cross section of the simply supported T-beam has the inertial properties I_y= 18.7 in.^{4} and I_x = 112.6 in.^{4}. The load P is applied at mid-span, inclined at 30^0 to the vertical and passing through the centroid ***C*** of the cross section.

(a) Find the angle between the neutral axis and the horizontal.

(b) If the working bending stress is 12 ksi, find the largest allowable value of the load **P**.





Unsymmetrical Bending of Unsymmetrical Sections

- In unsymmetrical sections principal axes do not coincide with centriodal axes.
- So there will always be unsymmetrical bending even though the loading plan is parallel to (or passing through) the centroidal axes.
- All the geometric parameters will be with respect to the principal axes. i.e.,
  - Second moment of areas (I_x, I_y, I_{xy})
  - Loading plane
  - Orientation of the N.A.
  - Deflection

Transformation of Second Moment of Area

$$y' = bd - cb$$

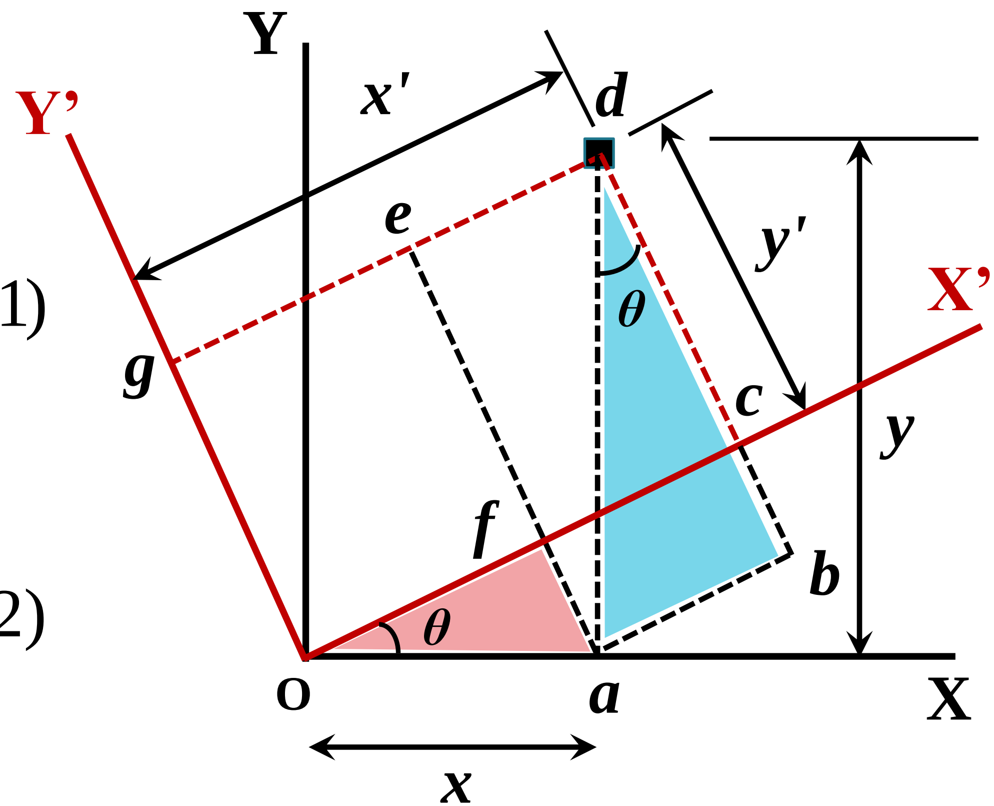
$$y' = bd - fa$$

$$y' = y\cos\theta - x\sin\theta \qquad (1)$$

$$x' = ge + ed$$

$$x' = of + ab$$

$$x' = x\cos\theta + y\sin\theta \qquad (2)$$



$$I_{y'x'} = \int_A y'x'dA \qquad (3)$$

$$I_{x'} = \int_A y'^2 dA \qquad (4)$$

$$I_{y'} = \int_A x'^2 dA \qquad (5)$$

$$(3) \Rightarrow \quad I_{x'y'} = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta)dA$$

$$I_{x'y'} = \int (xy\cos^2\theta + y^2\sin\theta\cos\theta - x^2\sin\theta\cos\theta - xy\sin^2\theta)dA$$

$$I_{x'y'} = \int (\cos^2\theta - \sin^2\theta)xydA + \int \sin\theta\cos\theta y^2 dA - \int \sin\theta\cos\theta x^2 dA$$

$$I_{x'y'} = I_{xy}\cos 2\theta + \frac{1}{2}I_x \sin 2\theta - \frac{1}{2}I_y \sin 2\theta$$

$$I_{x'y'} = I_{xy}\cos 2\theta + \frac{1}{2}(I_x - I_y)\sin 2\theta \qquad (6)$$

$\therefore \quad \sin 2\theta = 2\sin\theta\cos\theta$

$\therefore \quad \cos 2\theta = \cos^2\theta - \sin^2\theta$

$\therefore \quad \cos^2\theta = \left(\frac{1+\cos 2\theta}{2}\right)$

$\therefore \quad \sin^2\theta = \left(\frac{1-\cos 2\theta}{2}\right)$

for principal axes $\quad I_{x'y'} = 0$

$$(6) \Rightarrow \quad 0 = I_{xy}\cos 2\theta + \frac{1}{2}(I_x - I_y)\sin 2\theta$$

$$\tan 2\theta = -\frac{2I_{xy}}{(I_x - I_y)} \qquad (7)$$

$$\theta_p = -\frac{1}{2}\tan^{-1}\left[\frac{2I_{xy}}{(I_x - I_y)}\right] \qquad (8)$$

$\boldsymbol{\theta_p}$ is the orientation of the principal axes from the +ve centroidal x-axes

$$(4) \Rightarrow \quad I_{x'} = \int (y\cos\theta - x\sin\theta)^2 dA$$

$$I_{x'} = \int (y^2\cos^2\theta + x^2\sin^2\theta - 2xy\sin\theta\cos\theta)dA$$

$$I_{x'} = \cos^2\theta \int y^2 dA + \sin^2\theta \int x^2 dA - 2\sin\theta\cos\theta \int xy dA$$

$$I_{x'} = \left(\frac{1+\cos 2\theta}{2}\right) I_x + \left(\frac{1-\cos 2\theta}{2}\right) I_y - I_{xy}\sin 2\theta$$

$$I_{x'} = \frac{1}{2}\left(I_x + I_y\right) + \frac{1}{2}\left(I_x - I_y\right)\cos 2\theta - I_{xy}\sin 2\theta \qquad (9)$$

The Eqn. (9) can further be simplified by substituting the value of I_{xy} from the Eqn. (7)

$$(7) \Rightarrow \quad \tan 2\theta = -\frac{2I_{xy}}{(I_x - I_y)}$$

$$\therefore \quad I_{xy} = -\frac{1}{2}\tan 2\theta (I_x - I_y)$$

$$(9) \Rightarrow \quad I_{x'} = \frac{1}{2}\left(I_x + I_y\right) + \frac{1}{2}\left(I_x - I_y\right)\cos 2\theta + \frac{1}{2}\tan 2\theta\left(I_x - I_y\right)\sin 2\theta$$

$$I_{x'} = \frac{1}{2}\left(I_x + I_y\right) + \frac{1}{2}\left(I_x - I_y\right)\cos 2\theta + \frac{1}{2}\frac{\sin^2 2\theta}{\cos 2\theta}\left(I_x - I_y\right)$$

$$I_{x'} = \frac{1}{2}\left(I_x + I_y\right) + \frac{1}{2}\left(I_x - I_y\right)\cos 2\theta + \frac{1}{2}\left[\frac{(1-\cos^2 2\theta)}{\cos 2\theta}\right]\left(I_x - I_y\right)$$

$$I_{x'} = \frac{1}{2}\left(I_x + I_y\right) + \frac{1}{2}\left(I_x - I_y\right)\cos 2\theta + \frac{1}{2}\left(I_x - I_y\right)\sec 2\theta - \frac{1}{2}\left(I_x - I_y\right)\cos 2\theta$$

$$I_{x'} = \frac{1}{2}\left(I_x + I_y\right) + \frac{1}{2}\left(I_x - I_y\right)\sec 2\theta \qquad (10)$$

Similarly solving the Eqn. (5) we may have the following Solution for $\boldsymbol{I_{y'}}$.

$$(5) \Rightarrow \quad I_{y'} = \int (x\cos\theta + y\sin\theta)^2 \, dA$$

$$I_{y'} = \int (x^2\cos^2\theta + y^2\sin^2\theta + 2\sin\theta\cos\theta . xy)dA$$

$$I_{y'} = \cos^2\theta \int x^2 dA + \sin^2\theta \int y^2 dA + 2\sin\theta\cos\theta \int xy dA$$

$$I_{y'} = \left(\frac{1+\cos 2\theta}{2}\right) I_y + \left(\frac{1-\cos 2\theta}{2}\right) I_x + I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{1}{2}\left(I_x + I_y\right) - \frac{1}{2}\left(I_x - I_y\right)\cos 2\theta + I_{xy}\sin 2\theta \qquad (11)$$

$$(11) \Rightarrow \quad I_{y'} = \frac{1}{2}(I_x + I_y) - \frac{1}{2}(I_x - I_y)\cos 2\theta - \frac{1}{2}\tan 2\theta (I_x - I_y)\sin 2\theta$$

$$I_{y'} = \frac{1}{2}(I_x + I_y) - \frac{1}{2}(I_x - I_y)\cos 2\theta - \frac{1}{2}\frac{\sin^2 2\theta}{\cos 2\theta}(I_x - I_y)$$

$$I_{y'} = \frac{1}{2}(I_x + I_y) - \frac{1}{2}(I_x - I_y)\cos 2\theta - \frac{1}{2}\left[\frac{(1-\cos^2 2\theta)}{\cos 2\theta}\right](I_x - I_y)$$

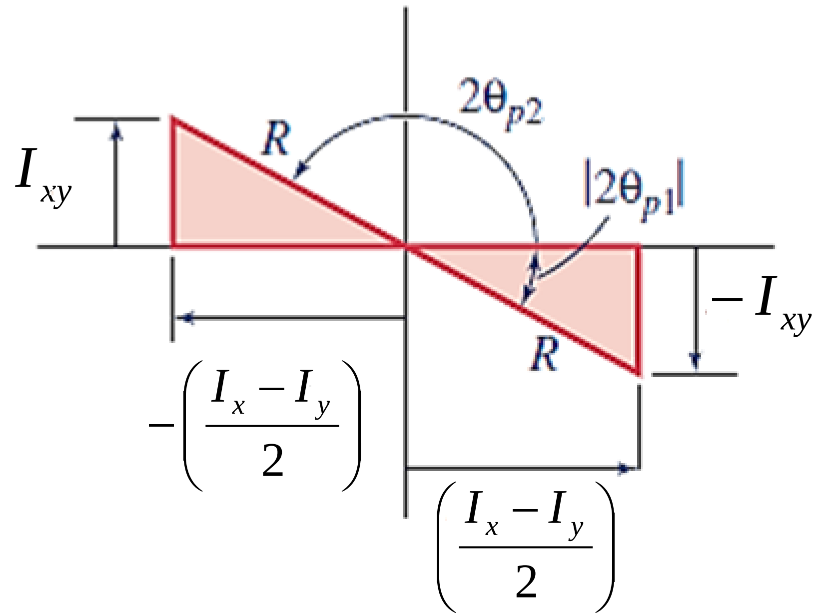
$$I_{y'} = \frac{1}{2}(I_x + I_y) - \frac{1}{2}(I_x - I_y)\cos 2\theta - \frac{1}{2}(I_x - I_y)\sec 2\theta - \frac{1}{2}(I_x - I_y)\cos 2\theta$$

$$I_{y'} = \frac{1}{2}(I_x + I_y) - \frac{1}{2}(I_x - I_y)\sec 2\theta \qquad (12)$$

Geometric Method:

$$(7) \Rightarrow \quad \tan 2\theta = -\frac{2I_{xy}}{(I_x - I_y)}$$

$$R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I^2_{xy}}$$



Substituting the value of ***sin2θ*** and ***cos2θ*** in Eqn. (10) and (12) or in Eqn. (9) and (11)

$$\therefore \quad \sin 2\theta = \pm \frac{I_{xy}}{R}$$

$$\therefore \quad \cos 2\theta = \pm \frac{(I_x + I_y)}{2R}$$

$$\left.\begin{matrix} I_{x'} \\ I_{y'} \end{matrix}\right\} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I^2_{xy}} \qquad (13)$$

- Second Moment of Area (I_x and I_y) is always a positive quantity
- Product Moments of Area (I_{xy}) may be positive or negative depending upon the geometry of the section.
- For any section $I_x + I_y = I_{x'} + I_{y'}$ but maximum and minimum values are different.
- Eqn. (7) may also be derived by differentiating the Eqn. (9) and (11) with respect to ***θ*** and equating them to zero as $I_{x'}$ and $I_{y'}$ have the maximum and minimum values about the principal axes.

Unsymmetrical Bending of Unsymmetrical Sections

In unsymmetrical sections principal axes do not coincide with In unsymmetrical bending the neutral axis of the x-section does not coincides with the axis of loading

Procedure to Solve:

1. Find out the centroid of the cross section and draw the axes ***x*** and ***y***
2. Calculate the I_x and I_y and I_{xy}.
3. Determine the orientation of Principal Axes (θ_p) by following Eqn.

$$\theta_p = -\frac{1}{2}\tan^{-1}\left[\frac{2I_{xy}}{(I_x - I_y)}\right]$$

4. Calculate the Principal moment of inertia, $I_{x'}$ and $I_{y'}$. By any of the following set of the Equations

$$\left.\begin{matrix} I_{x'} \\ I_{y'} \end{matrix}\right\} = \frac{1}{2}\left(I_x + I_y\right) \pm \frac{1}{2}\left(I_x - I_y\right)\cos 2\theta \mp I_{xy}\sin 2\theta$$

OR

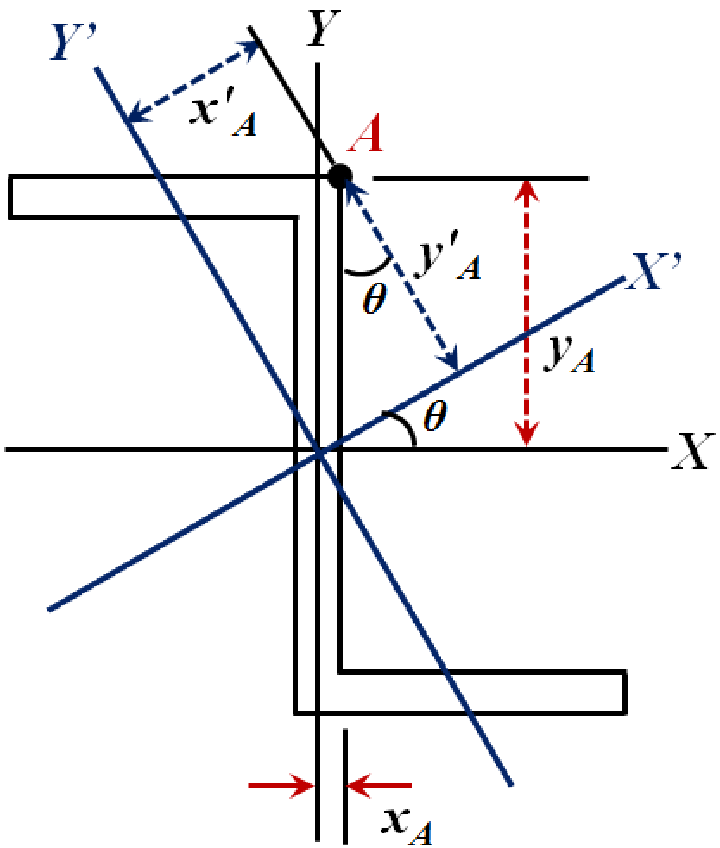
$$\left.\begin{matrix} I_{x'} \\ I_{y'} \end{matrix}\right\} = \frac{1}{2}\left(I_x + I_y\right) \pm \frac{1}{2}\left(I_x - I_y\right)\sec 2\theta$$

OR

$$\left.\begin{matrix} I_{x'} \\ I_{y'} \end{matrix}\right\} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I^2{}_{xy}}$$

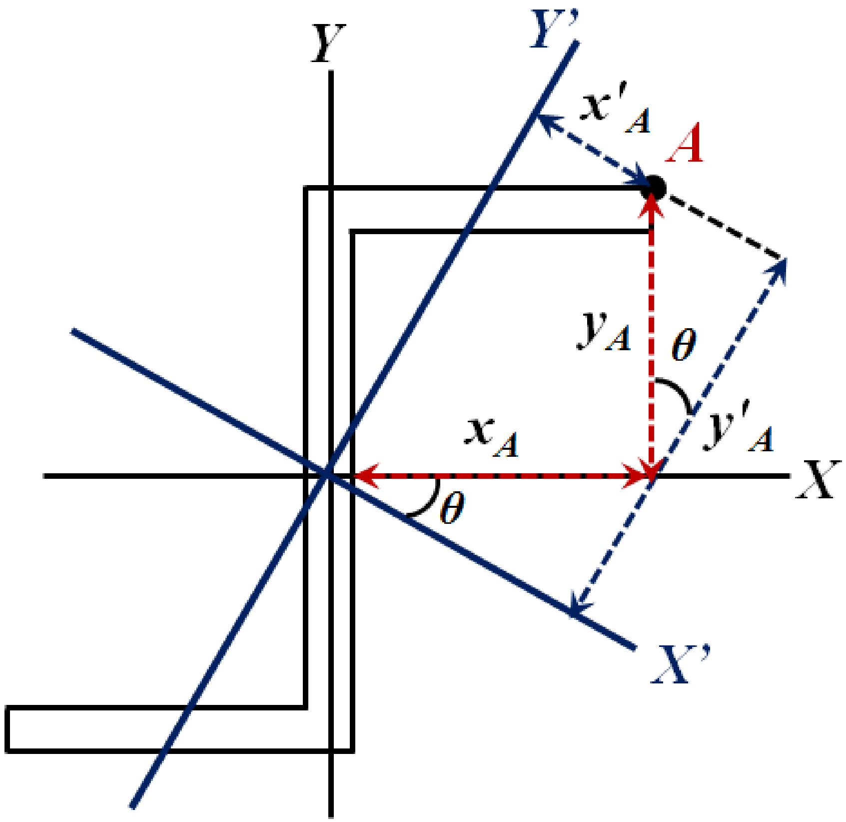
5. Determine the inclination (**α**) of the applied loading or resultant moment with respect to the principal axes.
6. Resolve the loading along the principal axes.
7. Determine the Coordinates of the points under consideration on the cross-section with respect to Principal Axes (i.e., ***y’*** and ***x’***).
8. Use the Flexure formula and principal of Super position to determine the stress at any specifies point

Coordinate with respect to Principal Axes:



$$y'_A = y_A \cos\theta - x_A \sin\theta$$

$$x'_A = x_A \cos\theta + y_A \sin\theta$$



$$y'_A = y_A \cos\theta + x_A \sin\theta$$

$$x'_A = x_A \cos\theta - y_A \sin\theta$$

In above expression insert the ***x*** and ***y*** values with respect to their coordinate sign, however insert the absolute value in the flexure equation to determine the bending stress.

Deflection (Alternatively)

since bending always occurs about the N.A., the deflection equation can be written in the form

$$\delta = \frac{PL^3}{3EI_{N.A}} \qquad \textit{for cantilever beam at free end}$$

where IN.A. is the second moment of area about the N.A. and W' is the component of the load perpendicular to the N.A. The value of IN,A. may be found either graphically using

$$I_{N.A} = \frac{1}{2}\left[\left(I_x + I_y\right) + \left(I_x - I_y\right)\cos 2\beta\right] - I_{xy}\sin 2\beta$$

$$OR \qquad I_{N.A} = \frac{1}{2}\left[\left(I_x + I_y\right) + \left(I_x - I_y\right)\sec 2\beta\right]$$

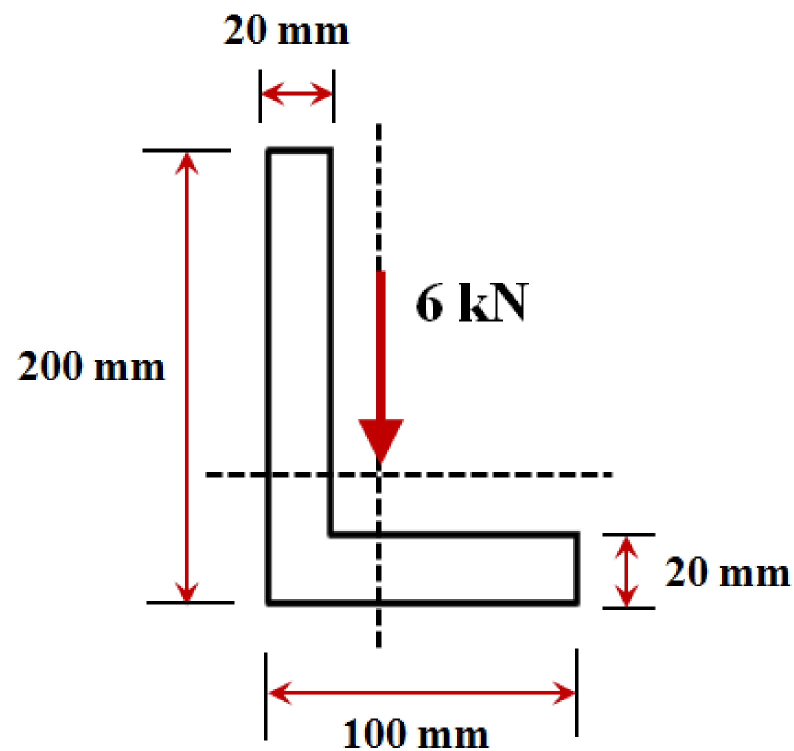
where ***β***, is the angle between the N.A. and the principal x-axis.

Example Problem 3:

A **200x100x20** mm Angle section is used as a cantilever beam of **3.0 m** long with 200 mm leg in vertical direction. It supports a load of **6 kN** at free end of beam. Compute the following

1. Maximum bending stress in the beam
2. Orientation of N.A.
3. Maximum deflection

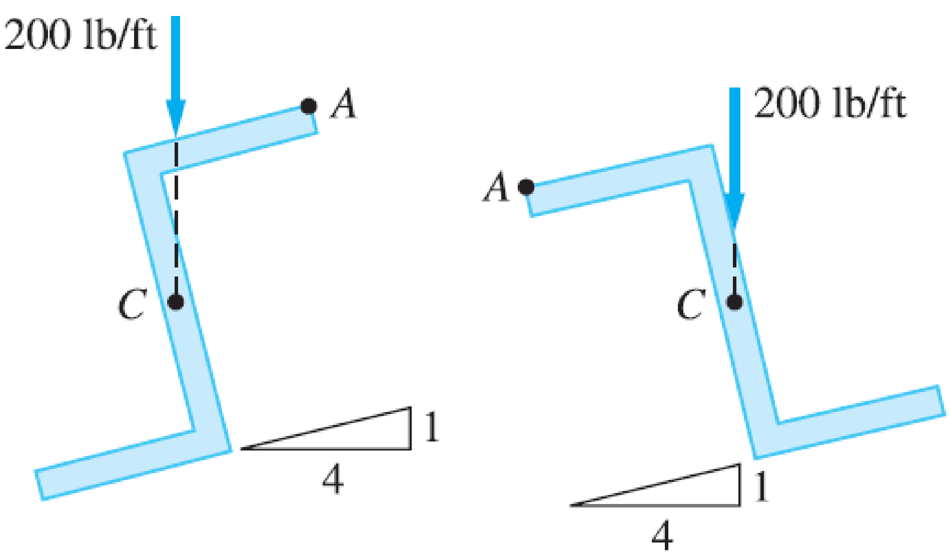
Also plot the stress profile



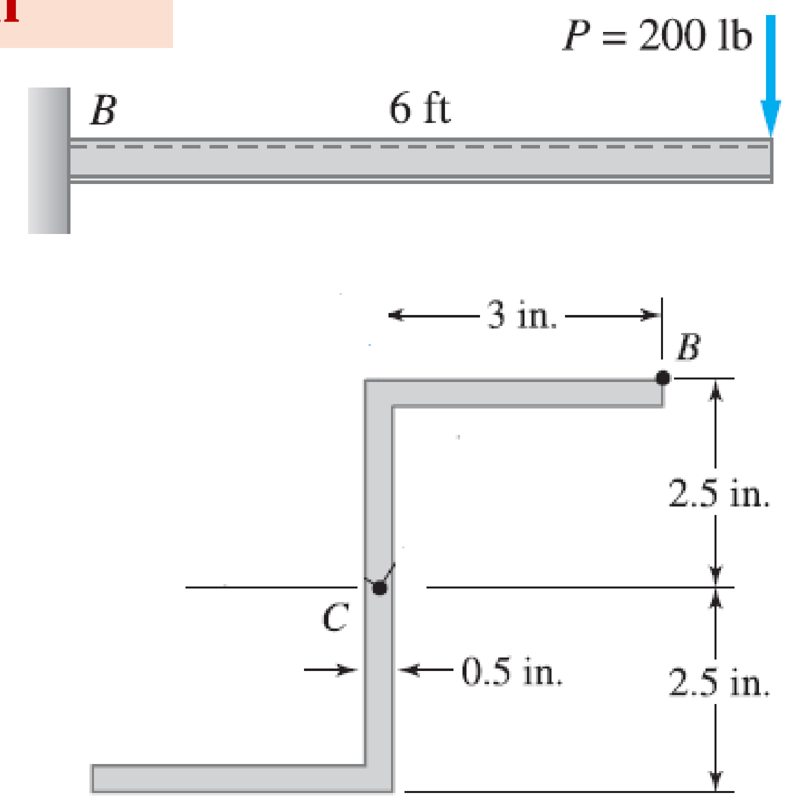
Prob. # 11.27: (Mech. of Material by Andrew Pytel 2nd Ed.)

The Z-section described in Figure below is used as a simply supported roof purlin, 12 ft long, carrying a distributed vertical load of 200 lb/ft. The slope of the roof is 1:4, as indicated in the figure. Determine the maximum bending stress at corner **A** of the purlin for the orientations **(a)** and **(b)**.

Please bring the solution in the next class
Attendance is conditional to solution



(a) (b)



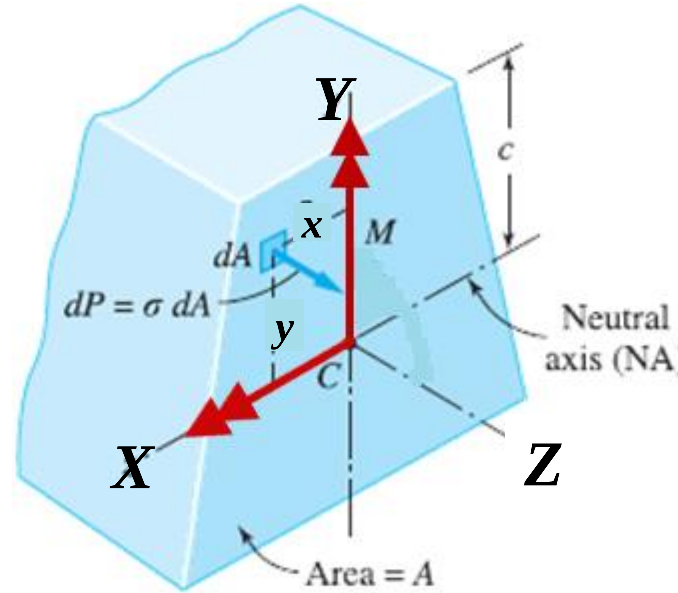
Radius of curvature Method

- Principal of superposition is most useful when the principal axes are known or can be found easily by calculation or inspection.
- It is also possible to calculate stresses with respect to a set of non–principal axes.
- Using the Principal of Superposition method, deflections can be found easily by resolving the applied lateral forces into components parallel to the principal axes and separately calculating the deflection components parallel to these axes.
- The total deflection at any point along the beam is then found by combining the components at that point into a resultant deflection vector. Note that the resulting deflection will be perpendicular to the neutral axis of the section at that point.

- The Radius of Curvature method (General Bending Theory) is useful if the principal axes are not easily found but the components I_x, I_y and I_{xy} of the inertia tensor can be readily determined.
- In this method all the parameter are used with respect to the centroidal axes
- By this method deflections cannot be determined by this method.

DERIVATION

Let consider a resistive force ***dP*** acting at a differential area ***dA*** due to the moment M_x & M_y.

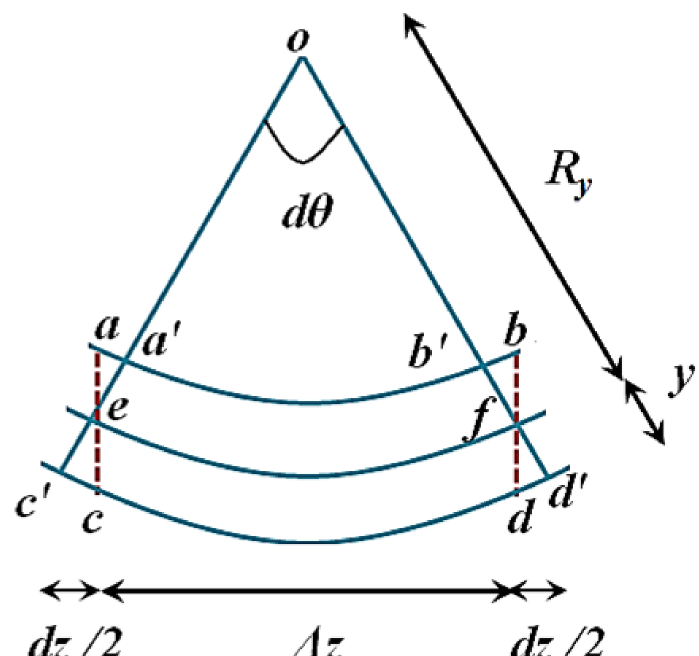


Strain due to M_x-

$$\varepsilon_z = \frac{c'd' - cd}{cd} = \frac{c'd' - ef}{ef}$$

$$\varepsilon_z = \frac{(R_y + y)d\theta - R_y d\theta}{R_y d\theta} = \frac{y}{R_y}$$

$$\varepsilon_z = y.K_y \qquad (1)$$



Strain due to M_y-

$$\varepsilon_z = \frac{x}{R_x} = x.K_x \qquad (2)$$

$$\therefore \quad K_y = \frac{1}{R_y}$$

$$\therefore \quad K_x = \frac{1}{R_x}$$

Total Strain:

$$\varepsilon_z = y.K_y + x.K_x$$

$$\therefore \quad \sigma_z = E.\varepsilon_z$$

$$\sigma_z = E(y.K_y + x.K_x) \qquad (3)$$

Here, $\boldsymbol{R_x}$ & $\boldsymbol{R_y}$ are radius of curvatures and $\boldsymbol{K_x}$ & $\boldsymbol{K_y}$ are the curvatures in $\boldsymbol{x}$ and $\boldsymbol{y}$ direction, respectively

Applying $(\Sigma F)_z = 0$:

$$\int_A dP = \int_A (\sigma_z.dA) = \int_A E(y.K_y + x.K_x)dA = 0$$

$$EK_y \int_A y.dA + EK_x \int_A x.dA = 0 \qquad (4)$$

In eqn. (***4***) ***E*** cannot be zero, also $\boldsymbol{K_x}$ & $\boldsymbol{K_y}$ cannot be zero as beam is bending. So eqn. (***4***) is only valid if ***x*** and ***y*** are zero.

Applying $(\Sigma M)_y = 0$:

$$M_y = -\int_A dP.x = -\int_A (\sigma_z.dA).x = -\int_A E(y.K_y + x.K_x)x.dA$$

$$M_y = -EK_y \int_A xy\,dA - EK_x \int_A x^2\,dA = -E(K_y I_{xy} + K_x I_y) \qquad (5)$$

Applying $(\Sigma M)_z = 0$:

$$M_x = -\int_A dP.y = -\int_A (\sigma_z.dA)y = -\int_A E(y.K_y + x.K_x)y.dA$$

$$M_x = -EK_y \int_A y^2\,dA - EK_x \int_A xy\,dA = -E(K_y I_x + K_x I_{xy}) \qquad (6)$$

Generally, $\boldsymbol{M_x}$ & $\boldsymbol{M_y}$ are known and $\boldsymbol{K_x}$ & $\boldsymbol{K_y}$ are to be determined.

For K_y:

$$M_x I_y = -EK_y I_x I_y - EK_x I_{xy} I_y$$ Multiplying Eqn. (6) by I_y

$$M_y I_{xy} = -EK_y I_{xy}^2 - EK_x I_y I_{xy}$$ Multiplying Eqn. (5) by I_{xy}

$$K_y = \frac{M_x I_y - M_y I_{xy}}{E(I_{xy}^2 - I_x I_y)} \qquad (7)$$ Subtracting Eqn. (5) from Eqn. (6)

For K_z:

$$M_x I_{xy} = -EK_y I_x I_{xy} - EK_x I_{xy}^{\;2}$$ Multiplying Eqn. (6) by $\boldsymbol{I_{xy}}$

$$M_y I_x = -EK_y I_x I_{xy} - EK_x I_y I_x$$ Multiplying Eqn. (5) by $\boldsymbol{I_x}$

$$K_x = \frac{M_y I_x - M_x I_{xy}}{E(I_{xy}^{\;2} - I_x I_y)} \qquad (8)$$

Subtracting Eqn. (6) from Eqn. (5)

Substituting the values of $\boldsymbol{K_x}$ and $\boldsymbol{K_y}$ in Eqn. (3)

$$(3) \Rightarrow \quad \sigma_z = E(K_x.x + K_y.y)$$

$$\sigma_z = E\left[\frac{M_y I_x - M_x I_{xy}}{E(I_{xy}^{\;2} - I_x I_y)}.x + \frac{M_x I_y - M_y I_{xy}}{E(I_{xy}^{\;2} - I_x I_y)}.y\right]$$

$$\sigma_z = \frac{(M_y I_x - M_x I_{xy})x + (M_x I_y - M_y I_{xy})y}{E(I_{xy}^{\;2} - I_x I_y)} \qquad (9)$$

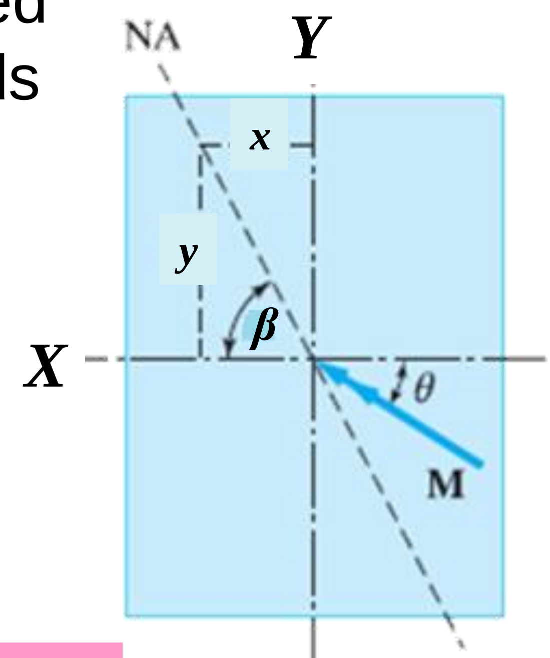
Inclination of the Neutral Axes (N.A.)

Equation of Neutral Axes can be determined by setting $\boldsymbol{\sigma_x} = \mathbf{0}$ in the eqn. **(3)**, which yields

$$0 = E(K_x.x + K_y.y) \qquad (3)$$

$$K_y.y = -K_x.x \qquad \therefore \frac{y}{x} = \tan\beta$$

$$\frac{y}{x} = \tan\beta = -\frac{K_x}{K_y}$$



$$\beta = -\tan^{-1}\left[\frac{K_x}{K_y}\right] = -\tan^{-1}\left[\frac{M_y I_x - M_x I_{xy}}{M_x I_y - M_y I_{xy}}\right] \qquad (10)$$

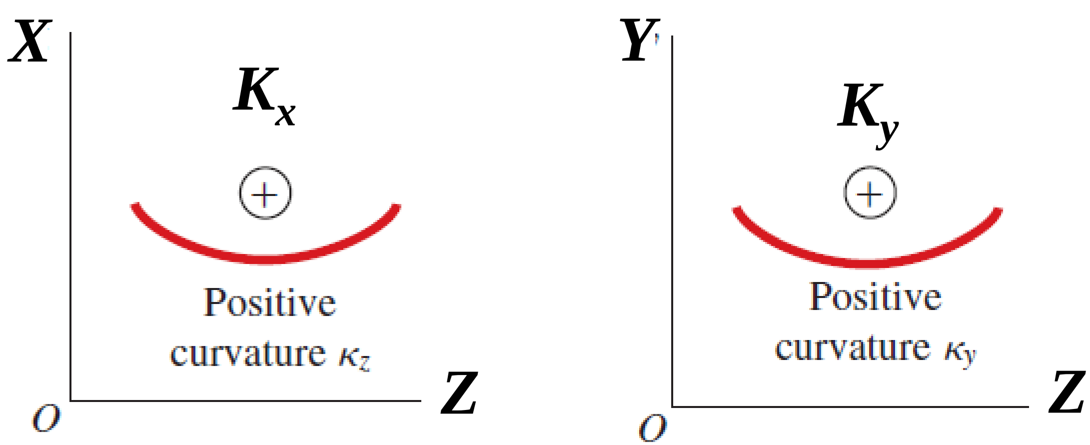
- The negative sign indicates the angle is in clock-wise
- Angle (β) is measured from the positive axes with respect to the local centroidal axes)

- This is the general solution and can be applied to any section (Symmetric or Unsymmetrical)
- Eqn. (7) and (8) are derived by assuming the positive bending (Tension at the bottom fiber)

  i.e., upward loading for the cantilever and downward loading for simply supported beam.

- For Negative bending tension at top fibers multiplied the Eqn. (7) and (8) by (**-1**).

  i.e., downward loading for the cantilever and upward loading for simply supported beam.



Case-I $M_y = 0$

$$K_y = \frac{M_x I_y - M_y I_{xy}}{E(I_{xy}^{\ 2} - I_x I_y)} = \frac{M_z I_y}{E(I_{xy}^{\ 2} - I_x I_y)}$$

$$K_x = \frac{M_y I_x - M_x I_{xy}}{E(I_{xy}^{\ 2} - I_x I_y)} = \frac{-M_x I_{xy}}{E(I_{xy}^{\ 2} - I_x I_y)}$$

$$\beta = -\tan^{-1}\left[\frac{K_x}{K_y}\right] = -\tan^{-1}\left[\frac{-M_x I_{xy}}{M_x I_y}\right]$$

Case-II $M_x = 0$

$$K_y = \frac{M_x I_y - M_x I_{xy}}{E(I_{xy}^{\ 2} - I_x I_y)} = \frac{-M_y I_{xy}}{E(I_{xy}^{\ 2} - I_x I_y)}$$

$$K_x = \frac{M_y I_x - M_x I_{xy}}{E(I_{xy}^{\ 2} - I_x I_y)} = \frac{M_y I_x}{E(I_{xy}^{\ 2} - I_x I_y)}$$

$$\beta = -\tan^{-1}\left[\frac{K_x}{K_y}\right] = -\tan^{-1}\left[\frac{-M_y I_x}{M_y I_{xy}}\right]$$

Case-III, Symmetrical Section $I_{xy} = 0$

(Or about the Principal axes)

$$K_y = \frac{M_x I_y - M_y I_{xy}}{E(I_{xy}^2 - I_x I_y)} = -\frac{M_x}{EI_x}$$

$$K_x = \frac{M_y I_x - M_x I_{xy}}{E(I_{xy}^2 - I_x I_y)} = \frac{M_y}{EI_y}$$

$$\beta = -\tan^{-1}\left[\frac{K_x}{K_y}\right] = -\tan^{-1}\left[\frac{M_y I_x}{M_x I_y}\right]$$

Procedure of Radius of Curvature Method

1. Find out the centroid of the cross section and draw the axes ***z*** and ***y***
2. Calculate the I_x and I_y and I_{xy}.
3. Determine the components of the loading with respect the centroidal axes. *i.e.*, M_x and M_y.

4. Calculate the curvatures, K_x and K_y as per the sign convention. By the following expressions

$$K_y = \frac{M_x I_y - M_y I_{xy}}{E(I_{xy}^2 - I_x I_y)} \qquad K_x = \frac{M_y I_x - M_x I_{xy}}{E(I_{xy}^2 - I_x I_y)}$$

5. Calculate the stress at any point by using the following equation.

$$\sigma_z = E(x.K_x + y.K_y)$$

6. In above equation use the ***x*** and ***y*** values of any specified point along their coordinate sign with respect to the centroidal axes.

7. Determine the inclination of the N.A. by the following Equation.

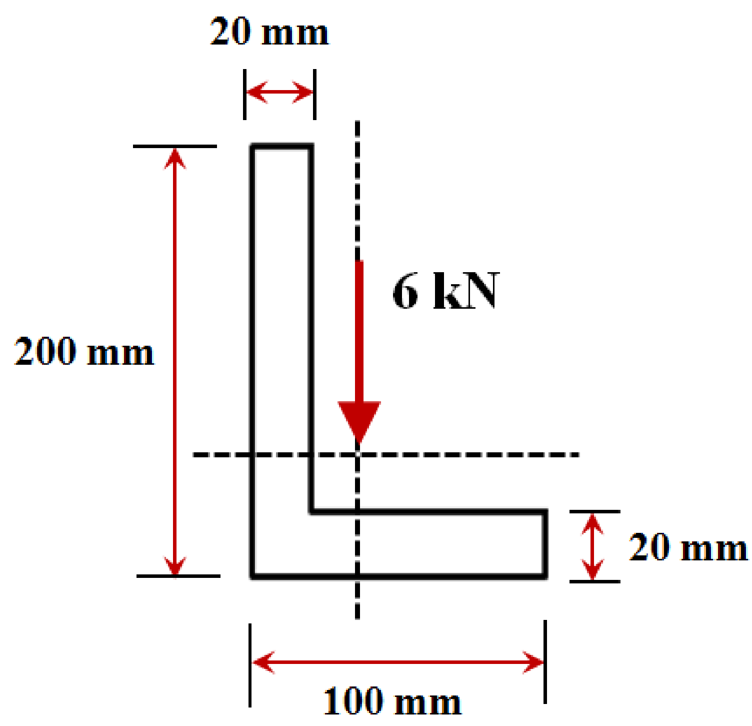
$$\beta = -\tan^{-1}\left[\frac{K_x}{K_y}\right]$$

Solved Example Problem 3, by Radius of curvature method

Cantilever beam

L = **3.0 m.**

Vertical Load P = 6 kN



Assignment Problem

Book: Mechanics of Materials 2nd Edition

By Andrew Pytel & Jaan Kiusalaas

By Method of super position

Problem 11.20 to 11.28

By Radius of curvature Method

Problem 11.04, 11.23, 11.26 to 11.27

Submission time = 2 weeks