Structural Dynamics

(Damped System)

Damped SDOF- General

- We have seen in the preceding chapter that the simple oscillator under idealized conditions of no damping, once excited, will oscillate indefinitely with a constant amplitude at its natural frequency.
- Experience indicates, however, that it is not possible to have a device which vibrates under these ideal conditions.
- Forces designated as frictional or damping forces are always present in any physical system undergoing motion.
- These forces dissipate energy; more precisely, the unavoidable presence of these frictional forces constitutes a mechanism through which the mechanical energy of the system, kinetic or potential energy, is transformed to other forms of energy such as heat.
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Damped SDOF- General

- The mechanism of this energy transformation or dissipation is quite complex and is not completely under- stood at this time. In order to account for these dissipative forces in the analysis of dynamic systems, it is necessary to make some assumptions about these forces, on the basis of experience.

Damped SDOF- Viscous Damping

- In considering damping forces in the dynamic analysis of structures, it is usually assumed that these forces are proportional to the magnitude of the velocity, and opposite to the direction of motion.
- This type of damping is known as viscous damping; it is the type of damping force that could be developed in a body restrained in its motion by a surrounding viscous fluid.
- There are situations in which the assumption of viscous damping is realistic and in which the dissipative mechanism is approximately viscous.
- Nevertheless, the assumption of viscous damping is often made regardless of the actual dissipative characteristics of the system.
- The primary reason for such wide use of this method is that it leads to a relatively simple mathematical analysis.

Damped SDOF- Eq. of motion

- Let us assume that we have modeled a structural system as a simple oscillator with viscous damping, as shown in Fig. 2.1(a).
- In this figure, m and k are respectively the mass and spring constant of the oscillator and c is the viscous damping coefficient.

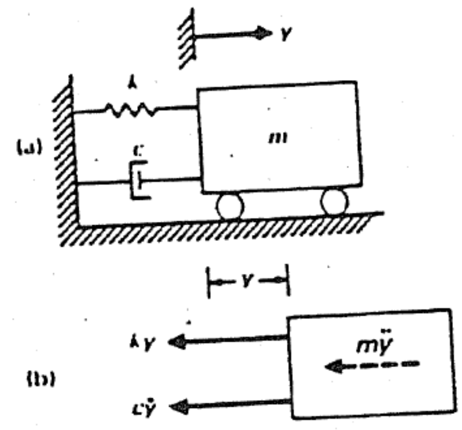

Fig. 2.1 (a) Viscous damped oscillator. (b) Free body diagram.

$$m\ddot{y} + c\dot{y} + ky = 0. \qquad (2.1)$$

Damped SDOF- Eq. of motion

- We proceed, as in the case of the un damped oscillator, to draw the free body diagram (FBD) and apply Newton's law to obtain the differential equation of motion.
- Figure 2.l(b) shows the FBD of the damped oscillator ir. which the inertial force $m\ddot{y}$ is also shown, so that we can use D'Alembert`s Principle.

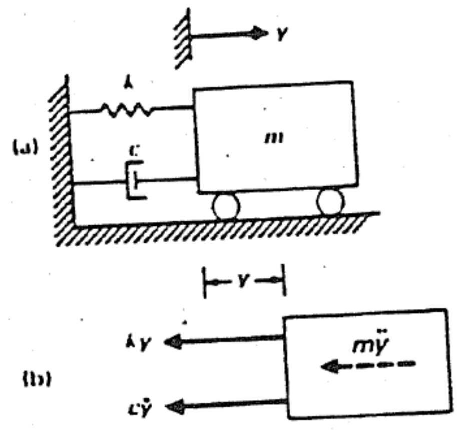

Fig. 2.1 (a) Viscous damped oscillator. (b) Free body diagram.

$$m\ddot{y} + c\dot{y} + ky = 0. \qquad (2.1)$$

Damped SDOF- Eq. of motion

- The summation of forces in the y direction gives the differential equation of motion.
- $m\ddot{y} + c\dot{y} + ky = 0$ (2.1)
- The reader may verify that a trial solution y =A sin (wt) or y = B cos (wt) will not satisfy eq. (2.l).
- However, the exponential function y = Ce^{pt}, does satisfy this equaiton.

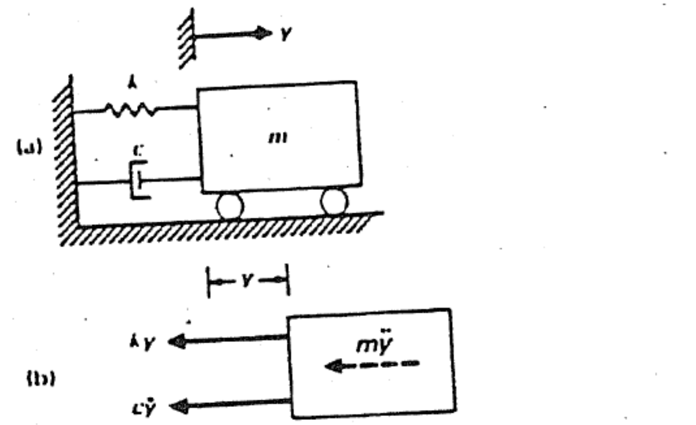

Fig. 2.1 (a) Viscous damped oscillator. (b) Free body diagram.

$$m\ddot{y} + c\dot{y} + ky = 0. \qquad (2.1)$$

Damped SDOF- Eq. of motion

- Substitution of this function into eq. (2.1) results in the equation

$$mCp^2 e^{pt} + cCp\, e^{pt} + kC\, e^{pt} = 0$$

- which, after cancellation of the common factors, reduces to an equation called the characteristic equation for the system,
- $mp^2 + cp + k = 0$ (2.2)
- The roots of this quadratic equation are
- $$\begin{matrix} p_1 \\ p_2 \end{matrix} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad (2.3)$$
- Thus the general solution of eq. (2.1) is given by the superposition of the two possible solutions, namely

$$y(t) = C_1\, e^{p_1 t} + C_2\, e^{p_2 t} \qquad (2.4)$$

Damped SDOF- Eq. of motion

$$y(t) = C_1 e^{p_1 t} + C_2 e^{p_2 t} \qquad (2.4)$$

- where C_1, and C_2 are constants of integration to be determined from the initial conditions.
- The final form of eq. (2.4) depends on the sign of the expression under the radical in eq. (2.3). Three distinct cases may occur.
- The quantity under the radical may either be zero, positive, or negative.
- The limiting case in which the quantity under the radical is zero is treated first.
- The damping present in this case is called critical damping.

Damped SDOF- Critically Damped System

- For a system oscillating with critical damping, as defined above, the expression under the radical in eq. (2.3) is equal to zero; that is,

$$\left(\frac{c_{cr}}{2m}\right)^2 - \frac{k}{m} = 0 \qquad (2.5)$$

$$c_{cr} = 2\sqrt{km} \qquad (2.6)$$

- where C_{cr} designates the critical damping value.
- Since the natural frequency of the un damped system is designated by $\omega = \sqrt{\frac{k}{m}}$, the critical damping coefficient given by eq. (2.6) may also be expressed in alternative notation as

$$c_{cr} = 2m\omega = \frac{2k}{\omega}. \qquad (2.7)$$

Damped SDOF- Critically Damped System

- In a critically damped system the roots of the characteristic equation are equal, and from eq. (2.3), they are

$$p_1 = p_2 = -\frac{c_{cr}}{2m}. \qquad (2.8)$$

- Since the two roots are equal, the general solution given by eq. (2.4) would provide only one independent constant of integration; hence, one independent
- solution, namely

$$y_1(t) = C_1\, e^{-(c_{cr}/2m)t}. \qquad (2.9)$$

- Another independent solution may be found by using the function

$$y_2(t) = C_2\, t e^{-(c_{cr}/2m)t}. \qquad (2.10)$$

Damped SDOF- Critically Damped System

- This equation, as the reader may verify, also satisfies the differential equation eq. (2.1).
- The general solution for a critically damped system is then given by the superposition of these two solutions.

$$y(t) = (C_1 + C_2 t)\, e^{-(c_{cr}/2m)t}. \qquad (2.11)$$

Damped SDOF- Over damped System

- It should be noted that, for the over-damped or the critically damped system, the resulting motion is not oscillatory; the magnitude of the oscillations decays exponentially with time to zero.
- Figure2.2 depicts graphically the response for the simple oscillator with critical damping.
- The response of the over-damped system is similar to the motion of the critically damped system of fig. 2.2 but he return toward the neutral potion requires more time as the damping is increased.

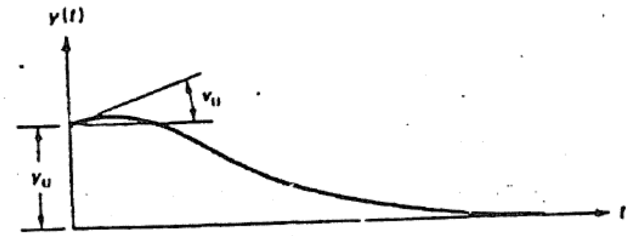

Fig. 2.2 Free-vibration response with critical damping.

Damped SDOF- Under Damped System

- When the value of the damping coefficient is less than the critical value ($c<c_{cr}$) which occurs when the expression under the radical sign is negative, the roots of th characteristic eq. (2.3) are complex conjugates, so that.

$$\begin{matrix} p_1 \\ p_2 \end{matrix} = -\frac{c}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}, \qquad (2.13)$$

- Whre $i = \sqrt{-1}$ is the imaginary unit.
- For this , it is convenient to make use of Euler's equation which relate exponential and trigonometric functions namely.

$$\begin{aligned} e^{ix} &= \cos x + i \sin x, \\ e^{-ix} &= \cos x - i \sin x. \end{aligned} \qquad (2.14)$$

Damped SDOF- Under Damped System

- The substitution of the roots p1, and p2 from eq. (2.13) into eq.(2.4) together with the use of eq. (2.14) gives the following convenient form for the general solution of the underdamped system:

$$y(t) = e^{-(c/2m)t}(A \cos \omega_D t + B \sin \omega_D t), \qquad (2.15)$$

- where A and B are redefined constants of integration and ω_D, the damped frequency of the system, is given by

$$\omega_D = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \qquad (2.16)$$

$$\omega_D = \omega\sqrt{1 - \xi^2}. \qquad (2.17)$$

Damped SDOF- Under Damped System

- This last result is obtained after substituting, in eq. (2.16), the expression for the un-damped natural frequency

$$\omega = \sqrt{\frac{k}{m}} \tag{2.18}$$

- And defining the damping ratio of the system as

$$\xi = \frac{c}{c_{cr}}. \tag{2.19}$$

Damped SDOF- Under Damped System

- Finally, _when the initial conditions of displacement and velocity, y_o and v_o , are introduced, the constants of integration can be evaluated and substituted into eq. (2.15), giving

$$y(t) = e^{-\xi\omega t}\left(y_0 \cos \omega_D t + \frac{v_0 + y_0 \xi\omega}{\omega_D} \sin \omega_D t\right). \qquad (2.20)$$

- Alternatively, this expression can be written as

$$y(t) = C e^{-\xi\omega t} \cos(\omega_D t - \alpha) \qquad (2.21)$$

where

$$C = \sqrt{y_0^2 + \frac{(v_0 + y_0 \xi\omega)^2}{\omega_D^2}} \qquad (2.22)$$

and

$$\tan \alpha = \frac{(v_0 + y_0 \xi\omega)}{\omega_D y_0}. \qquad (2.23)$$

Damped SDOF- Under Damped System

- A graphical record of the response of an under damped system with initial displacement y_o but starting with zero velocity ($\nu_o = 0$) is shown in Fig. 2.3.
- It may be seen in this figure that the motion is oscillatory, but not periodic.

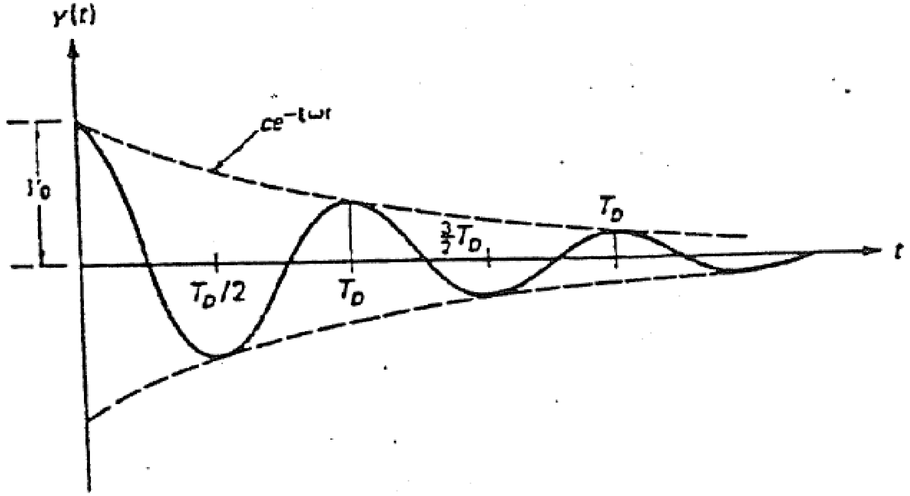

Fig. 2.3 Free vibration response for underdamped system.

Damped SDOF- Under Damped System

- A graphical record of the response of an under damped system The amplitude of vibration is not constant during the motion but decreases for successive cycles; nevertheless, the oscillations occur at equal intervals of time.
- This time interval is designated as the damped period of vibration and is given from eq. (2.17) by

$$T_D = \frac{2\pi}{\omega_D} = \frac{2\pi}{\omega\sqrt{1-\xi^2}}. \qquad (2.24)$$

- The value of the damping coefficient for real structures is much less than the critical damping coefficient and usually ranges between 2 to 20% of the critical damping value. Substituting for the maximum value $\zeta = 0.20$ into eq. (2.I7),
- $\omega_D = 0.98\omega$ (2.25)

Damped SDOF- Under Damped System

- It can be seen that the frequency of vibration for a system with as much as a 20% damping ratio is essentially equal to the un-damped natural frequency.
- Thus, in practice, the natural frequency for a clamped system may be taken to be equal to the un-damped natural frequency.

Damped SDOF- Lograthmic Decrement

- A practical method for determining experimentally the damping coefficient of a system is to initiate free vibration, obtain a record of the oscillatory motion such as the one shown in Fig. 2.4, and measure the rate of decay of the amplitude of motion.

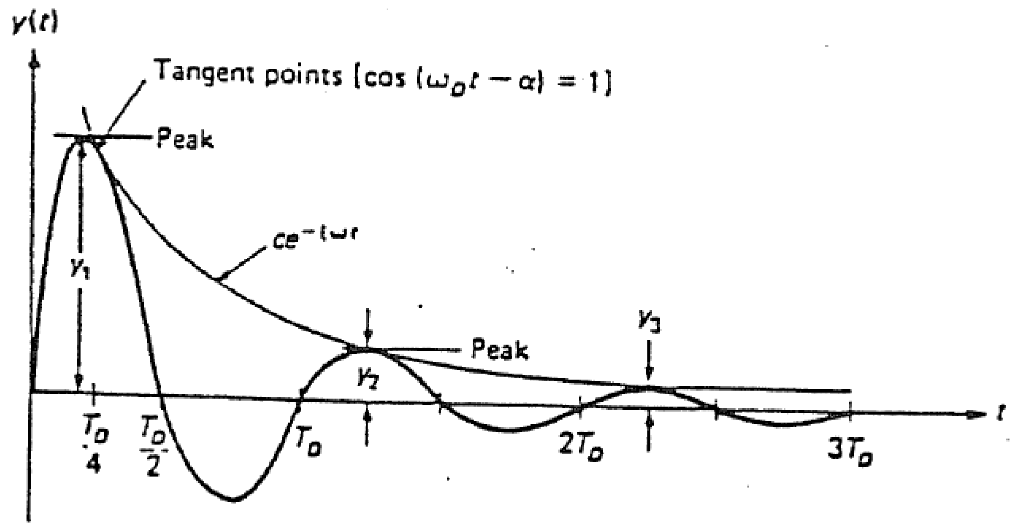

Fig. 2.4 Curve showing peak displacements and displacements at the points of tangency.

Damped SDOF- Lograthmic Decrement

- The decay may be conveniently expressed by the logarithmic
- decrement δ which is defined as the natural logarithm of the ratio of any two successive peak amplitudes y_1, and y_2 in free vibration, that is

$$\delta = \ln \frac{y_1}{y_2}. \tag{2.26}$$

- The evaluation of damping from the logarithmic decrement follows.
- Consider the damped vibration motion represented graphically 'in Fig. 2.4 and given analytically by eq. (2.21) as

$$y(t) = C e^{-\xi \omega t} \cos(\omega_D t - \alpha).$$

Damped SDOF- Lograthmic Decrement

- We note from this equation that, when the cosine factor is unity, the displacement is on points of the exponential curve $y(t) = Ce^{-\zeta wt}$ as shown in Fig. 2.4.
- However, these points are near but not equal to the positions of maximum displacement.
- The points on the exponential curve appear slightly to the right of the points of maximum amplitude.
- For most practical problems, the discrepancy is negligible and the displacement curve may be assumed to coincide at the peak amplitude, with the curve $y(t) = Ce^{-\zeta wt}$ so that we may write, for two consecutive peaks,y1, at time t1, and y2, at T_D seconds later as

$$y_1 = C e^{-\xi \omega t_1}$$

$$y_2 = C e^{-\xi \omega (t_1 + T_D)}.$$

Damped SDOF- Lograthmic Decrement

- Dividing these two peak amplitudes and taking the natural logarithm, we obtain

$$\delta = \ln \frac{y_1}{y_2} = \xi \omega T_D \quad (2.27)$$

- or by substituting, T_D, the damping period, from eq. (2.24),

$$\delta = 2\pi\xi/\sqrt{1-\xi^2}. \quad (2.28)$$

- As we can see, the damping ratio ζ can be calculated from eq. (2.28) after determining experimentally the amplitudes of two successive peaks of the system in free vibration.
- For small values of the damping ratio, eq. (2.28) can be approximated by eq. 2.29

$$\delta \simeq 2\pi\xi. \quad (2.29)$$

Damped SDOF- Example problem-1

- A vibrating system consisting of a weight of W= 10 lb and a spring with stiffness k = 20 lb/in is viscously damped so that the ratio of two consecutive amplitudes is 1.00 to 0.85.
- Determine:
- (a) the natural frequency of the un-damped system,
- (b) the logarithmic decrement,
- (c) the damping ratio,
- (d) the damping coefficient, and
- (e) the damped natural frequency.

Damped SDOF- Example problem-2

- A platform of weight W=4000 lb is being supported by four equal columns which are clamped to the foundation as well as to the platform. Experimentally it has been determined that a static force of F= 1000 lb applied horizontally to the platform produces a displacement of A = 0.10 in. It is estimated that damping in the structures is of the order of 5% of the critical damping.
- Determine for this structure the following:
- (a) un- damped natural frequency,
- (b) absolute damping coefficient,
- (c) logarithmic decrement, and
- (d) the number of cycles and the time required for the amplitude of motion to be reduced from an initial value of 0.l in to 0.01 in.