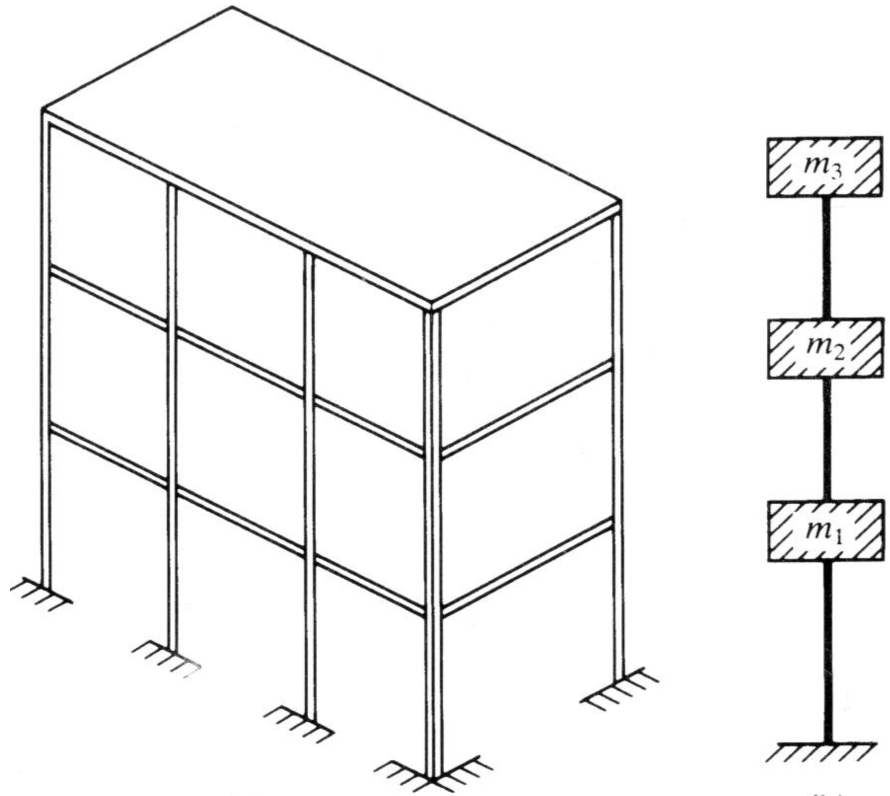
Multi-Degree of Freedom (MDOF) System

Multi-Degree of Freedom (MDOF) System

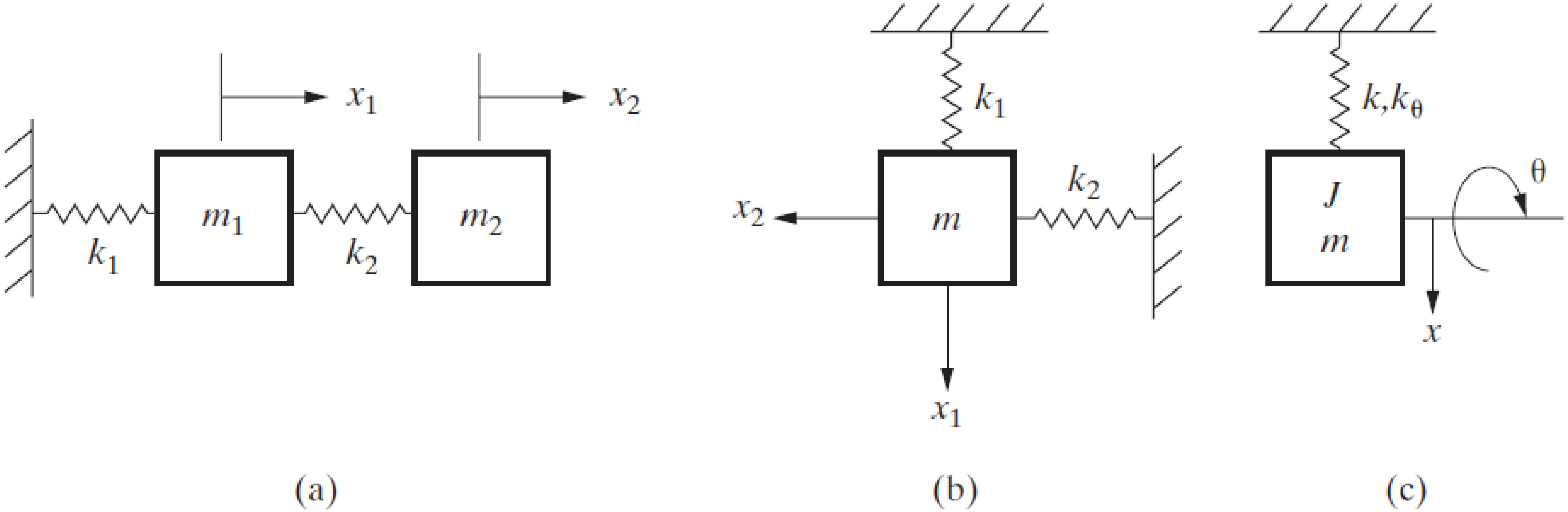
- **Lumped-mass or discrete-mass models:**



(a) (b)

Multi-Degree of Freedom (MDOF) System

- No. of DOF of system = No. of mass elements x number of motion types for each mass .



(a) A simple two-degree-of-freedom model consisting of two masses connected in series by two springs.

(b) A single mass with two degrees of freedom (i.e., the mass moves along both the x_1 *and* x_2 *directions*).

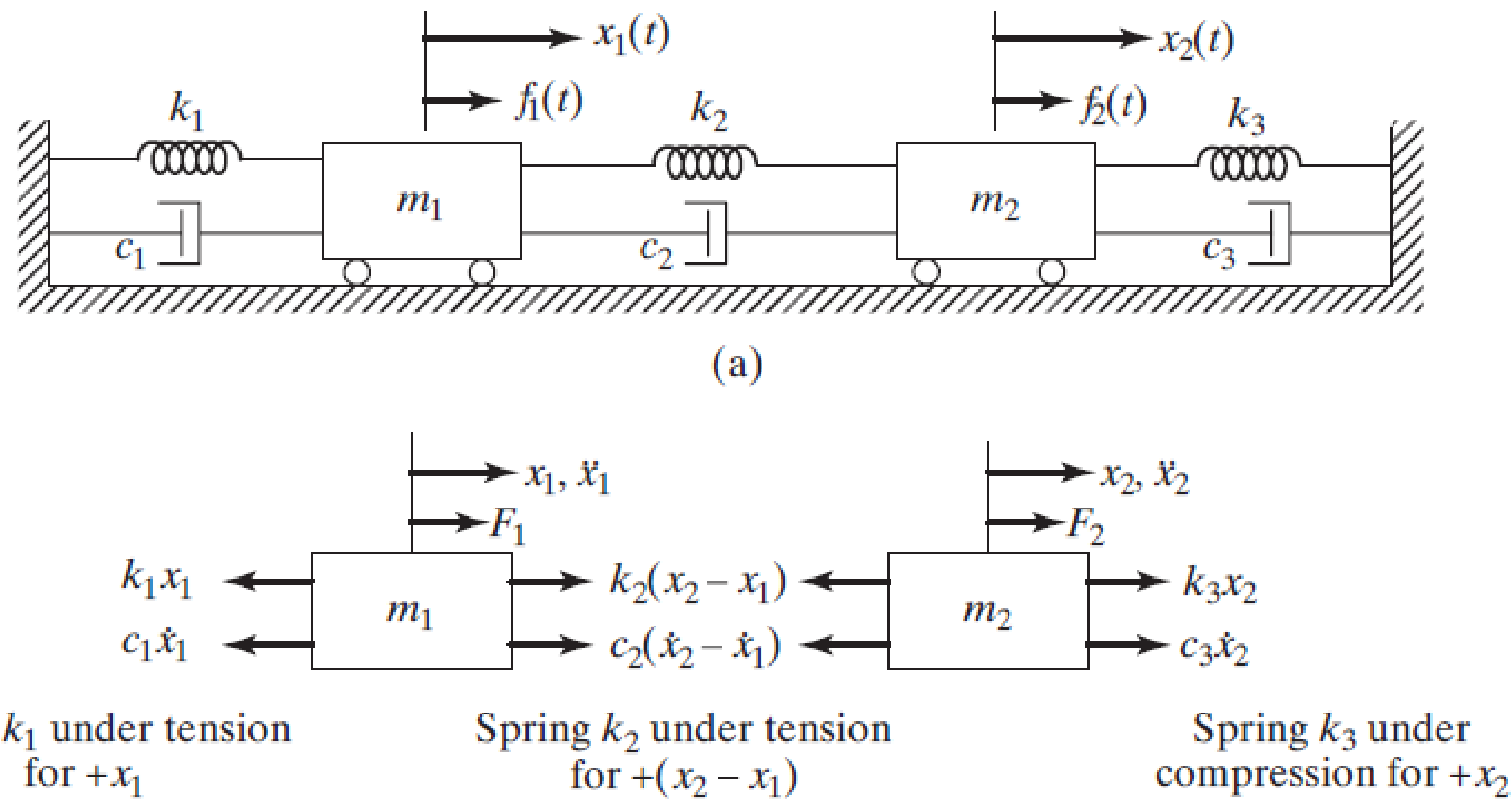
(c) A single mass with one translational degree of freedom and one rotational degree of freedom.

Multi-Degree of Freedom (MDOF) System

- **For each degree of freedom there exists an equation of motion – usually <u>coupled</u> differential equations.**
- **Coupled means that the motion in one coordinate system depends on the other.**
- **If harmonic solution is assumed, the equations produce n natural frequencies. (n= no. of DOF)**
- **The amplitudes of the n degrees of freedom are related by the *natural, principal or normal* mode of vibration.**
- **Under an arbitrary initial disturbance, the system will vibrate freely such that the n normal modes are superimposed.**
- **Under sustained harmonic excitation, the system will vibrate at the excitation frequency.**
- **Resonance occurs if the excitation frequency corresponds to one of the natural frequencies of the system**

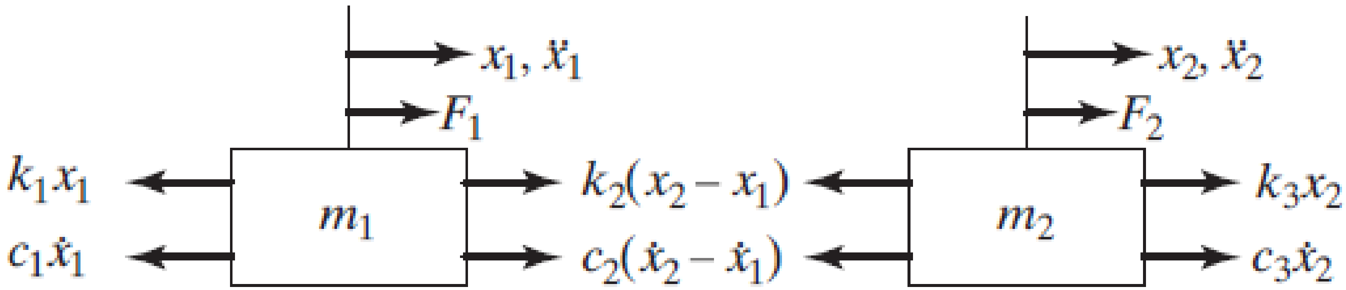
Multi-Degree of Freedom (MDOF) System

- Equations of motion
- Consider a viscously damped system:
- Motion of system described by position $x_1(t)$ and $x_2(t)$ of masses m_1 and m_2
- The free-body diagram is used to develop the equations of motion using Newton's second law



Multi-Degree of Freedom (MDOF) System

- **Equations of motion**



$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 - c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) = F_1$$
$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) + c_3\dot{x}_2 + k_3x_2 = F_2$$

or

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = F_2$$

- **The differential equations of motion for mass m_1 and mass m_2 are <u>coupled</u>.**

- **The motion of each mass is influenced by the motion of the other.**

Multi-Degree of Freedom (MDOF) System

- **Equations of motion**

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = F_1$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = F_2$$

- The coupled differential eqns. of motion can be written in matrix form:

$$[m]\ddot{\vec{x}}(t) + [c]\dot{\vec{x}}(t) + [k]\vec{x}(t) = \vec{F}(t)$$

where [m], [c] and [k] are the mass, damping and stiffness matrices respectively and are given by:

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \qquad [c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \qquad [k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

$\vec{x}(t)$, $\dot{\vec{x}}(t)$, $\ddot{\vec{x}}(t)$ and $\vec{F}(t)$ are the displacement, velocity, acceleration and force vectors

respectively and are given by :

$$\vec{x}(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \quad \dot{\vec{x}}(t) = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} \quad \ddot{\vec{x}}(t) = \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix} \quad \text{and} \quad \vec{F}(t) = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

- Note: the mass, damping and stiffness matrices are all square and symmetric $[m] = [m]^T$ and consist of the mass, damping and stiffness constants.

Multi-Degree of Freedom (MDOF) System

- **Free vibrations of undamped MDOF systems**
- **The eqns. of motion for a free and undamped two DoF system become:**

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$

$$m_2\ddot{x}_2 - k_2 x_1 + (k_2 + k_3)x_2 = 0$$

- **Let us assume that the resulting motion of each mass is harmonic: For simplicity, we will also assume that the response frequencies and phase will be the same:**

$$x_1(t) = X_1 \cos(\omega t + \phi) \quad \text{and} \quad x_2(t) = X_2 \cos(\omega t + \phi)$$

- **Substituting the assumed solutions into the eqns. of motion:**

$$\left[\left\{-m_1\omega^2 + (k_1 + k_2)\right\}X_1 - k_2 X_2\right]\cos(\omega t + \phi) = 0$$

$$\left[-k_2 X_1 + \left\{-m_2\omega^2 + (k_2 + k_3)\right\}X_2\right]\cos(\omega t + \phi) = 0$$

As these equations must be zero for all values of t, the cosine terms cannot be zero. Therefore:

$$\left\{-m_1\omega^2 + (k_1 + k_2)\right\}X_1 - k_2 X_2 = 0$$

$$-k_2 X_1 + \left\{-m_2\omega^2 + (k_2 + k_3)\right\}X_2 = 0$$

- **Represent two simultaneous algebraic equations with a trivial solution when X_1 and X_2 are both zero – no vibration.**

Multi-Degree of Freedom (MDOF) System

- Free vibrations of undamped systems
- Written in matrix form it can be seen that the solution exists when the determinant of the mass / stiffness matrix is zero:

$$\begin{bmatrix} \left\{-m_1\omega^2 + (k_1+k_2)\right\} & -k_2 \\ -k_2 & \left\{-m_2\omega^2 + (k_2+k_2)\right\} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

or

$$m_1 m_2 \omega^4 - \left\{(k_1+k_2)m_2 + (k_2+k_3)m_1\right\}\omega^2 + (k_1+k_2)(k_2+k_2) - k_2^2 = 0$$

- The solution to the *characteristic equation* yields the natural frequencies of the system.
- The roots of the characteristic equation are:

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1 m_2}\right\}$$

$$\pm \frac{1}{2}\left[\left\{\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1 m_2}\right\}^2 - 4\left\{\frac{(k_1+k_2)(k_2+k_3) - k_2^2}{m_1 m_2}\right\}\right]^{1/2}$$

- This shows that the homogenous solution is harmonic with natural frequencies ω_1 and ω_2

Multi-Degree of Freedom (MDOF) System

- Free vibrations of undamped systems
- The motion (free vibration) of each mass is given by:

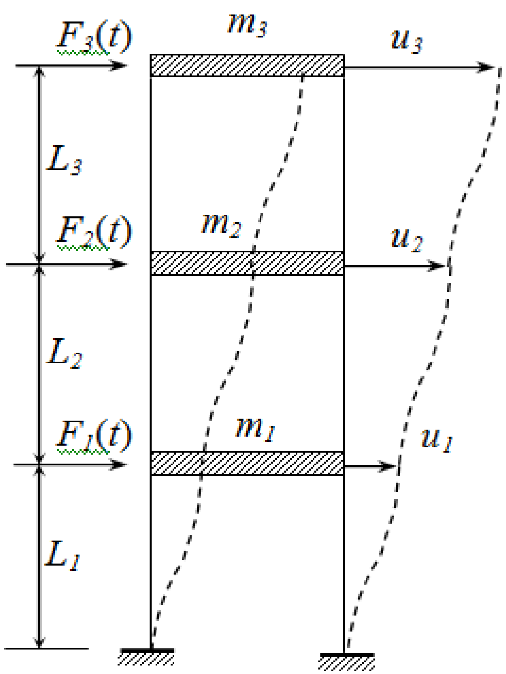
$$\vec{x}^{(1)}(t) = \begin{Bmatrix} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} \rightarrow \text{First mode}$$

$$\vec{x}^{(2)}(t) = \begin{Bmatrix} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{Bmatrix} = \begin{Bmatrix} X_1^{(2)} \cos(\omega_2 t + \phi_2) \\ r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{Bmatrix} \rightarrow \text{Second mode}$$

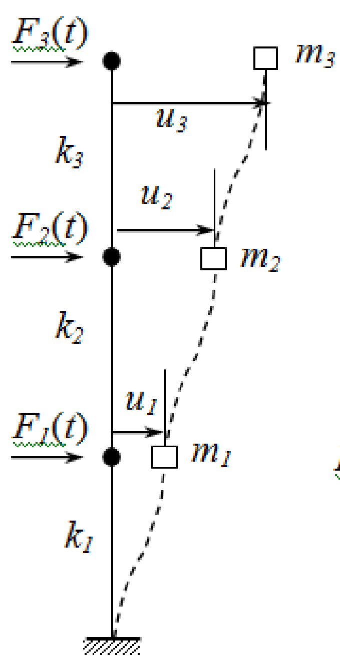
- The constants $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 *and* ϕ_2 are determined from the initial conditions.

Free Vibration Analysis For Multiple Degrees Of Freedom Structures

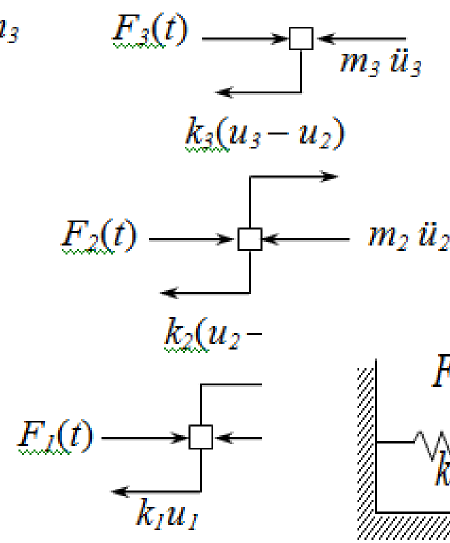
Various models to represent the shear buildings



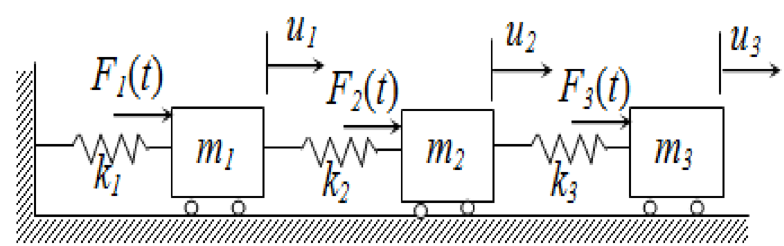
(a) Single Bay Representation



(b) Single Column Model



(a) Free Body Diagrams of Each Floor



Multi-mass Spring Model of a Shear Building.

Free Vibration Analysis For Multiple Degrees Of Freedom Structures

Shear Building:

- A structure in which there is no rotation of a horizontal section at the level of the floors.

- The following assumptions apply when modeling the structure using shear-building concept:
    I. The total mass of the structure is concentrated at the levels of the floors. In this way the actual structure with infinite number of degrees of freedom due to distributed mass is changed in to a lumped mass model with degrees of freedom equal in number to the lumped masses at the floors.
    II. The floors are considered infinitely rigid as compared to columns. Thus, the joints between the floors and the columns are fixed against rotation.
    III. The axial deformation of the columns is neglected. This means that the horizontal floors remain horizontal under the action of lateral loads.

Free Vibration Analysis For Multiple Degrees Of Freedom Structures

- Considering horizontal dynamic equilibrium of the free body diagrams of each of the three floors, gives:

$$m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2 u_2 - F_1(t) = 0$$

$$m_1\ddot{u}_1 + k_1 u_1 - k_2(u_2 - u_1) - F_1(t) = 0 \qquad \text{(I)}$$

$$m_2\ddot{u}_2 - k_2 u_1 + (k_2 + k_3)u_2 - k_3 u_3 - F_2(t) = 0$$

$$m_2\ddot{u}_2 + k_2(u_2 - u_1) - k_3(u_3 - u_2) - F_2(t) = 0 \qquad \text{(II)}$$

$$m_3\ddot{u}_3 - k_3 u_2 + k_3 u_3 - F_3(t) = 0$$

$$m_3\ddot{u}_3 + k_3(u_3 - u_2) - F_3(t) = 0 \qquad \text{(III)}$$

- The above system of equations may conveniently be written in matrix form as follows:

$$[M]\{\ddot{u}\} + [K]\{u\} = \{F\} \qquad \text{(IV)}$$

Where, [M] and [K] are the mass and stiffness matrices.

Free Vibration Analysis For Multiple Degrees Of Freedom Structures

- [*M*] and [*K*] are the mass and stiffness matrices, respectively, given by:

$$[M]=\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \qquad [K]=\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \qquad \text{(V)}$$

And $\{u\}$, $\{\ddot{u}\}$ and $\{F\}$ are the displacement, acceleration and force vector given by:

$$\{u\}=\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad \{\ddot{u}\}=\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix} \qquad \{F\}=\begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix} \qquad \text{(VI)}$$

- The **stiffness coefficient** (element k_{ij} of matrix [*K*]) is defined as the force produced at floor-*i* when a unit displacement is given to floor-*j*; all other floors being fixed at zero displacement.

Free Vibration Analysis For Multiple Degrees Of Freedom Structures

- Equation of free vibration is:

$$[M]\{\ddot{u}\} + [K]\{u\} = \{0\} \qquad \text{(VII)}$$

- Knowing that the vibration of the undamped system (no energy dissipation) will be simple harmonic motion, the general solution of this equation may be taken in terms of amplitude *a*, angular velocity ω, time *t* and phase angle α, as follows:

$$u_i = a_i \sin(\omega t - \alpha) \qquad i = 1, 2, \ldots, n \qquad \text{(VIII)}$$

- In matrix notation Eq. VIII becomes:

$$\{u\} = \{a\}\sin(\omega t - \alpha) \qquad \text{(IX)}$$

- Where a_i is the amplitude of motion of the *i*th coordinate and *n* is the number of degrees of freedom.

Free Vibration Analysis For Multiple Degrees Of Freedom Structures

- Considering Substituting Eq. IX in Eq. VII, we get:

$$-\omega^2[M]\{a\}\sin(\omega t-\alpha)+[K]\{a\}\sin(\omega t-\alpha)=\{0\}$$

or $([K]-\omega^2[M])\{a\}=\{0\}$ since $\sin(\omega t-\alpha)$ cannot be zero at all the times. (X)

or $[K]\{a\}=\omega^2[M]\{a\}$ similar to general equation form $A\,x=\lambda\,B\,x$ (XI)

$$\left|[K]-\omega^2[M]\right|=0 \qquad \text{(XII)}$$

Free Vibration Analysis For Multiple Degrees Of Freedom Structures

Eigen Values And Eigen Vectors:

- Let $A = [a_{jk}]$ be a given matrix and consider the vector equation $A\,x = \lambda\,x$, it is clear that the zero vector $x = 0$ is a solution for any value of λ.

- Value of λ for which t equation has a non-trivial solution $x \neq 0$ is called ***eigen-value*** or characteristic value or latent root of the matrix-*A*.

- The solutions $x \neq 0$ corresponding to *n* eigen-values of the equation are called ***eigen-vectors*** or characteristic vectors of *A* corresponding to particular eigen-values λ.

- The set of all eigen-values is called the ***spectrum*** of A.

Free Vibration Analysis For Multiple Degrees Of Freedom Structures

Natural Frequencies And Normal Modes:

- The non-trivial solution of Eq. X requires that the determinant of {*a*} must be equal to zero, i.e.

$$\left|[K]-\omega^2[M]\right|=0 \qquad \text{(XII)}$$

- When expanded, the above equation results in a polynomial of degree *n* in terms of ω^2, which is known as the ***characteristic equation*** of the system.

- This equation can be solved to get *n* real distinct values of ω^2 (ω_1^2, ω_2^2, …., ω_n^2), the positive square roots of which are called the ***angular natural frequencies*** (ω_1, ω_2, …., ω_n) of the structure.

- These frequencies may then be changed in to natural frequencies (f_1, f_2, …., f_n) having units of cycles per second.

$$f_i=\frac{\omega_i}{2\pi} \qquad T_i=\frac{1}{f_i} \qquad \text{(XIII)}$$

Free Vibration Analysis For Multiple Degrees Of Freedom Structures

- For each value of ω^2 satisfying the characteristic equation, Eq. XI can be solved for a_i, in terms of one reference value for any one constant out of the n-values.
- This is because that one of the equations is already used to calculate the value of ω^2 and hence two of the equations will become similar out of the set of n-equations.
- Usually the amplitude of first story is taken equal to unity and all other amplitudes are calculated with respect to it.

Normal Mode or Modal Shape of vibration

- Each set of a_i defines the relative amplitude and deformed shape of the frame with respect to a particular frequency and time period value.

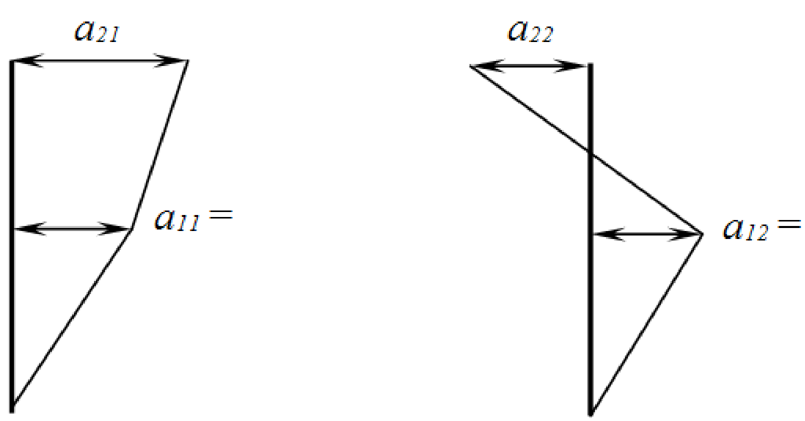
Fundamental Mode

- is used to refer to the mode associated with the lowest frequency, while the other modes are called ***harmonics*** or higher harmonics

Free Vibration Analysis For Multiple Degrees Of Freedom Structures

- The normal modes or modal shapes represent the n possible ways of simple harmonic motions of the structure that can occur in such a way that all the masses move in phase at the same frequency.

- The amplitude at the floor level-*i* for mode-*j* may be denoted by a_{ij}. For example, a_{21} denotes the relative amplitude of the second story when the structure vibrates freely at the fundamental natural frequency according to the fundamental mode.



(a) Fundamental Mode (b) Second Mode

Normal Modes for a two storey Building

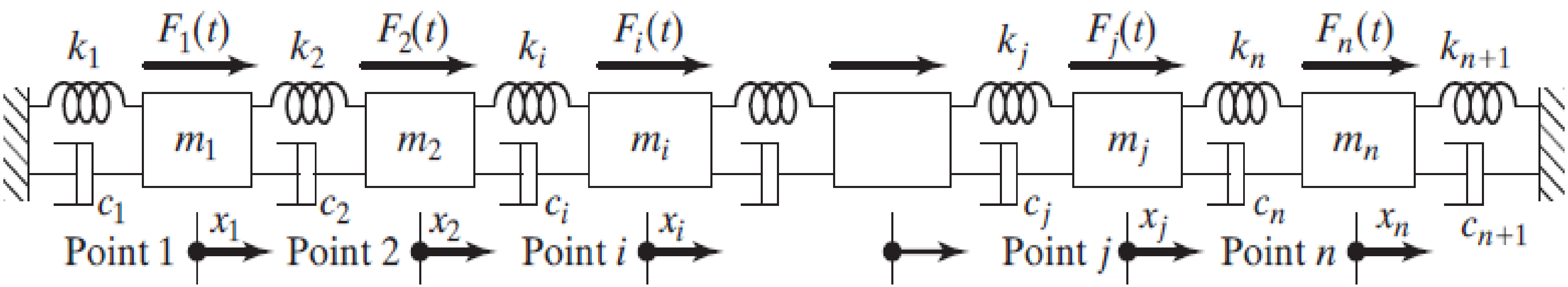
Multi Degree-of-Freedom systems

- **Equations of Motion – Newton's second law.**

1. **Define suitable coordinates to describe the position of each lumped mass in the model**
2. **Establish the static equilibrium of the system and determine the displacement of each lumped mass wrt to their respective static equilibrium position.**
3. **Draw the free-body diagram for each lumped mass in the model. Indicate the spring, damping and external forces on each mass element when a positive displacement and velocity is applied to each mass element.**
4. **Generate the equation of motion for each mass element by applying Newton's second law of motion with reference to the free-body diagrams:**

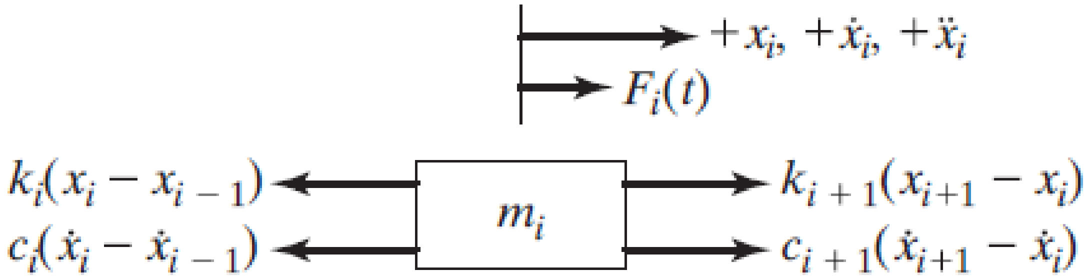
$$m_i \ddot{x}_i = \sum_j F_{ij} \quad (\text{ for mass } m_i) \quad \text{and} \quad J_i \ddot{\theta}_i = \sum_j M_{ij} \quad (\text{ for rigid body of inertia } J)$$

- **Example: Consider the specific MDoF system:**



Multi Degree-of-Freedom systems

- **Equations of Motion – Newton's second law.**



$$m_i\ddot{x}_i = -k_i\left(x_i - x_{i-1}\right) + k_{i+1}\left(x_{i+1} - x_i\right) - c_i\left(\dot{x}_i - \dot{x}_{i-1}\right) + c_{i+1}\left(\dot{x}_{i+1} - \dot{x}_i\right) + F_i \qquad for\ i = 1,2,3\ldots,n-1$$

Rearranging:

$$m_i\ddot{\boldsymbol{x}}_{\boldsymbol{i}} - c_i\dot{\boldsymbol{x}}_{\boldsymbol{i-1}} + \left(c_i + c_{i+1}\right)\dot{\boldsymbol{x}}_{\boldsymbol{i}} - c_{i+1}\dot{\boldsymbol{x}}_{\boldsymbol{i+1}} - k_i\boldsymbol{x}_{\boldsymbol{i-1}} + \left(k_i + k_{i+1}\right)\boldsymbol{x}_{\boldsymbol{i}} - k_{i+1}\boldsymbol{x}_{\boldsymbol{i+1}} = \boldsymbol{F}_{\boldsymbol{i}} \qquad for\ i = 1,2,3\ldots,n-1$$

- **Note that the system has both stiffness and damping coupling**
- **The equations of motion of masses m_1 *and* m_n at the extremities of the system are obtained by setting $i = 1$ & $x_{i-1} = 0$ and $i = n$ & $x_{n+1} = 0$**

$$m_1\ddot{\boldsymbol{x}}_{\boldsymbol{1}} + \left(c_1 + c_2\right)\dot{\boldsymbol{x}}_{\boldsymbol{1}} - c_2\dot{\boldsymbol{x}}_{\boldsymbol{2}} + \left(k_1 + k_2\right)\boldsymbol{x}_{\boldsymbol{1}} - k_2\boldsymbol{x}_{\boldsymbol{2}} = \boldsymbol{F}_{\boldsymbol{1}}$$

$$m_n\ddot{\boldsymbol{x}}_{\boldsymbol{n}} - c_n\dot{\boldsymbol{x}}_{\boldsymbol{n-1}} + \left(c_n + c_{n+1}\right)\dot{\boldsymbol{x}}_{\boldsymbol{n}} - k_n\boldsymbol{x}_{\boldsymbol{n-1}} + \left(k_n + k_{n+1}\right)\boldsymbol{x}_{\boldsymbol{n}} = \boldsymbol{F}_{\boldsymbol{n}}$$

- **In matrix form:**

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F}$$

Multi Degree-of-Freedom systems

- **Equations of Motion – Newton's second law.**
- Where the mass matrix [*m*], the damping matrix [c] and the stiffness matrix [*k*] are given by:

$$[m]=\begin{bmatrix} m_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & m_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & m_3 & \dots & 0 & 0 \\ . & & & & & . \\ . & & & & & . \\ . & & & & & . \\ 0 & 0 & 0 & \dots & 0 & m_n \end{bmatrix}$$

$$[c]=\begin{bmatrix} (c_1+c_2) & -c_2 & 0 & \dots & 0 & 0 \\ -c_2 & (c_2+c_3) & -c_3 & \dots & 0 & 0 \\ 0 & -c_3 & (c_3+c_4) & \dots & 0 & 0 \\ . & & & & . & . \\ . & & & & . & . \\ . & & & & . & . \\ 0 & 0 & 0 & \dots & -c_n & (c_n+c_{n+1}) \end{bmatrix}$$

Multi Degree-of-Freedom systems

- **Equations of Motion – Newton's second law.**

$$[k]=\begin{bmatrix} (k_1+k_2) & -k_2 & 0 & \dots & 0 & 0 \\ -k_2 & (k_2+k_3) & -k_3 & \dots & 0 & 0 \\ 0 & -k_3 & (k_3+k_4) & \dots & 0 & 0 \\ . & & & & . & . \\ . & & & & . & . \\ . & & & & . & . \\ 0 & 0 & 0 & \dots & -k_n & (k_n+k_{n+1}) \end{bmatrix}$$

- **And the displacement. Velocity, acceleration and excitation force vectors are given by:**

$$\vec{x}=\begin{Bmatrix} x_1(t) \\ x_2(t) \\ . \\ . \\ . \\ x_n(t) \end{Bmatrix} \qquad \dot{\vec{x}}=\begin{Bmatrix} \dot{x}_1(t) \\ x_2(t) \\ . \\ . \\ . \\ \dot{x}_n(t) \end{Bmatrix} \qquad \ddot{\vec{x}}=\begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ . \\ . \\ . \\ \ddot{x}_n(t) \end{Bmatrix} \qquad \vec{F}=\begin{Bmatrix} F_1(t) \\ F_2(t) \\ . \\ . \\ . \\ F_n(t) \end{Bmatrix}$$