Response to Nonperiodic Function

General Forcing Functions

SDOF systems – General forcing functions

- A non-periodic exciting force usually has a magnitude that varies with time; it acts for a specified period and then stops. The simplest form is the impulsive force, a force that has a large magnitude F and acts for a very short time

Methods for determining the response for General forcing functions are

- Representation of the excitation function with a ***Convolution integral***
- Using ***Laplace Transformations***
- Approximating F(t) with a suitable ***interpolation method*** then using a numerical procedure
- ***Numerical integration*** of the equations of motion.

SDOF systems – General forcing functions – Nonperiodic

- **Convolution integral**
- Consider one of the simplest nonperiodic exciting force: Impulsive force: which has a large magnitude F which acts for a very short time Δt.
- An impulse can be measured by the resulting change in momentum of the system:

$$Im\ pulse = F\Delta t = m\dot{x}_2 - m\dot{x}_1$$

where $\dot{x}_1$ *and* $\dot{x}_2$ *represent the velocity of the lumped mass before and after the impulse.*

- The magnitude of the impulse $F\Delta t$ is represented by

$$\underset{\sim}{F} = \int_{t}^{t+\Delta t} F\, dt$$

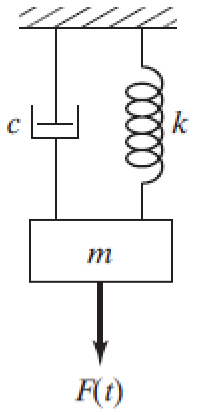
and a unit impulse is defined as

$$\underset{\sim}{f} = \lim_{\Delta t \to 0} \int_{t}^{t+\Delta t} F\, dt = Fdt = 1$$

- For *Fdt* to have a finite value, *F* approaches infinity as Δt nears zero.

SDOF systems – General forcing functions – Nonperiodic

- **Convolution integral – Impulse response**
- Consider a (viscously) damped SDOF (mass-spring-damper system) subjected to an impulse at *t=0.*



- For an underdamped system, the eqn. of motion is:

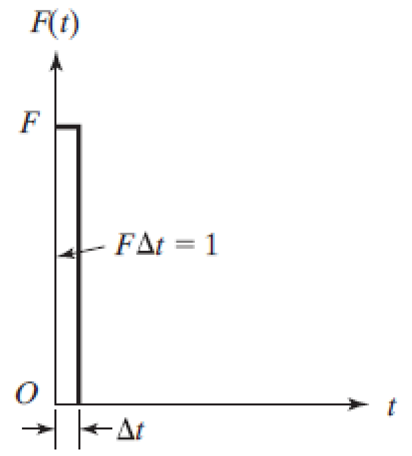
$$m\ddot{x}+c\dot{x}+kx=0$$

- And its solution:

$$x(t)=e^{-\zeta\omega_n t}\left\{x_0\cos(\omega_d t)+\frac{\dot{x}_0+\zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin(\omega_d t)\right\}$$

where

$$\zeta=\frac{c}{2m\omega_n} \qquad \omega_d=\omega_n\sqrt{1-\zeta^2}=\sqrt{\frac{k}{m}-\left(\frac{c}{2m}\right)^2} \qquad \omega_n=\sqrt{\frac{k}{m}}$$



- If, prior to the impulse load being applied, the mass is at rest, then:

$$x(t<0)=0 \;\; and \;\; \dot{x}(t<0)=0 \quad or \quad x(t=0^-)=0 \;\; and \;\; \dot{x}(t=0^-)=0$$

- The impulse-momentum equation gives:

$$\underset{\sim}{f}=1=m\dot{x}(t=0)-m\dot{x}(t=0^-)=m\dot{x}_0$$

- And the initial conditions are given by:

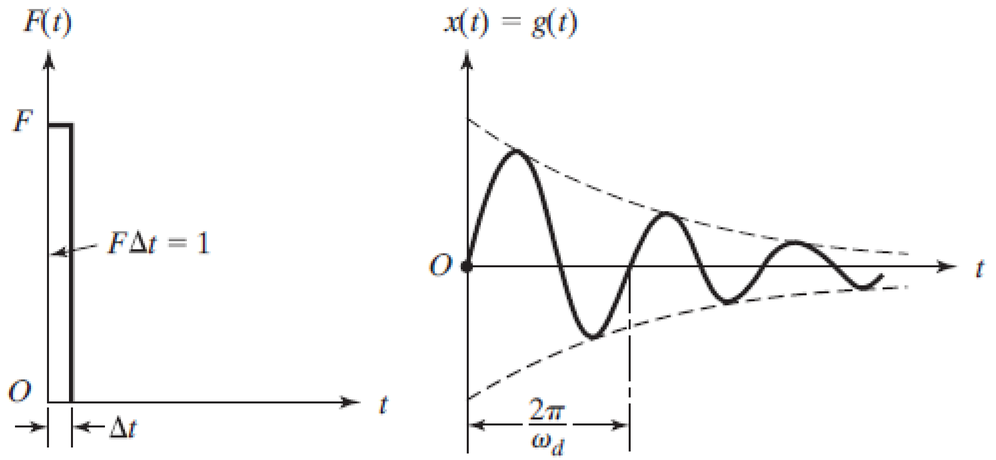
$$x(t=0)=x_0=0 \qquad and \qquad \dot{x}(t=0)=\dot{x}_0=\frac{1}{m}$$

SDOF systems – General forcing functions – Nonperiodic

- **Convolution integral – Impulse response**
- The solution reduces to:

$$x(t) = g(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d} \sin(\omega_d t)$$

- *g(t)* is the ***impulse response function*** an represents the response of a viscously damped single degree of freedom system subjected to a unit impulse.



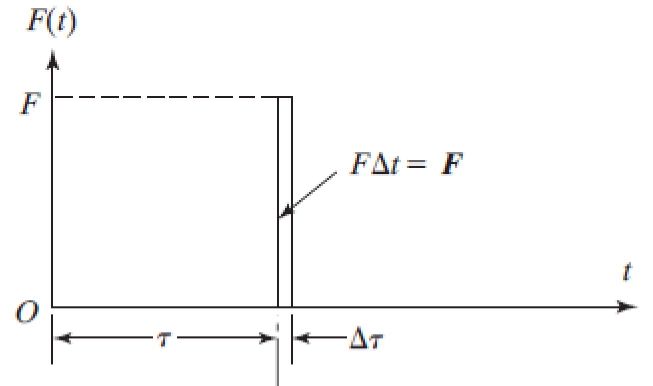
SDOF systems – General forcing functions – Nonperiodic

- **Convolution integral – Impulse response**
- If the magnitude of the impulse is $\underline{F}$ instead of unity, the initial velocity $x'_0 = F/m$ and the response becomes:

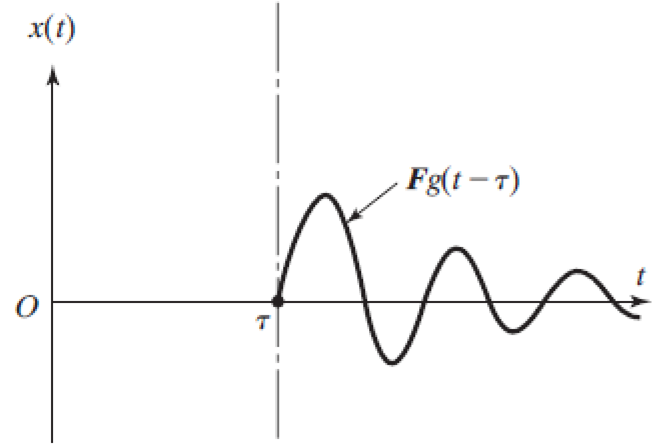
$$x(t) = \frac{\underset{\sim}{F} e^{-\zeta\omega_n t}}{m\omega_d} \sin(\omega_d t) = \underset{\sim}{F}\, g(t)$$

- If the impulse is applied to a stationary system at an arbitrary time $t = \tau$ the response is

$$x(t) = \underset{\sim}{F}\, g(t-\tau)$$



(a)



(b)

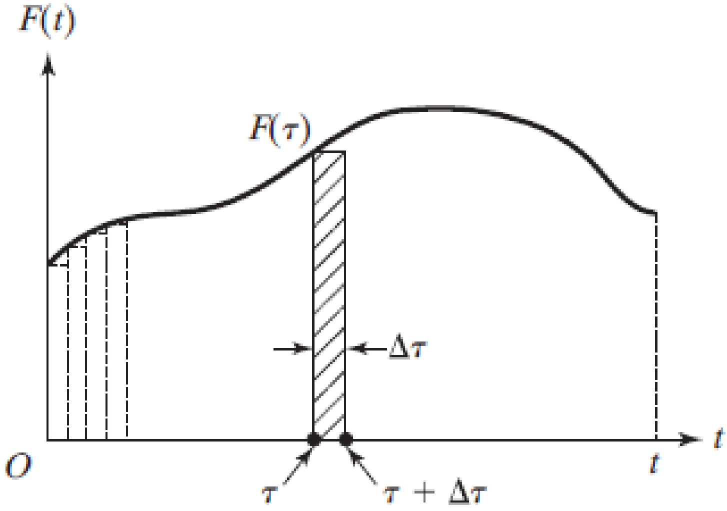
SDOF systems – General forcing functions – Nonperiodic

- **Convolution integral – Arbitrary exciting force**
- If we consider the arbitrary force to comprise of a series of impulses of varying magnitudes such that at time τ, the force *F(τ)* acts on the system for a short period $\Delta\tau$.
- The impulse acting at $t = \tau$ is given by $F(\tau)\Delta\tau$.
- At any time *t* the elapsed time is $t - \tau$
- The system response at *t* due to the impulse is

$$x(t) = \underset{\sim}{F}\, g(t-\tau) = F(\tau)\Delta\tau\, g(t-\tau)$$

- The total response at time *t* is determined by summing the responses caused by the impulses acting al all times τ:

$$x(t) = \sum F(\tau)\, g(t-\tau)\, \Delta\tau$$



Making $\Delta\tau \to 0$ the response can be expressed as :

$$x(t) = \int_0^t F(\tau)\, g(t-\tau)\, d\tau$$

Substituting the impulse response function $g(t-\tau)$:

$$x(t) = \frac{1}{m\omega_d}\int_0^t F(\tau) e^{-\zeta\omega_n(t-\tau)} \sin\left[\omega_d (t-\tau)\right] d\tau \leftarrow \text{Convolution or Duhamel integral}$$

- This solution does not account for initial conditions.
- Can be integrated explicitly or numerically depending on *F(t)*