Response to Nonperiodic Function

General Forcing Functions

SDOF systems – General forcing functions

• A non-periodic exciting force usually has a magnitude that varies with time; it acts for a specified period and then stops. The simplest form is the impulsive force, a force that has a large magnitude F and acts for a very short time

Methods for determining the response for General forcing functions are

- Representation of the excitation function with a *Convolution integral*
- Using *Laplace Transformations*
- Approximating F(t) with a suitable *interpolation method* then using a numerical procedure
- *Numerical integration* of the equations of motion.

- **Convolution integral**
- Consider one of the simplest nonperiodic exciting force: Impulsive force: which has a large magnitude F which acts for a very short time Δt . est nonperiodic exciting force: Impulsive force: which has a lar
 i a very short time Δt .
 *red by the resulting change in momentum of the system:
* $2 - m\dot{x}_1$ *

<i>he velocity of the lumped mass before and after the imp* Consider one of the simplest nonperiodic exciting force: Impulsi
magnitude F which acts for a very short time Δt .
An impulse can be measured by the resulting change in momen:
 $Im pulse = F \Delta t = m\dot{x}_2 - m\dot{x}_1$
where \dot{x}_1 and *Volution integral*
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Im pulse = F Δt *= mx*
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- An impulse can be measured by the resulting change in momentum of the system:

 $\dot{x}_2 - m\dot{x}$

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₁ and \dot{x}_2

The magnitude of the impulse $F\Delta t$ is represented by

$$
E = \int_{t}^{t+2t} F dt
$$

 $\mathbf{r} = \int_{t}^{\infty} \mathbf{r} \, dt$
and a unit impulse is def.

The magnitude of the impulse *F*_{Δt} is represented by\n
$$
E = \int_{t}^{t + \Delta t} F \, dt
$$
\nand a unit impulse is defined as\n
$$
t + \Delta t
$$
\n
$$
f = \lim_{\Delta t \to 0} \int_{t}^{t + \Delta t} F \, dt = F dt = 1
$$
\nFor *F dt* to have a finite value, *F* approaches infinity as Δt nears zero.

- **Convolution integral – Impulse response**
- Consider a (viscously) damped SDOF (mass-spring-damper system) subjected to an
impulse at *t*=0.
• For an underdamped system, the eqn. of motion is:
• $m\ddot{x} + c\dot{x} + kx = 0$
And its solution:
 $x(t) = e^{-\zeta \omega_n t} \left\{ x_0 \cos(\omega_d t)$ impulse at *t=0.*
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And its solution:
 $x(t) = e^{-\zeta \omega_n t} \left\{ x_0 \cos (\omega_d t) + \frac{\dot{x}_0 + \zeta \omega_n x_0}{\sqrt{1 - \zeta^2} \omega_n} \sin (\omega_d t) \right\}$
where

 $\leftarrow \Delta t$

where

$$
x(t) = e^{-\zeta \omega_n t} \left\{ x_0 \cos (\omega_d t) + \frac{\dot{x}_0 + \zeta \omega_n x_0}{\sqrt{1 - \zeta^2} \omega_n} \sin (\omega_d t) \right\}
$$

\n
$$
\zeta = \frac{c}{2m\omega_n} \qquad \omega_d = \omega_n \sqrt{1 - \zeta^2} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \qquad \omega_n = \sqrt{\frac{k}{m}}
$$

\n
$$
x(t < 0) = 0 \text{ and } \dot{x}(t < 0) = 0 \qquad \text{or} \qquad x(t = 0^-) = 0 \text{ and } \dot{x}(t = 0^-) = 0
$$

\n
$$
\text{impulse-momentum equation gives:} \qquad \text{Equation (1) for } t = 0 \text{ and } t = 0 \text{ and } t = 1 \text{ and } t = 1 \text{ and } t = 1 \text{ and } t = 0 \text{ and } t = 0
$$

• If, prior to the impulse load being applied, the mass is at rest, then:

$$
x(t<0)=0
$$
 and $\dot{x}(t<0)=0$ or $x(t=0^-)=0$ and $\dot{x}(t=0^-)=0$

• The impulse-momentum equation gives:

$$
t < 0 = 0 \text{ and } x(t < 0) = 0 \text{ or } x(t = 0)
$$

multiple-momentum equation gives:

$$
f(t = 0) - m\dot{x}(t = 0) = m\dot{x}
$$

And the initial conditions are given by:

$$
f = 1 = m\dot{x}(t = 0) - m\dot{x}(t = 0^{-}) = m\dot{x}_0
$$

e initial conditions are given by:

$$
x(t = 0) = x_0 = 0 \qquad and \qquad \dot{x}(t = 0) = \dot{x}_0 = \frac{1}{m}
$$

- **Convolution integral – Impulse response**
- The solution reduces to:

$$
x(t) = g(t) = \frac{e^{-\zeta \omega_n t}}{m\omega_d} \sin(\omega_d t)
$$

• *g(t)* is the *impulse response function* an represents the response of a viscously damped single degree of freedom system subjected to a unit impulse.

• **Convolution integral – Impulse response**

If the magnitude of the impulse is *instead of unity, the*

initial velocity
$$
x'_0 = F/m
$$
 and the response becomes:
\n
$$
x(t) = \frac{Fe^{-\zeta \omega_n t}}{m\omega_d} \sin(\omega_d t) = E g(t)
$$

• If the impulse is applied to a stationary system at an arbitrary time $t = \tau$ the response is

$$
x(t) = F g(t-\tau)
$$

• **Convolution integral – Arbitrary exciting force**

- If we consider the arbitrary force to comprise of a series of impulses of varying magnitudes such that at time τ , the force $F(\tau)$ acts on the system for a short period $\Delta \tau$.
- The impulse acting at $t = \tau$ is given by $F(\tau) \Delta \tau$.
- At any time *t* the elapsed time is $t \tau$
- The system response at *t* due to the impulse is

 $x(t) = F g(t-\tau) = F(\tau) \Delta \tau g(t-\tau)$

• The total response at time *t* is determined by summing the responses caused by the impulses acting al all times τ : $x(t) = E g(t-\tau) = F(\tau) \Delta \tau g(t-\tau)$
The total response at time *t* is determined by
summing the responses caused by the impulses
acting al all times τ :
 $x(t) = \sum F(\tau) g(t-\tau) \Delta \tau$
Making $\Delta \tau \rightarrow 0$ the response can be exp ressed as : *x*(*t*) = *E g*(*t* - *τ*) = *F*(*t*)
tal response at time *t* is
ing the responses cause
al all times *τ*:
 $x(t) = \sum F(\tau) g(t - \tau)$
 $x(t) = \sum F(\tau) g(t - \tau)$ *x*(*t*) = $\sum_{t=1}^{T} F(\tau) g(t-\tau)$
 x(*t*) = $\sum_{t=1}^{T} F(\tau) g(t-\tau)$
 x(*t*) = $\int_{t=1}^{T} F(\tau) g(t-\tau) d\tau$ ise at *t* due to the impulse is
 $t-\tau$) = $F(\tau)\Delta\tau g(t-\tau)$

at time *t* is determined by

pnses caused by the impulses
 τ :
 τ) $g(t-\tau)\Delta\tau$

response can be exp ressed as : bonses caused by the impulses
 τ :
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 τ :
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 τ :
 τ is $g(t-\tau) \Delta \tau$
 τ is $g(t-\tau) \Delta \tau$ in response at *t* due to the impulse is
 $= E g(t-\tau) = F(\tau) \Delta \tau g(t-\tau)$

sponse at time *t* is determined by

ne responses caused by the impulses

times τ :
 $= \sum F(\tau) g(t-\tau) \Delta \tau$
 $\rightarrow 0$ the response can be exp ressed as : sponse at time t is determined by

times τ :
 $=\sum F(\tau) g(t-\tau) \Delta \tau$
 $\rightarrow 0$ the response can be expressed as :
 t
 $=\int_{0}^{t} F(\tau) g(t-\tau) d\tau$ $\sum\limits_{ }^{ }$

 $\varDelta\tau\rightarrow0$

$$
x(t) = \sum F(\tau) g(t - \tau) \Delta \tau
$$

\nMaking $\Delta \tau \rightarrow 0$ the response can be expressed as :
\n
$$
x(t) = \int_{0}^{t} F(\tau) g(t - \tau) d\tau
$$

\n
$$
Substituting the impulse response function g(t - \tau):
$$
\n
$$
x(t) = \frac{1}{m\omega_d} \int_{0}^{t} F(\tau) e^{-\zeta \omega_n (t - \tau)} \sin \left[\omega_d (t - \tau)\right] d\tau
$$

 τ $\overline{}$

$$
x(t) = \int_{0}^{t} F(\tau) g(t-\tau) d\tau
$$

$$
x(t) = \int_{m\omega_d}^{t} F(\tau) g(t-\tau) d\tau
$$

$$
x(t) = \frac{1}{m\omega_d} \int_{0}^{t} F(\tau) e^{-\zeta \omega_n(t-\tau)} \sin \left[\omega_d (t-\tau)\right] d\tau \leftarrow Convolution or Duhamel integral
$$

- This solution does not account for initial conditions.
- Can be integrated explicitly or numerically depending on *F(t)*

