Response of SDOF System to Harmonic Loading

Response of SDOF System to Harmonic Loading

- External energy supplied to system as applied force or imposed motion (displacement, velocity or acceleration)
- This section deals only with ***harmonic excitation*** which results in ***harmonic response***
- Harmonic forcing function takes the form:

$$F(t) = F_0\, e^{i(\omega t+\phi)} \quad or \quad F(t) = F_0 \cos(\omega t + \phi) \quad or \quad F(t) = F_0 \sin(\omega t + \phi)$$

- Where F_0 is the amplitude, ω the frequency and ϕ the phase angle.

- The response of a linear system subjected to harmonic excitation is also harmonic.

- The response amplitude depends on the ratio of the excitation frequency to the natural frequency.

- Some “common” harmonic forcing functions are:
  - Rotating machine / element with (large) residual imbalance
  - Vehicle travelling on pavement corrugations or sinusoidal surfaces

Response of SDOF System to Harmonic Loading

- Equation of motion when a force is applied to a viscously damped SDOF system is:

$$m\ddot{x} + c\dot{x} + kx = F(t) \qquad \leftarrow \textit{non homogeneous differential eqn.}$$

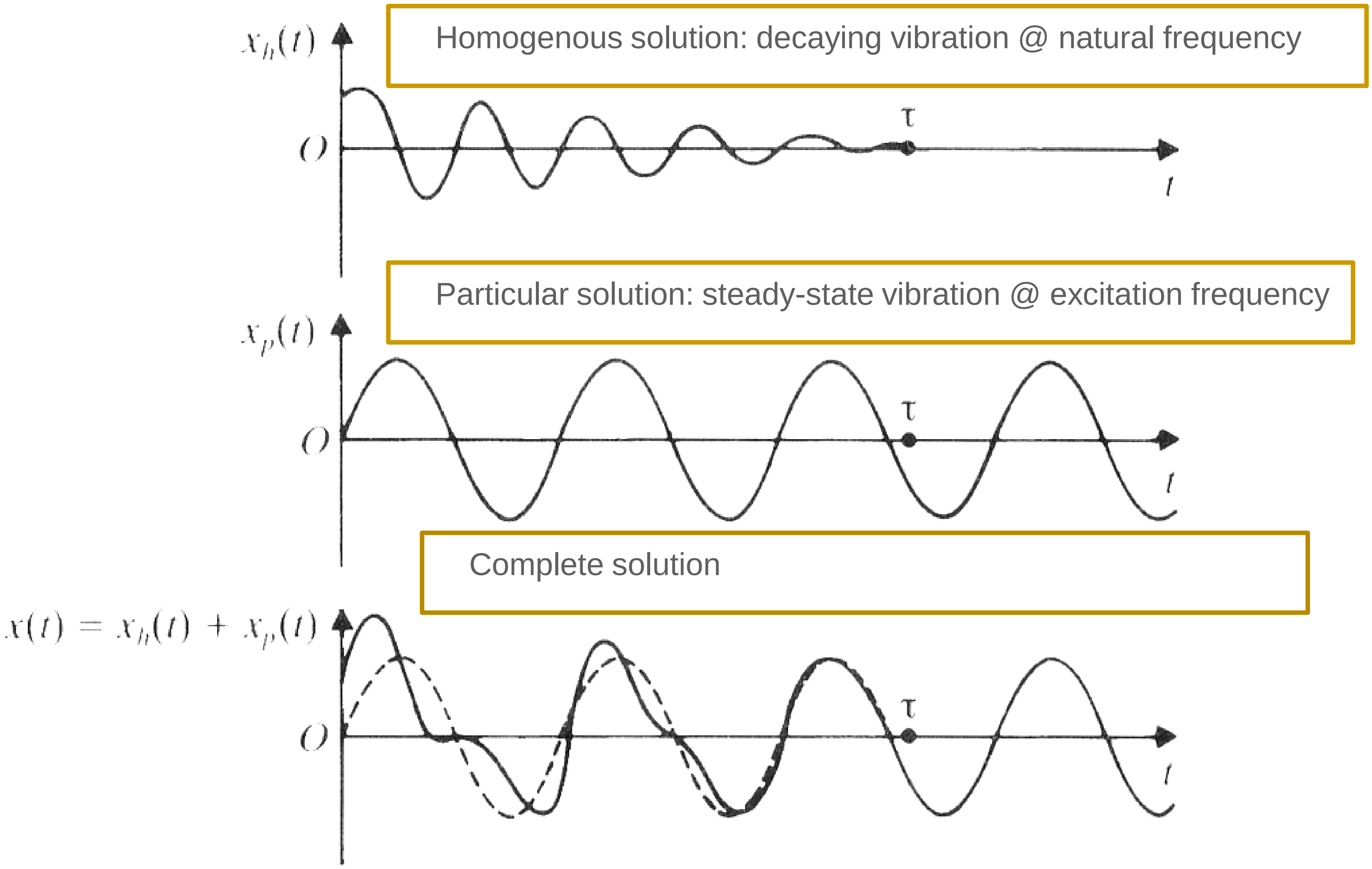
- The general solution to a nonhomogeneous DE is the sum if the homogeneous solution $x_h(t)$ and the particular solution $x_p(t)$.

- The homogeneous solution represents the solution to the free SDOF which is known to decay over time for all conditions (underdamped, critically damped and overdamped).

- The general solution therefore reduces to the particular solution $x_p(t)$ which represents the steady-state vibration which exists as long as the forcing function is applied.

Response of SDOF System to Harmonic Loading

- Example of solution to harmonically excited damped SDOF system:



Response of SDOF System to Harmonic Loading – undamped.

- Let the forcing function acting on the mass of an undamped SDOF system be:

$$F(t) = F_0 \cos(\omega t)$$

- The eqn. of motion reduces to:

$$m\ddot{x} + kx = F_0 \cos(\omega t)$$

- Where the homogeneous solution is:

$$x_h(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$$

$$\text{where } \omega_n = \sqrt{k/m}$$

- As the excitation is harmonic, the particular solution is also harmonic with the same frequency:

$$x_p(t) = X \cos(\omega t)$$

- Substituting $x_p(t)$ in the eqn. of motion and solving for X gives:

$$X = \frac{F_0}{k - m\omega^2}$$

- The complete solution becomes

$$x(t) = x_h(t) + x_p(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t) + \frac{F_0}{k - m\omega^2} \cos(\omega t)$$

Response of SDOF System to Harmonic Loading – undamped.

- Applying the initial conditions $x(t=0)=x_0 \quad and \quad \dot{x}(t=0)=\dot{x}_0$ gives:

$$C_1 = x_0 - \frac{F_0}{k-m\omega^2} \qquad and \qquad C_2 = \frac{\dot{x}_0}{\omega_n}$$

- The complete solution becomes:

$$x(t) = \left(x_0 - \frac{F_0}{k-m\omega^2} \right) cos(\omega_n t) + \left(\frac{\dot{x}_0}{\omega_n} \right) sin(\omega_n t) + \frac{F_0}{k-m\omega^2} cos(\omega t)$$

- The maximum amplitude of the steady-state solution can be written as:

$$\frac{X}{\delta_{st}} = \frac{1}{1-\left(\frac{\omega}{\omega_n} \right)^2} \qquad where\ \delta_{st} = \frac{F_0}{k}$$

- X/δ_{st} is the ratio of the dynamic to the static amplitude and is known as the ***amplification factor*** or ***amplification ratio*** and is dependent on the frequency ratio $r = \omega/\omega_n$.

Response of SDOF System to Harmonic Loading – undamped.

- When $\omega/\omega_n < 1$ the denominator of the steady-state amplitude is positive and the amplification factor increases as ω approaches the natural frequency ω_n. The response is ***in-phase*** with the excitation.

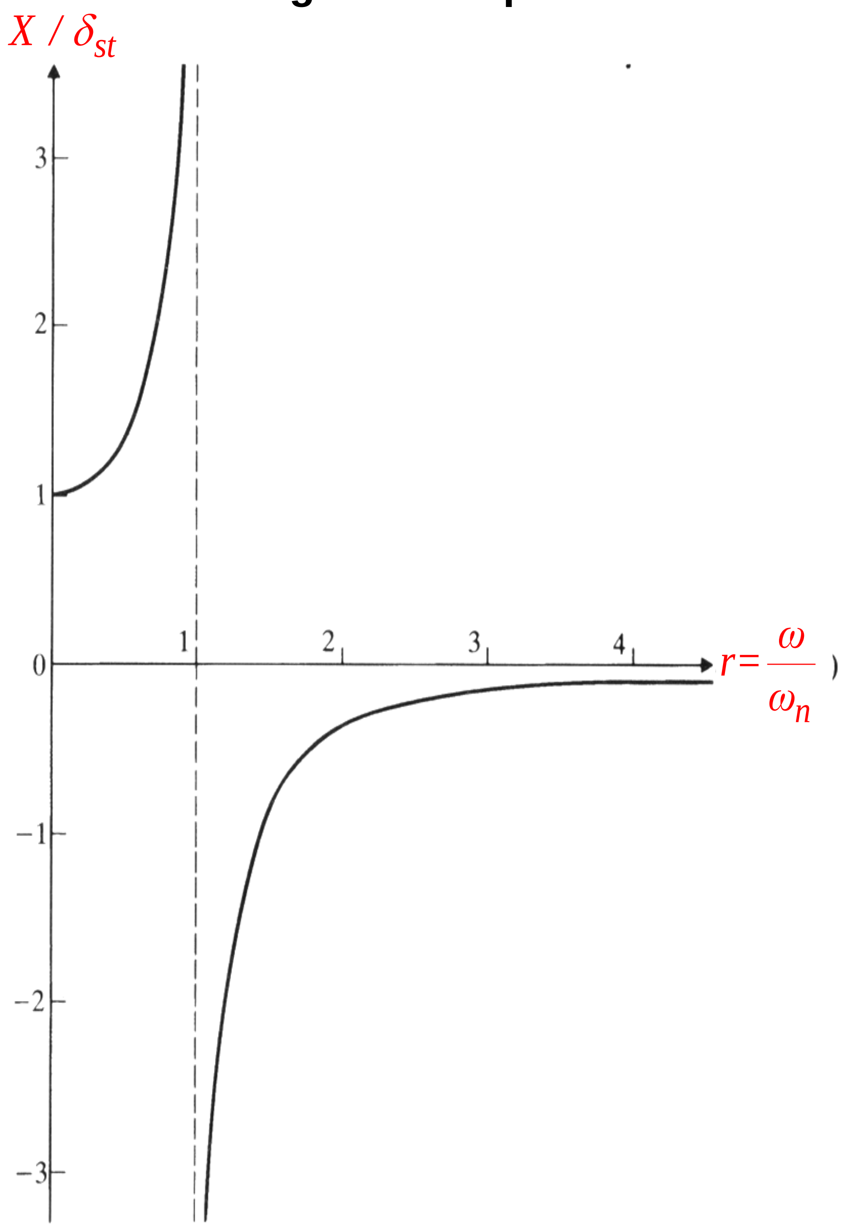
- When $\omega/\omega_n > 1$ the denominator of the steady-state amplitude is negative an the amplification factor is redefined as:

$$\frac{X}{\delta_{st}} = \frac{1}{\left(\dfrac{\omega}{\omega_n}\right)^2 - 1}$$

and the steady – state response becomes :

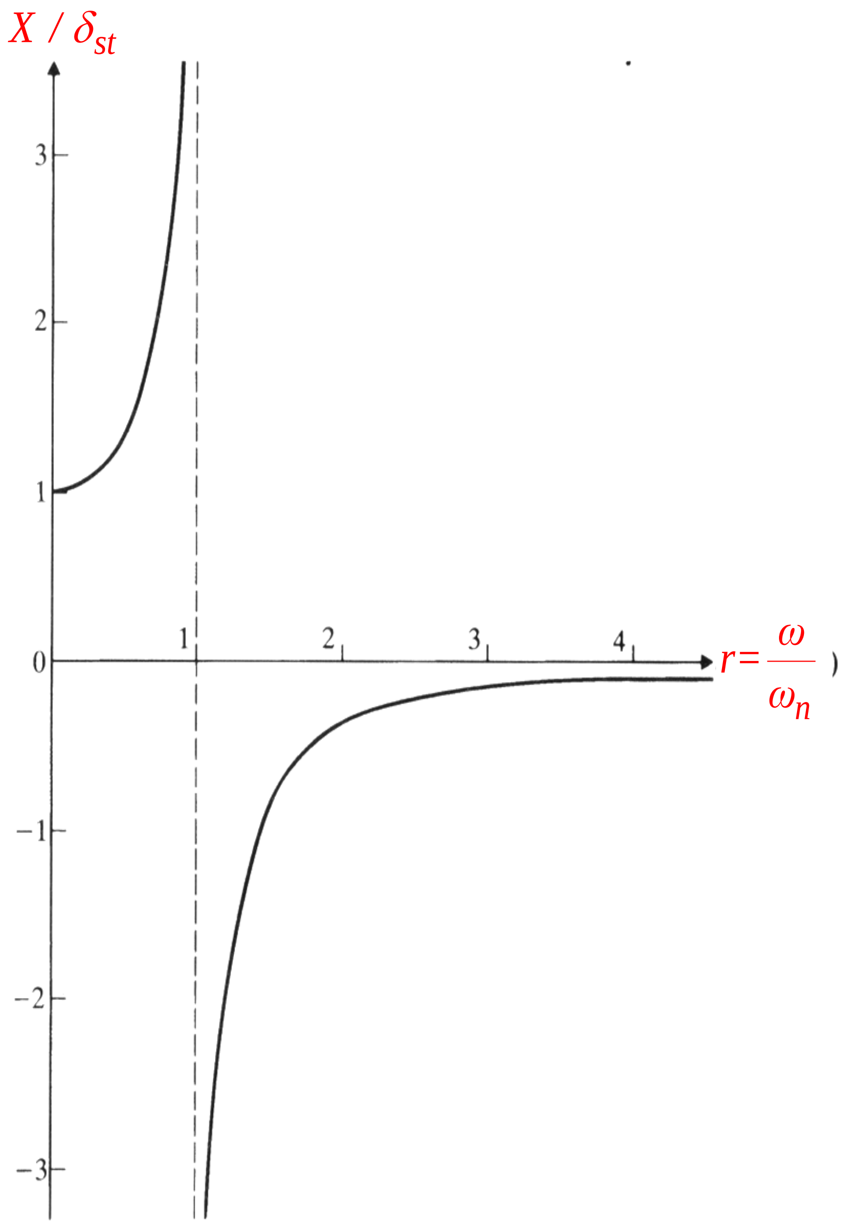
$$x_p(t) = -X\cos(\omega t)$$

which shows that the response is out-of-phase with the excitation and decreases ($\rightarrow$ zero) as ω increases ($\rightarrow \infty$)



Response of SDOF System to Harmonic Loading – undamped.

- When ω/ω_n = 1 the denominator of the steady-state amplitude is zero an the response becomes infinitely large. This condition when $\omega=\omega_n$ is known as resonance.



Response of SDOF System to Harmonic Loading – undamped.

- The complete solution

$$x(t)=\left(x_0-\frac{F_0}{k-m\omega^2}\right)cos(\omega_n t)\ +\ \left(\frac{\dot{x}_0}{\omega_n}\right)sin(\omega_n t)\ +\ \frac{F_0}{k-m\omega^2}cos(\omega t)$$

can be written as:

$$x(t)=Acos(\omega_n t+\phi)\ +\ \frac{\delta_{st}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\ cos(\omega t) \qquad for\ \omega/\omega_n<1$$

$$x(t)=Acos(\omega_n t+\phi)\ -\ \frac{\delta_{st}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\ cos(\omega t) \qquad for\ \omega/\omega_n>1$$

where A and ϕ are functions of x_0 and $\dot{x}_0$ as before.

- The complete solution is a sum of two cosines with frequencies corresponding to the natural and forcing (excitation) frequencies.

Response of SDOF System to Harmonic Loading – Damped.

- **Steady-state Solution**
- If the forcing function is harmonic:

$$F(t) = F_0 \cos(\omega t)$$

- The equation of motion of a SDOF system with viscous damping is:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t)$$

- The steady-state response is given by the particular solution which is also expected to be harmonic:

$$x_p(t) = X \cos(\omega t - \phi)$$

where the amplitude X and the phase angle ϕ are to be determined

Response of SDOF System to Harmonic Loading – Damped.

- Substituting x_p into the steady-state eqn. of motion yields:

$$X\left[\left(k-m\omega^2\right)cos(\omega t-\phi)-c\omega\, sin(\omega t-\phi)\right]=F_0\, cos(\omega t)$$

applying the trigonometric relationships :

$$cos(\omega t-\phi)=cos(\omega t)cos(\phi)+sin(\omega t)sin(\phi)$$

$$sin(\omega t-\phi)=sin(\omega t)cos(\phi)-cos(\omega t)sin(\phi)$$

we obtain :

$$X\left[\left(k-m\omega^2\right)cos(\phi)+c\omega\, sin(\phi)\right]=F_0$$

$$X\left[\left(k-m\omega^2\right)sin(\phi)-c\omega\, cos(\phi)\right]=0$$

which gives :

$$X=\frac{F_0}{\left[\left(k-m\omega^2\right)^2+(c\omega)^2\right]^{1/2}} \qquad and \qquad \phi=tan^{-1}\left(\frac{c\omega}{k-m\omega^2}\right)$$

for the particular solution

$$x_p(t)=X\, cos(\omega t-\phi)$$

Response of SDOF System to Harmonic Loading – Damped.

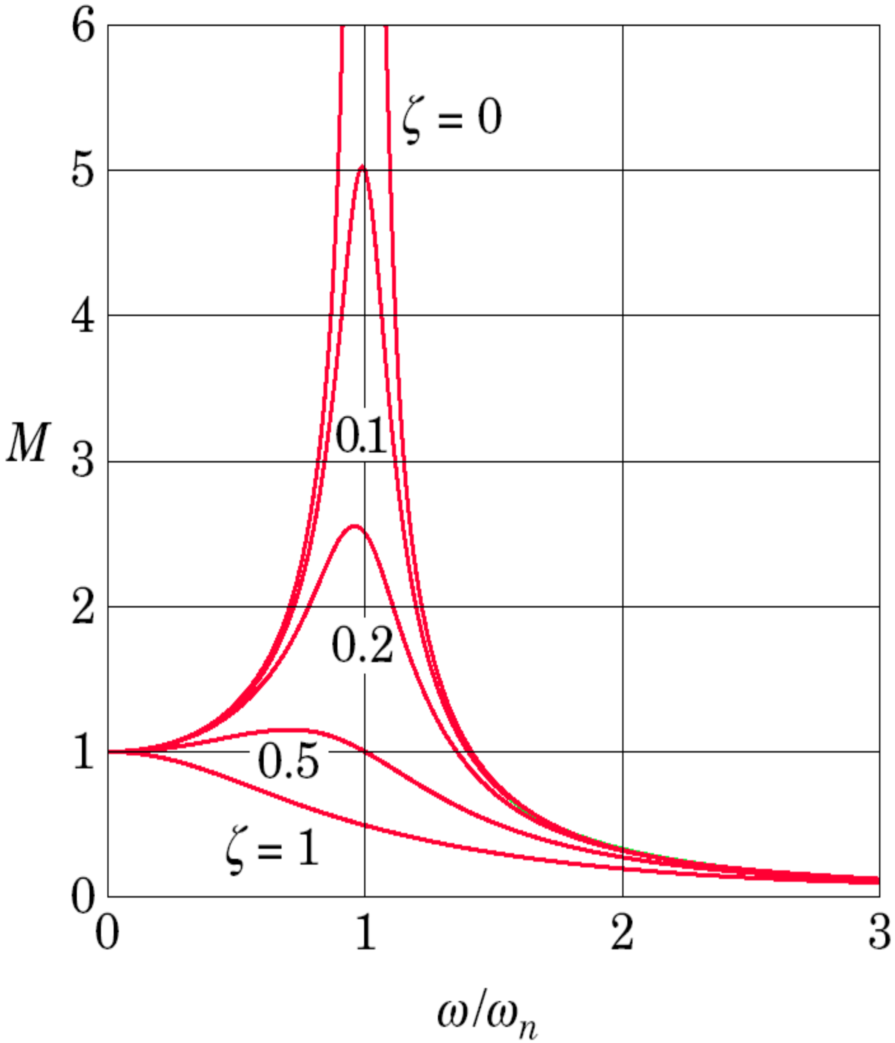
- Alternatively, the amplitude and phase can be written in terms of the frequency ratio $r = \omega/\omega_n$ and the damping coefficient ζ:

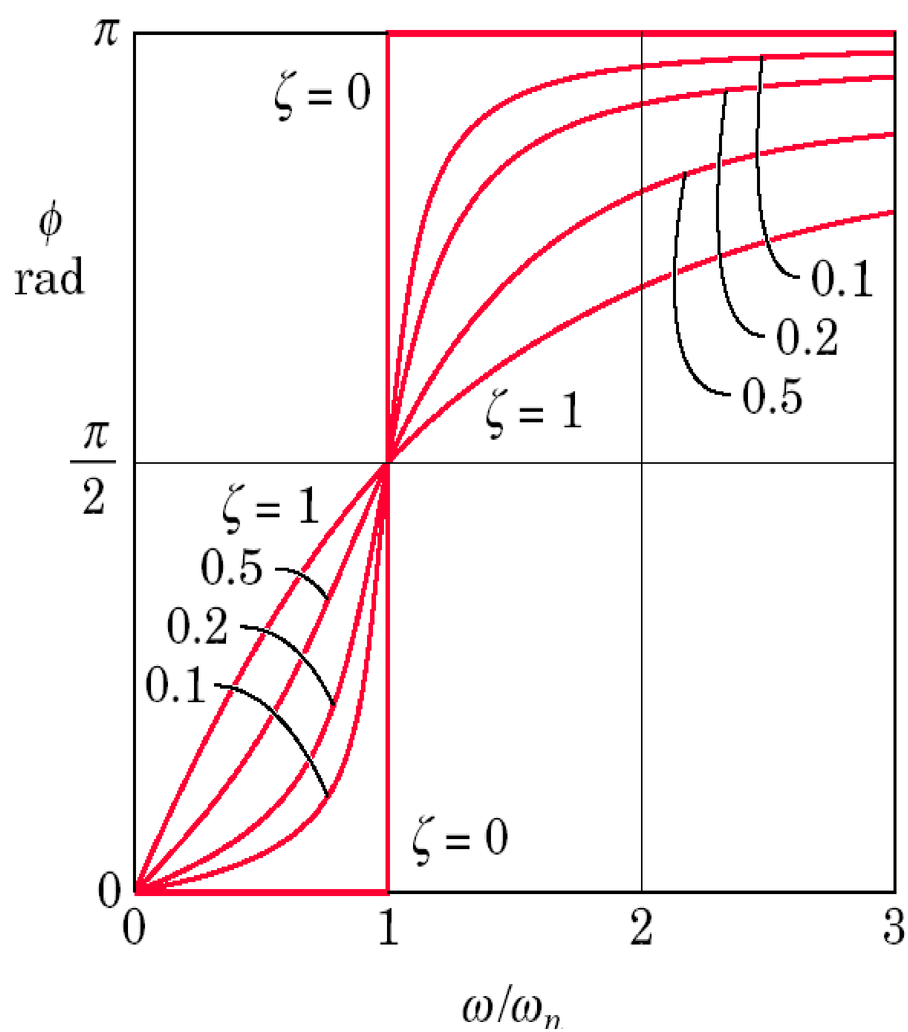
$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\left\{\left[1-r^2\right]^2 + \left[2\zeta r\right]^2\right\}^{1/2}}$$

$$\phi = a\tan\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right) = a\tan\left(\frac{2\zeta r}{1-r^2}\right)$$

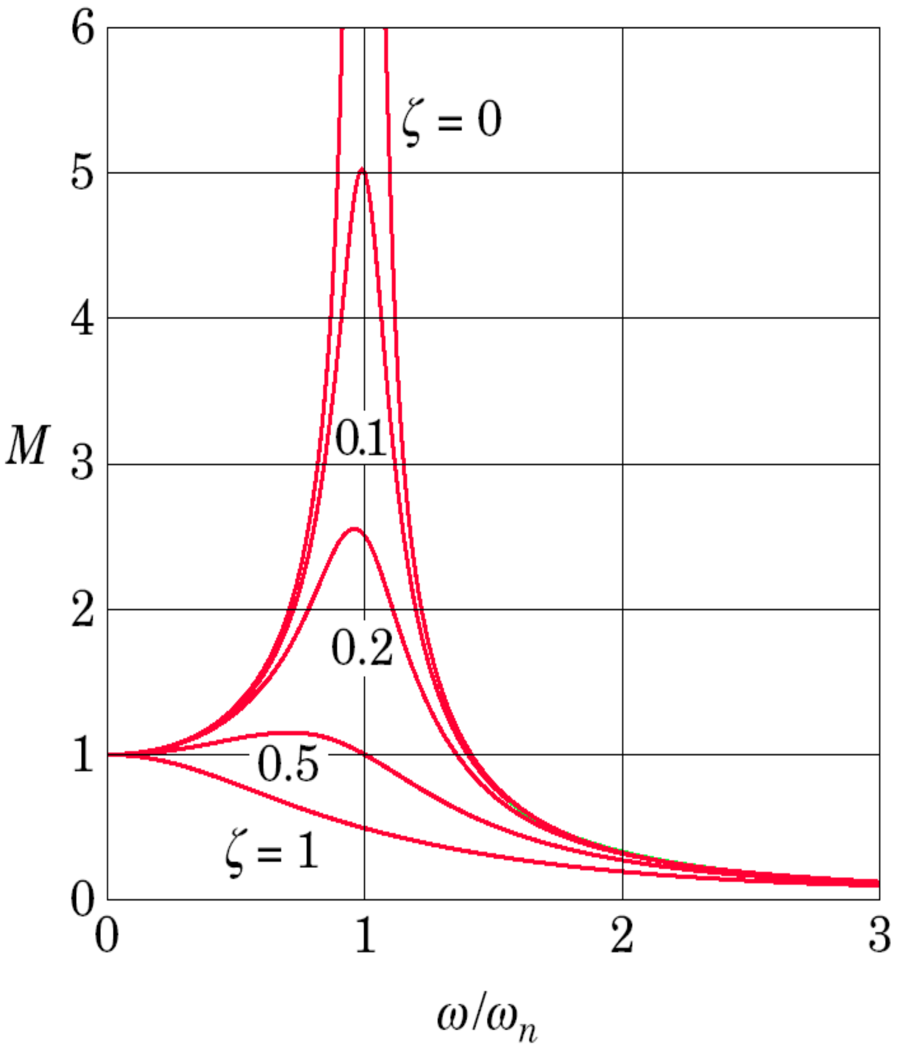
Response of SDOF System to Harmonic Loading – Damped.

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1-r^2\right]^2 + \left[2\zeta r\right]^2\right\}^{1/2}} \qquad \phi = a\tan\left(\frac{2\zeta r}{1-r^2}\right)$$



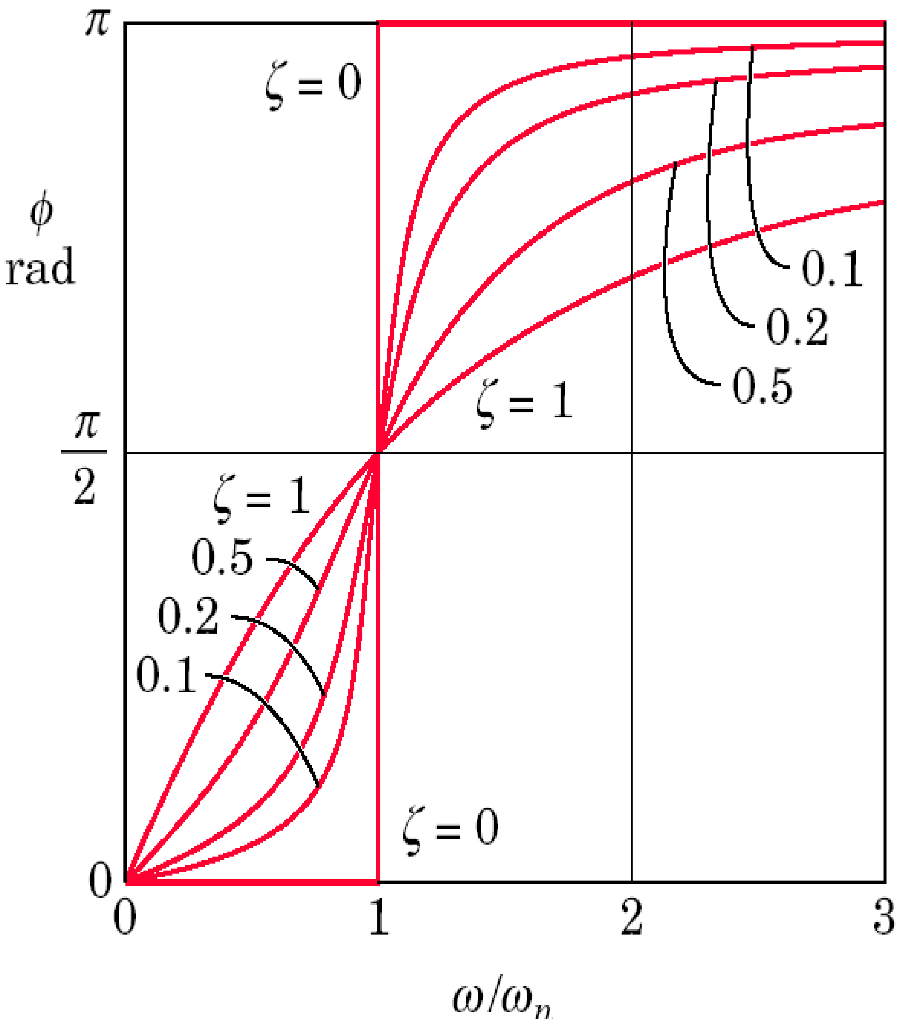


Response of SDOF System to Harmonic Loading – Damped.

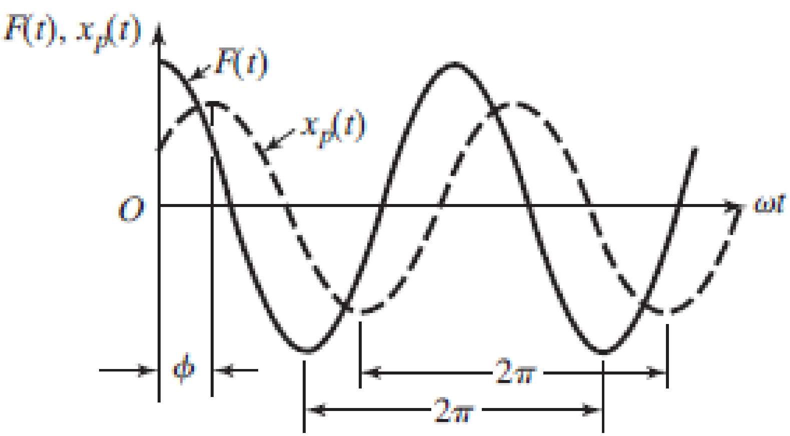


- The magnification ratio at all frequencies is reduced with increased damping.
- The effect of damping on the magnification ratio is greatest at or near resonance.
- The magnification ratio approaches 1 as the frequency ratio approaches 0.
- The magnification ratio approaches 0 as the frequency ratio approaches ∞

Response of SDOF System to Harmonic Loading – Damped.



- For undamped systems (*ζ = 0)* the phase angle is 0º (response in phase with excitation) for r<1 and 180º (response out of phase with excitation) for r>1.
- For damped systems (*ζ > 0)* when *r < 1* the phase angle is less than 90º and response lags the excitation and when *r >1* the phase angle is greater than 90º and the response leads the excitation (approaches 180º for large frequency ratios..
- For damped systems (*ζ > 0)* when *r =1* the phase lag is always 90º.



Response of SDOF System to Harmonic Loading – Damped.

- **Complete Solution**
- The complete solution is the sum of the homogeneous solution $x_h(t)$ and the particular solution $x_p(t)$:

$$x(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0) + X \cos(\omega t - \phi)$$

where $\omega_d = \omega_n \sqrt{1-\zeta^2}$, X *and* ϕ *are given as before, and* X_0 *and* ϕ_0 *are determined from the initial conditions*