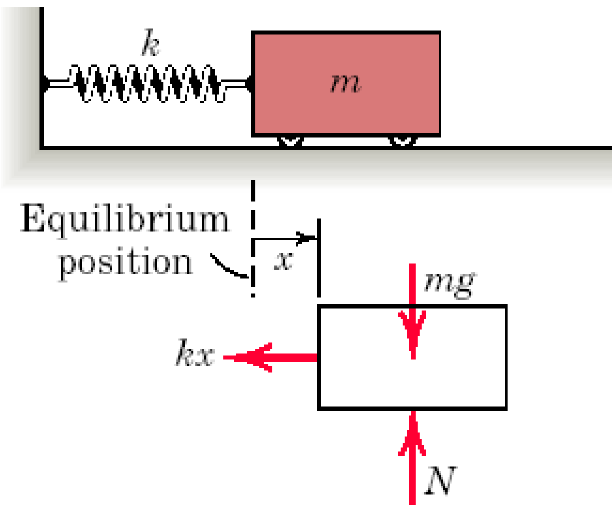
SINGLE DEGREE OF FREEDOM (SDOF) SYSTEM

Free undamped vibration single DoF

- Recall: Free vibrations → system given initial disturbance and oscillates free of external forces.
- Undamped: no decay of vibration amplitude
- Single DoF:
    - mass treated as rigid
    - Elasticity idealized by single spring
    - only one natural frequency.



- The equation of motion can be derived using
    - Newton's second law of motion
    - D'Alembert's Principle,
    - The principle of virtual displacements and,
    - The principle of conservation of energy.

Free undamped vibration single DoF

- Using Newton's second law of motion to develop the **equation of motion.**

1. Select suitable coordinates
2. Establish (static) equilibrium position
3. Draw free-body-diagram of mass
4. Use FBD to apply Newton's second law of motion:

"*Rate of change of momentum = applied force*"

$$F(t)=\frac{d}{dt}\left(m\frac{dx(t)}{dt}\right)$$

As m is constant

$$F(t)=m\frac{d^2x(t)}{dt^2}=m\ddot{x}$$

For rotational motion

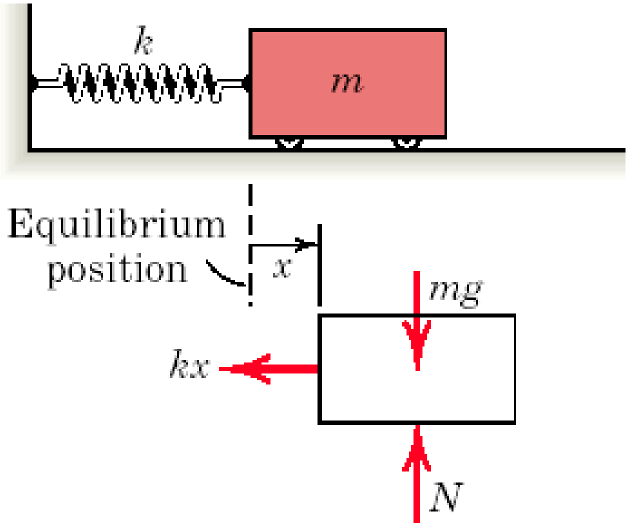
$$M(t)=J\ddot{\theta}$$

For the free, undamped single DoF system

$$F(t)=-kx=m\ddot{x}$$

or

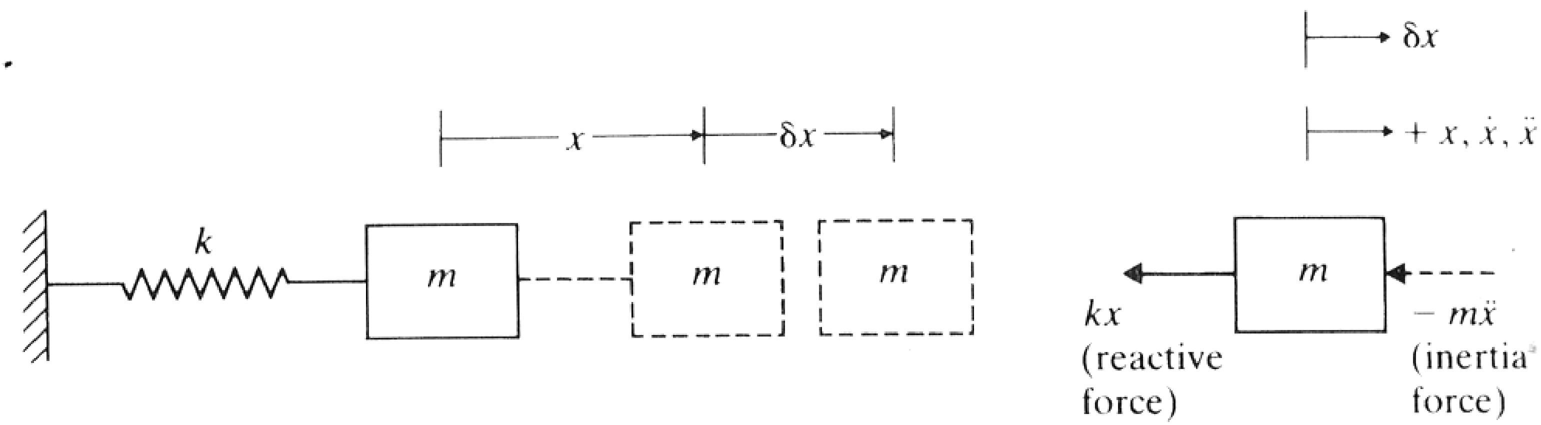
$$m\ddot{x}+kx=0$$



Free undamped vibration single DoF

Principle of virtual displacements:

- "When a system in equilibrium under the influence of forces is given a virtual displacement. The total work done by the virtual forces = 0"
- Displacement is imaginary, infinitesimal, instantaneous and compatible with the system



(a) Mass under a displacement x

(b) Free-body diagram

- When a virtual displacement *dx* is applied, the sum of work done by the spring force and the inertia force are set to zero:

$$-(kx)\delta x \; -(m\ddot{x})\delta x = 0$$

- Since $dx \neq 0$ the equation of motion is written as:

$$kx \; + m\ddot{x} = 0$$

Free undamped vibration single DoF

Principle of conservation of energy:

- No energy is lost due to friction or other energy-dissipating mechanisms.
- If no work is done by external forces, the system total energy = constant
- For mechanical vibratory systems:

$$KE + PE = constant$$

$$or$$

$$\frac{d}{dt}\left(KE + PE\right) = 0$$

- Since

$$KE = \frac{1}{2}m\dot{x}^2 \qquad and \qquad PE = \frac{1}{2}kx^2$$

$$then$$

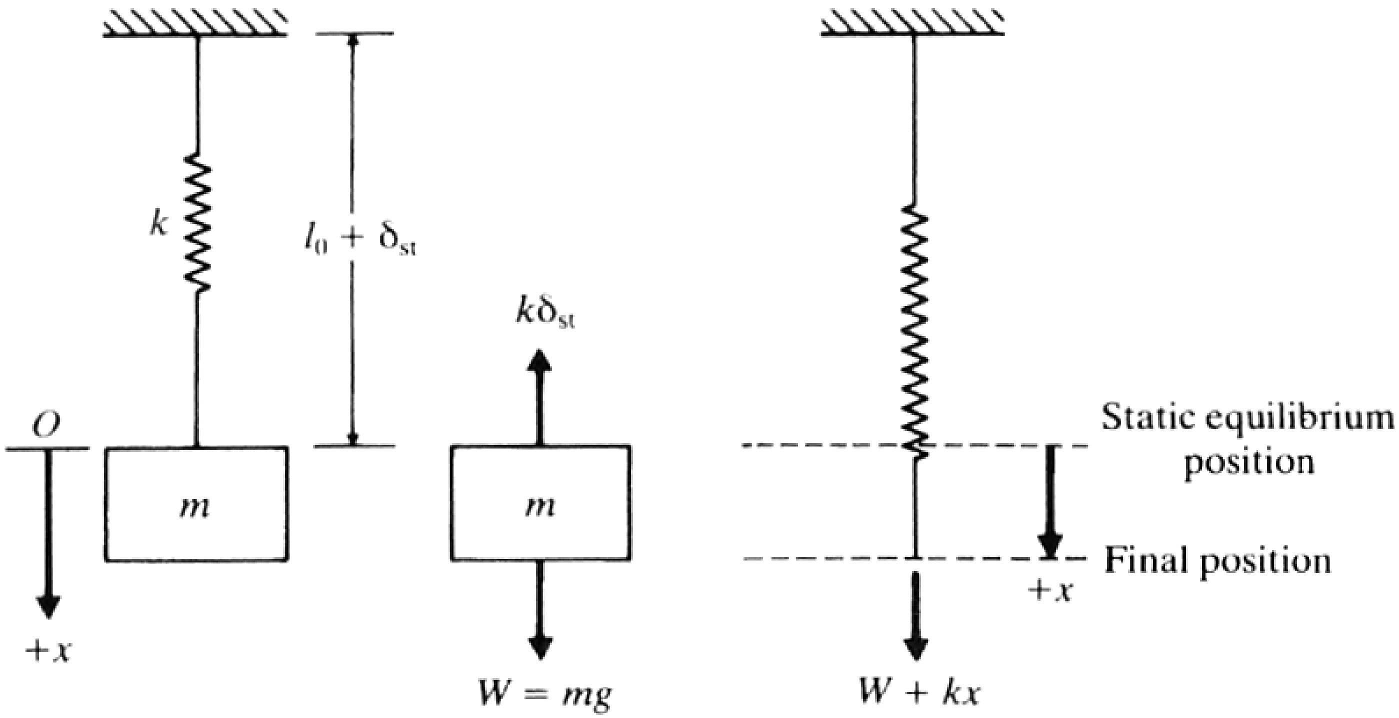
$$\frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$$

$$or$$

$$m\ddot{x} + kx = 0$$

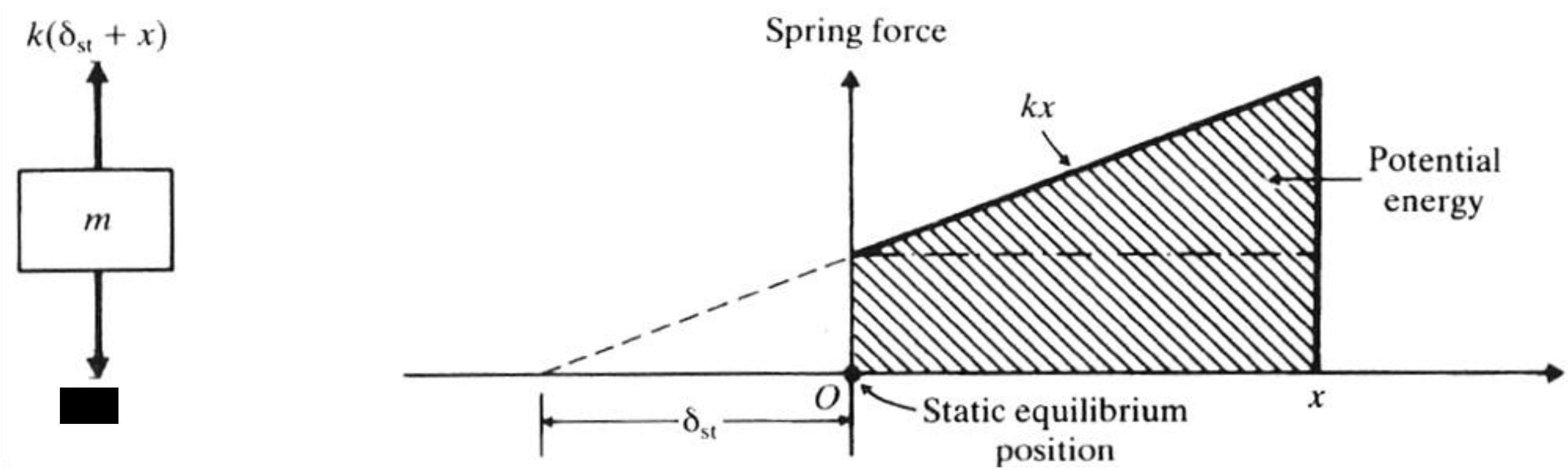
Free undamped vibration single DoF

Vertical mass-spring system:



Free undamped vibration single DoF

Vertical mass-spring system:



- From the free body diagram:, using Newton’s second law of motion:

$$m\ddot{x} = -k(x + \delta_{st}) + mg$$

$$since \quad k\delta_{st} = mg$$

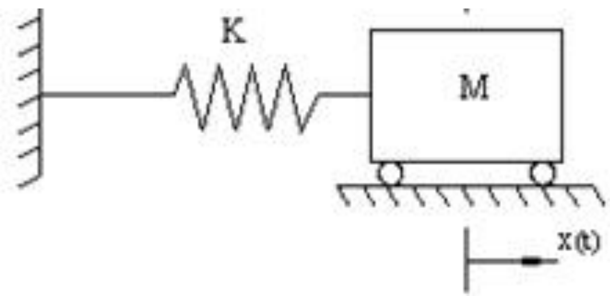
$$m\ddot{x} + kx = 0$$

- Note that this is the same as the eqn. of motion for the horizontal mass-spring system
- ∴ if x is measured from the static equilibrium position, gravity (weight) can be ignored
- This can be also derived by the other three alternative methods.

Free undamped vibration single DoF

- **The solution to the differential eqn. of motion.**
- As we anticipate oscillatory motion, we may propose a solution in the form:

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$$
$$or$$
$$x(t) = Ae^{i\omega_n t} + Be^{-i\omega_n t}$$
$$alternatively, if\ we\ let\ s = \pm i\omega_n$$
$$x(t) = Ce^{\pm st}$$



- By substituting for x(t) in the eqn. of motion:

$$C(ms^2 + k) = 0$$
$$since\quad c \neq 0,$$
$$ms^2 + k = 0 \qquad \leftarrow Characteristic\ equation$$
$$and$$
$$s = \pm i\omega_n = \pm\sqrt{\frac{k}{m}} \qquad \leftarrow roots = eigenvalues$$
$$or$$
$$\omega_n = \sqrt{\frac{k}{m}}$$

Free undamped vibration single DoF

- **The solution to the differential eqn. of motion.**
- Applying the initial conditions to the general solution: $x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$

$$x_{(t=0)} = A = x_0 \quad \textit{initial displacement}$$

$$\dot{x}_{(t=0)} = B\omega_n = \dot{x}_0 \quad \textit{initial velocity}$$

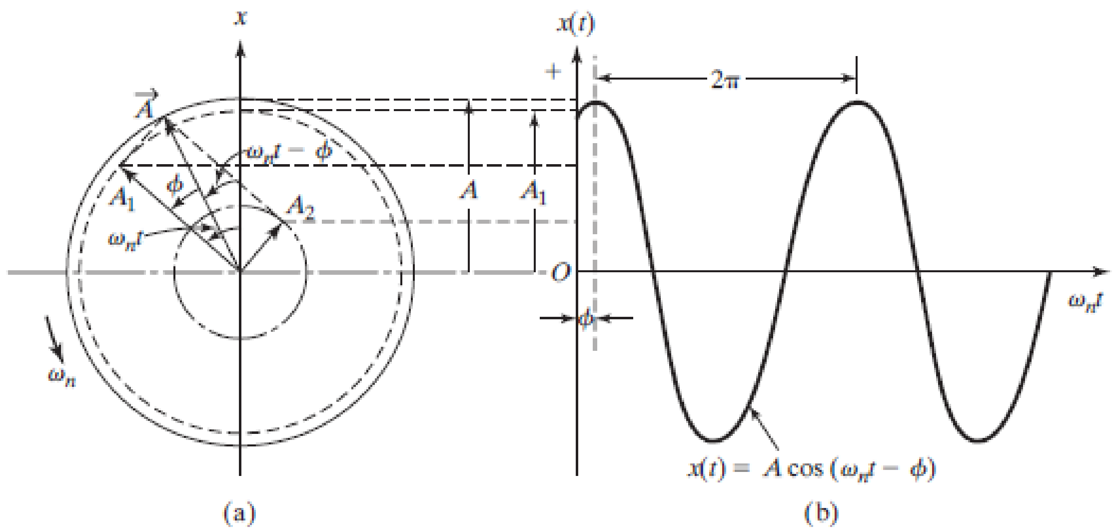
- The solution becomes:

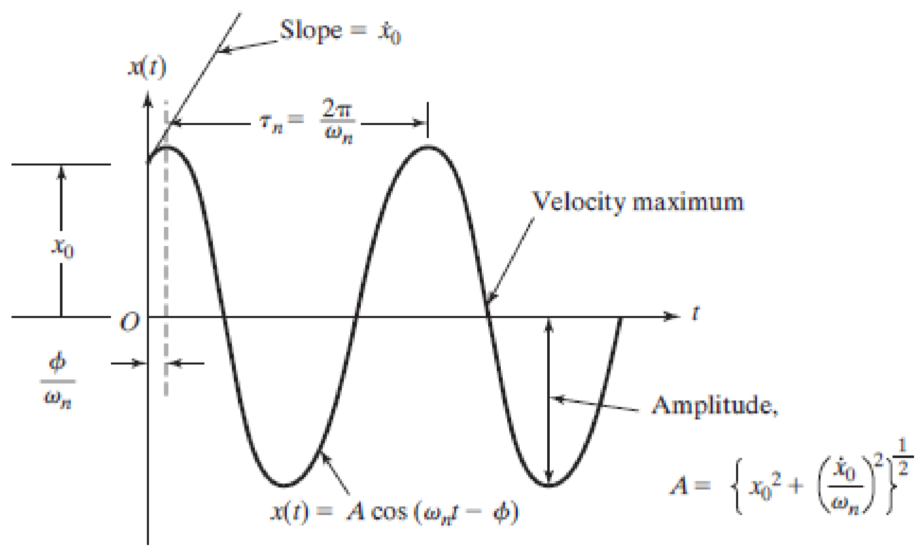
$$x(t) = x_0 \cos(\omega_n t) + \frac{\dot{x}_o}{\omega_n} \sin(\omega_n t)$$

$$\textit{if we let} \quad A_0 = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} \quad \textit{and} \quad \phi = a\tan\left(\frac{x_0 \omega_n}{\dot{x}_o} \right) \quad \textit{then}$$

$$x(t) = A_0 \sin(\omega_n t + \phi)$$

- This describes motion of harmonic oscillator:
  - Symmetric about equilibrium position
  - Thru equilibrium: velocity is maximum & acceleration is zero
  - At peaks and valleys, velocity is zero and acceleration is maximum
  - $\omega_n = \sqrt{(k/m)}$ is the natural frequency





FREE VIBRATION OF UNDAMPED SINGLE-DEGREE-OF-FREEDOM SYSTEMS

Free undamped vibration single DoF

- Note: for vertical systems, the natural frequency can be written as:

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\text{since} \quad k = \frac{mg}{\delta_{st}}$$

$$\omega_n = \sqrt{\frac{g}{\delta_{st}}} \quad or \quad f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$$

Free undamped vibration single DoF

- **Torsional vibration.**
- Approach same as for translational system. Laboratory exercise.

Free undamped vibration single DoF

- **Compound pendulum.**
- Given an initial angular displacement or velocity, system will oscillate due to gravitational acceleration.
- Assume rigid body → single DoF

Restoring torque:

$mgd \sin\theta$

$\therefore$ *Equation of motion:*

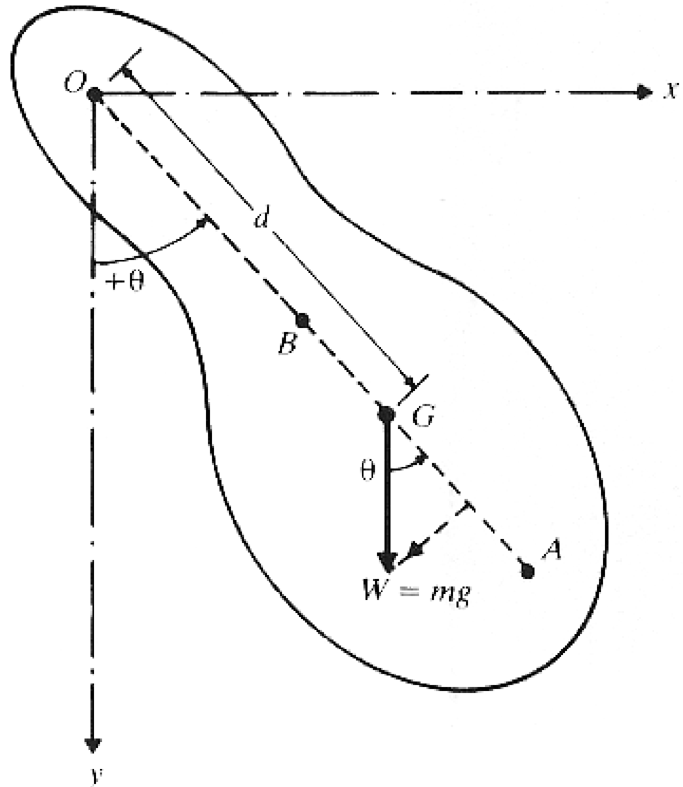
$J_o\ddot{\theta} + mgd \sin\theta = 0 \quad \leftarrow \text{nonlinear } 2^{nd} \text{ order ODE}$

Linearity is approximated if $\sin\theta \approx \theta$ *Therefore:*

$J_o\ddot{\theta} + mgd\theta = 0$

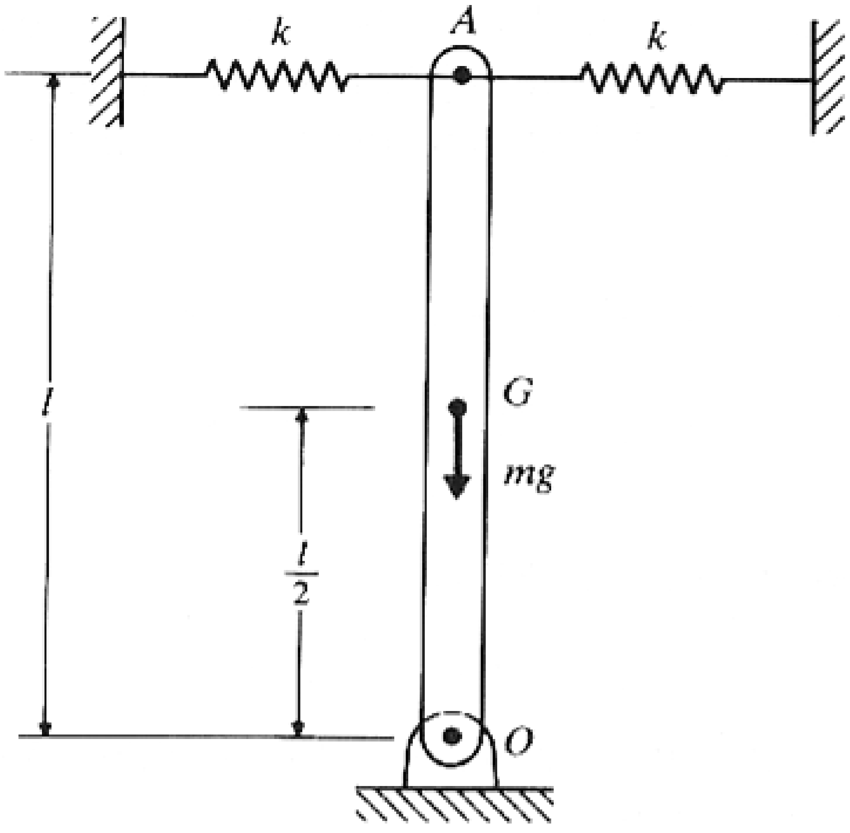
Natural frequency:

$$\omega_n = \sqrt{\frac{mgd}{J_o}}$$



Free undamped vibration single DoF

- **Stability.**
- Some systems may have inherent instability



Free undamped vibration single DoF

- **Stability.**
- Some systems may have inherent instability
- When the bar is deflected by θ,

The spring force is :

$$2kl\sin\theta$$

The gravitational force thru G is :

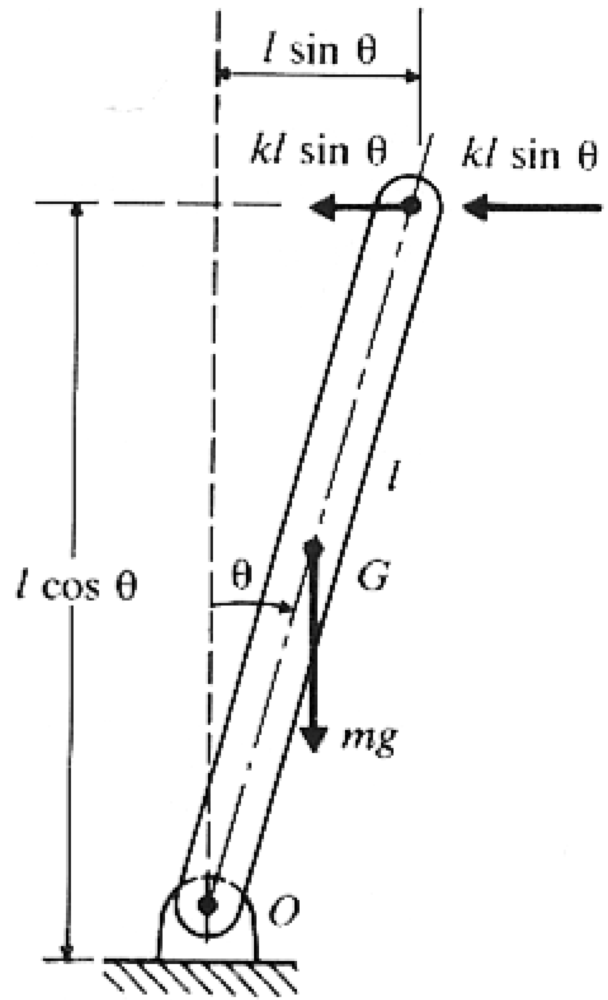
$$mg$$

The inertial moment about O due to the angular acceleration $\ddot{\theta}$ is :

$$J_o\ddot{\theta} = \frac{ml^2}{3}\ddot{\theta}$$

The eqn. of motion is written as :

$$\frac{ml^2}{3}\ddot{\theta} + (2kl\sin\theta)\,l\cos\theta - mg\frac{l}{2}\sin\theta = 0$$



Free undamped vibration single DoF

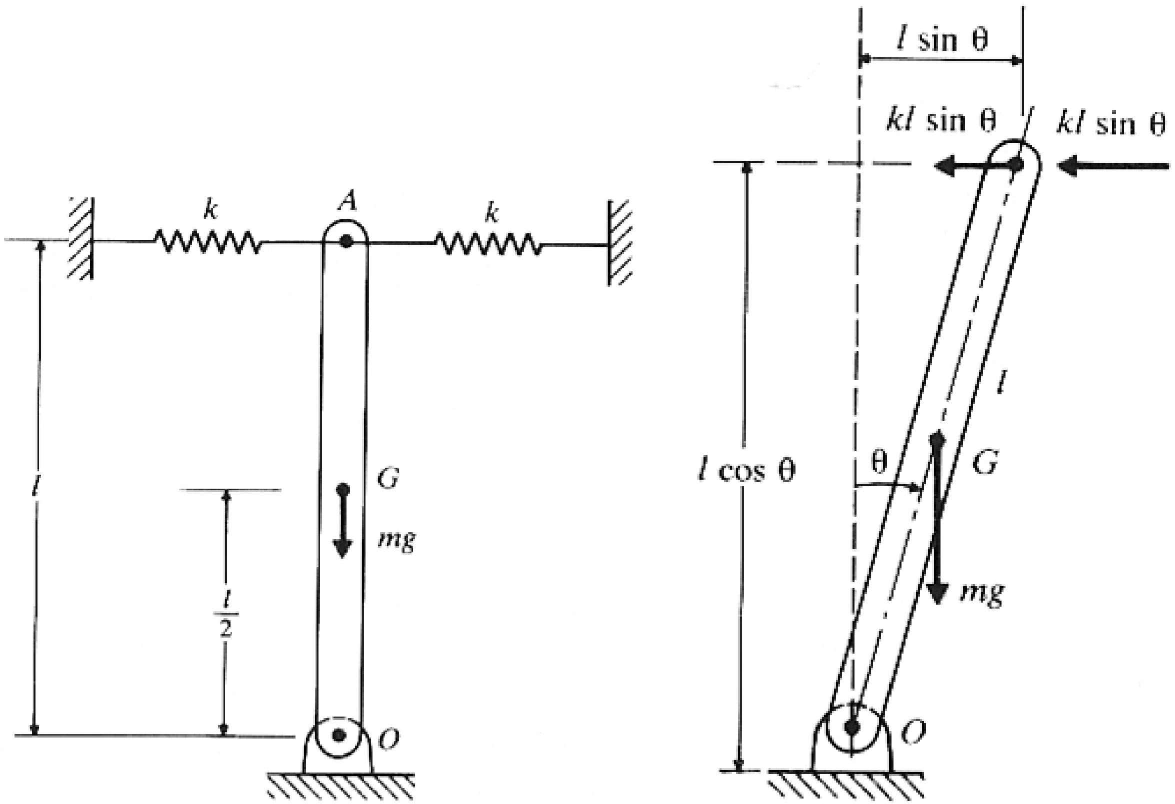
For small oscillations, $\sin\theta = \theta$ and $\cos\theta = 1$. *Therefore*

$$\frac{ml^2}{3}\theta + 2kl^2\theta - \frac{mgl}{2}\theta = 0$$

or

$$\ddot{\theta} + \left(\frac{12kl^2 - 3mgl}{2ml^2}\right)\theta = 0$$

$$\omega_{n=}\sqrt{\left(\frac{12kl^2 - 3mgl}{2ml^2}\right)}$$



Free single DoF vibration + viscous damping

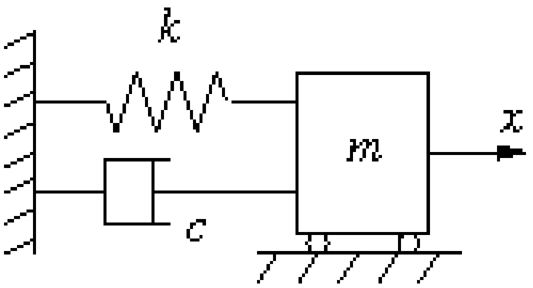
- Recall: viscous damping force ∝ velocity:

$$F = -c\dot{x} \qquad c = \text{damping constant or coefficient } [Ns/m]$$

Applying Newton's second law of motion to obtain the eqn. of motion:

$$m\ddot{x} = -c\dot{x} - kx \qquad or \qquad m\ddot{x} + c\dot{x} + kx = 0$$

If the solution is assumed to take the form:



$$x(t) = Ce^{st} \qquad where\ s = \pm i\omega_n$$

$$then: \quad \dot{x}(t) = sCe^{st} \quad and \quad \ddot{x}(t) = s^2Ce^{st}$$

Substituting for $x, \dot{x}$ and $\ddot{x}$ in the eqn. of motion

$$ms^2 + cs + k = 0$$

The root of the characteristic eqn. are:

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

The two solutions are:

$$x_1(t) = C_1e^{s_1t} \qquad and \qquad x_2(t) = C_2e^{s_2t}$$

Free single DoF vibration + viscous damping

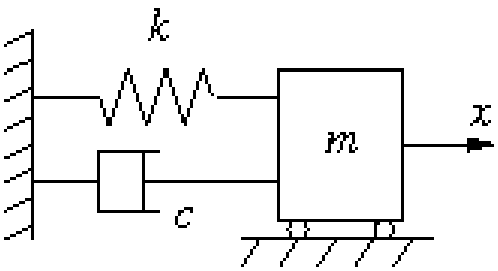
- The general solution to the Eqn. Of motion is:

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

or

$$x(t) = C_1 e^{\left\{-\frac{c}{2m}+\sqrt{\left(\frac{c}{2m}\right)^2-\left(\frac{k}{m}\right)}\right\}t} + C_2 e^{\left\{-\frac{c}{2m}-\sqrt{\left(\frac{c}{2m}\right)^2-\left(\frac{k}{m}\right)}\right\}t}$$

where C_1 and C_2 are arbitrary constants determined from the initial conditions.



Free single DoF vibration + viscous damping

- **Critical damping (c_c):** value of c for which the radical in the general solution is zero:

$$\left(\frac{c_c}{2m}\right)^2-\left(\frac{k}{m}\right)=0 \qquad or \qquad c_c=2m\sqrt{\frac{k}{m}} = 2m\omega_n = 2\sqrt{km}$$

- **Damping ratio (ζ):** damping coefficient : critical damping coefficient.

$$\zeta=\frac{c}{c_c} \qquad or \qquad \frac{c}{2m}=\frac{c}{c_c}\frac{c_c}{2m}=\zeta\omega_n$$

The roots can be re-written :

$$s_{1,2}=-\frac{c}{2m}\pm\sqrt{\left(\frac{c}{2m}\right)^2-\left(\frac{k}{m}\right)} = \left(-\zeta\pm\sqrt{\zeta^2-1}\right)\omega_n$$

And the solution becomes :

$$x(t)=C_1e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}+C_2e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

- The response x(t) depends on the roots s_1 and s_2 $\rightarrow$ the behaviour of the system is dependent on the damping ratio ζ**.**

Free single DoF vibration + viscous damping

$$x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

- When ζ <1, the system is underdamped. (ζ^2-1) is negative and the roots can be written as:

$$s_1 = \left(-\zeta + i\sqrt{1-\zeta^2}\right)\omega_n \quad \text{and} \quad s_2 = \left(-\zeta - i\sqrt{1-\zeta^2}\right)\omega_n$$

And the solution becomes :

$$x(t) = C_1 e^{\left(-\zeta + i\sqrt{1-\zeta^2}\right)\omega_n t} + C_2 e^{\left(-\zeta - i\sqrt{1-\zeta^2}\right)\omega_n t}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{ C_1 e^{\left(i\sqrt{1-\zeta^2}\right)\omega_n t} + C_2 e^{\left(-i\sqrt{1-\zeta^2}\right)\omega_n t} \right\}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{ (C_1 + C_2)\cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + i(C_1 - C_2)\sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right) \right\}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{ C_1' \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + C_2' \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right) \right\}$$

$$x(t) = X e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right) \quad \text{or} \quad x(t) = X_0 e^{-\zeta\omega_n t} \cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi_o\right)$$

Where C'_1, C'_2; X, ϕ and X_o, ϕ_o are arbitrary constant determined from initial conditions.

Free single DoF vibration + viscous damping

$$x(t) = e^{-\zeta\omega_n t}\left\{C_1' \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + C_2' \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

- For the initial conditions:

$$x(t=0) = x_0 \quad and \quad \dot{x}(t=0) = \dot{x}_0$$

Then

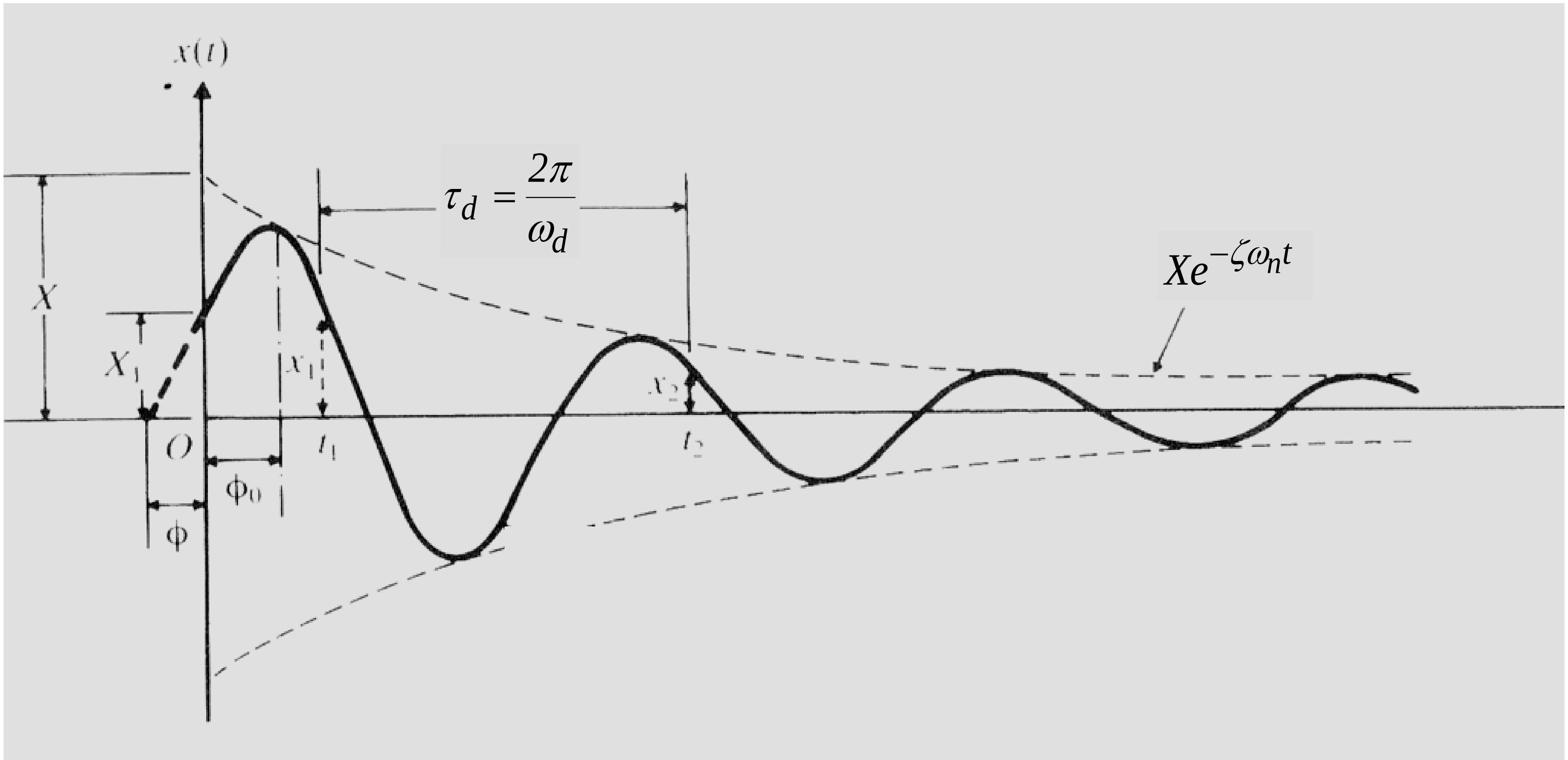
$$C_1' = x_0 \quad and \quad C_2' = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}$$

Therefore the solution becomes

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\left(\sqrt{1-\zeta^2}\,\omega_n t\right) + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)\right\}$$

- This represents a decaying (damped) harmonic motion with angular frequency $\sqrt{(1-\zeta^2)}\omega_n$ also known as the damped natural frequency. The factor $e^{-()}$ causes the exponential decay.

Free single DoF vibration + viscous damping



Exponentially decaying harmonic – free SDoF vibration with viscous damping .

Underdamped oscillatory motion and has important engineering applications.

Free single DoF vibration + viscous damping

$$x(t) = Xe^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right) \quad or \quad x(t) = X_0 e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi_o\right)$$

The constants (X, ϕ) *and* (X_0, ϕ_0) *representing the magnitude and phase become:*

$$X = X_0 = \sqrt{\left(C_1'\right)^2 + \left(C_2'\right)^2}$$

$$\phi = a\tan\left(\frac{C_1'}{C_2'}\right) \quad and \quad \phi_0 = a\tan\left(-\frac{C_2'}{C_1'}\right)$$

Free single DoF vibration + viscous damping

- When $\zeta = 1$, $c = c_c$, system is critically damped and the two roots to the eqn. of motion become:

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n$$

and solution is

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$

Applying the initial conditions $x(t=0) = x_0$ *and* $\dot{x}(t=0) = \dot{x}_0$ *yields*

$$C_1 = x_0$$

$$C_2 = \dot{x}_0 + \omega_n x_0$$

The solution becomes :

$$x(t) = \left[x_0 + \left(\dot{x}_0 + \omega_n x_0\right)t\right]e^{-\omega_n t}$$

- As $t \to \infty$, the exponential term diminished toward zero and depicts ***aperiodic*** motion

Free single DoF vibration + viscous damping

- When $\zeta > 1$, $c > c_c$, system is overdamped and the two roots to the eqn. of motion are real and negative:

$$s_1 = \left(-\zeta + \sqrt{\zeta^2 - 1} \right) \omega_n < 0$$

$$s_2 = \left(-\zeta - \sqrt{\zeta^2 - 1} \right) \omega_n < 0$$

with $s_2 \square s_1$ *and the initial conditions* $x(t=0) = x_0$ *and* $\dot{x}(t=0) = \dot{x}_0$

the solution becomes :

$$x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1} \right) \omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1} \right) \omega_n t}$$

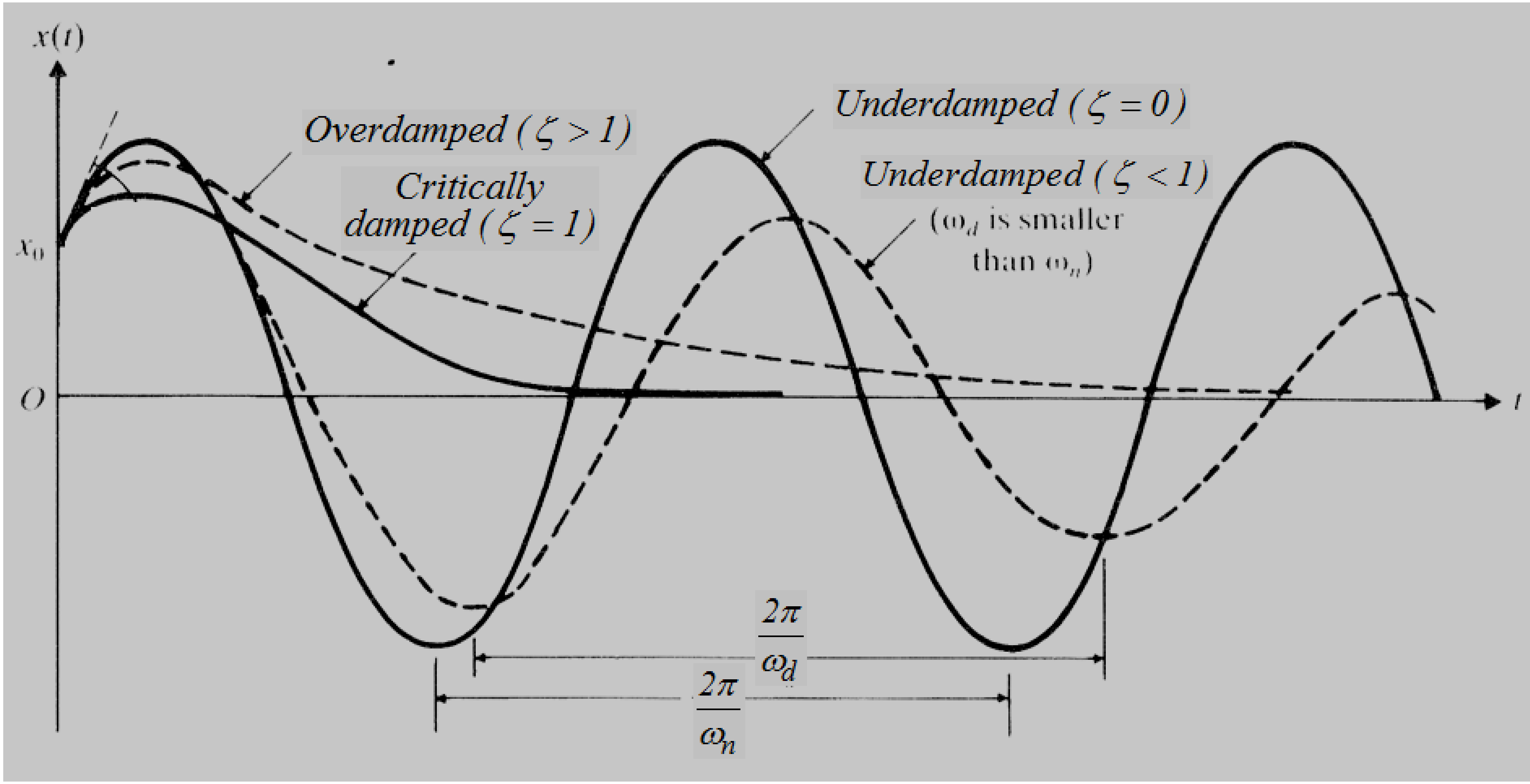
where

$$C_1 = \frac{x_0 \omega_n \left(-\zeta + \sqrt{\zeta^2 - 1} \right) + \dot{x}_0}{2 \omega_n \sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-x_0 \omega_n \left(-\zeta - \sqrt{\zeta^2 - 1} \right) - \dot{x}_0}{2 \omega_n \sqrt{\zeta^2 - 1}}$$

Which shows ***aperiodic*** motion which diminishes exponentially with time.

Free single DoF vibration + viscous damping



Critically damped systems have lowest required damping for aperiodic motion and mass returns to equilibrium position in shortest possible time.