### **SINGLE DEGREE OF FREEDOM (SDOF) SYSTEM**

- Recall: Free vibrations  $\rightarrow$  system given initial disturbance and oscillates free of external forces.
- Undamped: no decay of vibration amplitude
- Single DoF:
	- mass treated as rigid
	- Elasticity idealized by single spring
	- only one natural frequency.
- The equation of motion can be derived using
	- Newton's second law of motion
	- D'Alembert's Principle,
	- The principle of virtual displacements and,
	- The principle of conservation of energy.



- Using Newton's second law of motion to develop the **equation of motion.**
	- 1. Select suitable coordinates
	- 2. Establish (static) equilibrium position
	- 3. Draw free-body-diagram of mass
	- 4. Use FBD to apply Newton's second law of motion:

"*Rate of change of momentum = applied force"*

$$
F(t) = \frac{d}{dt} \left( m \frac{dx(t)}{dt} \right)
$$

As m is constant

$$
F(t) = m \frac{d^2x(t)}{dt^2} = m\ddot{x}
$$

For rotational motion

$$
M(t)=J\ddot{\theta}
$$

For the free, undamped single DoF system

$$
F(t) = -kx = m\ddot{x}
$$
  
or  

$$
m\ddot{x} + kx = 0
$$



#### **Principle of virtual displacements:**

- "When a system in equilibrium under the influence of forces is given a virtual displacement. The total work done by the virtual forces  $= 0$ "
- Displacement is imaginary, infinitesimal, instantaneous and compatible with the system



• When a virtual displacement *dx* is applied, the sum of work done by the spring force and the inertia force are set to zero:  $-( kx ) \delta x - ( m\ddot{x} ) \delta x = 0$ 

$$
-(kx)\delta x - (m\ddot{x})\delta x = 0
$$

Since  $dx \neq 0$  the equation of motion is written as:

 $kx + m\ddot{x} = 0$ 

#### **Principle of conservation of energy:**

- No energy is lost due to friction or other energy-dissipating mechanisms.
- If no work is done by external forces, the system total energy = constant
- For mechanical vibratory systems:

$$
KE + PE = constant
$$
  
or  

$$
\frac{d}{dt}(KE + PE) = 0
$$

**Since** 

$$
KE = \frac{1}{2}m\dot{x}^2 \quad and \quad PE = \frac{1}{2}kx^2
$$
  
then  

$$
\frac{d}{dt}(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2) = 0
$$
  
or  

$$
m\ddot{x} + kx = 0
$$

**Vertical mass-spring system:**



#### **Vertical mass-spring system:**



• From the free body diagram:, using Newton's second law of motion:

$$
m\ddot{x} = -k(x + \delta_{st}) + mg
$$
  
sin ce  $k\delta_{st} = mg$   

$$
m\ddot{x} + kx = 0
$$

- Note that this is the same as the eqn. of motion for the horizontal mass-spring system
- $\therefore$  if x is measured from the static equilibrium position, gravity (weight) can be ignored
- This can be also derived by the other three alternative methods.

- **The solution to the differential eqn. of motion.**
- As we anticipate oscillatory motion, we may propose a solution in the form:<br> $x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$

$$
x(t) = A\cos(\omega_n t) + B\sin(\omega_n t)
$$
  
or  

$$
x(t) = Ae^{i\omega_n t} + Be^{-i\omega_n t}
$$
  
alternatively, if we let  $s = \pm i\omega_n$   

$$
x(t) = Ce^{\pm st}
$$





$$
C(ms2 + k) = 0
$$
  
since  $c \neq 0$ ,  
 $ms2 + k = 0$   $\leftarrow$  Characteristic equation  
and  
 $s = \pm i\omega_n = \pm \sqrt{\frac{k}{m}}$   $\leftarrow$  roots = eigenvalues

$$
s = \pm i\omega_n = \pm \sqrt{\frac{k}{m}} \quad \leftarrow roots = eigenvalues
$$

*or*

$$
\omega_n = \sqrt{\frac{k}{m}}
$$

- **The solution to the differential eqn. of motion.**
- Applying the initial conditions to the general solution: *x*(*t*) =  $A cos(\omega_n t) + B sin(\omega_n t)$

*t* ions to the general solution:  $x(t) = Ac$ <br>  $x(t) = A = x_0$  *initial displacement*  $(x_{(t=0)}) = A = x_0$  *initial displaceme*<br>  $\dot{x}_{(t=0)} = B\omega_n = \dot{x}_0$  *initial velocity* s to the general solut<br> $y=0$  = A =  $x_0$  *initi*  $(t_{00}) = A = x_0$  *initial displacen*<br> $t_{00} = B\omega_n = \dot{x}_0$  *initial velocit* 

• The solution becomes:

$$
= B\omega_n = \dot{x}_0 \quad \text{initial velocity}
$$
\n
$$
x(t) = x_0 \cos(\omega_n t) + \frac{\dot{x}_0}{\omega_n} \sin(\omega_n t)
$$
\n
$$
\text{if we let} \quad A_0 = \left[ x_0^2 + \left( \frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} \quad \text{and} \quad \phi = a \tan \left( \frac{x_0 \omega_n}{\dot{x}_0} \right) \quad \text{then}
$$
\n
$$
x(t) = A_0 \sin(\omega_n t + \phi)
$$

- This describes motion of harmonic oscillator:
	- Symmetric about equilibrium position
	- Thru equilibrium: velocity is maximum & acceleration is zero
	- At peaks and valleys, velocity is zero and acceleration is maximum
	- $\omega_n = \sqrt{k/m}$  is the natural frequency

*Single Degree-of-Freedom systems*



**FREE VIBRATION OF UNDAMPED SINGLE-DEGREE-OF-FREEDOM SYSTEMS**

• Note: for vertical systems, the natural frequency can be written as:

$$
\omega_n = \sqrt{\frac{k}{m}}
$$
  
\nsinc*e*  $k = \frac{mg}{\delta_{st}}$   
\n
$$
\omega_n = \sqrt{\frac{g}{\delta_{st}}} \quad or \quad f_n = \frac{1}{2\pi} \sqrt{\frac{g}{\delta_{st}}}
$$

#### • **Torsional vibration.**

• Approach same as for translational system. Laboratory exercise.

- **Compound pendulum.**
- Given an initial angular displacement or velocity, system will oscillate due to gravitational acceleration.
- Assume rigid body  $\rightarrow$  single DoF

*Restoring torque:*

*mgd sin*  $\theta$ 

*Equation of motion :*  $\mathbb{R}^2$ 

*o n d nonlinear2 order ODE L*. Equation of motion :<br>  $J_o\ddot{\theta} + mgd \sin \theta = 0 \Leftrightarrow$  nonlinear  $2^{nd}$  order OD<br>
Linearity is approximated if  $\sin \theta \approx \theta$  Therefore mgd sin  $\theta$ <br>  $\therefore$  *Equation of motion* :<br>  $J_o \ddot{\theta} + mgd \sin \theta = 0 \quad \leftarrow$  nor.  $\int e^{2\pi} e^{2\pi} e^{2\pi} e^{2\pi}$ <br> $\theta \approx \theta$  Therefore : sin  $\theta$ <br>vation of motion :<br>+ mgd sin  $\theta = 0$   $\leftarrow$  nonling  $\leftarrow$ 

*sin :*  $\theta$  + mgd sin  $\theta$  = 0  $\leftarrow$ <br>nearity is approximated<br> $\ddot{\theta}$  + mgd  $\theta$  = 0  $\approx$ 

$$
J_o\theta + mgd \sin \theta
$$
  
Linearity is appi  

$$
J_o\ddot{\theta} + mgd\theta = 0
$$
  
Natural frequency

*enc y :*

$$
\omega_n = \sqrt{\frac{mgd}{J_o}}
$$



- **Stability.**
- Some systems may have inherent instability



- **Stability.**
- Some systems may have inherent instability
- When the bar is deflected by  $\theta$ ,

**Stability.**<br>Some systems may<br>When the bar is def<br>*The spring force is :*<br>2kl sin  $\theta$ *2kl sin*  $2$ kl sin $\theta$ 

```
The gravitational force thru G is :
```
*mg*

*The gravitational force thru G is :<br>mg*<br>*The inertial moment about O due to the angular acceleration*  $\ddot{\theta}$  *is :*  $\theta$ 



*The eqn. of motion is written as :* 

$$
J_o \ddot{\theta} = \frac{ml^2}{3} \ddot{\theta}
$$
  
The eqn. of motion is written as :  

$$
\frac{ml^2}{3} \ddot{\theta} + (2kl \sin \theta) l \cos \theta - mg \frac{l}{2} \sin \theta = 0
$$



## **Free undamped vibration single DoF e undamped vibration single**<br> $\theta = \theta$  and  $cos \theta = 1$ . Therefore

**Free undamped vibration**<br>*For small oscillations,*  $sin \theta = \theta$  *and*  $cos \theta = 1$ *. The refore*

For small oscillations, 
$$
sin \theta = \theta
$$
 and  $cos \frac{ml^2}{3} \theta + 2kl^2 \theta - \frac{mgl}{2} \theta = 0$ 

*or*



Recall: viscous damping force  $\infty$  velocity:

 $\lfloor Ns/m \rfloor$ **Free sin**<br>viscous damping for<br> $=-cx$   $c = damping$ <br> $Newton's second law$ riscous damping force  $\propto$  velocity:<br>  $-c\dot{x}$   $c = damping \ constant \ or \ coefficient \ [Ns/m$ <br>
Newton's second law of motion to obtain the eqn. of<br>  $=-c\dot{x}-kx$  or  $m\ddot{x}+c\dot{x}+kx=0$ <br>
tion is assumed to take the form :  $=$ *F* =  $-c\dot{x}$ <br>*F* =  $-c\dot{x}$ *II:* viscous dampin<br>  $F = -c\dot{x}$   $c = da\dot{x}$ <br> *m* $\ddot{x} = -c\dot{x} - kx$  or<br> *solution* is assumed **Free single DoF vil**<br>amping force  $\propto$  velocity<br>*c* = *damping constant or read to the set of mping constant c*<br>*r mix* + *cx* + *kx*<br>*r mix* + *cx* + *kx*<br>*kto take the form* **comation + visco**<br>*coefficient* [*Ns / m*]<br>*cobtain the ean of* **Free single DoF vibration + viscous dar**<br> *Applying Newton's second law of motion to obtain the eqn.of motion :*<br>  $\vec{m} = -c\dot{x}$  *c* = *damping constant or coefficient* [*Ns/m*]<br> *Applying Newton's second law of motion t*  $F = -c\dot{x}$  (<br>Applying Newton's<br> $m\ddot{x} = -c\dot{x} - kx$ <br>*f* the solution is ass  $s =$  damping constant or<br>  $s = s$  sec ond law of motion to<br>  $s = s$  or  $m\ddot{x} + c\dot{x} + kx = s$ <br>  $s$  sumed to take the form :

Newton's second law of motion to obtain<br>  $=-c\dot{x}-kx$  or  $m\ddot{x}+c\dot{x}+kx=0$ <br>
tion is assumed to take the form :<br>  $=Ce^{st}$  where  $s = \pm i\omega_n$ 

 $m\ddot{x} = -c\dot{x} - kx$  or  $m\ddot{x} + c\dot{x} + kx = 0$ 

*If the solution is assumed to take the form:* 

\n
$$
\text{ving Newton's second law of } m
$$
\n $m\ddot{x} = -cx - kx$ \n $\text{or } m\ddot{x} + cz$ \n

\n\n $\text{solution is assumed to take the}$ \n $x(t) = Ce^{st}$ \n $\text{where } s = \pm i\omega_n$ \n

\n\n $\dot{x}(t) = \omega_0 e^{st}$ \n $\text{and } \ddot{x}(t)$ \n

 $= sCe^{st}$  $m\ddot{x} = -c\dot{x} - kx$  or<br> *If the solution is assumed to*<br>  $x(t) = Ce^{st}$  where<br>
then :  $\dot{x}(t) = sCe^{st}$  and<br>
Substituting for x, x and x i  $d \quad \ddot{x}(t) =$ =  $Ce^{st}$  where  $s = \pm i\omega_n$ <br>
(t) =  $sCe^{st}$  and  $\ddot{x}(t) =$ <br>
1g for x,  $\dot{x}$  and  $\ddot{x}$  in the eqn.<br>  $+ cs + k = 0$ <br>
f the characteristic eqn. are  $\frac{1}{2}$ <br> $\frac{2}{e^{st}}$ *Substituting for x, x and x in the eqn.of motion*<br>  $s(t) = Ce^{st}$  where  $s = \pm i\omega_n$ <br> *Substituting for x, x and x in the eqn.of motion*<br>  $\cos^2 t + \cos t = 0$ *Then*  $\dot{x}(t) = sCe^{st}$  *and*  $\ddot{x}(t) = s^2$ <br> *The root of the characteristic eqn. are :*<br> *The root of the characteristic eqn. are :*  $x\ddot{x} + c\dot{x} + kx =$ <br>  $x \ddot{x} + \dot{x} \dot{y} + kx \dot{z}$ <br>  $\therefore \dot{x} \dot{y} + \dot{z} \dot{z} + kx \dot{z} \dot{z}$ <br>  $\therefore \dot{x} \dot{y} + kx \dot{z} \dot{z} + kx \dot{z} \dot{z}$ <br>  $\therefore \dot{y} \dot{z} + kx \dot{z} \dot{z} + kx \dot{z} \dot{z}$  $x(t) = Ce^{st}$ <br> $\dot{x}(t) = sCe^{st}$ <br>ituting for x, x a<br> $ms^2 + cs + k = 0$ <br>oot of the character

$$
ms^2 + cs + k = 0
$$

Substituting for x, 
$$
\dot{x}
$$
 and  $\ddot{x}$  in the eqn. of motion  
\n $ms^2 + cs + k = 0$   
\nThe root of the characteristic eqn. are:  
\n $s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$   
\nThe two solutions are:  
\n $x_1(t) = C_1 e^{s_1 t}$  and  $x_2(t) = C_2 e^{s_2 t}$ 

$$
x_1(t) = C_1 e^{s_1 t} \qquad and \qquad x_2(t) = C_2 e^{s_2 t}
$$



# **Free single DoF vibration + viscous damping**<br>
ion to the Eqn. Of motion is:<br>  $e^{s_1t} + C_2e^{s_2t}$

**1** Free single DoF viboral solution to the Eqn. Of motio  $x(t) = C_1 e^{S_1 t} + C_2 e^{S_2 t}$ • The general solution to the Eqn. Of motion is:

**Free single**  
eral solution to the Eqn  

$$
x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}
$$

*or*

or  
\n
$$
x(t) = C_1 e^{S_1 t} + C_2 e^{S_2 t}
$$
\n
$$
x(t) = C_1 e^{-\frac{1}{2m} \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}t} + C_2 e^{-\frac{1}{2m} \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}t}
$$
\nwhere  $C_1$  and  $C_2$  are arbitrary constants  
\ndet or mined from the initial conditions

 $x(t) = C_1 e^{(-t)} +$ <br>where  $C_1$  and  $C_2$  are arbitrary constant<br>det er mined from the initial conditions. *n .*



**• Critical damping (c<sub>c</sub>):** value of c for which the radical in the general solution is zero:

**Free single DoF vibration + viscous damping**  
bing (c<sub>c</sub>): value of c for which the radical in the general solution is zero:  

$$
\left(\frac{c_c}{2m}\right)^2 - \left(\frac{k}{m}\right) = 0 \qquad or \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n = 2\sqrt{km}
$$

• **Damping ratio ():** damping coefficient : critical damping coefficient.

$$
2m \int (m)^{-\sigma} \sigma r \qquad c_c = 2m \sqrt{m} = 2m \sigma_n = 2 \sqrt{m}
$$
  
io (ζ): damping coefficient : critical damping coefficient.  

$$
\zeta = \frac{c}{c_c} \qquad or \qquad \frac{c}{2m} = \frac{c}{c_c} \frac{c_c}{2m} = \zeta \omega_n
$$
  
The roots can be re-written :

 $\overline{\phantom{0}}$ 

$$
\zeta = \frac{C}{c_c} \quad \text{or} \quad \frac{C}{2m} = \frac{C}{c_c} \cdot \frac{C}{2m} = \zeta \omega_n
$$
\nThe roots can be re-written:

\n
$$
s_{1,2} = -\frac{C}{2m} \pm \sqrt{\left(\frac{C}{2m}\right)^2 - \left(\frac{k}{m}\right)} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) \omega_n
$$
\nAnd the solution becomes :

\n
$$
\left(\sqrt{C^2 - 1}\right)^2 = \sqrt{\frac{C^2}{2m}}
$$

$$
S_{1,2} = 2m^{-1}\sqrt{2m} \quad (m) = (-5 - \sqrt{5})^{2} \quad (m)
$$
  
And the solution becomes :  

$$
x(t) = C_1 e^{(-5 + \sqrt{5^2 - 1})\omega_n t} + C_2 e^{(-5 - \sqrt{5^2 - 1})\omega_n t}
$$

• The response x(t) depends on the roots  $s_1$  and  $s_2 \rightarrow$  the behaviour of the system is dependent on the damping ratio **.**

Free single DoF vibration + viscous damping  
\n
$$
\frac{x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}}
$$
\nsystem is underdamped. ( $\zeta^2$ -1) is negative and the roots can be  
\n
$$
(-\zeta + i\sqrt{1 - \zeta^2})\omega_n \quad \text{and} \quad s_2 = (-\zeta - i\sqrt{1 - \zeta^2})\omega_n
$$

• When  $\zeta$  <1, the system is underdamped.  $(\zeta^2-1)$  is negative and the roots can be written as:

Free single DoF vibration + VISCOUS damping  
\n
$$
x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}
$$
\n
$$
x(t) = C_1 e^{\left(-\zeta + i\sqrt{1 - \zeta^2}\right)} \omega_n \qquad \text{and} \qquad s_2 = \left(-\zeta - i\sqrt{1 - \zeta^2}\right) \omega_n
$$
\nAnd the solution becomes :  
\n
$$
x(t) = C_1 e^{\left(-\zeta + i\sqrt{1 - \zeta^2}\right)} \omega_n t + C_2 e^{\left(-\zeta - i\sqrt{1 - \zeta^2}\right)} \omega_n t
$$

*bec omes :*

Then 
$$
\zeta
$$
  $\zeta$ , the system is underaamped.  $(\zeta^{-1})$  is negative and the roots can be written as:  
\n
$$
s_1 = \left(-\zeta + i\sqrt{1 - \zeta^2}\right)\omega_n \quad \text{and} \quad s_2 = \left(-\zeta - i\sqrt{1 - \zeta^2}\right)\omega_n
$$
\nAnd the solution becomes :  
\n
$$
x(t) = C_1 e^{\left(-\zeta + i\sqrt{1 - \zeta^2}\right)}\omega_n t + C_2 e^{\left(-\zeta - i\sqrt{1 - \zeta^2}\right)}\omega_n t
$$
\n
$$
x(t) = e^{-\zeta\omega_n t} \left\{ C_1 e^{\left(i\sqrt{1 - \zeta^2}\right)}\omega_n t + C_2 e^{\left(-i\sqrt{1 - \zeta^2}\right)}\omega_n t \right\}
$$
\n
$$
x(t) = e^{-\zeta\omega_n t} \left\{ (C_1 + C_2) \cos\left(\sqrt{1 - \zeta^2}\omega_n t\right) + i(C_1 - C_2) \sin\left(\sqrt{1 - \zeta^2}\omega_n t\right) \right\}
$$
\n
$$
x(t) = e^{-\zeta\omega_n t} \left\{ C_1' \cos\left(\sqrt{1 - \zeta^2}\omega_n t\right) + C_2' \sin\left(\sqrt{1 - \zeta^2}\omega_n t\right) \right\}
$$
\n
$$
x(t) = X e^{-\zeta\omega_n t} \sin\left(\sqrt{1 - \zeta^2}\omega_n t + \phi\right) \quad \text{or} \quad x(t) = X_0 e^{-\zeta\omega_n t} \cos\left(\sqrt{1 - \zeta^2}\omega_n t - \phi_0\right)
$$

Where C'<sub>1</sub>, C'<sub>2</sub>; X,  $\phi$  and X<sub>o</sub>,  $\phi_o$  are arbitrary constant determined from initial conditions.

Free single DoF vibration + viscous damping  
\n
$$
x(t) = e^{-\zeta \omega_n t} \left\{ C_1' \cos \left( \sqrt{1 - \zeta^2} \omega_n t \right) + C_2' \sin \left( \sqrt{1 - \zeta^2} \omega_n t \right) \right\}
$$
\nconditions:  
\n
$$
x(t=0) = x_0 \text{ and } \dot{x}(t=0) = \dot{x}_0
$$

• For the initial conditions:

$$
x(t=0) = x_0 \quad and \quad \dot{x}(t=0) = \dot{x}_0
$$

*Then*

e initial conditions:  
\n
$$
x(t=0) = x_0
$$
 and  $\dot{x}(t=0) = \dot{x}_0$   
\nThen  
\n $C'_1 = x_0$  and  $C'_2 = \frac{\dot{x}_0 + \zeta \omega_n x_0}{\sqrt{1 - \zeta^2} \omega_n}$   
\nTherefore the solution becomes

$$
C'_{1} = x_{0} \quad and \quad C'_{2} = \frac{\dot{x}_{0} + \zeta \omega_{n} x_{0}}{\sqrt{1 - \zeta^{2}} \omega_{n}}
$$
  
zero the solution becomes  

$$
x(t) = e^{-\zeta \omega_{n} t} \left\{ x_{0} \cos \left( \sqrt{1 - \zeta^{2}} \omega_{n} t \right) + \frac{\dot{x}_{0} + \zeta \omega_{n} x_{0}}{\sqrt{1 - \zeta^{2}} \omega_{n}} \sin \left( \sqrt{1 - \zeta^{2}} \omega_{n} t \right) \right\}
$$

• This represents a decaying (damped) harmonic motion with angular frequency  $\sqrt{(1-\zeta^2)\omega_{\sf n}}$  also known as the damped natural frequency. The factor  $e^{(i)}$  causes the exponential decay.



Exponentially decaying harmonic – free SDoF vibration with viscous damping . Underdamped oscillatory motion and has important engineering applications.

le Degree-of-Freedom systems  
\nFree single DoF vibration + viscous damping  
\n
$$
x(t) = Xe^{-\zeta \omega_n t} \sin\left(\sqrt{1-\zeta^2} \omega_n t + \phi\right) \quad or \quad x(t) = X_0e^{-\zeta \omega_n t} \cos\left(\sqrt{1-\zeta^2} \omega_n t - \phi_0\right)
$$
\nThe constant is  $(X, \phi)$  and  $(X_0, \phi_0)$  representing the magnitude and phase become :

$$
u(t) = Xe^{-\zeta \omega_0 t} \sin\left(\sqrt{1 - \zeta^2 \omega_0 t + \phi}\right) \quad \text{or} \quad x(t) = X_0e^{-\zeta \omega_0 t} \cos\left(\sqrt{1 - \frac{\zeta^2 \omega_0 t}{c}}\right)
$$
  
constants ( X,  $\phi$ ) and ( X<sub>0</sub>,  $\phi$ ) representing the magnitude and phase  

$$
X = X_0 = \sqrt{\left(\frac{C_1}{C_1}\right)^2 + \left(\frac{C_2}{C_2}\right)^2}
$$

$$
\phi = a \tan\left(\frac{C_1}{C_2}\right) \quad \text{and} \quad \phi_0 = a \tan\left(-\frac{C_2}{C_1}\right)
$$

• When  $\zeta = 1$ , c=c<sub>c</sub>, system is critically damped and the two roots to the eqn. of motion become:

**Free single DoF vibration + visc**  
\n
$$
c_c
$$
, system is critically damped and the two  
\n $s_I = s_2 = -\frac{c_c}{2m} = -\omega_n$   
\nand solution is  
\n $x(t) = (C_I + C_2 t)e^{-\omega_n t}$   
\nApplying the initial conditions  $x(t = 0) = x$ 

$$
x(t) = (C_1 + C_2 t) e^{-\omega_n t}
$$

*i*<sub>0</sub> and  $\dot{x}(t=0) = \dot{x}_0$  $s_1 = s_2 = -\frac{c}{2m} = -\omega_n$ <br>and solution is<br> $x(t) = (C_1 + C_2 t)e^{-\omega_n t}$ <br>Applying the initial conditions<br> $C_1 = x_0$ *x*(*t* = 0) = *x*<sub>0</sub> *and i*(*t* = 0) = *i*<sub>(</sub> *and*  $\dot{x}(t=0) = \dot{x}_0$  *yields*  $(0, 0) = x_0$  and  $\dot{x}(t = 0) = \dot{x}_0$  yields

and solution is  
\n
$$
x(t) = (C_1 + C_2t)e^{-\omega_n t}
$$
\nApplying the initial conditions  $x(t)$   
\n
$$
C_1 = x_0
$$
\n
$$
C_2 = \dot{x}_0 + \omega_n x_0
$$
\nThe solution becomes :

$$
C_1 = x_0
$$
  
\n
$$
C_2 = \dot{x}_0 + \omega_n x_0
$$
  
\n*olution becomes :*  
\n
$$
x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t]e^{-\omega_n t}
$$

• As t $\rightarrow \infty$ , the exponential term diminished toward zero and depicts **aperiodic** motion

• When  $\zeta > 1$ , c>c<sub>c</sub>, system is overdamped and the two roots to the eqn. of motion are real and negative: **ee single DoF vibration + viscous dar**<br>tem is overdamped and the two roots to the e<br> $s_I = \left(-\zeta + \sqrt{\zeta^2 - I}\right)\omega_n < 0$ 

**Free single DOF VIDration + VISCOUS damping**  
\n<sub>c</sub>, system is overdamped and the two roots to the eqn. of m  
\n
$$
s_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n < 0
$$
\n
$$
s_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n < 0
$$
\nwith  $s_2 \square$   $s_1$  and the initial conditions  $x(t = 0) = x_0$  and the solution becomes :

itial conditions  $x(t = 0)$ <br> $\overline{a^2-1}$   $\int_0^{\infty} \rho_n t$  $2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n < 0$ <br>  $2 \square$  *S<sub>1</sub>* and the initial conditions  $x(t = 0) = x_0$  and  $\dot{x}(t = 0) = \dot{x}_0$ <br> *2 D S<sub>1</sub>* and the initial conditions  $x(t = 0) = x_0$  and  $\dot{x}(t = 0) = \dot{x}_0$ *ial conditions*  $x(t=0) = x_0$ <br> $\overline{I}$  $\int \omega_n t + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})} \omega_n t$  $s_1 = \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n < 0$ <br>  $s_2 = \left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n < 0$ <br>  $s_2 \Box$   $s_1$  and the initial conditions  $x(t = 0) = x_0$  and  $\dot{x}(t = 0) = \dot{x}_0$ <br>
dution becomes :  $x_2 \Box$  *s<sub>1</sub>* and the initial conditions  $x(t = 0) = x_0$  and  $\dot{x}(t = 0)$ <br>lution becomes :<br> $x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})} \omega_n t + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})} \omega_n t$ *the solution becomes :*  $(0, t) = x_0$  and  $\dot{x}(t = 0) = \dot{x}_0$ 

$$
s_2 \Box \quad s_1 \text{ and the initial conditions } x(t=0) = x_0
$$
\n
$$
\text{lution becomes:}
$$
\n
$$
x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}
$$
\n
$$
s_2 \omega_n \left(-\zeta + \sqrt{\zeta^2 - 1}\right) + \dot{x}_0
$$
\n
$$
c_1 = \Box
$$

*w here*

$$
x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}
$$
  
\ne  
\n
$$
C_1 = \frac{x_0 \omega_n \left(-\zeta + \sqrt{\zeta^2 - 1}\right) + \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}
$$
  
\n
$$
C_2 = \frac{-x_0 \omega_n \left(-\zeta - \sqrt{\zeta^2 - 1}\right) - \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}}
$$

Which shows *aperiodic* motion which diminishes exponentially with time.



**Free single DoF vibration + viscous damping**

Critically damped systems have lowest required damping for aperiodic motion and mass returns to equilibrium position in shortest possible time.