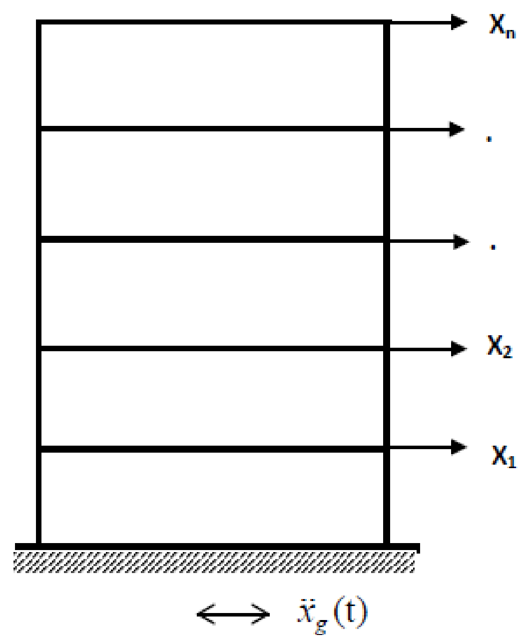
Response Spectrum Method for MDOF System

Response Spectrum Method for MDOF System

Multi degree of freedom (MDOF) systems are usually analyzed using Modal Analysis. A typical MDOF system with '*n*' degree of freedom is shown in Figure 1 . This system when subjected to ground motion undergoes deformations in number of possible ways. These deformed shapes are known as modes of vibration or mode shapes. Each shape is vibrating with a particular natural frequency. Total unique modes for each MDOF system are equal to the possible degree of freedom of system. The equations of motion for MDOF system is given by

$$[m]\{\ddot{x}(t)\} + [c]\{\dot{x}(t)\} + [k]\{x(t)\} = -[m]\{r\}\ddot{x}_g(t)$$



MDOF system with '*n*' degrees-of-freedom.

Response Spectrum Method for MDOF System

$$[m]\{\ddot{x}(t)\} + [c]\{\dot{x}(t)\} + [k]\{x(t)\} = -[m]\{r\}\ddot{x}_g(\mathrm{t})$$

where, $[m]$ = Mass matrix ($n \times n$); $[k]$ = Stiffness matix ($n \times n$); $[c]$ = Damping matrix ($n \times n$); $\{r\}$ = Influence coefficient vector ($n\times1$); $\{x(t)\}$= relative displacement vector; $\{\dot{x}(t)\}$= relative velocity vector, $\{\ddot{x}(t)\}$= relative acceleration vector, and $\ddot{x}_g(t)$= earthquake ground acceleration.

The undamped eigen values and eigen vectors of the MDOF system are found form the characteristic equation

$$\{[k] - \omega_i^2[m]\}\phi_i = 0 \qquad i = 1, 2, 3, \ldots, n$$

$$\det\left|\{[k] - \omega_i^2[m]\}\right| = 0$$

where,

ω_i^2 = eigen values of the ith mode

ϕ_i = eigen vector or mode shape of the ith mode

ω_i = natural frequency in the ith mode.

Response Spectrum Method for MDOF System

Let the displacement response of the MDOF system is expressed as

$$\{x(t)\} = [\phi]\{y(t)\}$$

where $\{y(t)\}$ represents the modal displacement vector, and $[\phi]$ is the mode shape matrix given by

$$[\phi] = [\phi_1, \phi_2 ,\dots\dots\dots \phi_n]$$

Substituting $\{x\} = [\phi]\{y\}$ in equation (4.18) and pre-multiply by $[\phi]^T$

$$[\phi]^T [m][\phi]\{\ddot{y}(t)\} + [\phi]^T [c][\phi]\{\dot{y}(t)\} + [\phi]^T [k][\phi]\{y(t)\} = -[\phi]^T [m]\{r\}\ddot{x}_g(t)$$

The above equation reduces to

$$[M_m]\{\ddot{y}(t)\} + [C_d]\{\dot{y}(t)\} + [K_d]\{y(t)\} = -[\phi]^T [m]\{r\}\ddot{x}_g(t) \quad \text{—— 1}$$

Response Spectrum Method for MDOF System

where,

$[\phi]^T[m][\phi] = [M_m]$ = generalized mass matrix

$[\phi]^T[c][\phi] = [C_d]$ = generalized damping matrix

$[\phi]^T[k][\phi] = [K_d]$ = generalized stiffness matrix

By virtue of the properties of the $[\phi]$, the matrices $[M_m]$ and $[K_d]$ are diagonal matrices. However, for the classically damped system (i.e. if the $[C_d]$ is also a diagonal matrix), the equation (1) reduces to the following equation

$$\ddot{y}_i(t) + 2\xi_i\omega_i\dot{y}_i(t) + \omega_i^2 y_i(t) = -\Gamma_i\,\ddot{x}_g(t) \qquad (i = 1, 2, 3, \ldots, n)$$

where,

$y_i(t)$ = modal displacement response in the ith mode,

ξ_i = modal damping ration in the ith mode, and

Γ_i = modal participation factor for ith mode expressed by

$$\Gamma_i = \frac{\{\phi_i\}^T[m]\{r\}}{\{\phi_i\}^T[m]\{\phi_i\}}$$

Response Spectrum Method for MDOF System

The maximum displacement response of the structure in the i[th] mode is

$$x_{i,\max} = \phi_i \, \Gamma_i \; S_d \; (\xi_i, \omega_i) \qquad (i = 1, 2, \ldots\ldots, n)$$

The maximum acceleration response of the structure in the i[th] mode is

$$\{ \ddot{x}_a \}_{i,\max} = \{\phi_i\} \Gamma_i S_{pa} (\xi_i, \omega_i) \qquad (i = 1, 2, \ldots\ldots, n)$$

The required response quantity of interest, r_i i.e. (displacement, shear force, bending moment etc.) of the structure can be obtained in each mode of vibration using the maximum response obtained in equations . However, the final maximum response, $r_{\max}$ shall be obtained by combining the response in each mode of vibration using the modal combinations rules.

Response Spectrum Method for MDOF System

Modal Combination Rules

The commonly used methods for obtaining the peak response quantity of interest for a MDOF system are as follows:

- Absolute Sum (ABSSUM) Method,
- Square root of sum of squares (SRSS) method, and
- Complete quadratic combination (CQC) method

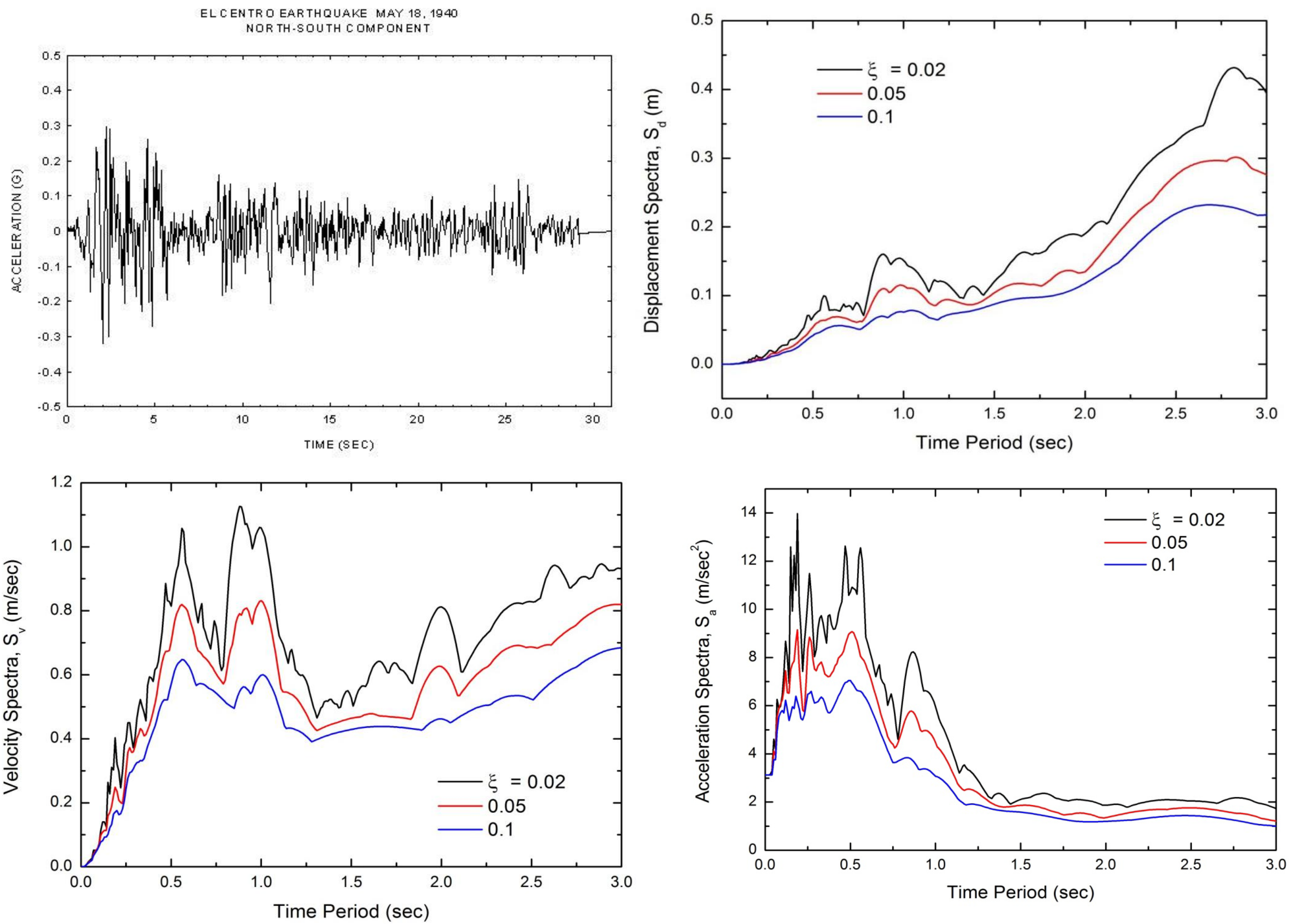
In ABSSUM method, the peak responses of all the modes are added algebraically, assuming that all modal peaks occur at same time. The maximum response is given by

$$r_{\max} = \sum_{i=1}^{n} |r_i|$$

In the SRSS method, the maximum response is obtained by square root of sum of square of response in each mode of vibration and is expressed by

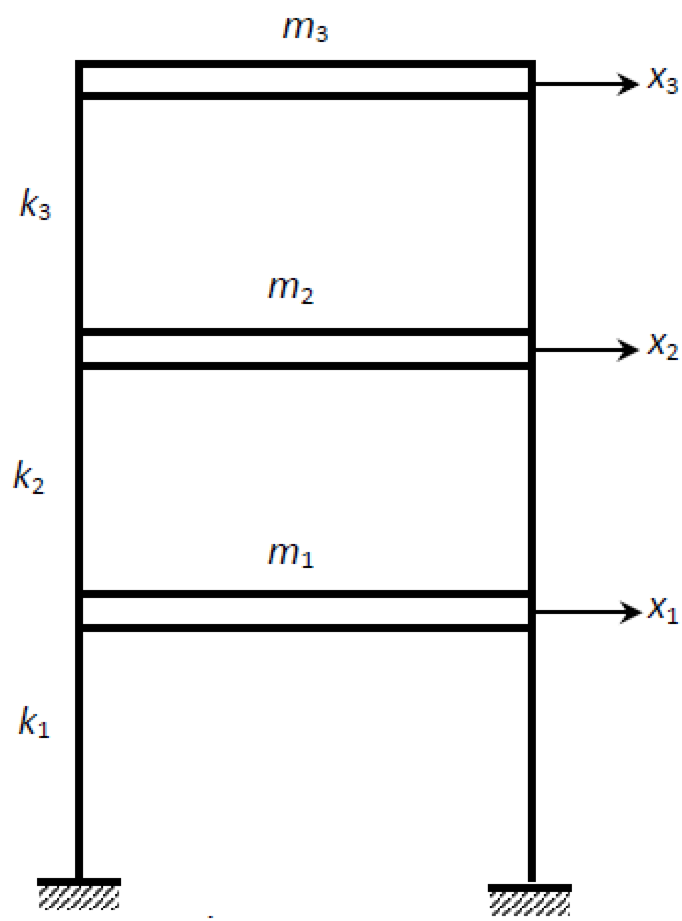
$$r_{\max} = \sqrt{\sum_{i=1}^{n} r_i^2}$$

Response spectra of El-Centro, 1940 earthquake ground motion.



Example 2

A three-story building is modeled as 3-DOF system and rigid floors as shown in Figure Determine the top floor maximum displacement and base shear due to El-Centro, 1940 earthquake ground motion using the response spectrum method. Take the inter-story lateral stiffness of floors i.e. $k_1 = k_2 = k_3 = 16357.5 \times 10^3$ N/m and the floor mass $m_1 = m_2 = 10000$ kg and $m_3 = 5000$ kg.



Example 2

$$[m]=\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}=\begin{bmatrix} 10000 & 0 & 0 \\ 0 & 10000 & 0 \\ 0 & 0 & 5000 \end{bmatrix}$$

and the stiffness matrix,

$$[k]=\begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}=\begin{bmatrix} 32715 & -16357.5 & 0 \\ -16357.5 & 32715 & -16357.5 \\ 0 & -16357.5 & 16357.5 \end{bmatrix}$$

Finding eigen values and eigen vectors using the equation

$$\left\{[k]-\omega^2[m]\right\}\{\phi\}=0$$

$$\det\left|[k]-\omega^2[m]\right|=0$$

Example 2

$$\det\left|16357.5\times10^3\begin{bmatrix}2 & -1 & 0\\ -2 & 2-2\lambda & -1\\ 0 & -1 & 1-\lambda\end{bmatrix}-\omega^2\times5000\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\right|=0$$

$$\lambda=\frac{\omega^2\times5000}{16357.5\times10^3}$$

$$\det\begin{vmatrix}2-2\lambda & -1 & 0\\ -1 & 2-2\lambda & -1\\ 0 & -1 & 1-\lambda\end{vmatrix}=0$$

$$(2-2\lambda)\left[(2-2\lambda)(1-\lambda)-1\right]+\left[-1+\lambda\right]=0$$

$$(2-2\lambda)\left[2-2\lambda-2\lambda+2\lambda^2-1\right]-1+\lambda=0$$

$$2-2\lambda\left[2\lambda^2-4\lambda+1\right]+\lambda-1=0$$

$$4\lambda^2-8\lambda+2-4\lambda^3+2\lambda^2-2\lambda+\lambda-1=0$$

$$-4\lambda^3+12\lambda^2-9\lambda+1=0$$

$$\lambda_1=0.134,\qquad \lambda_2=1\,,\qquad \lambda_3=1.866$$

Implying that

$$\omega_1=20.937\,rad/\sec\qquad \omega_2=57.2\,rad/\sec\qquad \omega_3=78.13\,rad/\sec$$

Example 2

On substituting ω^2 in the characteristic equation,

For Mode 1

$$\begin{bmatrix} 2-2\times0.134 & -1 & 0 \\ -1 & 2-2\times0.134 & -1 \\ 0 & -1 & 1-0.134 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{bmatrix} = 0$$

Assuming $\Phi_{11} = 1$,

$$2-2\times0.134 = \phi_{21}$$

$$\phi_{21} = 1.732 \text{ and } \phi_{31} = 2.0$$

$$\{\phi_1\} = \begin{Bmatrix} 1 \\ 1.732 \\ 2.0 \end{Bmatrix}$$

Example 2

For Mode 2

$$\begin{bmatrix} 2-2 & -1 & 0 \\ -1 & 2-2 & -1 \\ 0 & -1 & 1-1 \end{bmatrix} \begin{bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{bmatrix} = 0$$

$$\{\phi_2\} = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

For Mode 3

$$\begin{bmatrix} 2-2\times1.866 & -1 & 0 \\ -1 & 2-2\times1.866 & -1 \\ 0 & -1 & 1-1.866 \end{bmatrix} \begin{bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{bmatrix} = 0$$

$$\{\phi_3\} = \begin{Bmatrix} 1 \\ -1.733 \\ 2.0 \end{Bmatrix}$$

Example 2

Modal Participation Factors

$$\Gamma_1 = \frac{\{\phi_1\}^T \{m\}\{r\}}{\{\phi_1\}^T \{m\}\{\phi\}} = \frac{\begin{bmatrix}1 & 1.733 & 2\end{bmatrix}\times 5000\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}}{\begin{bmatrix}1 & 1.733 & 2\end{bmatrix}\times 5000\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 1.733\\ 2\end{bmatrix}}$$

$$\Gamma_1 = 0.622$$

Similarly,

$$\Gamma_2 = \frac{\{\phi_2\}^T \{m\}\{r\}}{\{\phi_2\}^T \{m\}\{\phi_2\}} = 0.333$$

$$\Gamma_3 = \frac{\{\phi_2\}^T \{m\}\{r\}}{\{\phi_3\}^T \{m\}\{\phi\}} = 0.045$$

Example 2

1st Mode Response

$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{20.937} = 0.30 \text{ sec}$$

$$\xi_1 = 0.02$$

From the response spectra

$$S_{d1} = 0.01902m$$

Top floor displacement $= \Gamma_1 \times \phi_{31} \times S_{d1} = 2 \times 0.622 \times 0.01902$

$$= 0.0236m$$

Base shear $= k \times \phi_{11} \times \Gamma_1 \times S_{d1} = 16.357 \times 10^6 \times 1 \times 0.622 \times 0.01902$

$$= 193 \; kN$$

Example 2

2**^{nd}* ***Mode Response

$$T_2 = \frac{2\pi}{57.2} = 0.11\,\text{sec}$$

$$\xi_2 = 0.02$$

from response spectra

$$S_{d2} = 0.00231m$$

Top floor displacement $= \phi_{32} \times \Gamma_2 \times S_{d2} = -1 \times 0.333 \times 0.00231$

$= -7.69 \times 10^{-4} m$

Base shear $= k \times \phi_{12} \times \Gamma_2 \times S_{d2} = 16357 \times 10^3 \times 1 \times 0.333 \times 0.00231 = \ 12.58kN$

Example 2

3^rd^ Mode Response

$$T_3 = \frac{2\pi}{78.13} = 0.08\sec$$

$$\xi_3 = 0.02$$

from response spectra

$$S_{d3} = 9.77\times10^{-4}\,m$$

Top floor displacement $= \phi_{33}\times\Gamma_3\times S_{d3}$

$$= 2\times0.045\times9.77\times10^{-4}\,m$$

$$= 8.793\times10^{-5}\,m$$

Base shear $= k\times\phi_{13}\times\Gamma_3\times S_{d3}$

$$= 16357.5\times10^3\times1\times0.045\times9.77\times10^{-4}$$

$$= 0.719\ kN$$

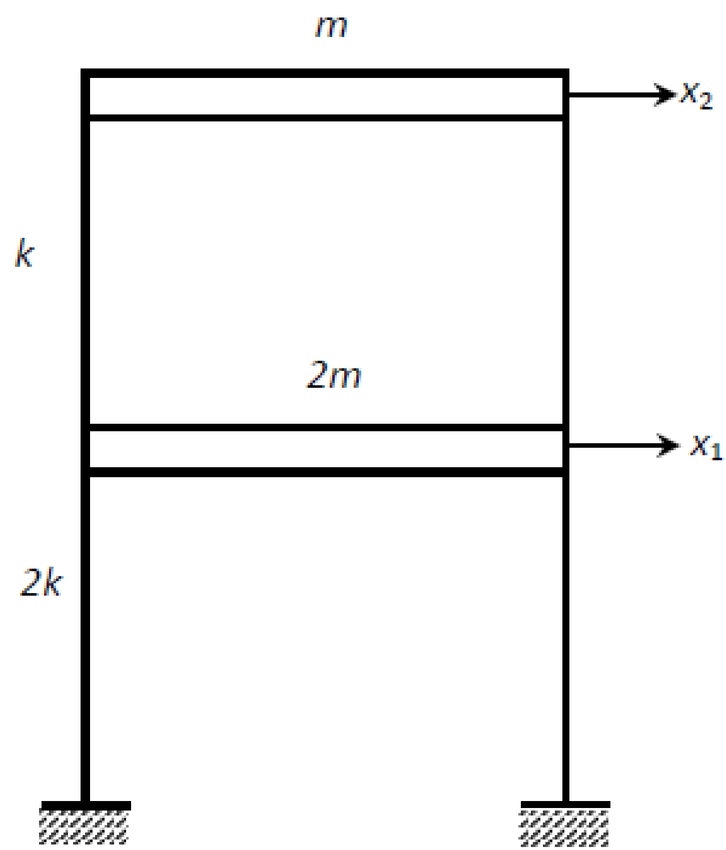
Example 2

Peak responses using the SRSS modal combination rule are given below

Mode	Top floor displacement (mm)	Base shear (kN)
1	23.6	193
2	-0.769	12.58
3	0.0879	0.719
SRSS	23.6	193.41
Exact Response (from time history analysis)	23.4	196.4

ASSIGNMENT

A two-story building in Lahore is modeled as 2-DOF system and rigid floors on stiff soil as shown in the Figure. Determine the top floor maximum displacement and base shear using UBC-97 response spectra. Take the inter-story stiffness, *$k = 200 \times 10^3$ N/m and the floor mass, m = 2500 kg.*



UBC Design Response Spectra

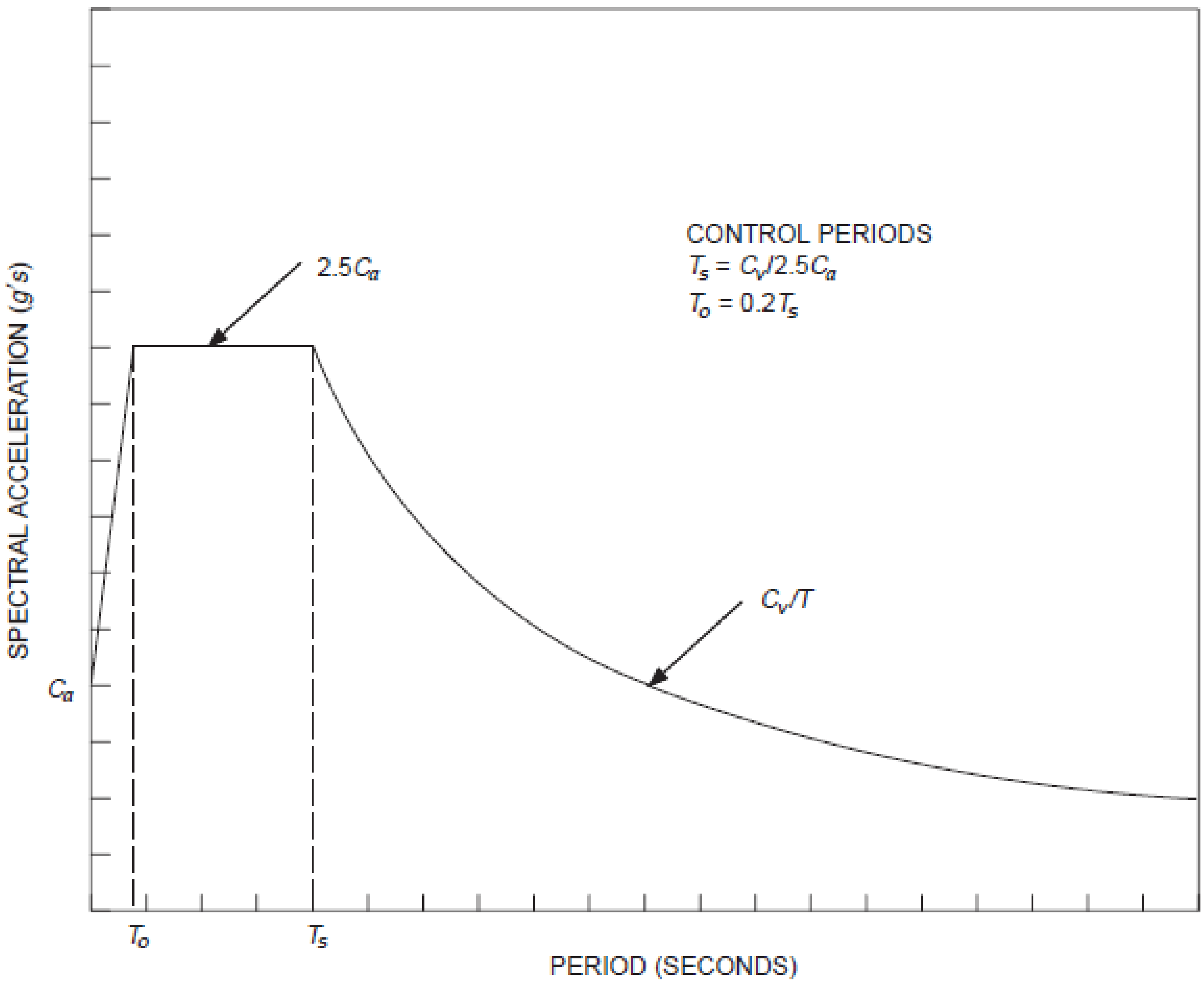


FIGURE 16-3—DESIGN RESPONSE SPECTRA

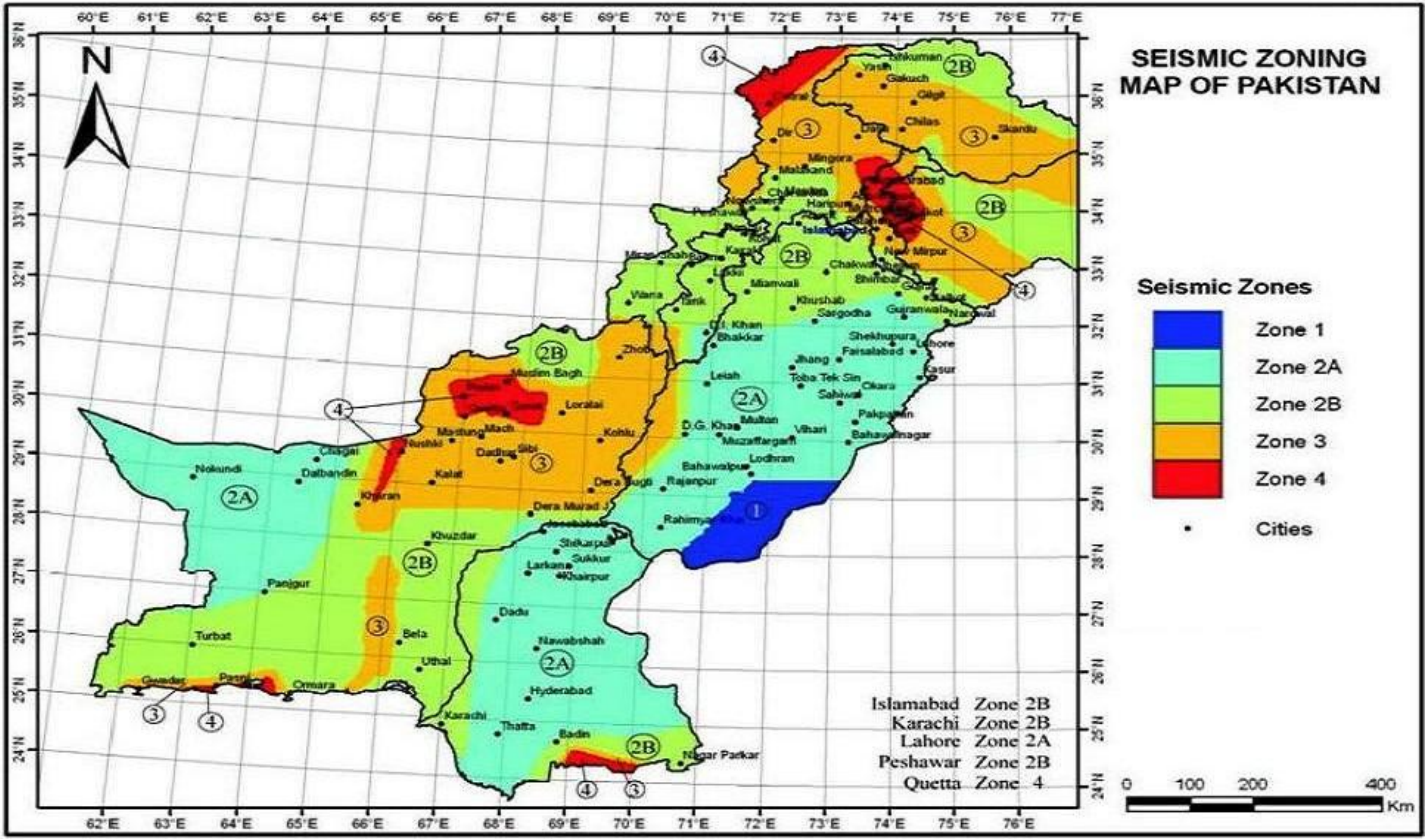

SEISMIC ZONING
MAP OF PAKISTAN
Seismic Zones
Zone 1
Zone 2A
Zone 2B
Zone 3
Zone 4
Cities
Islamabad Zone 2B
Karachi Zone 2B
Lahore Zone 2A
Peshawar Zone 2B
Quetta Zone 4
0 100 200 400
Km
N
Karachi
Hyderabad
Thatta
Badin
Nagar Parkar
Nawabshah
Dadu
Larkana
Sukkur
Khairpur
Shikarpur
Khuzdar
Bela
Uthal
Ormara
Gwadar
Pasni
Turbat
Panjgur
Nokkundi
Dalbandin
Chagai
Kharan
Nushki
Kalat
Mastung
Mach
Sibi
Loralai
Kohlu
Zhob
Muslim Bagh
Dera Bugti
Dera Murad J.
Rajanpur
D.G. Khan
Multan
Vihari
Lodhran
Bahawalpur
Bahawalnagar
Muzaffargarh
Leiah
Bhakkar
Jhang
Faisalabad
Toba Tek Sin
Okara
Sahiwal
Pakpattan
Kasur
Lahore
Shekhupura
Sargodha
Khushab
Mianwali
Gujranwala
Narowal
Chakwal
Bhimbar
New Mirpur
Islamabad
Wana
Tank
D.I. Khan
Peshawar
Mingora
Malakand
Chitral
Dir
Dasu
Chilas
Gilgit
Gakuch
Yasin
Skardu
1
2A
2B
3
4
60°E
61°E
62°E
63°E
64°E
65°E
66°E
67°E
68°E
69°E
70°E
71°E
72°E
73°E
74°E
75°E
76°E
77°E
24°N
25°N
26°N
27°N
28°N
29°N
30°N
31°N
32°N
33°N
34°N
35°N
36°N

TABLE 16-I—SEISMIC ZONE FACTOR Z

ZONE	1	2A	2B	3	4
Z	0.075	0.15	0.20	0.30	0.40

NOTE: The zone shall be determined from the seismic zone map in Figure 16-2.

TABLE 16-J—SOIL PROFILE TYPES

SOIL PROFILE TYPE	SOIL PROFILE NAME/GENERIC DESCRIPTION	AVERAGE SOIL PROPERTIES FOR TOP 100 FEET (30 480 mm) OF SOIL PROFILE		
		Shear Wave Velocity, $\bar{v}_s$ feet/second (m/s)	Standard Penetration Test, $\bar{N}$ [or $\bar{N}_{CH}$ for cohesionless soil layers] (blows/foot)	Undrained Shear Strength, $\bar{s}_u$ psf (kPa)
S_A	Hard Rock	> 5,000 (1,500)	—	—
S_B	Rock	2,500 to 5,000 (760 to 1,500)		
S_C	Very Dense Soil and Soft Rock	1,200 to 2,500 (360 to 760)	> 50	> 2,000 (100)
S_D	Stiff Soil Profile	600 to 1,200 (180 to 360)	15 to 50	1,000 to 2,000 (50 to 100)
S_E[1]	Soft Soil Profile	< 600 (180)	< 15	< 1,000 (50)
S_F	Soil Requiring Site-specific Evaluation. See Section 1629.3.1.			

[1]Soil Profile Type S_E also includes any soil profile with more than 10 feet (3048 mm) of soft clay defined as a soil with a plasticity index, $PI > 20$, $w_{mc} \geq 40$ percent and $s_u < 500$ psf (24 kPa). The Plasticity Index, PI, and the moisture content, w_{mc}, shall be determined in accordance with approved national standards.

TABLE 16-Q—SEISMIC COEFFICIENT C_a

SOIL PROFILE TYPE	SEISMIC ZONE FACTOR, *Z*				
	$Z = 0.075$	$Z = 0.15$	$Z = 0.2$	$Z = 0.3$	$Z = 0.4$
S_A	0.06	0.12	0.16	0.24	$0.32N_a$
S_B	0.08	0.15	0.20	0.30	$0.40N_a$
S_C	0.09	0.18	0.24	0.33	$0.40N_a$
S_D	0.12	0.22	0.28	0.36	$0.44N_a$
S_E	0.19	0.30	0.34	0.36	$0.36N_a$
S_F	See Footnote 1				

[1]Site-specific geotechnical investigation and dynamic site response analysis shall be performed to determine seismic coefficients for Soil Profile Type S_F.

TABLE 16-R—SEISMIC COEFFICIENT C_v

SOIL PROFILE TYPE	SEISMIC ZONE FACTOR, *Z*				
	$Z = 0.075$	$Z = 0.15$	$Z = 0.2$	$Z = 0.3$	$Z = 0.4$
S_A	0.06	0.12	0.16	0.24	$0.32N_v$
S_B	0.08	0.15	0.20	0.30	$0.40N_v$
S_C	0.13	0.25	0.32	0.45	$0.56N_v$
S_D	0.18	0.32	0.40	0.54	$0.64N_v$
S_E	0.26	0.50	0.64	0.84	$0.96N_v$
S_F	See Footnote 1				

[1]Site-specific geotechnical investigation and dynamic site response analysis shall be performed to determine seismic coefficients for Soil Profile Type S_F.