Structural Mechanics (CE- 312)
Cylinders and Pressure Vassals

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CYLINDERS

- Cylinders are generally meant to contain fluids (liquid & gas).
- Cylinders must be strong enough to bear all the stress/pressure subjected by the containing fluid otherwise they will burst.
- In order to make them safe and durable, we should either choose a stronger material on increase the thickness.

Examples of cylinders are

- **Cylindrical vassal**: tankers petroleum and water etc.
- Pipes, conduits or ducts in which fluid is flowing under pressure, a shallow length of these ducts and pips act as a cylinder

Various Types Cylinders and Shells

Types of Cylinders

1. Thin cylinders
2. Thick cylinders

1. THIN CYLINDERS

These are the cylinders which has diameter more than 20 times of the thickness of the wall (or shell).

when a thin cylinder is subjected to internal pressure, three mutually perpendicular stresses are set up in the cylinder material, namely the *Circumferential or Hoop Stress*, *Longitudinal Stress* and *Radial Stress*.

Characteristics/Assumption of Thin Cylinders

The following assumptions are made in order to derive the expression for stress and strains in thin cylinders.

1. The ratio between inside diameter (***d***) and thickness (***t***) is more than 20.

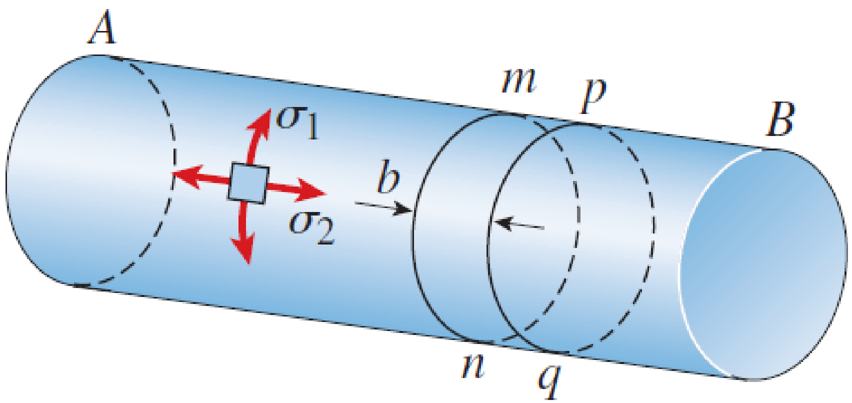
$$d_i / t > 20 \quad or \quad r_i / t > 10$$

$$t < d_i / 20 \quad or \quad t < r_i / 10$$

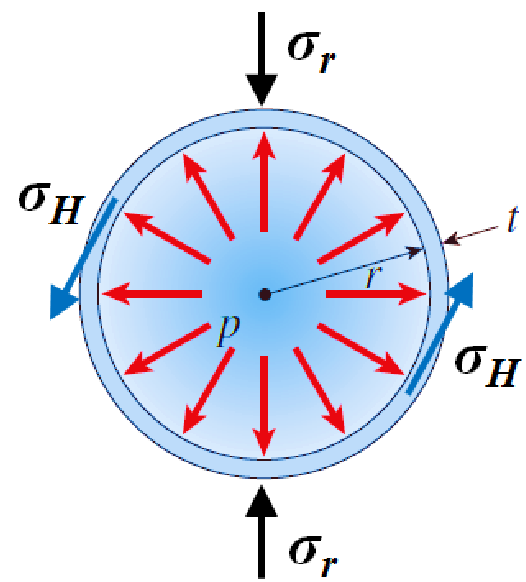
2. For above condition the stresses between the inner and outer surfaces of the wall vary by less than 5% for larger radius this error is even more less.
3. The magnitude of the radial stress is so small in comparison with Hoop and Longitudinal stress that it can be ignored.
4. The stresses (Hoop and Longitudinal) are uniformly distributed through the thickness of wall.
5. The ends of the cylinders are not supported from the sides.
6. The weight of the cylinder and fluid contained inside are not taken into account.
7. The atmosphere pressure is taken as the reference pressure

STRESSES IN THIN CYLINDERS

Consider a thin seamless cylindrical cylinder of nominal diameter ***d*** and thickness ***t***, contain some fluid at an internal pressure ***p***. The two ends of the cylinder are closed with walls perpendicular to shell or cylinder.



1. *Radial Stress* (σ_r)

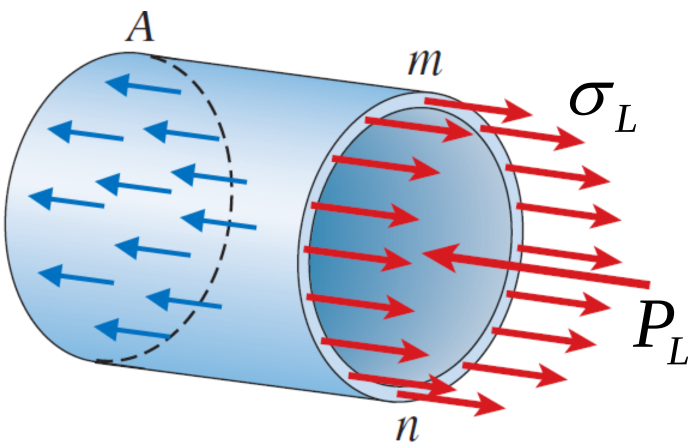


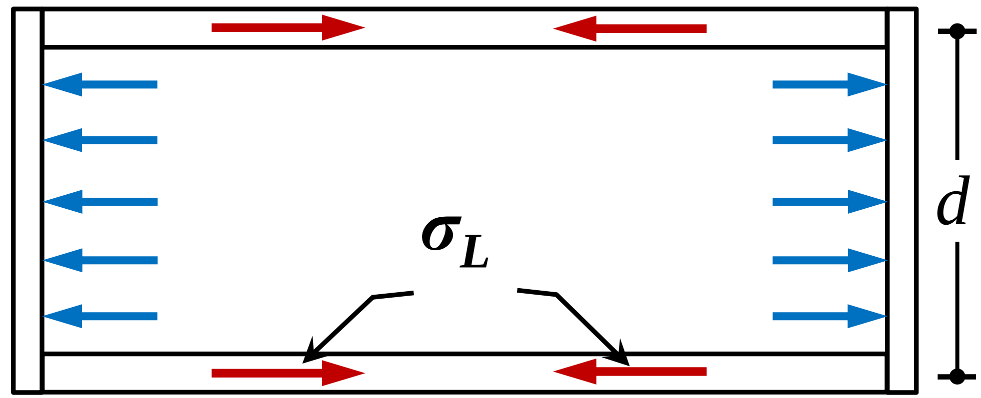
Radial Stress is tangential at any point of the wall thickness of the cylinder and always equal and opposite to the internal pressure p_i.

$$\sigma_r = -p_i$$

2. Longitudinal Stress (σ_L)

Assuming any cutting section along the length, the resisting force acting at cutting wall section must be equal to the force produced by the bursting pressure at the cylinder wall



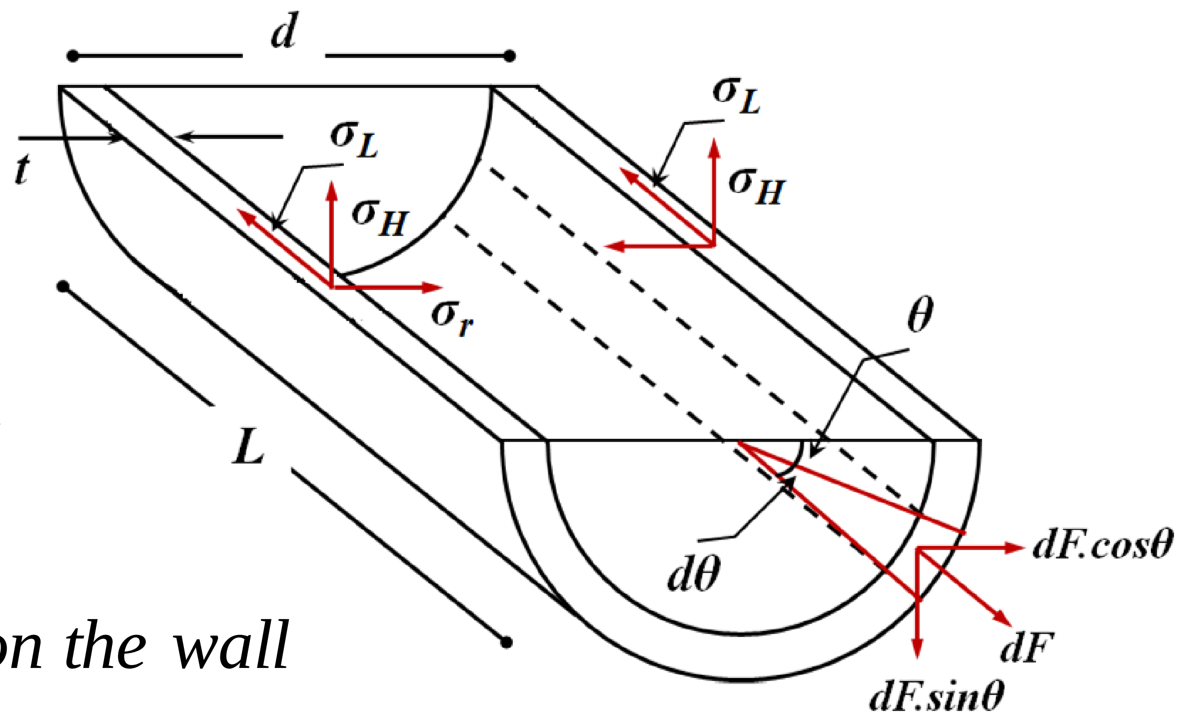


$$Bursting \quad force = Resisting \quad forece$$

$$p.\frac{\pi d^2}{4} = \sigma_L.(\pi dt) \quad \Rightarrow \quad \sigma_L = \frac{pd}{4t}$$

3. Hoop Stress or Transversal Stress (σ_H)

These are also called the circumferential stresses. Assuming a cutting section along/parallel to the cylinder length from the center. Transversal force at cutting section must be balanced by the vertical component of the force induced by the internal pressure.



$$dF = Pressure \times Area$$

$$dF = p(rd\theta \times L)$$

$$dF = Normal\ \ force\ on\ the\ wall$$

*Horizontal components of the force (**dF**) will be balanced by the components on the opposite sides*

Total resultant vertical force can be calculated as following

$$F_y = \int_0^{\pi} dF \sin\theta = \int_0^{\pi} (prLd\theta)\sin\theta$$

$$F_y = pLr\int_0^{\pi} \sin\theta d\theta = pL\frac{d}{2}\left|-\cos\theta\right|_0^{\pi}$$

$$F_y = pL\frac{d}{2}(2) = pLd$$

Alternatively

Force induced by internal pressure

F_y = *Pressure* x *Projected Area*

= p x $d.L$ = pdL

The resisting force perpendicular to the cutting wall section is given as

$$Resisting \quad force = (\sigma_H \times L.t)2$$

Applied Force = Resisting Force

$$F_y = Resisting \quad force$$

$$pLd = (\sigma_H \times L.t)2$$

$$\sigma_H = \frac{pd}{2t}$$

EFFECT OF END PLATES AND JOINTS

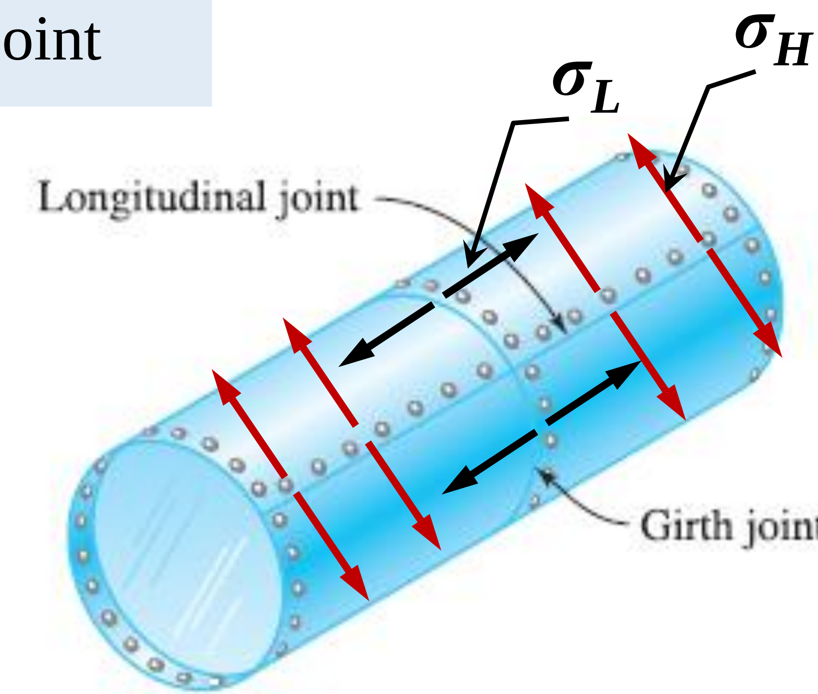
In general the strength of components reduce with the presence of joint and end plates. The effect of joint is taken into account by considering the joint efficiency factor into the equations.

η_L = Efficacy of longitudinal joint

η_c = Efficacy of circumferential joint

$$\sigma_H = \frac{pd}{2t\eta_L}$$

$$\sigma_L = \frac{pd}{4t\eta_c}$$



Example Problem # 1

A thin cylindrical vessel of 2.0 m diameter and 4.0 m length contains a particular gas at a pressure of 1.65 N/mm^2. If the permissible tensile stress of the material of the shell is 150 N/mm^2, find the maximum thickness required.

Data

Permissible tensile stress, σ_{all} = 150 N/mm^2 (Mpa)

L = 4.0 m

d = 2.0 m

p = 1.65 N/mm^2

t = ?

Example Problem # 2

A cylindrical compressed air drum is 2.0 m in diameter with plates 12.5 mm thick. The efficiencies of the longitudinal (η_L) and circumferential (η_c) joints are 85% and 45% respectively. If the tensile stress in the plating is to be limited to 100 MPa, find the maximum safe air pressure.

Data

Permissible tensile stress, $\boldsymbol{\sigma_{all}}$ = 100 MPa

L = 4.0 m $\qquad$ η_L = 85 %

t = 12.5 mm $\qquad$ η_c = 45 %

p = ?

CHANGE IN CYLINDER DIMENSION

a) Change in Length

The change in length of cylinder may be determined from the total longitudinal strain, neglecting the radial strain.

$$Longitudinal\ strain = \varepsilon_L - \nu.\varepsilon_H = \frac{1}{E}(\sigma_L - \nu\sigma_H)$$

$$Change\ in\ length = Longitudinal\ strain + Original\ length$$

$$\delta_L = \frac{1}{E}(\sigma_L - \nu\sigma_H) \times L$$

$$\delta_L = \frac{1}{E}\left(\frac{pd}{4t} - \nu\frac{pd}{2t}\right) \times L$$

$$\delta_L = \frac{pd}{4tE}(1 - 2\nu) \times L$$

b) Change in Diameter

The change in diameter of cylinder is given as following

Change in Diameter = Diameter Strain x Original Diameter

Now the change in the diameter may be determined from the consideration of circumferential change. The stress acting around a circumference σ_H or *Hoop Stress* gives rise to the *Circumferential* or *Hoop Strain* , ε_H.

$$Change\ in\ circumference = Circumferencial\ \ strain \times Original\ \ circumference$$

$$Change\ in\ circumference = \varepsilon_H \times \pi d$$

$$New\ circumference = Original\ \ circumference \times Change\ in\ circumference$$

$$New\ circumference = \pi d + \varepsilon_H \times \pi d = \pi d(1+\varepsilon_H)$$

New diameter can be obtained dividing the *new circumference* by π.

$$New\ diameter = d(1+\varepsilon_H)$$

$$Change\ in\ diameter = d \times \varepsilon_H$$

$$Original\ diameter = d$$

$$Diameter\ Strain\ ,\varepsilon_d = \frac{d\varepsilon_H}{d} \Rightarrow \varepsilon_d = \varepsilon_H$$

Therefore,

$$Change\ in\ diameter = Diameter\ strain \times Original\ diameter$$

$$Change\ in\ diameter = \left(\varepsilon_H - \nu\varepsilon_L\right) \times d$$

$$\delta_d = \frac{1}{E}\left(\sigma_H - \nu\sigma_L\right) \times d = \frac{1}{E}\left(\frac{pd}{2t} - \nu\frac{pd}{4t}\right) \times d$$

$$\delta_d = \frac{pd}{4tE}\left(2 - \nu\right) \times d$$

c) Change in Internal Volume

The change in internal volume may be determined by the following expression.

$$Volumetric\ Strain = \frac{Change\ in\ volume}{Original\ volume} \Rightarrow \varepsilon_v = \frac{\delta V}{V}$$

$$Volum\ of\ Cylinder = \frac{\pi d^2}{4} \times L$$

$$\text{Change in volume}, \delta V = \frac{\partial V}{\partial d} + \frac{\partial V}{\partial L} = \frac{\partial}{\partial d}\left(\frac{\pi d^2}{4} \times L\right) + \frac{\partial}{\partial L}\left(\frac{\pi d^2}{4} \times L\right)$$

$$\delta V = \frac{\pi}{4} L.2d\delta d + \frac{\pi d^2}{4}\delta L$$

$$\text{Thus} \qquad \varepsilon_V = \frac{\delta V}{V} = \frac{\frac{\pi}{4} L.2d\delta d + \frac{\pi d^2}{4}\delta L}{\frac{\pi d^2}{4} \times L} = \frac{2\delta d}{d} + \frac{\delta L}{L}$$

$$\varepsilon_V = 2\varepsilon_H + \varepsilon_L$$

Substituting the value of $\boldsymbol{\varepsilon_H}$ and $\boldsymbol{\varepsilon_L}$

$$\varepsilon_V = 2\frac{pd}{4tE}(2-\nu) + \frac{pd}{4tE}(1-2\nu)$$

$$\varepsilon_V = \frac{pd}{4tE}(4-2\nu+1-2\nu) = \frac{pd}{4tE}(5-4\nu)$$

$$\therefore \qquad \varepsilon_H = \frac{pd}{4tE}(2-\nu) \times d$$

$$\therefore \qquad \varepsilon_L = \frac{pd}{4tE}(1-2\nu) \times L$$

Example Problem # 3

A cylindrical shell, 0.8 m in a diameter and 3 m long is having 10 mm wall thickness. If the shell is subjected to an internal pressure of 2.5 N/mm^2, determine

(a) change in diameter,

(b) change in length, and

(c) change in volume.

Take E = 200 GPa and Poisson's ratio = 0.25.

Data

Diameter of the shell, d = 0.8 m = 800 mm.

Thickness of the shell, t = 10 mm.

Internal pressure, p = 2.5 N/mm^2.

δd, δL *and* δV = ?

VESSELS SUBJECTED TO FLUID PRESSURE

The fluid change in volume as the pressure is increased, which must be taken into account while calculating the amount of fluid which must be pumped into the cylinder to raise the pressure by a specified amount.

Now, the bulk modulus, ***K*** is defined as following

$$Bulk\ Modulus\ , K = \frac{Volumetric\ stress}{Volumetric\ strain}$$

$$K = \frac{p}{\varepsilon_V} = \frac{p}{\delta V / V} = \frac{pV}{\delta V}$$

$$Change\ (reduction)\ in\ Fluid\ Volume\ under\ pressure = \delta V = \frac{pV}{K}$$

The extra fluid require to raise the pressure must take up this volume together with the increase in the internal volume of the cylinders.

Extra Fluid Require to raise the cylinder pressure by ***p*** = Increase in the volume of cylinder + Change (reduction) in the fluid volume

Extra Fluid Require to raise the cylinder pressure by ***p***

$$= \left[2\varepsilon_H + \varepsilon_L\right].V + \frac{pV}{K}$$

$$= \frac{pd}{4tE}\left[5 - 4\nu\right].V + \frac{pV}{K}$$

Example Problem # 4

A copper tube of 50 mm diameter and 1200 mm length has a thickness of 1.2 mm with closed ends. It is filled with water at atmospheric pressure. Find the increase in pressure when an additional volume of 32 cc of water is pumped into the tube. Take *E* for copper = 100 GPa, Poisson's ratio = 0.3 and *K* for water = 2000 N/mm^2.

STRESSES IN SPHERICAL CYLINDERS

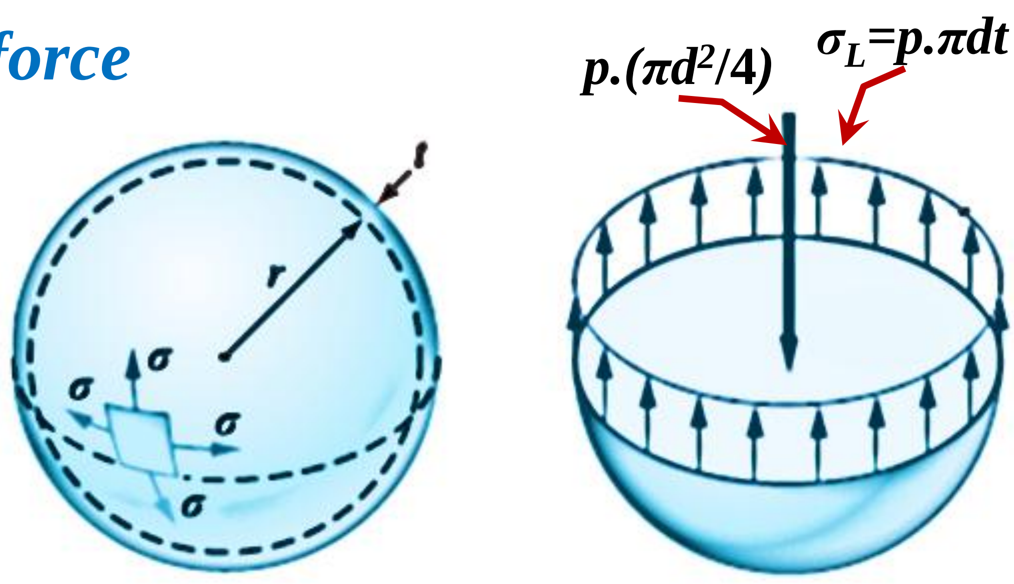
Because of the symmetry of the sphere, two mutually perpendicular *hoop or circumferential stresses* (**σ_H**) of equal value and a *radial stress* will be set up owing to internal pressure. For the thin spherical cylinder (**$t < d_i/20$**) radial stress is neglected. Thus, the stress system is one of equal biaxial hoop stresses.

Considering, the equilibrium of the half-sphere

Bursting force = Resisting force

$$p.\frac{\pi d^2}{4} = \sigma_H.(\pi dt)$$

$$\sigma_H = \frac{pd}{4t}$$



CHANGE IN VOLUME OF SPHERICAL CYLINDER

As for the cylinder,

Change in volume = Original volume x *Volumetric strain*

But,

Volumetric strain = Sum of three mutually perpendicular strains (in this case all equal)

$$Volumetric\ \ strain \quad = \quad 3\varepsilon_d = 3\varepsilon_H$$

$$Volumetric\ \ strain \quad = \quad \frac{3}{E}\left(\sigma_H - \nu\sigma_H\right) \qquad\qquad \therefore \sigma_L = \sigma_H$$

$$\varepsilon_V \quad = \quad \frac{3pd}{4tE}\left(1-\nu\right)$$

$$Change\ in\ volume = Volumetric\ \ strain + Original\ volume$$

$$\varepsilon_V \quad = \quad \frac{3pd}{4tE}\left(1-\nu\right) \times V$$

$$Volume\ of\ sphere, V = \frac{4}{3}\pi r^3 = \frac{4}{24}\pi d^3$$

Example Problem # 5

(a) A sphere, **1 m internal diameter** and **6mm wall thickness**, is to be pressure-tested for safety purposes with water as the pressure medium. Assuming that the sphere is initially filled with water at atmospheric pressure, what extra volume of water is required to be pumped in to produce a pressure of **3 MPa** gauge? For water, ***K* = 2.1 GPa**.

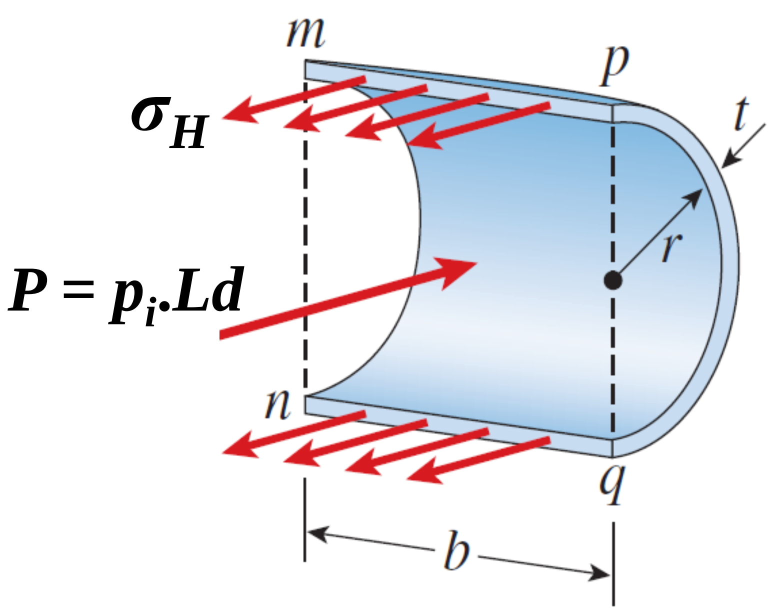
(b) The sphere is now placed in service and filled with gas until there is a **volume change of 72×10^{-6} m^3**. Determine the pressure exerted by the gas on the walls of the sphere.

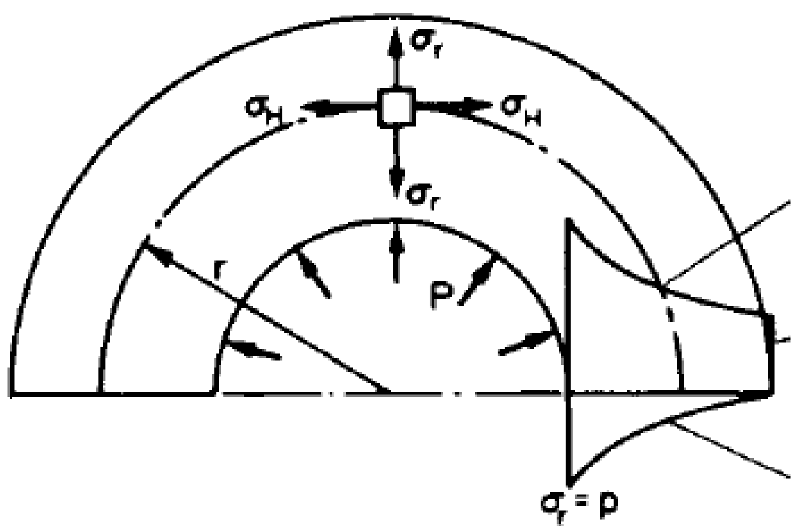
(c) To what value can the gas pressure be increased before failure occurs according to the maximum principal stress theory of elastic failure? For the material of the sphere **E = 200 GPa**, ***v* = 0.3** and the yield stress σ_y, in simple tension = **280 MPa.**

Thick Cylinders

Difference between thin and thick cylinders		
	Thin Cylinders	**Thick Cylinders**
1	The wall thickness is less than one-tenth (1/10) of inner radius of cylinder	The wall thickness is more than or equal one-tenth (1/10) of inner radius of cylinder.
2	The radial (shear) stress is neglected.	The radial (shear) stress is considered.
3	The hoop stress is assumed to be uniformly distributed over the wall thickness.	The hoop stress varies parabolic ally over the wall thickness.
4	**Examples:** Tires, gas and water storage tank.	**Examples:** Gun barrels, high pressure vassal in oil-refining industries.

	Thin Cylinders	Thick Cylinders
5	Analytical treatment is simple and approximate.	Analytical treatment is complicate and accurate.
6	Thin cylinders are statically determinate.	Thick cylinders are statically indeterminate.
7	State of the stress is ***Membrane***.	State of the stress is ***Triaxial***.





STRESSES IN THIN CYLINDERS

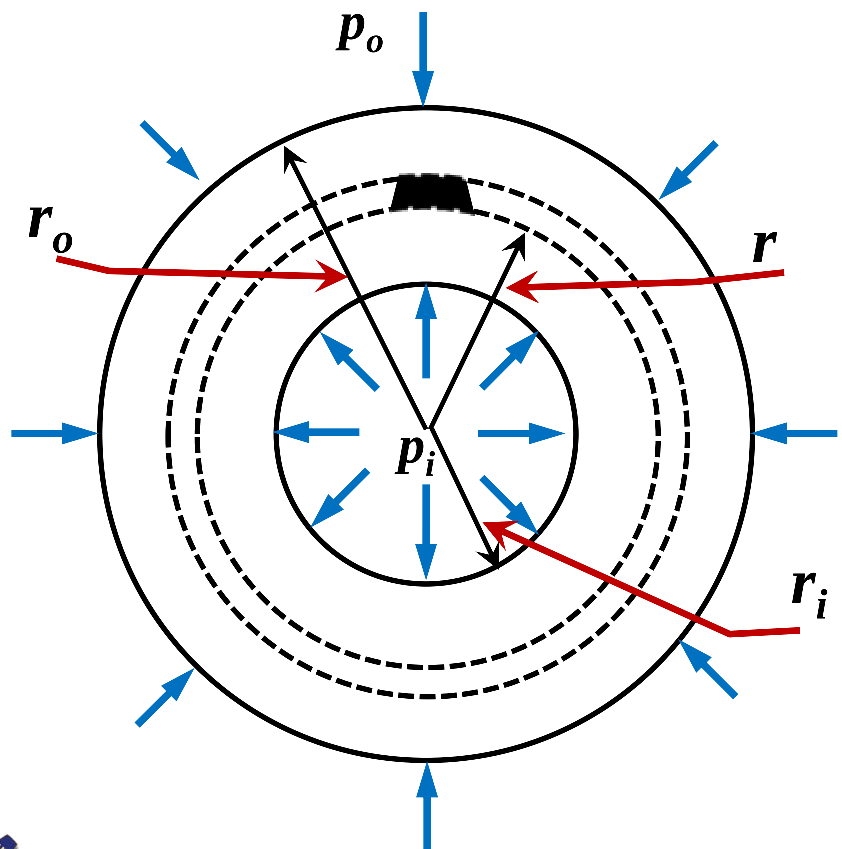
Assumption

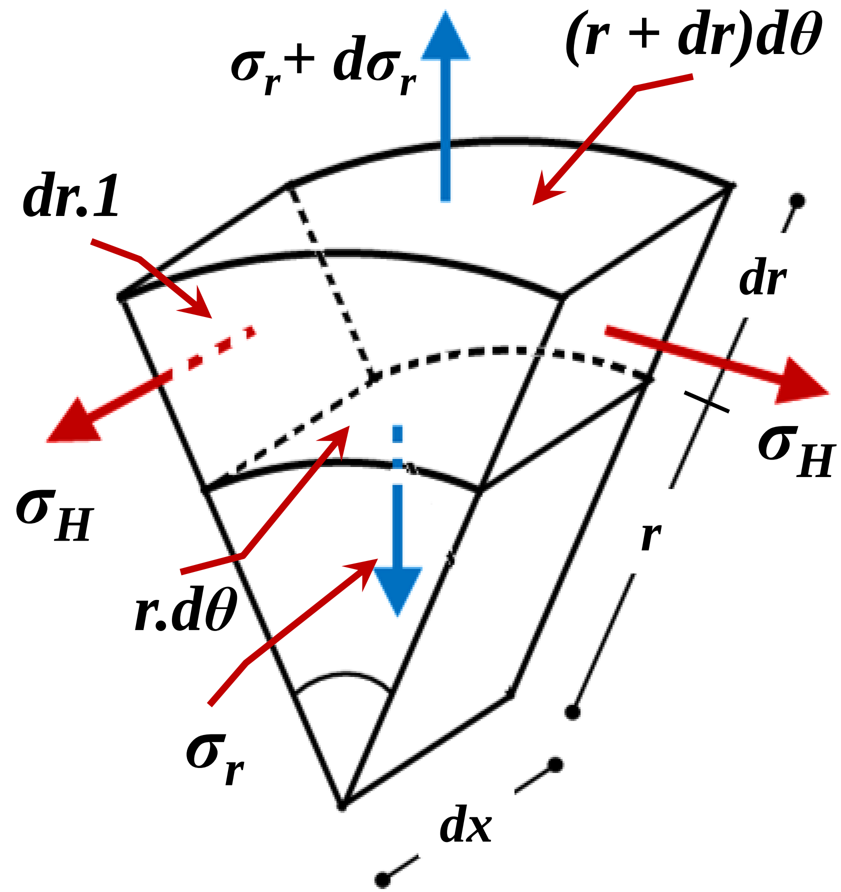
1. The ratio between inside diameter (***d***) and thickness (***t***) is less than 20.
2. The material of cylinder is homogeneous and isotropic.
3. plane sections perpendicular to the longitudinal axes of cylinder remain plane even after the application of the internal pressure. The implies that the longitudinal strain is same at all points of the cylinder.
4. All fibers of material are free to expand or contract independently without being confined by adjacent fibers.

GABRIAL LAME THEORY

Consider a thick walled open ends cylinder. It is loaded by internal pressure p_i and external pressure p_o. It has inner radius r_i and outer radius r_o.

In order to derive the expression for internal stresses, consider an annular cylinder of radius ***r***, radial thickness ***dr*** and longitudinal thickness ***dx* =1**. on any small element of the ring, σ_r and σ_H will be the radial and hoop stresses, respectively.





$$\sum Fy = 0$$

$$(\sigma_r + d\sigma_r)(r + dr)d\theta - \sigma_r(rd\theta.1) - 2\sigma_H(dr.1)\sin(d\theta/2) = 0$$

$$\sigma_r.r + \sigma_r.dr + d\sigma_r.r + d\sigma_r dr - \sigma_r.r - \sigma_H.dr = 0$$

$$\sigma_r.dr + d\sigma_r.r - \sigma_H.dr = 0$$

$$\frac{d\sigma_r}{dr}.r + \sigma_r - \sigma_H = 0 \qquad (1)$$

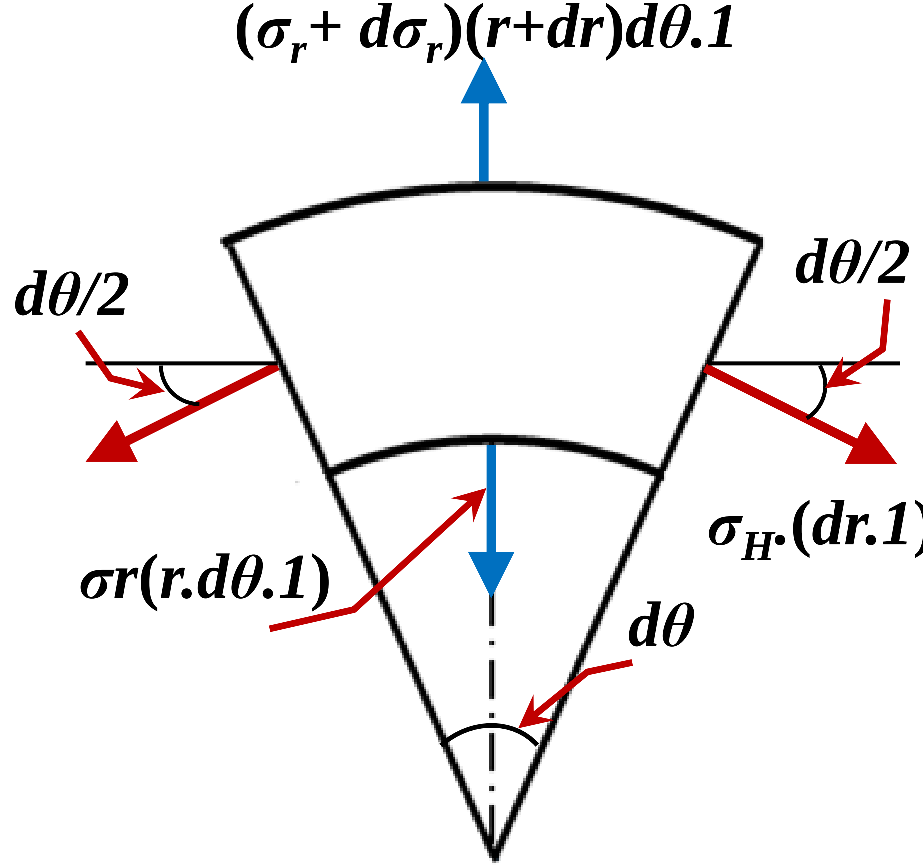
For very small angle

$$\sin\frac{d\theta}{2} = \frac{d\theta}{2} \qquad (radian)$$

product of the derivative

also neglected

i.e, $\quad d\sigma_r.dr = 0$



Considering the assumption that plane sections remain plane. *i.e.*, longitudinal strain is constant.

$$Longitudinal\ strain = \varepsilon_L - \nu\varepsilon_H - \nu\varepsilon_r = \frac{1}{E}\left[\sigma_L - \nu\sigma_H - \nu\sigma_r\right]$$

$$Longitudinal\ strain = \frac{1}{E}\left[\sigma_L - \nu\left(\sigma_H + \sigma_r\right)\right] = Constant$$

As ***υ*** and ***E*** are constant for a given material, thus

$$\sigma_H + \sigma_r = 2A = Constant\,(Lame\ Constant)$$

$$\sigma_H = 2A - \sigma_r \qquad (2)$$

Substituting the value of $\boldsymbol{\sigma_H}$ in Eqn. (1)

$$(1) \Rightarrow \quad \frac{d\sigma_r}{dr}.r + \sigma_r - \left(2A - \sigma_r\right) = 0$$

$$\frac{d\sigma_r}{dr}.r + 2\sigma_r = 2A$$

$$\frac{d\sigma_r}{dr}.r = 2\left(A - \sigma_r\right)$$

$$\frac{d\sigma_r}{\left(A - \sigma_r\right)} = \frac{2dr}{r}$$

Integrating

$$\ln\left(A - \sigma_r\right)\left(-1\right) = 2\ln r + C$$

$$-\ln\left(A - \sigma_r\right) = \ln r^2 + C$$

$$\frac{d\sigma_r}{(A-\sigma_r)} = \frac{2dr}{r}$$

Integrating

$$\ln(A-\sigma_r)(-1) = 2\ln r + C$$

$$\ln(A-\sigma_r) + \ln r^2 = -C$$

$$\ln(A-\sigma_r)r^2 = -C$$

$$(A-\sigma_r)r^2 = e^{-C}$$

$$\therefore \quad \ln x = \log_e x = a$$

$$e^a = x$$

$$\therefore \quad \ln x = \log_e x = a \quad \Rightarrow \quad e^a = x$$

Let

$$e^{-C} = constant = B$$

$$(A-\sigma_r)r^2 = B$$

$$\sigma_r = A - \frac{B}{r^2} \qquad (3)$$

$$(2) \Rightarrow$$

$$\sigma_H = A + \frac{B}{r^2} \qquad (4)$$

The Eqn. (3) and (4) area called the Lame's Equation for radial and hoop stresses, respectively. Constant A & B are compute based on the end/boundary conditions.

It is important to note that radial stress is compressive and hoop stress is tensile in nature. And also their algebraic summation is always a constant over the wall thickness

Example Problem # 6

The internal and external diameters of a ***thick hollow cylinder*** are 80 mm and 120 mm respectively. It is subjected to an external pressure of 40 N/mm^2 and an internal pressure of 120 N/mm^2. Calculate the circumferential stress at the external and internal surfaces and determine the radial and circumferential stresses at the mean radius.

Data

d_i = 80 mm , d_o = 120 mm

p_i = 120 N/mm^2 , p_o = 400 N/mm^2

$(\sigma_H)_o$, $(\sigma_H)_i$ *and* $(\sigma_H)_{mean}$ = ?

$(\sigma_r)_{mean}$ = ?

Example Problem # 7

The cylinder of a hydraulic press has an internal diameter of 0.3 m and is to be designed to withstand a pressure of 10 MPa without the material being stressed over 20 MN/m^2. Determine the thickness of the metal and the hoop stress on the outer side of the cylinder.

Data

d_i = 0.3 m = 300 mm

σ_{all} = 20 MPa

p_i = 10 MPa ,

Thickness , t = ?

$(\sigma_H)_o$ = ?